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New deformed Heisenberg algebra from the μ-deformed model of dark matter

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Recently, the μ-deformation-based approach to modeling dark matter, which exploits μ-deformed thermodynamics, was extended to the study of galaxy halo density profile and of the rotation curves of a number of (dwarf or low brightness) galaxies. For that goal, μ-deformed analogs of the Lane–Emden equation (LEE) have been proposed, and their solutions describing density profiles obtained. There are two seemingly different versions of μ-deformed LEE which possess the same solution, and so we deal with their equivalence. From the latter property we derive new, rather unusual, μ-deformed Heisenberg algebra (HA) for the position and momentum operators, and present the μ-HA in few possible forms (each one at $\mu \to 0$ recovers usual HA). The generalized uncertainty relation linked with the new μ-HA is studied, along with its interesting implications including the appearance of the quadruple of both maximal and minimal lengths and momenta.

KEYWORDS

deformed BEC model of dark matter, deformed lane-emden equation, deformed heisenberg algebra, generalized uncertainty relation, maximal/minimal length uncertainty

1 Introduction

The suggestion of the existence of minimal nonzero (uncertainty of) length linked with generalized uncertainty principle (GUP) or relation (GUR) has been advanced in the context of string theory and quantum gravity (Gross and Mende, 1988; Amati et al., 1989; Adler and Santiago, 1999; Scardigli, 1999; Maziashvili, 2006), see also (Chang et al., 2011) and the reviews (Garay, 1995; Hossenfelder, 2013). It was shown to follow from modified or deformed

Extension (Kempf et al., 1995) of the Heisenberg algebra (HA). It is worth to mention that the concept of maximum observable momenta can play as well important role, see, e.g., (Ali et al., 2009). Such a quantity was predicted, in particular, within the doubly special relativity theory suggesting rather simple (with terms linear and quadratic in momentum) modification of the right hand side of commutators (Magueijo and Smolin, 2002; Magueijo and Smolin, 2005). Further it became clear that besides such a minimal extension of the original HA, a lot of generalizations are possible, suggesting diverse ways to generalize (or deform) the HA. As a tools to classify diverse forms of deformed HA, the concept of deformation function(s) is of importance, see, e.g., (Saavedra and Utreras, 1981; Jannussis, 1993; Gavrilik et al., 2010; Dorsch and Nogueira, 2012; Maslowski et al., 2012; Gavrilik and Kachurik, 2016a). Clearly, the choice of such function must determine the corresponding GUR. As usual, most of the authors deal with position-momentum commutation relations of deformed HA that involve particular function of X, p and deformation parameter(s) in its right hand side (Jannussis, 1993; Dorsch and Nogueira, 2012; Maslowski et al., 2012).

It is also possible that both the right and left hand sides of defining commutation relation are appropriately deformed (Gavrilik and Kachurik, 2012), although such approach may be overlapping with the case of standard commutator and the terms containing XP and PX in its right-hand side as it was considered in (Quesne and Tkachuk, 2007).

In the present paper, a special form of deformed HA will be derived in the context of the so-called μ-deformation based approach aimed to model (Gavrilik et al., 2018; Gavrilik et al., 2019) basic properties of dark matter that surrounds dwarf galaxies, and its consequences analyzed.

The case of GUR related with the minimal $(\Delta X)_{\min}$ is the best known and well studied one. In relation with this, due to the conjugated roles of position and momentum, the concept of $(\Delta P)_{\min}$ has appeared. As it was demonstrated in (Kempf, 1997), a single theory—single extended or generalized HA (GHA) and the corresponding GUR do exist which can jointly accommodate the both special quantities, $(\Delta X)_{\min}$ and $(\Delta P)_{\min}$.

Then, an interesting question arises whether it is possible that the opposite concept of *maximal* uncertainties for the momentum and/or the position does exist. Quite recently, it was shown in some papers that such a possibility indeed can be realized (Pedram, 2012; Perivolaropoulos, 2017; Bensalem and Bouaziz, 2019; Skara and Perivolaropoulos, 2019; Hamil and Lutfuoglu, 2021; Bensalem and Bouaziz, 2022; Pramanik, 2022). Moreover, as a generalization of the already mentioned unified treatment of A. Kempf, in the work (Perivolaropoulos, 2017) of L. Perivolaropoulos, it was explicitly shown that one can provide a theory (based on appropriate generalization of HA) which incorporates the whole quadruple of $(\Delta X)_{\min}$, $(\Delta P)_{\min}$, $(\Delta P)_{\max}$, and $(\Delta X)_{\max}$.

Usual treatments in the most of papers are in a sense model-independent, implying a kind of universality. That means, physical meaning of $(\Delta X)_{\min}$, $(\Delta P)_{\min}$, $(\Delta P)_{\max}$, and $(\Delta X)_{\max}$ is rather universal and depends on Planck length or its inverse, i.e., Planck energy scale (Planck mass).

On the contrary, our treatment is based on (related with) special deformed HA deduced in the framework of particular model of dark matter. It is remarkable that all the four quantities: $(\Delta X)_{\min}$, $(\Delta P)_{\min}$, $(\Delta P)_{\max}$, and $(\Delta X)_{\max}$ do appear. So it is clear and natural that the physical meaning of this quadruple is tightly linked with physics of the model, i.e., with properties of the halo of DM hosted by dwarf galaxies.

For our case (connection with DM) some motivation was due to the work (Perivolaropoulos, 2017), since therein the cosmology-related uncertainty relation was explored, along with clear meaning of maximal length: as suggested in (Perivolaropoulos, 2017), this quantity can be naturally interpreted as *cosmological horizon*.

The uncertainty relation in its initial form due to Heisenberg is linked with the standard commutation relation and *is shared* by different states. Unlike, for all the deformed versions of HA, explicit dependence of GUR on particular state does appear—for deformed oscillators this was noticed in the pioneer papers (Biedenharn, 1989; Macfarlane, 1989). In our present paper, just this fact/property is in the focus and exploited to full extent.

Unlike the approach perceived in (Harko, 2011) and some other papers also exploring galaxy rotation curves with the use of the well-known Lane–Emden equation (LEE), in our line of research we deal with the (μ-)Bose-condensate model of dark matter (Gavrilik et al., 2018), and with such tool as μ-deformed analogs (Gavrilik et al., 2019) of LEE. In general, as it is well-known, deformation of an object under study is not unique, and in (Gavrilik et al., 2019) we encountered two different possible forms of μ-deformed LEE, with the corresponding different sets of solutions, one of which being the deformed function $\sin_\mu(kr)/(kr)$. In the present work, the third form of LEE will be introduced that nevertheless possesses the indicated solution as well. Just from the requirement of equivalence of two seemingly different deformed versions of LEE, the new μ-deformed HA can be deduced and its basic properties and consequences explored.

