frontiers | Frontiers in Education

TYPE Original Research
PUBLISHED 20 October 2022
DOI 10.3389/feduc.2022.1040805

RETRACTED: A modified approach to professional learning communities in mathematics: Fostering teacher reflection around formative assessments of students' thinking

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SPECIALTY SECTION
This article was submitted to Teacher Education, a section of the journal Frontiers in Education

RECEIVED 09 September 2022
ACCEPTED 05 October 2022
PUBLISHED 20 October 2022

CITATION
Needs M, Bahr DL, McMillan BG, Nixon RS, Braden QH and Royster R (2022) A modified approach to professional learning communities in mathematics: Fostering teacher reflection around formative assessments of students' thinking. *Front. Educ.* 7:1040805.
doi: 10.3389/feduc.2022.1040805

A team of three third grade teachers utilized a modified approach to Professional Learning Communities (PLCs) based on principles and procedures that characterize Lesson Study to collaborate about their mathematics instruction. They gathered to design mathematical tasks and anticipate the thinking those tasks might elicit. Subsequent to facilitating lessons based on those tasks, they gathered again to compare the thinking they observed to the thinking they anticipated they would see, and then designed additional tasks informed by their observations. This paper reports on an investigation conducted by one of the teachers who assumed the role of a native, participant researcher as she qualitatively studied the nature of the teachers' reflections as they collaborated on five occasions. The six domains of Mathematical Knowledge for Teaching (MKT) were used as a conceptual framework for analysis, particularly in looking for connections the teachers were making between or among MKT domains. Our analysis revealed that the teachers learned to consistently engage in very complex thinking that involved interconnected webs representing multiple MKT domains. Furthermore, evidence suggests that the construction of these webs influenced changes in teacher perspectives on the nature of mathematics teaching and learning and produced an increased interest and ability in "making serious use of student thinking," (p. 11). Such an approach to conducting PLCs appears to possess some potential as a grass roots means of promoting mathematics education reform.

KEYWORDS

PLC, reflection < teacher education, mathematics education, professional development, inservice teacher education

10.3389/feduc.2022.1040805

Introduction

Teacher reflection has been commonly viewed as an important vehicle for encouraging teaching improvement since the 1970's when teachers began to be viewed as reflective professionals who construct conceptions of their practice through reflection (Schon, 1983). In harmony with Dewey's (1933) warning against a mechanical approach to teaching and teacher preparation, Zeichner (1983) called reflective-based teacher preparation and development an inquiry-oriented teacher education "which prioritizes the development of inquiry about teaching and about the contexts in which teaching is carried out" (p. 5).

Current practice around Professional Learning Communities (PLCs; DuFour et al., 2016) is touted as one means of encouraging teachers to reflect on their practice in a manner that engages them in self-driven improvement. PLCs provide a vehicle for collaborative unit and lesson design and a means for examining the effect of the resulting instruction. They comprise a five-step process:

1. Pre-instruction PLC. Teams of teachers plan units of instruction as guided by their core standards; operationalize those standards by developing, finding, and/or adapting a commonly administered end of unit summative assessment; then design specific lessons within the unit often aided by the curriculum available.
2. Deliver Tier 1 Instruction. Each member of the team teaches the lessons within the unit to their students. The predominant mode of instruction in the PLC literature is direct instruction.
3. Administer Common End-of-Unit Assessment. Following the final lesson within the unit, each member of the team administers and scores the common assessment.
4. Post-instruction PLC. The team discusses the data from the common assessment to determine which students met or exceeded predetermined success criteria and which need further remedial support. Students are then grouped according to the type of instruction they need—extension, practice, medium support (2), and intensive support (Tier 3).
5. Post-instruction. Deliver the proposed type of instruction to the identified groups. Direct instruction also typically characterizes the instruction in this step among all groups.

However, there seems to be incongruities between each step of the interactive PLC–instructional process and modern conceptions of mathematics instruction (e.g., National Council of Teachers of Mathematics [NCTM], 2014). First, the pre-instruction PLC does not make serious use of anticipated student thinking that could surface in lesson implementation. Second, inquiry-based instruction is not promoted in the PLC literature, contradicting perspectives associated with modern mathematics education reform. Third, the grouping of students based on assessment data stares in the face of decades of research that has demonstrated that ability grouping in mathematics instruction is ineffective at best and often harmful to students (Schaub and Baker, 1994; Braddock and Slavin, 1995; Stigler and Hiebert, 1997; Okano and Tsuchiya, 1999; Boaler et al., 2000; Boaler and William, 2001; Castle et al., 2005).

