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Integral form of the beam envelope equation for electron beam propagation in vacuum and its relation to the K-V equation

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The study of the beam envelope radius—a parameter characterizing the transverse size and evolution of a particle beam along its propagation path—is fundamental to the particle accelerators application and the execution of space-borne experiments employing artificial relativistic electron beams. In this paper, we investigate the propagation of electron beams in vacuum and derive an integral form of the beam envelope equation. This equation is equivalent to the simplified differential form of the Kapchinsky-Vladimirsky (K-V) equation excluding the effects of external forces and radial emittance. The integral equation is validated by the widely used ASTRA (A Space Charge Tracking Algorithm) simulation code. The effect of electron energy on beam envelope radius is uncertain and depends on whether internal force or external force dominates. When external force dominates, a decrease in electron energy results in a smaller beam envelope radius. Conversely, when internal force dominates, an increase in electron energy leads to a smaller beam envelope radius. This study is a bridge between integral form and differential form of the envelope equation, and will provide a better understanding for the K-V equation and a scientific basis for researching beam propagation technology.

KEYWORDS

electron beam, beam envelope equation, K-V equation, electron energy, beam radius

Key points

- An integral form of the beam envelope equation for electron beam propagation in a vacuum is derived.
- This equation is equivalent to the simplified differential form of the K-V equation excluding the effects of external forces and radial emittance.
- The effect of electron energy on beam radius depends on whether internal or external forces dominate.

1 Introduction

An electron beam is a collection of energetic electrons, typically generated either naturally or artificially (Banks and Raitt, 1988; Winckler, 1980), which are manipulated by electromagnetic fields and directed through a vacuum or plasma (Neubert and Gilchrist, 2002a; Borovsky et al., 2020). Electron beams are found in many space environments, such as the solar wind (Arshad et al., 2014; Sun et al., 2020), magnetic reconnection outflow region (Åsnes et al., 2008), and auroras (Banks and Raitt, 1988), particularly in areas where energetic electrons are accelerated by waves. Electron beams can interact with magnetic fields and background plasma (Druyvesteyn, 1938; Mustafaev, 2001; Arshad and Mirza, 2014; Reeves et al., 2020), playing a significant role in a variety of space phenomena.

Electron beams have proven to be a powerful tool in space science research. During the 1970s and 1980s, keV electron beams were injected from balloons and sounding rockets to probe the fundamental physical processes in space physics. These experiments were used for applications such as mapping magnetic field lines in the Earth's magnetosphere [e.g., Hendrickson et al., 1975, 1976; Winckler et al., 1975], exciting artificial auroras, studying beam-plasma interactions [e.g., Gendrin, 1974; Cambou et al., 1978, 1980], and investigating wave generation and amplification as well as instabilities [e.g., Monson et al., 1976; Dechambre et al., 1980]. They also provided insights into spacecraft charging [e.g., Mullen et al., 1986; Sasaki et al., 1986, 1988; Banks et al., 1990], as well as military applications. From the late 1990s to the present, the development of electron accelerator technology (Szuszczewicz, 1985; Lewellen and Buechler, 2019) enabled the injection of relativistic electron beams into the space environment from spacecraft [e.g., Neubert et al., 1996; Krause, 1998, 1999; Gilchrist et al., 2001; Neubert and Gilchrist, 2002b, Neubert and Gilchrist, 2004; Miars et al., 2020; Reeves et al., 2020; Xue et al., 2023; Fang et al., 2024]. Results from these studies indicate that relativistic beams are more stable than keV beams, due to a combination of factors including the higher relativistic electron mass, lower beam densities, and reduced spacecraft charging effects.

In Earth's auroras, magnetic reconnection generates high-energy electron beams that are accelerated and injected along magnetic field lines into regions such as the ionosphere, forming well-defined beam structures. Electron beams possess high kinetic energy and excellent directional properties, enabling them to interact with spacecraft in orbit and produce space radiation effects, such as surface charging, internal charging, and total ionizing dose effect (Zheng et al., 2019; Bodeau and Baker, 2021). These effects can lead to malfunction in spacecraft, posing significant threats to human spaceflight (Hastings, 1995; Castello et al., 2018). To mitigate or prevent the hazards associated with electron beams, scientists have investigated their generation, propagation, and interaction in the space environment through theoretical analyses and numerical simulations. It is essential for beam envelope control to understand the evolution of electron beam propagation. A critical question in the study of electron beam propagation is identifying the factors that influence the evolution of the electron beam envelope and understanding their quantitative effects.

A significant amount of pioneering work has been conducted to address this question. In 1959, Kapchinsky and Vladimirov (K-V) made a major breakthrough in beam physics by deriving the envelope equation for a continuous beam with a uniform charge density and an elliptical cross-section (Kapchinsky and Vladimirov, 1995). This equation accounts for the effects of the self-electric and self-magnetic fields associated with the beam's space charge and current, as well as external forces. Comparing with the kinetic analysis of single particle, the electron beam tends to diverge owing to space charge forces between two equal-charge particles. Their work became a cornerstone in the field, with profound implications for beam analysis and design. In 1971, Sacherer expanded on this foundation by introducing the concept of root-mean-square (RMS) emittance (Sacherer, 1971). He demonstrated that the K-V equation is not limited to beams with uniform charge density, but is also valid for any charge distribution with elliptical symmetry, provided that the beam boundary and emittance are defined. For convenience, the normalized beam emittance (Lawson, 1988), which represents the product of the beam radius and divergence angle, is used in the K-V equation. The influence of internal forces associated with the beam's space charge and current remained unclear until the concept of perveance was introduced (Chen and Davidson, 1993), providing a quantitative framework for understanding their role in the K-V equation. When electron beams are injected into plasma instead of a vacuum, the K-V equation must be modified to account for charge neutralization factor (Neubert and Gilchrist, 2002a), as plasma electrons will respond to the beam and move away from it. It is important to note that the K-V equation is a second-order differential equation, and obtaining an analytical expression for the variation of the beam envelope radius with respect to the propagation direction is challenging.