The paper is structured as follows. In **Section 2**, some basics of μ-deformation and μ-deformed calculus are presented. In **Section 3.1** we describe relevant deformed analogs of LEE and, from the condition of their equivalence, obtain the μ-analog of HA which is the central object of this work. The corresponding GUR which involves the parameter μ is derived, and its main properties are explored in **Section 3.2**, including the appearance of minimal and maximal uncertainties of both position and momentum. **Section 3.3** is devoted to discussion of implications of these quantities for dark matter. The paper is ended with concluding remarks.

2 Deformed functions and calculus

2.1 Basis functions

The so-called μ-bracket of a number or operator X,

$$[X]_\mu \equiv \frac{X}{1+\mu X}; \qquad [X]_\mu \to X, \text{ if } \mu \to 0, \tag{1}$$

and the related μ-deformed oscillator have been introduced 3 decades ago in (Jannussis, 1993). More recently, there appeared some papers (Gavrilik et al., 2010; Gavrilik and Mishchenko, 2012; Gavrilik et al., 2013) in which the μ-deformation based approach was initiated and developed.

For our purposes we define the μ-deformed trigonometric function (see (Gavrilik et al., 2013; Gavrilik et al., 2019) and references therein) as

$$\sin_\mu x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{[2n+1]_\mu!}, \qquad \cos_\mu x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{[2n]_\mu!}, \tag{2}$$

where $[n]_\mu! = [1]_\mu\, [2]_\mu \ldots [n]_\mu$. Clearly, at $\mu \to 0$ one recovers customary sine and cosine.

For our purposes, we introduce the μ-deformed analogs of spherical Bessel functions, namely,

$$j_0^{(\mu)}(x) = \frac{\sin_\mu x}{x}, \qquad y_0^{(\mu)}(x) = \frac{\cos_\mu x}{x}. \tag{3}$$

At $\mu = 0$ these reduce to the familiar Bessel functions.

The physical motivation for introducing these functions is two-fold: the first one in Eq. (3) describes the density profile of the dark matter halo and also leads to the rotation curves within the μ-deformed extension (Gavrilik et al., 2019) of the Bose-condensate model, while both functions, taken jointly, are of importance for constructing the representation space of the position and momentum operators, see **Sections 3.1–3.2** below.

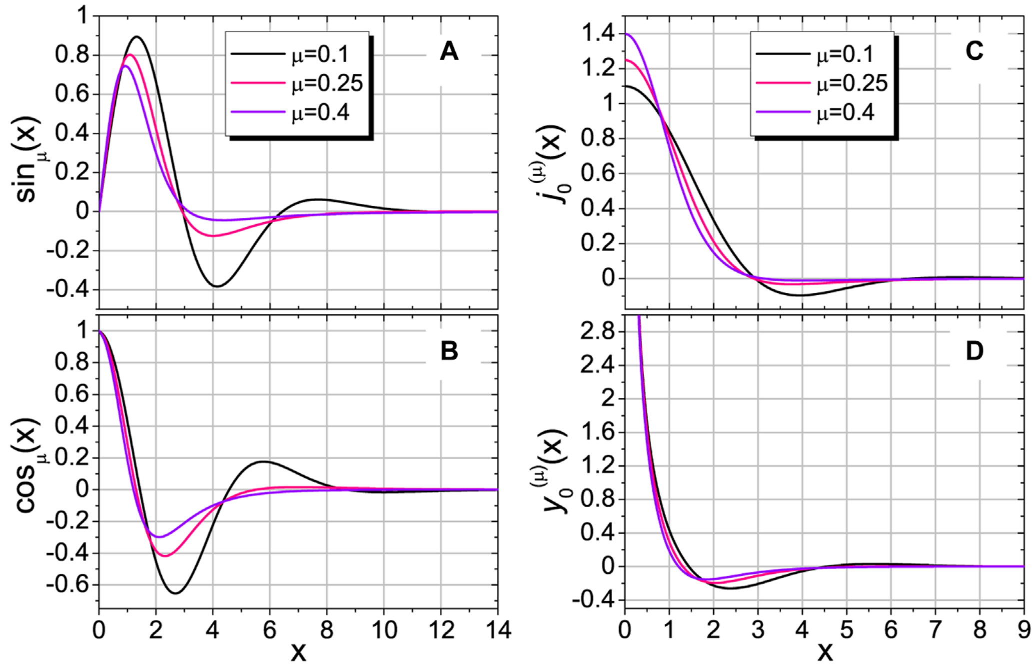

FIGURE 1
Basic μ-deformed trigonometric **(A, B)** and spherical **(C, D)** functions.

Since the applied deformation concerns mainly the basic trigonometric functions, let us study $\sin_\mu x$ and $\cos_\mu x$ in detail. Contracting the corresponding series to the Gaussian hypergeometric function, we can then represent them in the analytic form

$$\sin_\mu x = I(x,\mu)\sin\varphi(x), \qquad \cos_\mu x = I(x,\mu)\cos\varphi(x) \tag{4}$$

where

$$I(x,\mu) \equiv \left(1+\mu^2x^2\right)^{-\frac{1+\mu}{2\mu}}, \qquad \varphi(x) \equiv \frac{1+\mu}{\mu}\arctan(\mu x). \tag{5}$$

Therefore, in the case of μ-deformation, the main trigonometric identity is written as follows:

$$\sin^2_\mu x + \cos^2_\mu x = I^2(x,\mu), \qquad I(x,\mu) \le 1. \tag{6}$$

The behavior of the μ-deformed trigonometric and spherical functions is shown in Figure 1.

In principle, it is possible to express the deformed trigonometric functions in terms of the μ-deformed exponential function. The μ-analogs of exponential and logarithmic functions are

$$\mathrm{e}_\mu(x) = (1-\mu x)^{-\frac{1+\mu}{\mu}}, \qquad \ln_\mu(x) = \frac{1}{\mu}\left(1 - x^{-\frac{\mu}{1+\mu}}\right), \tag{7}$$

which give us the known functions at $\mu \to 0^-$ due to the asymptotic formulas:

$$\mathrm{e}(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n, \qquad \ln(x) = \lim_{n\to\infty} n\left(x^{1/n}-1\right). \tag{8}$$

Note that the μ-deformed functions exhibit a non-trivial property at $\mu > 0$:

$$\begin{aligned} \left(\mathrm{e}_\mu(x)\right)^n &= \mathrm{e}_\mu\left(\frac{1-(1-\mu x)^n}{\mu}\right), \\ \ln_\mu(x^n) &= \frac{1-\left(1-\mu\ln_\mu(x)\right)^n}{\mu}. \end{aligned} \tag{9}$$

Focusing on the problems with spherical symmetry, we need to define an inner product $\langle f|g\rangle$ in terms of which the real functions $u_1(x) = j_0^{(\mu)}(x)$ and $u_2(x) = y_0^{(\mu)}(x)$ become orthonormal on finite interval $x \in [0;\ R(\mu)]$:

$$\langle f|g\rangle = \int_0^{R(\mu)} f^*(x)\,g(x)\,w_\mu(x)\,\mathrm{d}x, \qquad \langle u_i|u_j\rangle = \delta_{i,j}, \tag{10}$$

where the asterisk means complex conjugation; the Latin indexes i, j run from 1 to 2.