There are problems with the overall PLC process as well. Post-common assessment instruction that is based on a summative assessment does little to support a teacher's instructional decisions while teaching a unit. In this traditional PLC model, instead of using formative assessment to support students throughout a unit, support comes after Tier 1 instruction is completed. Thus, it does not provide within-unit opportunities for teacher learning that formative assessments can also provide. Another problem stems from the underlying expectation that some students will fail rather than anticipating what they are actually capable of learning. It represents an ambulatory approach to learning rather than a guardrail approach, i.e., rather than preventing initial failure to learn, it anticipates failure and includes plans to deal with that failure.

An alternative approach to Professional Learning Communities

We propose an alternative approach to PLCs that emphasizes teacher learning through reflection, the letter "L" in "PLC," that aligns with modern conceptions of mathematics education, and that could even reduce the need for remedial Tier 2 instruction. Our proposal is grounded on four lines of mathematics education research: the Comprehensive Mathematics Instructional Framework (CMI; Hendrickson et al., 2008), Instructional Rounds (City, 2011) and the related practice of Lesson Study (Hurd and Lewis, 2011), and Generative Growth (Franke et al., 2001).

The comprehensive mathematics instructional framework

The CMI Framework is a response to Chazan and Ball's (1999) frustration that educators are often left "with no framework for the kinds of specific, constructive pedagogical moves that teachers might make" (p. 2). It comprises three major components: a Teaching Cycle, a Learning Cycle, and a Continuum of Mathematical Understanding. The Teaching Cycle is based on work out of Michigan State University (Schroyer and Fitzgerald, 1986) and begins by a teacher presenting a worthwhile task (launch), allowing students time to productively struggle (National Governors Association Center for Best Practices, and Council of Chief State School Officers, 2010) with the mathematics of the task (explore), and ends with a teacher-orchestrated class discussion in which student thinking is shared and examined by students (discuss). Formative assessment is a critical component of all three stages of the Teaching Cycle and is accomplished through the wise use of open-ended questioning.

The framework clearly outlines that each stage of the Learning Cycle (develop understanding, solidify understanding, and practice understanding) will look different depending on the purpose of the lesson. Those purposes vary based on teachers' continuous assessment of the details of student thinking, where the thinking fits within the progression of solution strategies (Empson and Levi, 2011; Carpenter et al., 2015) and how that thinking coincides with the Learning Cycle. Student understanding progresses from initially surfaced thinking in a math task (develop understanding) to a more solid understanding through the examination and extension of the surfaced thinking (solidify understanding), and then is refined and generalized (practice understanding).

The third component of the CMI Framework is the Continuum of Mathematical Understanding which comprises three distinct domains: conceptual understanding of mathematics, procedural understanding of mathematics, and representational understanding of mathematics. The thinking within those three domains changes as it moves through the phases of the Learning Cycle—indeed, as students make connections within and across domains (Hendrickson et al., 2008).

Instructional rounds and lesson study

City (2011) described Instructional Rounds as "a disciplined way for educators to work together to improve instruction." Instructional Rounds serve to support teachers in reflecting and learning from their own practice through an inquiry process which involves "assembling a network of teachers, defining problems of practice, observing in classrooms, and debriefing those observations, and identifying the next level of work" (City, 2011, p. 2).

Lesson study (Hurd and Lewis, 2011) is a widely used process that resembles Instructional Rounds collaboration because it involves teams of teachers finding, adapting, or constructing a worthwhile mathematical task, i.e., curriculum planning, then making "serious use of student thinking" (Ball, 2001, p. 11) by anticipating the thinking they believe the task will elicit. The team uses that anticipated thinking to design a lesson and then one member of the team delivers the lesson to her or his students while the rest of the team observes. After the lesson has been taught, the team meets to debrief the lesson plan during which they compare the observed thinking to the anticipated thinking and adjust the lesson plan accordingly for use by the other members of the team. Debriefing sessions also may include subsequent lesson planning. Franke et al. (1998) called the improvement process fostered by such collaborative reflection that focuses on student thinking "generative growth." She later reported (Franke et al., 2001) that generative growth results in enhanced teachers' knowledge of mathematics and of students' mathematical thinking as well as changes in their practice that place student thinking front and center in instructional design and implementation.