Although K-V equation is a powerful tool for studying the evolution of electron beam envelope, the integral equation is more convenient to describe the variation of the beam envelope radius with respect to the propagation direction. Some theoretical investigations into the integral equation for electron beam propagation have contributed to deriving the expression for perveance in the K-V equation. Bekefi et al. (1980) derived the integral expression for the beam radius at a distance from the source using relativistic particle dynamics. However, their integral expression did not account for the slope of the trajectory at the initial position (Bekefi et al., 1980; Vinokurov, 2001). Reiser added the initial slope to the integral expression, but he overlooked the case in which the slope of the trajectory becomes negative when the beam converges (Reiser, 2008).

Despite the progress made in these studies, further research is still needed on the integral equation governing electron beam propagation. In this paper, we examine the propagation of electron beams in a vacuum and derive an integral form of the beam envelope equation. This equation is equivalent to the simplified differential form of the K-V equation, excluding the effects of external forces and radial emittance. Additionally, we discover that the impact of electron energy on the beam envelope radius is uncertain and depends on whether internal or external forces dominate. The structure of this paper is organized as follows: Section 2 presents the derivation of the integral form of the beam envelope equation and the validation of integral equation by ASTRA simulation code. In Section 3, we provide the relationship between the integral form and

the K-V equation. Finally, Section 4 offers a briefly discussion of the conclusions.

2 The electron beam envelope integral equation

This section presents the derivation of the integral form equation and benchmarks the theoretical electron beam envelope models against numerical simulations performed with ASTRA (A Space Charge Tracking Algorithm). In addition, the quantitative impact of several factors on the electron beam envelope radius is provided.

2.1 Derivation of the electron beam envelope integral equation

The following simplifying assumptions have been made in deriving the equations: (1) The electron density is assumed to be uniform within the cylindrical beam and zero outside it; (2) The transverse velocity component of the electrons is assumed to be small compared to the axial velocity, meaning the angle with the axis (slope) is small; (3) The flow is laminar, meaning all beam particles follow trajectories that do not intersect.

Based on the first assumption, the electron beam is treated as a long cylinder with a maximum radius r_b and a number density n_0 . The assumption of uniform charge density results in space-charge forces that are linearly proportional to the distance from the beam's center. This linear relationship significantly simplifies the equations of motion, making them analytically solvable. Without the assumption of uniform charge density, the charge density $\rho(r)$ varies with radial distance r . This variation alters the calculation of the space charge force. Under the uniform charge density assumption, the radial space charge force $F_r = qE = qE_r = \frac{q\rho_0 r}{2\epsilon_0}$ exhibits a simple linear dependence on r . In contrast, for a Gaussian charge density distribution defined as $\rho(r) = \rho_0 e^{-\frac{r^2}{2\sigma^2}}$, where ρ_0 is the central charge density and is σ the standard deviation, $F_r = qE_r = \frac{q}{\epsilon_0} \int_0^r \rho(r') r' dr'$ leads to nonlinear behavior. Such nonlinear forces cause particle trajectories to deviate from simple linear oscillations, which can increase beam diffusion or emittance growth and complicate analytical solutions. However, real beams often exhibit non-uniform density distributions, such as Gaussian or parabolic distributions, which lead to nonlinear space-charge forces that the model fails to account for (Arshad et al., 2017a; Arshad 2018; Arshad and Poedts, 2020). Consequently, this limitation restricts the model's accuracy and applicability, particularly in scenarios involving high-intensity beams, non-uniform density distributions, or dynamic conditions where nonlinear phenomena—such as emittance growth, filamentation, or instabilities—play a significant role (Arshad et al., 2017b; Arshad et al., 2022; Arshad et al., 2011). Nevertheless, the model retains practical value for how to moderate intensity beams, where space-charge effects are minimal, the uniform density assumption remains reasonably valid, and the linear approximation offers an adequate description.

As indicated by the second assumption, the velocity of the electrons in the beam, v , is primarily directed along the axial direction (i.e., $v_r \ll v$, $v_\theta \ll v$, $v_z \approx v$, v_r is the radial component

of velocity, v_θ is the angular component of velocity, and v_z is the axial component of velocity.). Thus, the charge density is $\rho_0 = -en_0$ and the beam current $I_0 = -en_0\pi r_b^2 v$, where e refers to the charge of the electron. The radial component of electric field is obtained using Gauss's law (Buchholz, 1986), $\int \epsilon_0 E \cdot dS = \int \rho dV$, which yields

$$E_r = \frac{\rho_0 r_b}{2\epsilon_0} \quad (1)$$

$$E_r = \frac{I_0}{2\pi\epsilon_0 r_b v} \quad (2)$$

The azimuthal component of magnetic field is obtained using Ampère's circuital law (Cavalleri et al., 1996), $\int B \cdot dl = \mu_0 I$, which gives

$$B_\theta = \frac{\mu_0 I_0}{2\pi r_b} \quad (3)$$

E_r and B_θ refer to the electric and magnetic fields outside the beam. We now examine the motion of a beam particle in this field, using only the radial force equation (Reiser, 2008)

$$\frac{d}{dt} \left(\gamma m_0 \frac{dr_b}{dt} \right) = -e(E_r - v_z B_\theta) \quad (4)$$

where we neglect the force term $-er \frac{dB_\theta}{dt}$ on the grounds that $r \frac{dB_\theta}{dt}$ is negligibly small, and γ is constant since there is no external acceleration. γ refers to the Lorentz factor, defined as $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, where $\beta = \frac{v}{c}$ and c denotes the speed of light. Substituting E_r from Equations 1, 2, B_θ from Equation 3, $v_z = v = \beta c$, and the expression of I_0 , Equation 4 becomes Equations 5, 6

$$\frac{d^2 r_b}{dt^2} = \frac{e^2 n_0 r_b}{2\epsilon_0 m_0 \gamma^3} \quad (5)$$

$$\frac{d^2 r_b}{dt^2} = -\frac{e I_0}{2\pi\epsilon_0 m_0 r_b \beta c \gamma^3} \quad (6)$$