For the orthogonality of $\sin\varphi$ and $\cos\varphi$ on the interval $\varphi \in [0;\ \pi]$, we constitute *ad hoc*

$$w_\mu(x)\,\mathrm{d}x = \frac{2}{\pi}x^2I^{-2}(x,\mu)\,\mathrm{d}\varphi(x), \qquad \pi = \varphi(R(\mu)), \tag{11}$$

and obtain

$$w_\mu(x) = \frac{2(1+\mu)}{\pi}x^2\left(1+\mu^2x^2\right)^{\frac{1}{\mu}}, \qquad R(\mu) = \frac{1}{\mu}\tan\frac{\mu\pi}{1+\mu}, \tag{12}$$

where $R(\mu)$ coincides with the first zero of $\sin_\mu x$.

Expanding these as.

$$w_\mu(x) = \frac{2}{\pi}x^2\left[1+(1+x^2)\mu + \left(x^2+\frac{x^4}{2}\right)\mu^2 + O(\mu^3)\right], \tag{13}$$

$$R(\mu) = \pi - \pi\mu + \left(\pi + \frac{\pi^3}{3}\right)\mu^2 + O(\mu^3), \tag{14}$$

We see that the known quantities are restored at $\mu = 0$.

TABLE 1 The μ-deformed derivatives.

	f(x)	$D_x^{(\mu)} f(x)$
1	x^n	$[n]_\mu x^{n-1}$
2	$\mathrm{e}_\mu(px)$	$p\mathrm{e}_\mu(px)$
3	$\ln_\mu(x)$	$(1+\mu-\mu^2)^{-1}x^{-2+\frac{1}{1+\mu}}$
4	$\sin_\mu(x)$	$\cos_\mu(x)$
5	$\cos_\mu(x)$	$-\sin_\mu(x)$
6	$j_0^{(\mu)}(x)$	$\frac{1+\mu}{1-\mu}\,y_0^{(\mu)}(x) - \frac{1-\mu^2x^2}{(1-\mu)x}\,j_0^{(\mu)}(x)$
7	$y_0^{(\mu)}(x)$	$-\frac{1+\mu}{1-\mu}\,j_0^{(\mu)}(x) - \frac{1-\mu^2x^2}{(1-\mu)x}\,y_0^{(\mu)}(x)$

2.2 Deformed differential calculus

We would like to define the μ-deformed derivative $D_x^{(\mu)}$ with respect to the positive variable x, and its inverse. Let the functions $f(x)$ and $\phi(x)$ admit expansion in the Taylor series and satisfy the relation

$$D_x^{(\mu)} f(x) = \phi(x). \tag{15}$$

The actions of $D_x^{(\mu)}$ and antiderivative $\left(D_x^{(\mu)}\right)^{-1}$ are respectively given as.

$$\phi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[f(x) - x^{-\frac{1}{\mu}}\int_0^x f'(s)\, s^{\frac{1}{\mu}}\,\mathrm{d}s\right], \tag{16}$$

$$f(x) = \mu x\,\phi(x) + \int_0^x \phi(s)\,\mathrm{d}s + f(0), \tag{17}$$

Were the prime means ordinary differentiation.

We see that $\phi(x) = f'(x)$ at $\mu \to 0$ due to vanishing $(s/x)^{1/\mu}$ for $s < x$. By definition, the derivative $D_x^{(\mu)}$ lowers the exponent of the monomial x^n by one, namely, $D_x^{(\mu)} x^n = [n]_\mu x^{n-1}$. However, the operator $D_x^{(\mu)}$ violates the Leibniz rule: $D_x^{(\mu)}(f(x)\,g(x)) \neq g(x)D_x^{(\mu)} f(x) + f(x)D_x^{(\mu)} g(x)$.

The μ-deformed derivatives of some functions are collected in Table 1. To derive expressions 4–7, we have used the known auxiliary integrals:

$$\int \sin^{p-1}x \left\{\begin{matrix} \sin((p+1)x) \\ \cos((p+1)x) \end{matrix}\right\} \mathrm{d}x = \frac{1}{p}\sin^p x \left\{\begin{matrix} \sin(px) \\ \cos(px) \end{matrix}\right\}. \tag{18}$$

On the base of relations 4–5 (not 6–7) in Table 1, we define the Hermitian momentum operator $\hat{P}$ as

$$\hat{P} = -\frac{\mathrm{i}}{x} D_x^{(\mu)} x, \tag{19}$$

so that $\langle u_i|\hat{P}|u_i\rangle = 0$, and $\langle u_1|\hat{P}|u_2\rangle = \langle u_2|\hat{P}^*|u_1\rangle = \mathrm{i}$ (see Eq. 10), using the imaginary unit i. This operator will play an important role in the study of the deformed Heisenberg algebra further on.

To demonstrate the action of $\hat{P}$ on some functions, note that $\hat{P}x^n = -\mathrm{i}[n+1]_\mu x^{n-1}$ for $n \geq 0$, and then

$$\hat{P}\psi_p(x) = p\psi_p(x), \qquad \psi_p(x) = \frac{\mathrm{e}_\mu(\mathrm{i}px)}{x}. \tag{20}$$

In addition, we consider the radial part $\Delta_r^{(\mu)}$ of μ-deformed Beltrami–Laplace operator and its inverse (up to the additive constant $C \sim f(0)$).

$$\Delta_r^{(\mu)} f(r) \equiv \frac{1}{r^2} D_r^{(\mu)}\left(r^2 D_r^{(\mu)} f(r)\right), \tag{21}$$

$$\left(\Delta_r^{(\mu)}\right)^{-1} f(r) = \mu^2 r^2 f(r) + (1+\mu)\int_0^r f(s)\, s\,\mathrm{d}s - \frac{1-\mu}{r}\int_0^r f(s)\, s^2\,\mathrm{d}s + C. \tag{22}$$