An alternative approach

Using this literature review as conceptual grounding, the following steps outline an alternative approach to PLCs for the purposes of promoting meaningful teacher reflection in service of enhancing student achievement through inquiry-based Tier 1 instruction.

1. A team of teachers (PLC) design a task together that will surface developing understanding.
2. The PLC anticipates student thinking based on the progressions of student strategies and in connection to all three domains of mathematical understanding–conceptual, procedural, representational—which informs the rest of lesson design.
3. Rather than a formal, summative assessment at the end of instruction (unit), the common assessment consists of a task that is launched for both instructional and assessment purposes and questioning that occurs as students engage with the task (in other words, formative assessment) (This step does not suggest that common end-of-unit assessments are not also valuable.).
4. The post instruction debriefing (PLC) involves categorizing observed student thinking based on a progression of student understanding, then comparing it to the thinking the teachers anticipated they would see.
5. The PLC uses the observed thinking to plan subsequent tasks and lessons to build solidify and practice Understanding.

This approach to PLCs focuses on the instruction of a current unit rather than the instruction that has just been completed. Therefore, because it involves anticipating student thinking about a task that is about to be launched, it provides an environment for the improvement of the instruction around that task as teachers reflect on their own practice—the presentation of a task, the quality of the questions they intend to pose, and organizing and orchestrating discussion during which students engage in each other's thinking. It also paves the way for higher quality subsequent lessons because unlike moving to Tier 2 and Tier 3 instruction as a result of examining end-of-unit common assessment data, the PLC plans subsequent tasks that begin the process of examining and extending, then refining and generalizing student thinking that first surfaces at the beginning of a unit.

Lens, purpose, and research questions

The purpose of this study was to investigate the teacher reflections that surfaced as a third grade team utilized the alternative PLC process. The lens through which the study was conducted was provided by the Mathematical Knowledge for Teaching (MKT) model (Ball et al., 2008) which is portrayed in **Figure 1**.

Ball (2011) posits that "teaching mathematics is a special kind of mathematical work" (n.p.), and she and her co-creators of the MKT model sought to construct a practice-based theory of content knowledge *for* teaching. They investigated "the nature of professionally oriented subject matter knowledge in mathematics by studying actual mathematics teaching and identifying mathematical knowledge for teaching based on analyses of the mathematical problems that arise in teaching" (p. 389). Their research revealed two empirically discernable categories which appear as headings on the figure: subject matter knowledge and pedagogical content knowledge. Within each category are three domains listed and described here:

1. Common Content Knowledge. Mathematical knowledge that is known and used "in settings other than teaching" (Ball et al., 2008, p. 399) such as in knowing multiple algorithms for multi-digit subtraction.
2. Specialized Content Knowledge. Mathematical knowledge known only by teachers, such as knowing the relationship between multiplying and dividing whole numbers. It involves "nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive of specialized content knowledge" (p. 401).
3. Horizon Content Knowledge. "[A]n awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403) such as in recognizing the connections between multiplying whole numbers and multiplying fractions.
4. Knowledge of Content and Students. "[K]nowledge that combines knowing about students and knowing about mathematics" (p. 401). This knowledge allows teachers to anticipate and interpet student thinking.
5. Knowledge of Content and Teaching. "[C]ombines knowing about teaching and knowing about mathematics" (p. 401). Combining content and pedagogical knowledge produces a "mathematical knowledge of the design of instruction" (p. 401).
6. Knowledge of Content and Curriculum. Knowledge of available curricular materials and expectations in former and future grades. This allows a teacher to situate their instruction in the sequence experienced by students.

Using MKT as a lens, we first sought to characterize the reflections of the teachers in our study. Therefore, we asked the question: How did the teachers talk about issues relating to mathematics teaching and learning while reflecting on their practice as they participate in the alternative approach to PLCs? Inasmuch as this alternative PLC process is designed to promote teacher learning, a second question was addressed: How did the reflections of the teachers change over time—the two novice teachers on the team as well as the veteran team leader?