Using the relationship $\frac{d^2 r_b}{dt^2} = v_z \frac{d}{dz} \left(v_z \frac{dr_b}{dz} \right) = (\beta c)^2 \frac{d^2 r_b}{dz^2}$, Equation 6 becomes

$$\frac{d^2 r_b}{dz^2} = -\frac{e I_0}{2\pi\epsilon_0 m_0 r_b \beta^3 c^3 \gamma^3} \quad (7)$$

The perveance K , a dimensionless quantity, is defined by Chen and Davidson (1993) as Equations 8, 9

$$K = -\frac{e I_0}{2\pi\epsilon_0 m_0 c^3 \beta^3 \gamma^3} \quad (8)$$

$$K = \frac{2 I_0}{\beta^3 \gamma^3 I_A} \quad (9)$$

where $I_A = \frac{4\pi\epsilon_0 m_0 c^3}{-e}$ and $\beta\gamma = \sqrt{\left(1 + \frac{E}{m_0 c^2}\right)^2 - 1}$ and E is the kinetic energy. In terms of the perveance, as defined in Equation 9, Equation 7 can be expressed as

$$\frac{d^2 r_b}{dz^2} = \frac{K}{r_b} \quad (10)$$

It is important to note that $K > 0$ for electrons. Multiplying both sides of Equation 10 by $r' = \frac{dr_b}{dz}$, we obtain

$$r' r'' = \frac{K}{r_b} r' \quad (11)$$

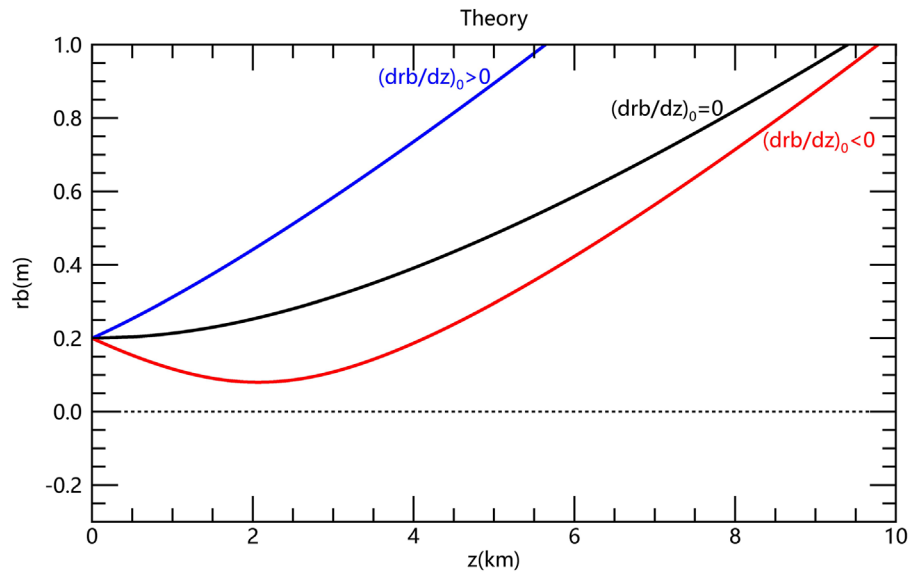


FIGURE 1

The theoretical electron beam envelope radius as a function of propagation distance for three initial slopes of the trajectory ($r'_0 = 10^{-4}$, $r'_0 = 0$ and $r'_0 = -10^{-4}$) with an energy of $E = 1$ MeV, current $I_0 = 1$ mA and initial beam radius $a = 0.2$ m.

By integrating Equation 11, we get

$$r'^2 - 2K \ln r_b = \text{const} \quad (12)$$

Evaluating Equation 12 with the initial boundary conditions $r_b = a$ and $r' = r'_0$ at $z = 0$ where a refers to initial beam radius and r'_0 refers to the initial change rate of the beam radius along the z -axis, we obtain $r'^2 - 2K \ln r_b = r_0'^2 - 2K \ln a$, or equivalently,

$$\frac{dr_b}{dz} = \pm \sqrt{r_0'^2 + 2K \ln \left(\frac{r_b}{a} \right)} \quad (13)$$

Next, we integrate Equation 13 by defining $u = \sqrt{\frac{r_0'^2}{2K} + \ln \left(\frac{r_b}{a} \right)}$, which yields Equations 14, 15

$$z = \pm \int_a^{r_b} \frac{dr_b}{\sqrt{r_0'^2 + 2K \ln \left(\frac{r_b}{a} \right)}} \quad (14)$$

$$z = \pm \frac{1}{\sqrt{2K}} \int_{\frac{r_0'^2}{2K}}^{\frac{r_0'^2}{2K} + \ln \left(\frac{r_b}{a} \right)} \frac{1}{u} \frac{dr_b}{du} du \quad (15)$$

From the expression of u , we have

$$\frac{dr_b}{du} = 2au \exp \left(u^2 - \frac{r_0'^2}{2K} \right) \quad (16)$$

Substituting Equation 16 into the Equation 15, we obtain

$$z = \pm a \sqrt{\frac{2}{K}} \exp \left(-\frac{r_0'^2}{2K} \right) \int_{\frac{r_0'^2}{2K}}^{\frac{r_0'^2}{2K} + \ln \left(\frac{r_b}{a} \right)} e^{u^2} du \quad (17)$$