It is easy to verify for positive n that

$$\Delta_r^{(\mu)} r^n = [n]_\mu \cdot [n+1]_\mu\, r^{n-2}; \qquad \left(\Delta_r^{(\mu)}\right)^{-1} r^n = \frac{r^{n+2}}{[n+2]_\mu \cdot [n+3]_\mu}, \quad C = 0. \tag{23}$$

We also verify that

$$\left(\Delta_r^{(\mu)}\right)^{-1} j_0^{(\mu)}(r) + j_0^{(\mu)}(r) = 0, \qquad C = -j_0^{(\mu)}(0), \tag{24}$$

by the use of the integrals

$$\int \cos^{p-1}x \left\{\begin{matrix} \sin((p+1)x) \\ \cos((p+1)x) \end{matrix}\right\} \mathrm{d}x = \frac{1}{p}\cos^p x \left\{\begin{matrix} -\cos(px) \\ \sin(px) \end{matrix}\right\}. \tag{25}$$

3 Deformed Heisenberg algebra and uncertainty principle

3.1 Deformed equations and Heisenberg algebra

Here we are going to present the equations of some models using deformed differential calculus. The main model for us, from which the deformed Heisenberg algebra will follow, is described by the deformation of the Lane–Emden equation (LEE) for finite density function $\rho(r)$ in the two possible formulations.

$$\left(\Delta_r^{(\mu)}\rho(r) + k^2\right)\rho(r) = 0, \tag{26}$$

$$\left(D_r^{(\mu)}D_r^{(\mu)} + g_\mu(r)\frac{2}{r}D_r^{(\mu)} + h_\mu(r)k^2\right)\rho(r) = 0, \tag{27}$$

Where

$$g_\mu(r) = \frac{1}{1-2\mu}\left(1 - \frac{1-\mu}{1+\mu}\mu^2k^2r^2\right), \qquad h_\mu(r) = \frac{1+2\mu}{1-2\mu} - 2\mu^2\frac{1-\mu^2k^2r^2}{(1+\mu)(1-2\mu)}. \tag{28}$$

Note that the version Eq. 26 of μ-deformed LEE was already dealt with earlier in (Gavrilik et al., 2019), whereas the version in Eq. 27 is completely new, unpublished one. As seen, $g_\mu(r) \to 1$ and $h_\mu(r) \to 1$ at $\mu \to 0$.

It is important that, due to the special form of $g_\mu(r)$ and $h_\mu(r)$, these two μ-deformed versions of LEE have the same physical solution $j_0^{(\mu)}(kr)$ (along with $y_0^{(\mu)}(kr)$) at $\mu < 0.5$, which means that the two versions are equivalent. To display this equivalence we have to explicitly transform Eq. 26 into Eq. 27. Setting $kr \equiv x$ for simplicity, we assume the permutation rule as $D_x^{(\mu)}\, x = \sigma(x)\, x\, D_x^{(\mu)} + \lambda(x)$, apply it twice to the operator $D_r^{(\mu)}\, r^2\, D_r^{(\mu)}$ of the μ-Laplace operator in Eq. 21, and find the functions $\sigma(x)$ and $\lambda(x)$. Then, the equivalence of Eq. 26 and Eq. 27 is seen, with the non-trivial commutation relation:

$$\sigma(x)\; x D_x^{(\mu)} - D_x^{(\mu)}\, x = -\lambda(x), \tag{29}$$

$$\begin{aligned} \sigma(x) &= \frac{1}{\sqrt{h_\mu}} = \left[\frac{(1-2\mu)(1+\mu)}{1+\mu(3+2\mu^3x^2)}\right]^{1/2}, \\ \lambda(x) &= \frac{2g_\mu}{(1+\sigma)h_\mu} = \frac{1+\mu-(1-\mu)\mu^2x^2}{\mu[2+\mu(1+\mu^2x^2)]}(1-\sigma). \end{aligned} \tag{30}$$

As result, we have come to *the new (μ-deformed) generalization of Heisenberg algebra*.

The functions $\lambda(x)$ and $\sigma(x)$ are real for $0<\mu<0.5$, tend to 1 at $\mu\to 0$, and are shown in Figure 2. We have $0\le\sigma(x)\le 1$, while the maximum of $\lambda(x)$ is determined by $\lambda(0)$ and is equal to

$$\lambda_{\max}(\mu) = \frac{1+\mu}{(2+\mu)\mu}\left(1-\sqrt{\frac{(1-2\mu)(1+\mu)}{1+3\mu}}\right). \tag{31}$$

Although the function $\lambda(x)$ has a tail in negative values for

$$x > x_{\max}(\mu) = \frac{1}{\mu}\sqrt{\frac{1+\mu}{1-\mu}}, \tag{32}$$

as shown in Figure 2, consideration of the problem over finite interval of $x\in[0;\,R(\mu)]$ with $R(\mu)\le x_{\max}(\mu)$ for $\mu\in(0;0.5]$ guarantees a positive value of $\lambda(x)$. Therefore, $R(\mu)$ varies between $R_{\min}\simeq 2.886$ and $R_{\max}=2\sqrt{3}\simeq 3.464$, and it is the minimum positive number that satisfies the condition $\sin_\mu R(\mu)=0$ (see Eq. 12).

It seems to be of interest to consider, elsewhere, the quantum-mechanical problem of the propagation of a particle, viewed as a spherical wave $\Psi(r)$ in a space curved due to μ-deformation. Without specifying the boundary condition, it can be formulated as follows:

$$\hat{P}^2\Psi(r) = k^2\Psi(r), \tag{33}$$

where the momentum operator Eq. 19 for $x=r$ is used. Let us remark again that the operator $D_r^{(\mu)}$ in $\hat{P}$ is a pseudohermitian one, see e.g., (Mostafazadeh, 2002; Bagchi and Fring, 2009; Gavrilik and Kachurik, 2016b; Gavrilik and Kachurik, 2019), but the "sandwiching" $\eta^{-1}D_r^{(\mu)}\eta$ with $\eta=r$ transforms it into Hermitian form as in Eq. 33.

In view of the definition Eq. 19 of the momentum operator, we formulate our μ-deformed Heisenberg algebra:

$$\sigma(x)\; x\hat{P} - \hat{P}x = \mathrm{i}\lambda(x). \tag{34}$$

In what follows, we will focus on the study of the uncertainty principle (relation) which follows from the algebra Eq. 34.