Researchers who have added to the knowledge base around MKT support the notion of a specialized type of knowledge for teaching and that it is useful to recognize that there are multiple categories or domains of that knowledge. However, they also emphasize the importance of mental connections between the knowledge within those categories and domains. That is, if, as Silverman and Thompson (2008) suggested, "an expert teacher is one with organized knowledge bases that can be quickly and easily drawn upon while being engaged in the act of teaching" (p. 501), a high value must be placed on "synthesized knowledge" (Hurrell, 2013, p. 56). This synthesized knowledge is constructed through "purposefully integrated experiences that allow teachers the opportunity to not only extend their mathematical and pedagogical understandings but also to create connections to create a new knowledge" (Hurrell, 2013, p. 56). Therefore, not only did we utilize the MKT model to analyze teacher reflections in their PLCs, but we also looked deeper at the connections teachers were making between the components of knowledge they were constructing across categories and domains. The third research question guiding this study was: While using the alternative PLC process, what was the nature of the mental connections teachers made between the types of Mathematical Knowledge of Teaching they were constructing?

Materials and methods

Participants

Participants for this study were the three members of the third-grade team at a middle-class, predominantly white elementary school in the Intermountain West that included the lead author. They are all white females, two in their first year of teaching, and one, the lead author, in her seventh, and their classes comprised 73 students. Their school district provided time each week for PLCs, and all participants participated in five PLC meetings led by the first author at the school

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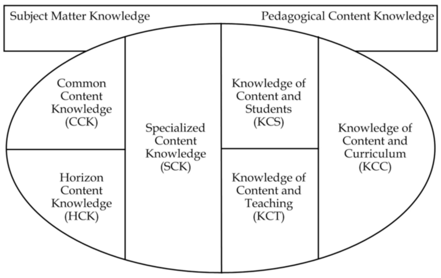

FIGURE 1
Domains of mathematical knowledge for teaching.

throughout 8 months of the 2021–2022 school year using the alternative PLC process.

Procedures for gathering and analyzing data

The PLC meetings were video recorded and transcribed for analysis. The first author assumed the role of a native, participant researcher (Gall et al., 2006). The biases inherent in such an approach that could threaten the validity of the study were exposed and reconciled as she and the second author worked together to establish a consistent and credible analysis. During the initial portion of each phase of analysis–as outlined below–they mutually created observation notes as they watched each recording and began coding which served to establish coding consistency. To provide credibility, they worked separately for the remainder of each phase, then compared analyses. If analysis comparison revealed differences, they negotiated until a consensus was reached.

During the first pass through the data, coding was guided by the six domains of the MKT model. The coding during the second pass fleshed out one of the six domains—Knowledge of Content and Teaching—using a component of the CMI framework, the stages of the Teaching Cycle. The most critical part of the launch phase is the presentation of a task, i.e., presenting the right task at the right time (Reys and Long, 1995). The quality of a teacher's engagement with students during the explore stage is highly dependent upon the quality of his or her questioning (National Council of Teachers of Mathematics [NCTM], 2014). The effectiveness of discussion orchestration depends to a great degree upon the selecting of what student thinking will be shared and in what order (Stein et al., 2008) and engaging the students with each other's mathematical ideas (Webb et al., 2014; Franke et al., 2015).

The third pass through the data involved interpretation of teacher reflections within each code. A fourth pass involved compiling a research text that organized segments and constructed text to connect ideas from all participants (Saldaña, 2015). The fifth and final pass was conducted to create an analytic memo that organized the codes from the observation notes and the transcripts into emerging themes across all collaborations in order to investigate the nature of the mental connections the teachers were making. As "coding is analysis" (Miles and Huberman, 1994, p. 56) analytic texts were created and refined during each pass through the data. A general history of the collaboration was constructed and then codes analyzed to determine the main findings to answer the question about the nature of teachers' reflection on their practice and how that reflection changed over time.

During this analysis, other themes arose related to the first and third research questions as we contrasted the teachers' reflections across PLCs. First, we noticed an increase in the number of comments made by the novice teachers. Therefore, the number of comments made by each member of the team during each PLC was counted. Then, to get a sense for the relative richness of those comments, the number of lines that comprised each comment was also counted. Furthermore, it was observed that there was a gradual shift in the roles that each teacher fulfilled as demonstrated by each teachers' use of questioning to prompt discussion. Therefore, the number of questions asked was counted as well. Finally, we noticed that there seemed to be changes in the number of MKT domains that were being addressed, so the number of comments related to each domain was counted. This quantification was followed by comparing the counts obtained from each team member across PLCs.