The Dawson integral is given by $F(w) = e^{-w^2} \int_0^w e^{u^2} du$. If the initial slope of the trajectory is greater than or equal to zero (i.e., $r'_0 \geq 0$), Equation 17 can be written as Equations 18, 19

$$z = a \sqrt{\frac{2}{K}} \exp \left(-\frac{r_0'^2}{2K} \right) \int_{\frac{r_0'^2}{2K}}^{\frac{r_0'^2}{2K} + \ln \left(\frac{r_b}{a} \right)} e^{u^2} du \quad (18)$$

$$z = \sqrt{\frac{2}{K}} \left[-aF \left(\frac{|r'_0|}{\sqrt{2K}} \right) + r_b F \left(\sqrt{\frac{r_0'^2}{2K} + \ln \left(\frac{r_b}{a} \right)} \right) \right] \quad (19)$$

using the expression for the Dawson integral. If the initial slope of the beam profile is negative ($r'_0 < 0$), the beam radius will initially decrease until it reaches a minimum value, where $r' = 0$. According to Equation 13, the minimum beam radius is given by $r_{\min} = a \exp \left(-\frac{r_0'^2}{2K} \right)$. Beyond this point, the beam radius will increase again. Under the conditions that the initial slope of the trajectory is negative (i.e., $r'_0 < 0$) and $r_b > r_{\min}$, Equation 17 can be written as Equations 20, 21

$$z = a \sqrt{\frac{2}{K}} \exp \left(-\frac{r_0'^2}{2K} \right) \int_{-\frac{r_0'}{\sqrt{2K}}}^{\sqrt{\frac{r_0'^2}{2K} + \ln \left(\frac{r_b}{a} \right)}} e^{u^2} du \quad (20)$$

$$z = \sqrt{\frac{2}{K}} \left[-aF \left(\frac{|r'_0|}{\sqrt{2K}} \right) + r_b F \left(\sqrt{\frac{r_0'^2}{2K} + \ln \left(\frac{r_b}{a} \right)} \right) \right] \quad (21)$$

where $F(w)$ represents the Dawson integral. When the initial slope of the trajectory is negative (i.e., $r'_0 < 0$) and $r_b < r_{\min}$, Equation 17 becomes

$$z = -a \sqrt{\frac{2}{K}} \exp \left(-\frac{r_0'^2}{2K} \right) \int_{-\frac{r_0'}{\sqrt{2K}}}^{\sqrt{\frac{r_0'^2}{2K} + \ln \left(\frac{r_b}{a} \right)}} e^{u^2} du \quad (22)$$

$$z = -\sqrt{\frac{2}{K}} \left[-aF \left(\frac{|r'_0|}{\sqrt{2K}} \right) + r_b F \left(\sqrt{\frac{r_0'^2}{2K} + \ln \left(\frac{r_b}{a} \right)} \right) \right] \quad (23)$$

By combining Equations 19, 21, 23, we obtain

$$z = \pm \sqrt{\frac{2}{K}} \left[-aF \left(\frac{|r'_0|}{\sqrt{2K}} \right) + r_b F \left(\sqrt{\frac{r_0'^2}{2K} + \ln \left(\frac{r_b}{a} \right)} \right) \right] \quad (24)$$

The negative sign in Equation 17 applies when $r'_0 < 0$ and $r_b < a \exp\left(-\frac{r_0^2}{2K}\right)$, while the positive sign applies in all other cases. Figure 1 illustrates the variation of beam radius (r_b) as a function of propagation distance (z) under three initial conditions: $r'_0 > 0$ (blue curve), $r'_0 = 0$ (black curve), and $r'_0 < 0$ (red curve) with an energy of $E = 1$ MeV, current $I_0 = 1$ mA and initial beam radius $a = 0.2$ m. The blue curve represents a scenario where the initial slope of trajectory is positive, resulting in a monotonic increase. A positive initial slope of trajectory ($r'_0 > 0$) indicates that particles possess outward-directed transverse velocities. This results in beam divergence during the initial propagation phase, owing predominantly to Coulomb repulsion from space charge forces. The black curve corresponds to a zero-initial slope, leading to a steady increase without an inflection point. In this situation, the beam possesses a perfectly parallel trajectory with a uniform transverse velocity distribution. The red curve reflects a negative initial slope, showing an initial decrease followed by an upward trend. A negative initial slope of trajectory ($r'_0 < 0$) is permissible when well-defined initial conditions are imposed, such as particles with initial velocities precisely aligned toward the beam axis.

2.2 Numerical verification of electron beam envelope equations with ASTRA

To compare and validate the theoretical electron beam envelope equations in integral form, the propagation of the electron beam is simulated using the ASTRA code (Floettmann, 2017). ASTRA, a space charge tracking algorithm developed by the German Electron Synchrotron Research Institute (DESY), employs the Particle-In-Cell (PIC) method to simulate collective self-field effects, such as space charge, in high-energy charged particle beams. The program incorporates comprehensive algorithms for generating initial particle distributions and performing particle tracking, making it a widely used tool in simulation studies of high-energy charged particles.

Figure 2 illustrates how ASTRA works. The program *generator* can be used to generate an initial particle distribution by specifying parameters such as the particle count, particle energy, and types of beam distributions (radial uniform/Gaussian/plateau) in the input file. The core program *Astra* then performs particle tracking under the influence of internally computed space-charge fields and/or external electromagnetic fields. Its computational workflow iterates through four integrated components: (1) Field Calculation, which solves the governing field equations on a discrete grid; (2) Field Interpolation, which interpolates both external and self-generated fields from the grid nodes to individual particle positions; (3) Particle Push, which integrates the equations of motion over a time step to update particle positions and momenta; and (4) Charge Deposition, which maps particle charges back onto the grid. This cycle repeats until the simulation concludes.