3.2 Generalized uncertainty principle

Denoting the standard deviations as

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}, \qquad \Delta P = \sqrt{\langle\hat{P}^2\rangle - \langle\hat{P}\rangle^2}, \tag{35}$$

we proceed to the analysis of the generalized uncertainty relation (GUP)

$$\Delta x\,\Delta P \ge \frac{1}{2}|\langle[x,\hat{P}]\rangle|, \tag{36}$$

where the commutator is taken from Eq. 34.

To gain insight into the general properties of Eq. 36 for the μ-deformed Heisenberg algebra Eq. 34, let us combine Eq. 34 with its Hermitian conjugate to obtain

$$[(1+\sigma(x))x,\hat{P}] = 2\mathrm{i}\lambda(x). \tag{37}$$

Applying Eq. 36 to this commutation relation, we get

$$\Delta[(1+\sigma)x]\;\Delta P \ge \langle\lambda(x)\rangle. \tag{38}$$

Taking into account that $1\ge\sigma(x)>0$ for $\mu<0.5$ in the left hand side, we come to the GUP

$$\Delta x\,\Delta P \ge \frac{1}{2}\langle\lambda(x)\rangle. \tag{39}$$

To evaluate the averages, we specify the states similarly to quantum ones. So, let us consider a normalized mixed state $|\xi\rangle$ for $\xi\in[0;2\pi)$ in a Hilbert space basis Eq. 3 endowed with the inner product from Eq. 10:

$$|\xi\rangle = \cos\xi|u_1\rangle + \sin\xi|u_2\rangle. \tag{40}$$

Here $u_1(x)=j_0^{(\mu)}(x)$ and $u_2(x)=y_0^{(\mu)}(x)$ as before.

In fact, the mixed state $|\xi\rangle$ represents a general solution to the μ-deformed LEE, given by Eq. 26 and Eq. 27. Since the μ-deformed LEE is formulated for the local density of matter and, therefore, basically differs from the complex-valued Schrödinger equation, it is natural to describe its solution from Eq. 40 in terms of real-valued functions. Although the state $|0\rangle$ for $\xi=0$, such that $\langle x|0\rangle=j_0^{(\mu)}(x)$, serves to describe the finite DM distribution in (Gavrilik et al., 2019), the case $\xi\ne 0$ admits the contribution of the cuspidal distribution $y_0^{(\mu)}(x)$ at $x\to 0$.

Thus, we define the mean:

$$\langle(\ldots)\rangle = \langle\xi|(\ldots)|\xi\rangle \tag{41}$$

for fixed $\xi\in[0;\,2\pi]$ and $0<\mu<0.5$.

In contrast to quantum mechanics, Eq. 41 suggests to evaluate a mean of some operator $(\ldots)$ in the basis generated by the μ-deformed LEE. There is no mathematical incorrectness in choosing basis functions coinciding with physical distributions. Only in turning to an interpretation, does one face the averaging (of powers) of the distribution function itself (this also happens in multifractal analysis).

The necessary matrix elements are given by.

$$\langle\xi|f(x)|\xi\rangle = A_f(\mu) - B_f(\mu)\cos(2\xi) + C_f(\mu)\sin(2\xi), \tag{42}$$

$$\langle\xi|f(x)|\xi+\pi/2\rangle = B_f(\mu)\sin(2\xi) + C_f(\mu)\cos(2\xi), \tag{43}$$

$$\begin{aligned} \begin{Bmatrix} A_f \\ B_f \\ C_f \end{Bmatrix} &= \frac{1}{\pi}\int_0^\pi \begin{Bmatrix} 1 \\ \cos(2\varphi) \\ \sin(2\varphi) \end{Bmatrix} f(X(\varphi))\,\mathrm{d}\varphi, \\ X(\varphi) &= \frac{1}{\mu}\tan\frac{\mu\varphi}{1+\mu}. \end{aligned} \tag{44}$$

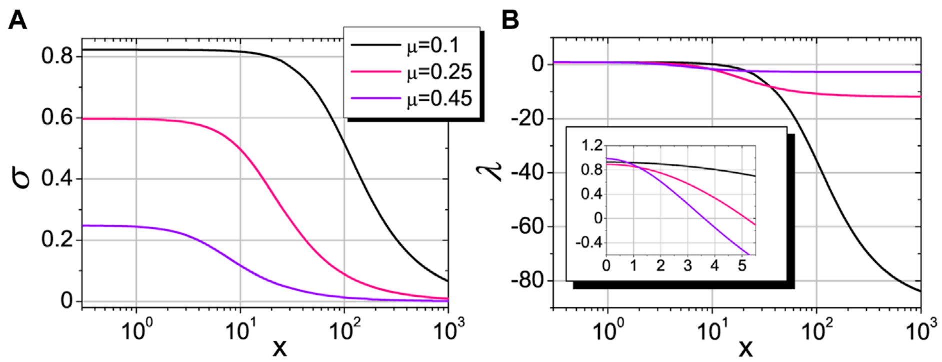

FIGURE 2
Functions $\sigma(x)$ **(A)** and $\lambda(x)$ **(B)** in a wide range of variable x and at fixed μ.

It is immediately seen that

$$\begin{aligned}\langle \hat{P}\rangle &= \langle \xi|\hat{P}|\xi\rangle = -\mathrm{i}\langle \xi|\xi+\pi/2\rangle = 0,\\ \langle \hat{P}^2\rangle &= \langle \xi|\hat{P}^2|\xi\rangle = -\langle \xi|\xi+\pi\rangle = 1,\end{aligned} \tag{45}$$

Therefore, the standard deviation $\Delta P = 1$ is fixed for the set of states $\{|\xi\rangle\}$.

On the other hand, let us introduce the functions

$$\Delta x(\xi,\mu) \equiv \sqrt{\langle \xi|x^2|\xi\rangle - (\langle \xi|x|\xi\rangle)^2}, \quad \Lambda(\xi,\mu) \equiv \langle \xi|\lambda(x)|\xi\rangle, \tag{46}$$

which represent the averages Δx and $\langle \lambda(x)\rangle$, respectively.

Thus, in the basis of the μ-deformed spherical waves, one has $\Delta P = 1$, and it is required that

$$\Delta x(\xi,\mu) \geq \frac{1}{2}\Lambda(\xi,\mu). \tag{47}$$

This relation can be analyzed with the help of **Figure 3A**.

Let us introduce the auxiliary momentum variance, accounting for Eq. 47:

$$\delta P(\xi,\mu) = \frac{\Lambda(\xi,\mu)}{2\Delta x(\xi,\mu)} \leq 1. \tag{48}$$

We see that $\delta P(\xi,\mu) \leq \Delta P = 1$ and $\delta P(\xi,\mu)\,\Delta x(\xi,\mu) = \Lambda(\xi,\mu)/2$ by definition. The behavior of $\delta P(\xi,\mu)$ is shown in **Figure 3B**.