It is important to note that each PLC we observed differed from the others in one or more ways. PLC 1 was a pre-teaching collaboration meeting about addition and subtraction, that is, a collaboration that focused on anticipating the student thinking that might surface in a future lesson. PLC 2 was a post-teaching collaboration in which the anticipated thinking from PLC 1 was compared to the thinking observed during the actual lesson. The lesson associated with the first two collaborations was an introductory lesson at the beginning of the unit designed to surface a substantial amount of varied student thinking (develop understanding). PLC 3 was also a post-teaching collaboration where the team compared the anticipated thinking from their lesson that surfaced understanding about fraction number sense to the observed thinking during the actual lesson. PLC 4 was a post-teaching collaboration also focused on fraction number sense except that the purpose of the lesson was to examine and extend thinking (solidifying understanding) surfaced from a previous lesson focused on developing understanding. PLC 5 was both a post-teaching collaboration and a pre-teaching collaboration relating to lessons about fraction equivalence. That is, the team compared the thinking they anticipated would surface from a lesson focused on developing understanding to the actual thinking observed during the lesson, then went on to anticipate student thinking for another lesson focused on solidifying understanding related to equivalence.

During the negotiations involved with analyzing these data, it was quite common to have to decide between two or more MKT domains. In many of those instances, it was agreed that a segment data could be coded with more than one domain, and, more importantly, that the teachers' reflections actually represented a mental connection, or interaction, between two types of thinking. Some of those connections were made between two domains or among all three domains within the same MKT category, and other connections were made across categories (e.g., CCK and SCK).

Findings

Research questions 1 and 2

In an effort to address the first research question about how the teachers talked about issues relating to mathematics teaching and learning, as well as the third question about how the reflections changed over time, a few quantitative analyses were conducted. The number of comments made by each teacher was counted as well as the number of lines that comprised each comment in order to obtain a measure of the relative richness of those comments.

Table 1 portrays the numbers of comments made by each teacher across all five PLCs as well as the total number of transcript lines that comprised those comments.

There was a considerable change in the difference between the number of comments made by Makayla, the team leader, and the other two team members. By PLC 5, all three teachers made about the same number of comments. Likewise, when the two novice teachers commented in the later PLCs, they not only made more comments, but they also had more to say.

In order to document the gradual shift in the roles that each teacher fulfilled, the number of questions asked to prompt discussion was counted as well as shown in Table 2.

TABLE 1 Comments made by each teacher during the five PLCs.

Teacher	PLC 1	PLC 2	PLC 3	PLC 4	PLC 5
Makayla	23.159	18.111	8.49	41.152	21.106
Heather	12.46	9.54	4.34	34.152	21.116
Kate	11.45	5.23	8.64	35.130	20.82

TABLE 2 Number of questions posed by each teacher during the five PLCs.

Teacher	PLC 1	PLC 2	PLC 3	PLC 4	PLC 5
Makayla	16	6	7	24	12
Heather	1	4	0	8	7
Kate	8	2	5	8	10

Over time, an increase in the number of questions the novice teachers asked in relation to the number of questions the team leader asked was observed. In the first PLC, most of the prompting for discussion and reflection was provided by the questions asked by the team leader. By the fifth PLC, the combined contribution to the discussion made by the prompting questions of the other team members exceeded that of the team leader.

Table 3 depicts changes in the number of MKT domains being addressed by the teachers across the five PLCs.

The most common MKT domain discussed was KCS while the least common was HCK. In fact, both KCS and KCT were the most discussed domains.

Research question 3

With regards to the third research question, we found several kinds of MKT connections across all five PLCs, but the findings discussed here will primarily focus on the first PLC, recognizing that it provides examples that appeared in all five PLCs. At the end of the section "Findings," however, other PLCs will be touched upon to provide evidence of connections within the other PLCs (In order to preserve the confidentiality of the two novice team members, pseudonyms were used in lieu of their real names. The team leader's name was not changed as she is the lead author. All three teachers provided formal consent for their use of their comments in this report.).

Connecting CCK, KCS, and SCK

This portion of PLC 1 begins with Makayla, the team leader inviting the teachers to use their CCK to solve a multi-digit addition task themselves in multiple ways. Interestingly, she also invited them to use their CCK in the service of KCS by indicating their own solutions will help them anticipate what their students might do.