In the ASTRA program, the 1 MeV electrons are assumed to be evenly spread out across the cross-sectional area. In a uniform distribution, the maximum radius r_b is $\sqrt{2}$ times the RMS beam radius. As shown in Figure 1, the initial beam radius is 0.2 m, which

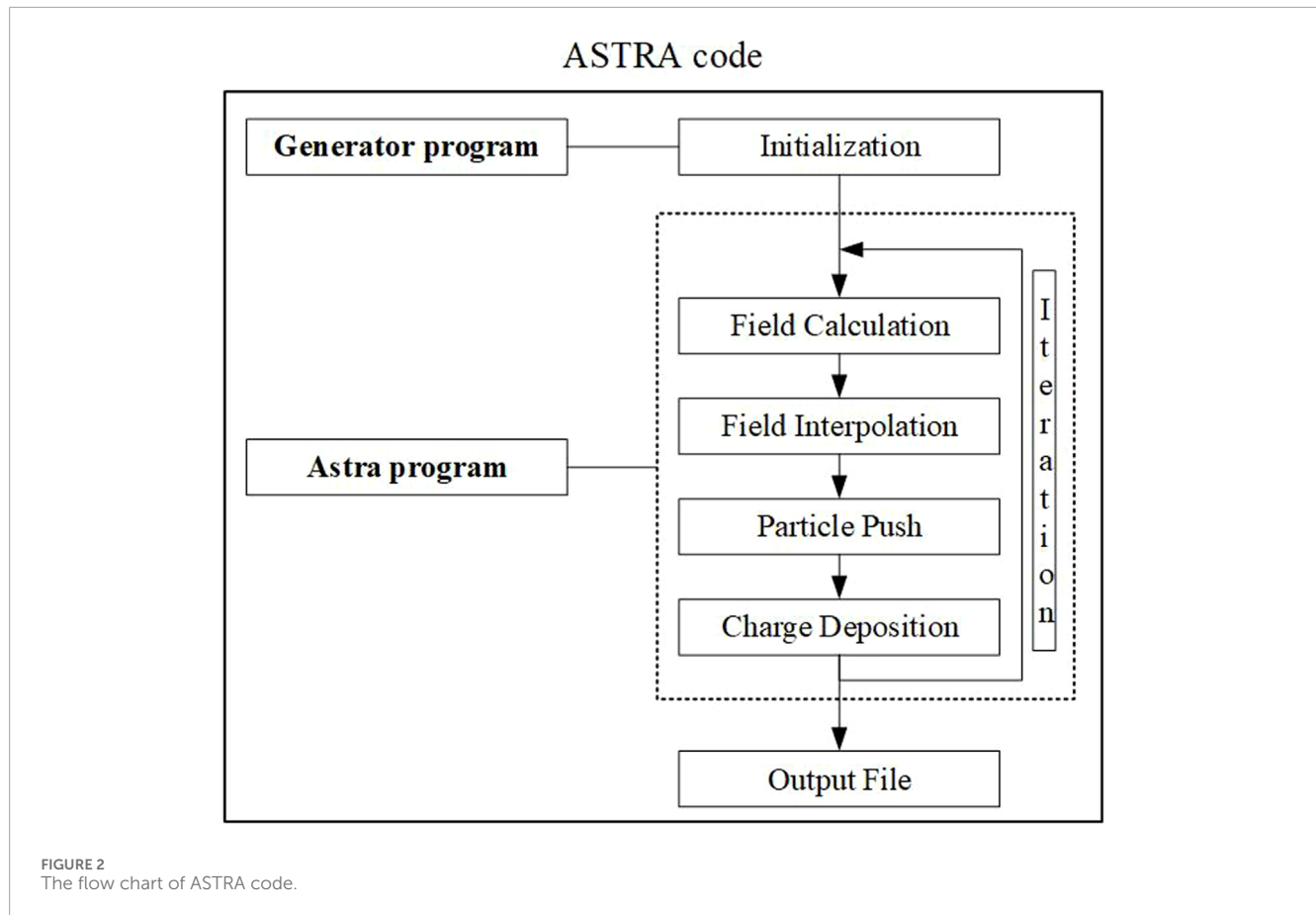
corresponds to an RMS beam radius of 141.4 mm. The total charge is 5.0 pC, and the total emission time is 5 ns, resulting in a current of 1 mA. Figure 3 illustrates the simulated results of the electron beam envelope radius as a function of propagation distance for three different initial trajectory slopes, obtained using the ASTRA program. The simulations were performed with the same beam energy, current, and initial beam radius as those used in Figure 1. The theoretical results presented in Figure 1 are also superimposed as solid lines in Figure 3. The trend of the simulated results (dashed lines) by ASTRA code in Figure 3 is consistent with theoretical results (solid lines). However, the exact simulated beam radius may differ slightly from the theoretical results, as the electron distribution along the axial direction can influence the evolution of the beam envelope radius. In deriving the equations in Equations 17, the electron distribution in the axial direction is not constrained.

2.3 Results and discussion

According to Equations 17, 5, the theoretical electron beam envelope radius depends on energy, current and initial beam radius. Figure 4 presents the variation of beam radius with respect to propagation distance for three different energy levels: $E = 1$ MeV (black curve), $E = 2$ MeV (blue curve), and $E = 4$ MeV (red curve). These results were obtained under the following conditions: an initial current of $I_0 = 1$ mA, an initial beam radius of $a = 0.2$ m, and a zero-initial slope of the trajectory ($r'_0 = 0$). As the propagation distance increases, the beam radius for the 1 MeV case expands rapidly, indicating significant beam divergence. For the 2 MeV case, the beam radius also increases, but at a slower rate compared to the 1 MeV case. In contrast, the beam radius for the 4 MeV case remains nearly constant, suggesting negligible beam expansion at this higher energy level.