The admissible domain of variety of the running values of Δx and ΔP is shown in **Figure 4**. We see that the black and pink curves are in antiphase regime, as it should be. For comparison, the violet curve describes the change in the deviation Δx according to the hyperbolic law in accordance with the standard Heisenberg algebra.

3.2.1 Alternative approach

To confirm the validity of Eq. 39 for the algebra Eq. 34 non-linear in x, it is worth to develop an alternative calculation scheme applicable to various ways of writing the commutator for x and $\hat{P}$. For instance, there is a possibility to rewrite relation Eq. 34 in equivalent form as

$$[x,\hat{P}] = \mathrm{i}\frac{2\lambda(x)}{1+\sigma(x)} + \frac{1-\sigma(x)}{1+\sigma(x)}\{x,\hat{P}\}, \tag{49}$$

where $\{x,\hat{P}\} \equiv x\hat{P} + \hat{P}x$ is anticommutator.

To analyze the GUP given by Eq. 36 for this commutation relation, we assume that the brackets $\langle(\dots)\rangle$ mean the quantum average over the state defined by the *real* wave function in the coordinate representation. Then, the action of the operator $\mathrm{i}\hat{P}$ on such a state results in a real-valued expression, what immediately yields

$$|\langle [x,\hat{P}]\rangle| = 2\left\langle \frac{\lambda(x)}{1+\sigma(x)}\right\rangle - \left\langle \frac{1-\sigma(x)}{1+\sigma(x)}\{x,\mathrm{i}\hat{P}\}\right\rangle, \tag{50}$$

when the positive first term on the right hand side dominates the second one.

In view of the inequality $|\langle \hat{A}\hat{B}\rangle| \leq |\langle \hat{A}\rangle|\,|\langle \hat{B}\rangle|$, we split the last term as

$$\left|\left\langle \frac{1-\sigma(x)}{1+\sigma(x)}\{x,\mathrm{i}\hat{P}\}\right\rangle\right| \leq \left\langle \frac{1-\sigma(x)}{1+\sigma(x)}\right\rangle |\langle \{x,\mathrm{i}\hat{P}\}\rangle|, \quad 0 \leq \sigma(x) \leq 1. \tag{51}$$

At this stage, we obtain

$$|\langle [x,\hat{P}]\rangle| \geq 2\left\langle \frac{\lambda(x)}{1+\sigma(x)}\right\rangle - \left\langle \frac{1-\sigma(x)}{1+\sigma(x)}\right\rangle |\langle \{x,\mathrm{i}\hat{P}\}\rangle|. \tag{52}$$

To evaluate $|\langle \{x,\mathrm{i}\hat{P}\}\rangle|$, we introduce the operators $\delta x = x - \langle x\rangle$ and $\delta\hat{P} = \hat{P} - \langle \hat{P}\rangle$, where the hat over $\hat{P}$ distinguishes the operator $\delta\hat{P}$ from the function δP in Eq. 48. Then one gets

$$\langle \{x,\mathrm{i}\hat{P}\}\rangle = 2\mathrm{i}\langle x\rangle\langle \hat{P}\rangle + \langle \{\delta x,\mathrm{i}\delta\hat{P}\}\rangle. \tag{53}$$

Due to the Cauchy–Schwartz inequality $|\langle \hat{A}\hat{B}\rangle|^2 \leq |\langle \hat{A}^2\rangle|\,|\langle \hat{B}^2\rangle|$, the following estimate holds:

$$|\langle \{\delta x,\mathrm{i}\delta\hat{P}\}\rangle| \leq 2\Delta x\,\Delta P. \tag{54}$$

Since the dark matter flux is assumed to be absent in the halo, one can put $\langle \hat{P}\rangle = 0$, which is confirmed by our direct calculations. Combining, we have the estimate

$$|\langle [x,\hat{P}]\rangle| \geq 2\left\langle \frac{\lambda(x)}{1+\sigma(x)}\right\rangle - 2\left\langle \frac{1-\sigma(x)}{1+\sigma(x)}\right\rangle \Delta x\,\Delta P. \tag{55}$$

Substituting it into Eq. 36 and accounting for $\langle 1\rangle = 1$, we arrive at.

$$\Delta x\,\Delta P \geq \frac{1}{2}\langle \lambda(x)\rangle_W, \tag{56}$$

$$\langle(\dots)\rangle_W \equiv \frac{\langle W(x)(\dots)\rangle}{\langle W(x)\rangle}, \quad W(x) = \frac{1}{1+\sigma(x)}, \tag{57}$$

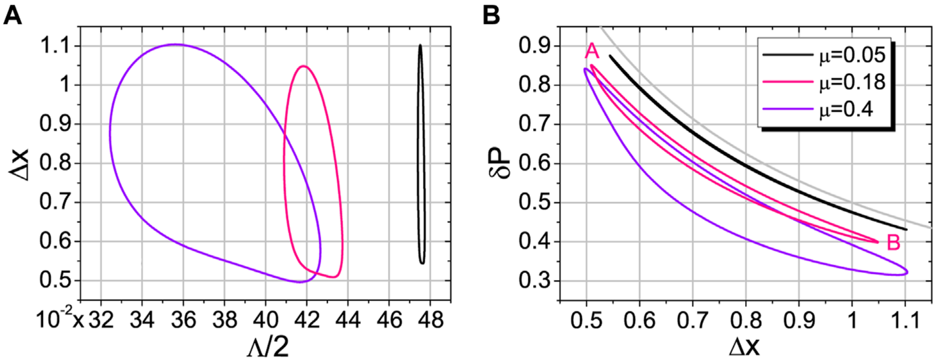

FIGURE 3
(A): Position deviation Δx (ξ,μ) *versus* mean $\langle\lambda(x)\rangle = \Lambda(\xi,\mu)$, which are computed in the state $|\xi\rangle$ at fixed μ **(B)**: Limiting momentum deviation $\delta P\ (\xi,\mu) = \Lambda/(2\Delta x)$ *versus* Δx (ξ,μ). Turning points of the pink banana-like curve are A (0.51;0.85), B (1.05;0.4). Grey line corresponds to $\delta P = 1/(2\Delta x)$.