> Makayla: let's take what like three minutes to solve it ourselves. Sure. Maybe the first way how we would solve it and a second way. How could we see them solve it?

TABLE 3 Mathematical Knowledge for Teaching domains addressed across all five PLCs.

MKT domain	PLC 1	PLC 2	PLC 3	PLC 4	PLC 5	Total
CCK	25	0	2	1	11	39
SCK	20	4	8	4	4	40
HCK	3	0	0	6	1	10
KCT	33	16	12	8	10	79
KCS	31	9	10	69	4	123
KCC	17	5	4	19	6	51

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Kate then shows some deep mathematical understanding of a specialized nature (SCK) as she recognizes that different representations still relate to the same basic strategy.

> Kate: Having a hard time. These, these are like all, pretty much the same strategy. They just look different.

After two of the teachers say they have three or four strategies, Heather shares her thinking.

> Heather: There. Okay. So I did typical standard algorithm cause I figured there's gonna be kids who know how to do it.

Interestingly, Heather's KCS informed her choice to use the standard algorithm, one of the strategies in her CCK repertoire, even though she was just asked to solve the problem for herself. Heather continues by sharing another strategy.

> Heather: Yeah. And then I also broke up. I, what I should have written up here was one hundred, 63 plus, and then a parenthesis like, uh, 10 plus 10 plus six to represent the 26. Yeah. So yeah, just added the tens and then the last little ones.

Kate also used the standard algorithm but then adjusts it by making flexible use of some of the digits, an invented strategy that is a component of her CCK.

> Kate: I use the standard algorithm 'cause my, my brain goes, I was taught mm-hmm and then I do the ones first. So three plus six equals nine and then the tens, 60 plus 20 plus 80, then added those two together. So 80 plus nine is 89 And then add the last hundred. So it's 180.

> Makayla: Oh, nice. How we've been doing it for yeah. Number talks. Mm-hmm I thought, oh, that's probably how the kids will do it. Smart thinking. We'll have to like note that one for anticipated thinking for kiddos. Yeah.

Makayla recognized the invented strategy in Kate's thinking (CCK) then remembered that she had seen students invent the same strategy from previous lessons (KCS). Heather then talked more about the standard algorithm and why she thought about it. She started by describing her strategy (CCK), described the thinking she has observed from her students (KCS), then made the connection between the two (SCK).

> Heather: I started with the standard algorithm, cuz that works best for me. Um, and then I've noticed some of my kids it's pretty much the same thing, but they're doing it, um, horizontally and they're just like drawing arrows to get to basically the same.

Using CCK-KCS-SCK connections to construct KCT

Other types of connections were made as the teachers continued reflecting in the same PLC. Heather could see the relationship between the standard algorithm (CCK) and the students' strategy (SCK). Her efforts to interpret student thinking (KCS) is connected in her mind with her understanding of the standard algorithm, i.e., CCK. In the following exchange, Kate's reflection also exemplified the same KCS-CCK relationship. She anticipated her students would surface very complex strategies (KCS), then switched to reflecting on her own CCK as she described her flexible use of the digits, then connected it to what she has seen her students do in previous lessons (SCK), but of course this connection enabled her to see the tasks in multiple lessons ("number talks") cohesively which enhances her knowledge of content and teaching (KCT).

> Kate: So they're gonna go. Okay. Well the 100 is gonna move over and then the six and the two we're gonna make that eight and three then six will be nine. And then I split it into tens in one. So a hundred plus 60 plus three or a hundred cents in ones, I guess. Um, plus 20 plus six. And then I added so the hundred stayed and then, um, grouped the tens and the ones. So it would be a hundred plus 80 plus nine. And then one of the things that I think my kids are probably gonna do, just cuz of how we've been doing the number talks is um, instead of 163 plus 26, they're gonna change that 26 to a 20 and add the six to, to the bigger number. And then they'll just pop that 20 on the end.

The CCK-KCS-SCK connection also fosters KCT in this reflection by Makayla—but not only KCT in terms of seeing relationships between tasks and lessons, but also organizing and sequencing the sharing of thinking that will take place as she orchestrated a discussion. Then she compared the way the teachers are representing their thinking to the way she anticipated her students will represent theirs.