Figure 5 shows how the beam radius changes with propagation distance for three different current values: $I_0 = 1$ mA (black curve), $I_0 = 5$ mA (blue curve), and $I_0 = 10$ mA (red curve), while keeping the energy constant at $E = 1$ MeV, the initial beam radius at $a = 0.2$ m, and the initial trajectory slope at $r'_0 = 0$. For all current values, the beam radius increases with propagation distance. However, the rate of increase is significantly influenced by the magnitude of the current. The $I_0 = 1$ mA curve shows a gradual increase in beam radius, whereas higher currents lead to more rapid expansion. The $I_0 = 5$ mA curve displays a steeper rise, and the $I_0 = 10$ mA curve exhibits the most pronounced expansion.

Figure 6 illustrates the effect of varying initial beam radii on the propagation of beam radius with respect to distance. The three cases considered are: $a = 0.2$ m (black curve), $a = 0.4$ m (blue curve), and $a = 0.6$ m (red curve), with the energy fixed at $E = 1$ MeV, the current at $I_0 = 1$ mA, and the initial slope of the trajectory set to $r'_0 = 0$. All three curves exhibit a nonlinear increase in beam radius as the propagation distance increases with the expansion being more pronounced for smaller initial beam radii. Specifically, the case with $a = 0.2$ m shows the most rapid expansion, as the Coulomb repulsion is stronger for smaller initial beam radii.



3 Relationship between the integral form and K-V equation

The derivative of both sides of the Equation 24 gives

$$dz = \pm \sqrt{\frac{2}{K}} \left[F(w) + r_b F'(w) \frac{dw}{dr_b} \right] dr_b \quad (25)$$

where $w = \sqrt{\frac{r_0^2}{2K} + \ln\left(\frac{r_b}{a}\right)}$. The derivative of the Dawson integral is given by

$$F'(w) = 1 - 2wF(w) \quad (26)$$

The derivative of w with respect to r_b is

$$\frac{dw}{dr_b} = \frac{1}{2r_b w} \quad (27)$$

Substituting Equation 26 and Equation 27 into Equation 25 yields Equation 13. The K-V equation is expressed as

$$\frac{d^2 r_b}{dz^2} = -k_0^2 r_b + \frac{K}{r_b} + \frac{\epsilon_r^2}{r_b^3} \quad (28)$$

where $k_0 = \frac{\Omega_L}{v} = -\frac{eB}{2m_0 c(\beta\gamma)}$ represents the focusing effect from the applied magnetic field, and ϵ_r is the radial emittance. Integrating Equation 28 gives

$$\frac{dr_b}{dz} = \pm \sqrt{r_0'^2 + 2K \ln\left(\frac{r_b}{a}\right) + k_0^2(a^2 - r_b^2) + \epsilon_r^2 \left(\frac{1}{a^2} - \frac{1}{r_b^2}\right)} \quad (29)$$

under the conditions of constant k_0 . If the radial emittance is zero and no external force is applied, Equation 29 simplifies to Equation 13. The integral Equation 24 is equivalent to the simplified differential form of the K-V equation, excluding the effects of external forces and radial emittance. The derivative $\frac{dr_b}{dz}$ should be zero at the minimum of beam radius, as deduced from Equation 29. The period of electron beam between two minima of beam radius is determined by several parameters, including the initial beam radius, the initial slope of the trajectory, the magnitude of magnetic field, the current and the electron energy.

It is worth noting that electron energy is included in the expression for both k_0 and K . The combined effect of electron energy depends on whether k_0 or K dominates. When k_0 dominates, a decrease in electron energy results in a smaller beam envelope radius. Conversely, when K dominates, an increase in electron energy leads to a smaller beam envelope radius.

As shown in Figure 7, the four colored curves (purple, cyan, green, and orange) represent the results derived from the K-V equation with a current $I_0 = 1$ mA, an initial beam radius $a = 0.2$ m, an initial slope $r_0' = 0$, radial emittance $\epsilon_r = 20$ mm·mrad and a magnetic field strength of $B = 2000$ nT for different energy ($E = 0.5$ MeV, $E = 1.0$ MeV, $E = 1.5$ MeV and $E = 2.0$ MeV). The four colored curves display oscillatory features, with both the oscillation period and amplitude varying with energy. This oscillatory behavior is also observed in the Beam PIC simulation (Jiao et al., 2022). Electron beams with higher energy exhibit a smaller beam envelope

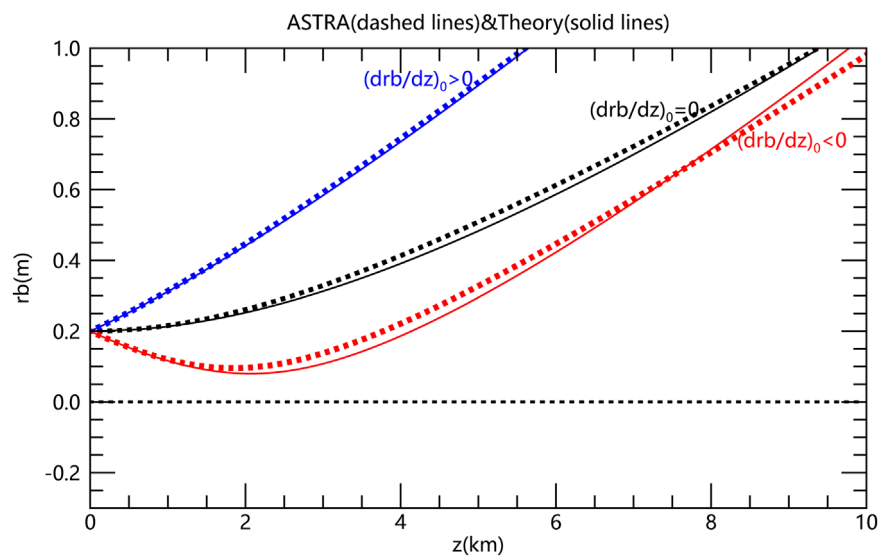


FIGURE 3

The electron beam envelope radius as a function of propagation distance for three initial slopes of the trajectory ($r'_0 = 10^{-4}$, $r'_0 = 0$ and $r'_0 = -10^{-4}$) with an energy of $E = 1$ MeV, current $I_0 = 1$ mA and initial beam radius $a = 0.2$ m simulated by ASTRA code (dotted lines) and compared with theoretical results (solid lines).