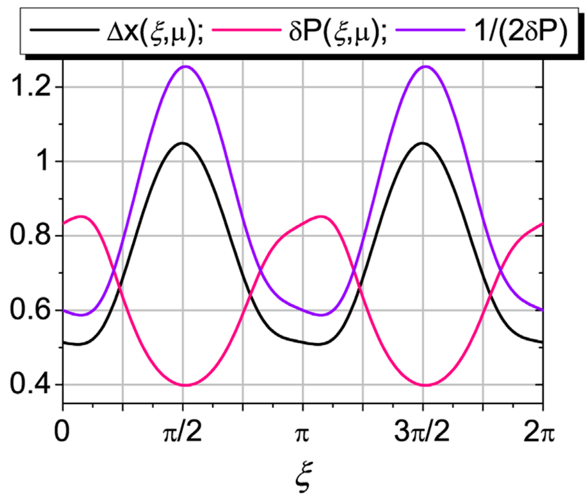

FIGURE 4
Dependence of the lower limit deviations Δx (ξ,μ) and δP (ξ,μ) on the state labeled by ξ at $\mu = 0.18$. The running deviations Δx and ΔP satisfy the GUP $\Delta x\Delta P \geq \Lambda/2$ and vary in the ranges $\Delta x\ (\xi,\mu) \leq \Delta x \leq R\ (\mu = 0.18) \simeq 2.89$ and $\delta P\ (\xi,\mu) \leq \Delta P \leq 1$.

Where the new mean $\langle(\ldots)\rangle_W$ with additional convex weighting function $W(x)$ arises.

For a given function $W(x)$ we get

$$\langle\lambda(x)\rangle_W = \langle\lambda(x)\rangle + \frac{\langle\delta W(x)\,\delta\lambda(x)\rangle}{\langle W(x)\rangle}, \tag{58}$$

where $\langle\delta W(x)\delta\lambda(x)\rangle$ is a covariance between the convex function $W(x)$ and concave $\lambda(x)$, and it determined by deviations like $\delta f(x) = f(x) - \langle f(x)\rangle$.

Since the function $\sigma(x)$ (and $W(x)$) changes only slightly over the interval $x \in [0;\ R(\mu)]$ in **Figure 2A**, it can be approximated by a constant close to $\sigma(0)$ when calculating integrals. This provides $\delta W(x) \to 0$ and numerically leads to expressions:

$$\left\langle\frac{\lambda(x)}{1+\sigma(x)}\right\rangle \simeq \frac{\langle\lambda(x)\rangle}{1+\langle\sigma(x)\rangle}, \qquad \left\langle\frac{1}{1+\sigma(x)}\right\rangle \simeq \frac{1}{1+\langle\sigma(x)\rangle}, \tag{59}$$

when we use $\langle(\ldots)\rangle = \langle\xi|(\ldots)|\xi\rangle$ in the range $0 < \mu < 0.5$.

This circumstance leads again to Eq. **39** for the states $|\xi\rangle$, that is just Eq. **47**.

Note that the appearance of the mean Eq. **57** is associated with the initial Eq. **49** for the commutation relation. In other cases, we may only encounter means of type Eq. **59**, where it would be justified to use the estimate $1 \geq \langle\sigma(x)\rangle$.

3.3 Application to dark matter

Let us remind the connection between the operators in terms of the dimensionless variable $x = kr$ and the operators of the physical radial coordinate r and the momentum $\hat{P}_r$:

$$r = \frac{x}{k}, \qquad \hat{P}_r = \hbar k\hat{P}, \tag{60}$$

where k is the parameter of Eqs **26** and **27** and has the dimension of inverse length; the operator $\hat{P}$ is given by Eq. **19**.

The most successful results of paper (Gavrilik et al., 2019) for describing the dark matter halo of dwarf galaxies based on the μ-deformed Lane–Emden equation were obtained in the following range of parameters:

$$\mu = 0.151\ldots0.18, \qquad k = 0.17\ldots2.64\ \mathrm{kpc}^{-1}. \tag{61}$$

Using the turning points $A\ ((\Delta x)_{\min};\ (\delta P)_{\max})$ and $B\ ((\Delta x)_{\max};\ (\delta P)_{\min})$ for fixed μ as in **Figure 3B**, we relate extreme physical values Δr and ΔP_r with dimensionless ones Δx and δP as.

$$\Delta r = \Delta x\left[\frac{k}{1\ \mathrm{kpc}^{-1}}\right]^{-1}\mathrm{kpc}, \tag{62}$$

$$\Delta P_r = \delta P\left[\frac{k}{1\ \mathrm{kpc}^{-1}}\right]\times 6.394\times10^{-27}\ \frac{\mathrm{eV}}{c}. \tag{63}$$

The calculation results are collected in **Table 2**. Therein, we present the obtained data for five dwarf galaxies (from the eight ones given in **Table 1** of (Gavrilik et al., 2019)), because just for these galaxies the μ-deformation based description of the rotation curves is most successful with respect to earlier approaches, as it provides the best agreement with observational data (certainly better then

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TABLE 2 The parameters for the dark matter halos of dwarf galaxies.

Galaxy	μ	k, kpc^{-1}	$(\Delta r)_{max}$, kpc	$(\Delta r)_{min}$, kpc	$(\Delta P_r)_{max}$, $10^{-27} eV/c$	$(\Delta P_r)_{min}$, $10^{-27} eV/c$
M81dwB	0.18	2.64	0.398	0.193	14.38	6.75
DDO 53	0.18	0.97	1.082	0.526	5.28	2.48
IC 2574	0.179	0.17	6.18	3.0	0.926	0.435
NGC 2366	0.178	0.37	2.84	1.38	2.02	0.946
HO I	0.151	1.27	0.830	0.402	6.98	3.33

if one uses the profile from the usual Bose-condensate model of DM being the solution of non-deformed Lane–Emden equation as in (Harko, 2011), or uses the famous Navarro–Frenk–White profile (Navarro et al., 1997)).

Of course, the remaining three galaxies can also be considered, but the choice of five ones is both sufficient, trustful, and best suited for our treatment and conclusions.

Note that both the Lane–Emden equation and its μ-deformed extensions determine the distribution function $\rho(r)$ of the matter, not the wave function of single particle. Therefore, the mean Eq. **41** is a quadratic form in the distribution, related here with $|0\rangle$ which differs by a multiplicative constant defining $\rho(0)$ (Gavrilik et al., 2019). Generally speaking, the state $|0\rangle$ may not determine the turning points of a banana-like curve in the $(\Delta x, \delta P)$ plane in **Figure 3B**, along with the extreme values of the deviations Δr and ΔP_r. Nevertheless, the mathematically correct mean Eq. **41** can be used to obtain new additional information about dark matter, even by means of considering the moments $\langle \rho^n \rangle$ of the distribution similarly to multifractal analysis. Besides, the extreme deviations set the limits for the fluctuations of physical quantities at $\xi = 0$.