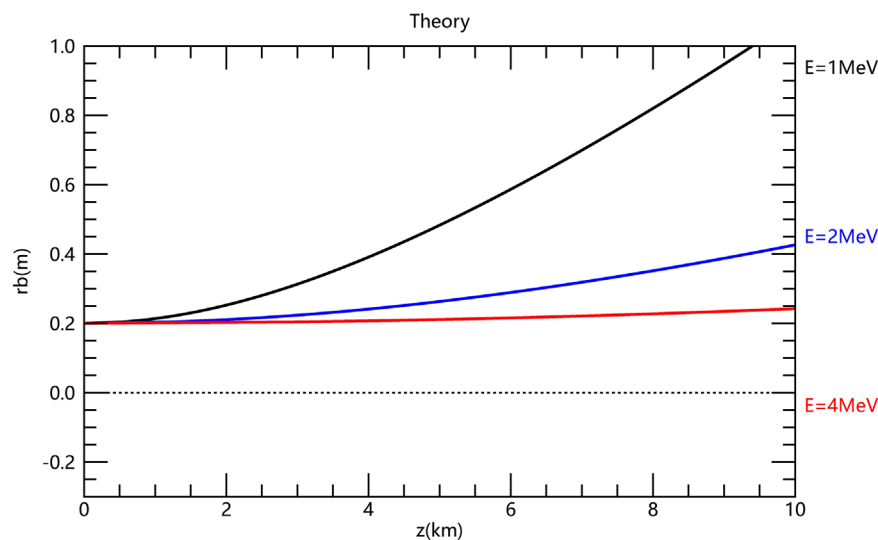


FIGURE 4

The theoretical electron beam envelope radius as a function of propagation distance for three energies ($E = 1$ MeV, $E = 2$ MeV and $E = 4$ MeV) with current $I_0 = 1$ mA, initial beam radius $a = 0.2$ m and initial slopes of the trajectory $r'_0 = 0$.

radius during the initial half period when K dominates. However, the maximum beam envelope radius does not vary monotonically with energy. Specifically, the electron beam with an energy of 1 MeV has the smallest maximum envelope radius compared to electron beams with other energies. The black curve represents the result obtained using the simplified K-V equation with a current $I_0 = 1$ mA, an initial beam radius $a = 0.2$ m but without r'_0 , ε_r or B . In contrast, the black curve does not display any oscillatory features and instead increases monotonically. This black curve is identical to the black curve in Figure 1, which indicates that the integral Equation 24

is equivalent to the simplified differential form of the K-V equation.

4 Conclusion

This paper presents a theoretical and analytical framework for studying electron beam transport. The integral equation for the electron beam envelope in a vacuum is derived. This equation accounts for both Coulomb repulsion and self-magnetic forces.

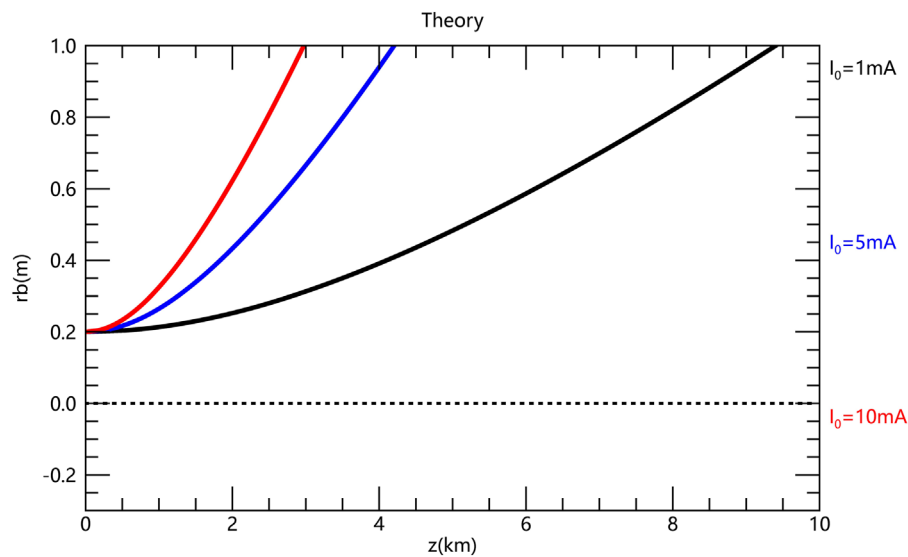


FIGURE 5

The theoretical electron beam envelope radius as a function of propagation distance for three currents ($I_0 = 1\text{mA}$, $I_0 = 5\text{mA}$ and $I_0 = 10\text{mA}$) with an energy $E = 1\text{MeV}$, initial beam radius $a = 0.2\text{m}$ and initial slopes of the trajectory $r'_0 = 0$.