Without a deep study of the structure of averages here, let us analyze the physical consequences of the data in **Table 2**. We see that in the non-relativistic theory the momentum deviation $\Delta P_r = m\Delta v_r$, where m is the particle mass, Δv_r is the deviation of particle radial velocity v_r. Since the original work (Gavrilik et al., 2019) was using bosons with $m \sim 10^{-22}$ eV$/c^2$, we obtain from **Table 2** that $\Delta v_r \sim 10^{-5} c$ in units of the speed of light c. Moreover, deviation of the kinetic energy $\Delta E_K = m(\Delta v_r)^2/2$ can be used to determine the effective temperature of dark matter, namely, $T_{\text{eff}} = (\Delta P_r)^2/(2m) \sim 10^{-32}$ eV. This value is much smaller than the critical temperature of the Bose–Einstein condensation, as it should be in such a paradigm.

Due to the GUP given by Eq. **39**, we relate the temperature T_{eff} of the spherical layer in the vicinity of $\langle r \rangle$ to its width $2\Delta r$:

$$(\Delta r)^2\, T_{\text{eff}} \geq \frac{\hbar^2}{8m} \langle \lambda \rangle^2. \tag{64}$$

This formula holds for a macroscopic system of finite volume when Δr does not exceed the radius of the system, and it shows that a smaller domain may have a higher temperature, and vice verse.

It is worth to note that the mean $\langle \lambda \rangle$ in Eq. **64** takes values in the limited interval $\langle \lambda \rangle = \Lambda(\xi, \mu) \in [\Lambda_{\text{min}}, \Lambda_{\text{max}}]$, where the positive Λ_{min} and Λ_{max} depend on μ (see Eq. **46**; **Figure 3A**). For $\mu = 0.18$, we have $\Lambda_{\text{min}} \simeq 0.818$ and $\Lambda_{\text{max}} \simeq 0.875$.

4 Concluding remarks

In this work we have studied unusual consequences of the new (μ-deformed) generalization of the Heisenberg algebra Eqs **29** and **34** which is special as it was derived within the extension of Bose-condensate dark matter model based on μ-deformation. From the generalized algebra we obtained non-trivial GUR that generates minimal and maximal uncertainties of both positions (minimal/maximal lengths) and momenta. The obtained GUR is strictly dependent on the states (labeled by ξ) of the system, and such dependence was exploited to full extent.

In **Table 2**, the galaxies M81dwB and IC 2574 look as the two "extreme" cases. Namely, for the latter we have the largest $(\Delta r)_{\text{max}}$ and $(\Delta r)_{\text{min}}$, while for the former these quantities show smallest values. Clearly, the situation concerning $(\Delta P_r)_{\text{max}}$ and $(\Delta P_r)_{\text{min}}$ is quite opposite. Noteworthy, the value of μ (strength of deformation) for M81dwB and IC 2574 is almost the same. The relations Eq. **62**, **63** show the defining role of the quantity k which involves scattering length a and particle mass m as $k \propto m^{3/2} a^{-1/2}$ (Harko, 2011).

For the considered galaxies (each labeled by its specific value of μ) we conclude: since the particle mass is same (namely, 10^{-22} eV$/c^2$), we have differing scattering lengths in halos of different galaxies (*vice versa*, would we assume same scattering length for all the five galaxies we would have somewhat differing masses of DM particle in different galaxies, though this second option seems to be less realistic). As already shown in (Harko, 2011), where the BEC DM model is also based on the LEE, there is no universality of model parameters when describing all admissible objects. In fact, this issue remains in our model, which improves the previously fitted rotation curves by including an additional parameter μ. Physically, we can only control the applicability conditions of our model: consider DM-dominated dwarf galaxies leaving aside their rigid rotation, which contributes to the distribution function (Zhang et al., 2018; Nazarenko, 2020). Therefore, giving clear physical meaning to differing scattering lengths in halos of different galaxies remains an interesting task for future study.

Note that the parameter k is related to the observed radius r_{gal} of the galactic halo by $kr_{\text{gal}} = R(\mu)$, where the right-hand side is determined by the parameter $R(\mu)$ from Eq. **12**, replacing $\pi = R\,(0)$ in the non-deformed case. We can easily find a small difference (of several percent) between the values of r_{gal} in the deformed and non-deformed cases, by comparing these with the galaxy radii from (Harko, 2011). However, the simulation of rotation curves is more successful in the deformed case, as shown in (Gavrilik et al., 2019).

It is worth to remark that the values of $(\Delta r)_{\text{max}}$ and $(\Delta r)_{\text{min}}$ for the two galaxies M81dwB and IC 2574, and the others in **Table 2**, reside well within the observed sizes of DM halos as it should. Accordingly, the values of $(\Delta P_r)_{\max}$ and $(\Delta P_r)_{\min}$ for these same galaxies lie in the ranges completely consistent with DM being in the (μ-Bose) condensate state. Clearly this is in agreement with the above reasonings concerning the effective temperature.

It is of interest to analyze possible special meaning of our results on the existence of finite $(\Delta r)_{\max}$ and $(\Delta r)_{\min}$ in the context of treatment in (Lee and Lim, 2010; Lee, 2016) of minimum length scale of galaxies (note that for the candidate length scales one can take into consideration such concepts as coherence length, Compton wavelength, quantum Jeans length scale, gravitational Bohr radius, and de Broglie wavelength, see (Lee and Lim, 2010) and references therein). Time dependence of some of these quantities, e.g., characteristic length scale $\tilde{\xi}$ (minimum size of DM dominated galaxies) is studied in (Lee, 2016). Let us quote one of the interesting predictions of this work: with the mass of DM particles chosen as $m = 5 \times 10^{-22}$ eV$/c^2$, it follows that $\tilde{\xi}(z=0) = 311.5$ pc while $\tilde{\xi}(z=5) = 81.2$ pc, i.e., early dwarf galaxies were significantly more compact. In view of the extremely tiny mass of the particle from dark sector, a question may arise of possible (inter)relation of this entity with the cosmic microwave background (CMB). The very first answer which comes to one's mind could be that no relation is possible, because of the absence of interaction between visible and dark sectors. However, when considered in the framework of doubly special relativity, the properties of the photon gas at these special conditions can appear, see (Chung et al., 2019), much more interesting and non-trivial. Noteworthy, the treatment in (Chung et al., 2019), on one hand, is potentially applicable for studying some unclear features of CMB, and, on the other hand, involves a kind of deformation which is very similar to the μ-deformation explored herein. We hope to address the details of all these intriguing issues elsewhere.

Data availability statement

The original contributions presented in the study are included in the article/**Supplementary Material**, further inquiries can be directed to the corresponding author.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Supplementary material

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fspas.2023.1133976/full#supplementary-material

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