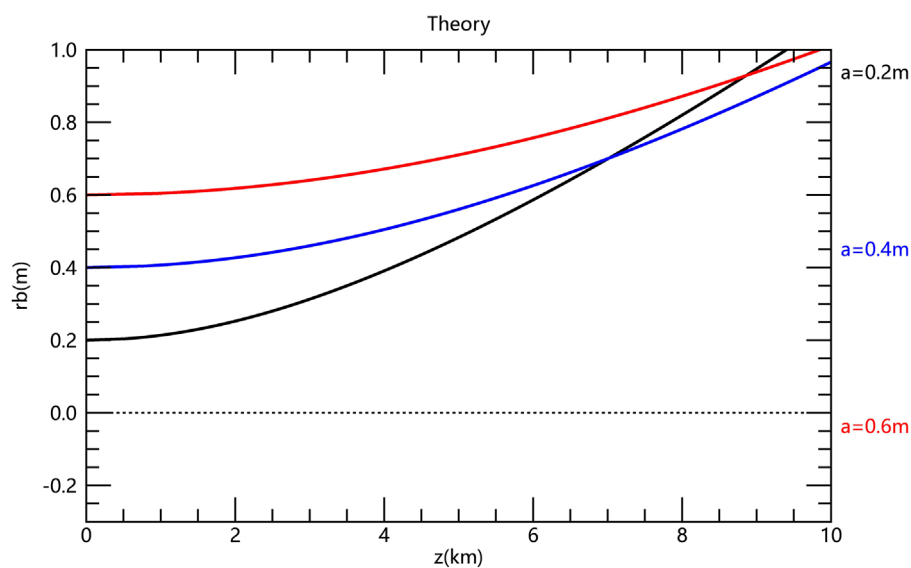


FIGURE 6

The theoretical electron beam envelope radius as a function of propagation distance for three initial beam radii ($a = 0.2\text{m}$, $a = 0.4\text{m}$ and $a = 0.6\text{m}$) with an energy $E = 1\text{MeV}$, current $I_0 = 1\text{mA}$ and initial slopes of the trajectory $r'_0 = 0$.

Although these two forces act in opposite directions, their combined effect causes the beam to diverge.

The radius of the electron beam envelope is determined by several factors, including the initial beam radius, the initial slope of the trajectory, the magnitude of magnetic field, the current and the electron energy. Specifically, a smaller initial radius leads to a larger beam envelope radius. If the initial slope of the trajectory is greater than or equal to zero, the electron beam will diverge. If the initial slope is negative, the beam will first converge and then diverge. The parameter k_0 , which quantifies the focusing strength of

the applied magnetic field, is proportional to the magnitude of the magnetic field. Thus, a stronger magnetic field causes the electron beam to converge. The perveance K , which quantifies the defocusing effect of the beam's equilibrium self-fields, is proportional to the current. Therefore, lower current causes the electron beam to converge. The net effect of electron energy on the beam envelope radius is determined by the relative dominance of k_0 or K . If k_0 dominates, lower electron energy yields a smaller radius; whereas when K dominates, higher electron energy reduces the beam envelope radius.

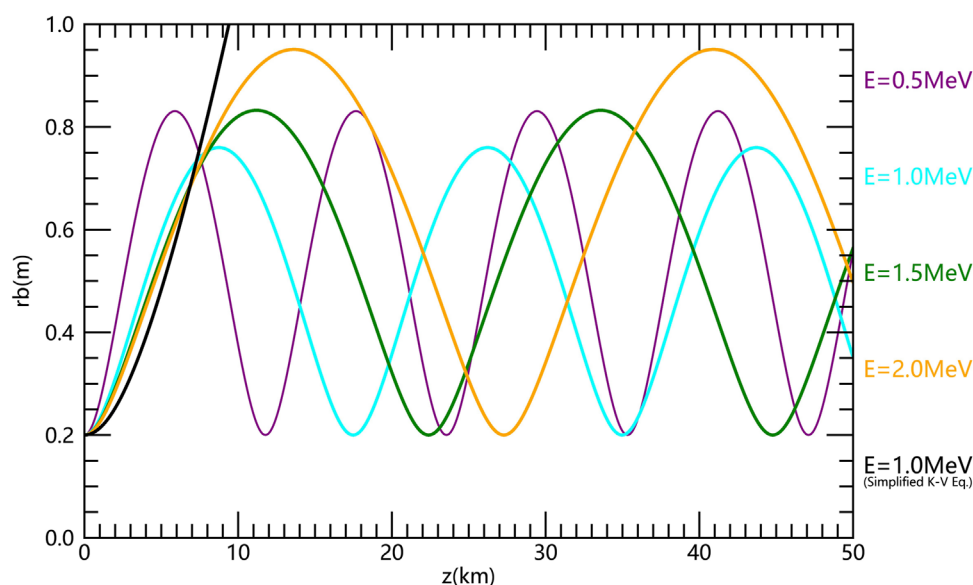


FIGURE 7

The electron beam envelope radius as a function of propagation distance for energies $E = 0.5$ MeV (purple curve), $E = 1.0$ MeV (cyan curve), $E = 1.5$ MeV (green curve) and $E = 2.0$ MeV (orange curve), as determined by the K-V equation. The black curve represents the result for $E = 1.0$ MeV, guided by simplified K-V equation without r_0' , k_0 or B .

This study is a bridge between integral form and differential form of the envelope equation, and will provide a better understanding for the K-V equation and a scientific basis for researching beam propagation technology. It is important to note that, in this study, we assume that the background space to be a vacuum environment, while the actual space environment typically includes background plasma. Consequently, future research will focus on the interaction between the electron beam and the background plasma, and how this interaction influences the evolution of the beam envelope radius.

Data availability statement

The datasets presented in this study can be found in online repositories. The names of the repository/repositories and accession number(s) can be found below: <https://doi.org/10.6084/m9.figshare.29821676.v1>.

Author contributions

CZ: Writing – review and editing, Writing – original draft. XZ: Validation, Writing – review and editing, Investigation. SF: Formal Analysis, Supervision, Writing – review and editing. HZ: Validation, Supervision, Writing – review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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