> Makayla: (While numbering the strategies) Yeah. Okay. So I think we'll definitely see standard algorithm. Mm-hmm I think we'll see this one too. Cuz my kids have done that one. Mm-hmm and I think we'll see the number talk one you brought up. I, I was thinking cause they solved like how I would solve, right? Yeah. But I know. Yeah. All kids that'll like switch, manipulate those numbers around. Yeah. And then I also think kids will do the second one. This one you did Heather. Okay. I don't know about you guys, but I could see my kids doing it, but I think they would write it a little bit differently.

Heather admitted to coming to an understanding of a student's strategy from the PLC (KCS).

Heather: I think they would too. I couldn't figure it out until I saw what you did.

Extending Mathematical Knowledge for Teaching across grades (HCK) while integrating the use of KCS, SCK, and KCT

Kate connected the strategies they were discussing with a method apparently taught in second grade—an example of Horizon Content Knowledge (HCK)—which then helped her anticipate another way her students might think about the problem (KCS), but she also saw the progression from one strategy to another (SCK) which then served to inform the way she sequenced the students' sharing in a discussion (KCT).

Kate: I was just thinking, I learned some second-grade teachers called the rollout method, but I've been trying to get them to use like expanded form addition. I don't know. There's a natural lead into it, but well that's what they're doing is writing it in standard form. Exactly.

Makayla also reflected on her HCK by connecting their anticipations (KCS) with the mathematics of future grades.

Makayla: That's gonna make more sense as they move throughout the grades.

Makayla then described thinking that she has seen before (KCS) that is partially conceived (CCK), connected that thinking to other strategies (SCK), talked about the mathematics she hoped that thinking will evolve into thus showing more of her CCK while also using that knowledge to sequence the sharing in an upcoming discussion (KCT).

Makayla: When I've had kids who like have a partial understanding of regrouping, someone mentioned having them try solving it this way. Cuz then they can see where the 10 goes because it's very explicit. Like here's the ones. When you make a 10, you need to move it to that tens place rather than just that little one on top mm-hmm yeah. So I think this will be the three strategies we'll see. That's great. Um, no more [shared strategies] than that. 1, 2, 3.

Heather sees connections between teacher strategies, an example of Specialized Content Knowledge (SCK), including the strategic use of expanded form, a solid strategy. Then she connected that knowledge (SCK) to her own knowledge of number lines (CCK) which in turn informed her anticipating (KCS).

Heather: Yeah. Similar to like what you did. Maybe it would be what Kate did. Yeah. Where you grouped your ones first mm-hmm. I love number lines. Yeah. Where they might add the ones, tens, hundreds. Or if they'll add, like I'm gonna do 163 plus 20 and then add six after that.

As the teachers were discussing the various strategies, number lines appeared to be a benchmark representation (CCK) because of the support it provides the teachers in thinking through several strategies (SCK).

Kate: That's my favorite strategy for everything. Like when in doubt use a number line.

Heather used her HCK in describing a student who uses "number points," a teacher-taught strategy from second grade that requires to write or visualize dots on numerals. The teachers have quite a conversation about the worth of number points as they consider the lack of understanding using them seems to imply in the minds of their students—using their CCK to inform their KCS. Given that conclusion about number points, Makayla suggested students who use number points be invited to solve problems with base 10 blocks (KCT) anticipating that the students will begin to conceptualize what is happening with the numbers (KCS). Makayla's suggestion also implied the use of SCK because of the connections she had made between the strategies in her own mind.

Heather: I've been having a student use number points. Like they're what is that? So they do number points. Like they should be like six plus 1, 2, 3, cuz there's three points on that number. Oh how do I keep them moving on, to get away from that? So they have to like touch the number in order to use it.

Makayla: Interesting. I wonder if they have a hard time like conceptualizing what the quantity of three is or what the quantity of six is. So maybe what we want to give that student is blocks.

Kate: Mm-hmm. That might help. So they could visually see. Okay. Mm-hmm cause touch points. I know they do that a lot in the younger grades.

Heather: Cause that's what she [the student] says. "I wanna use touch points cuz that's how I was taught." And I say, okay mm-hmm touch Points could be confusing too. And you're gonna lose track of how many points are on that.

Makayla: Well it's like does a touch point actually represent a quantity of something? No. Touchpoint is like how you memorize, like writing. Yeah. Yeah.

Now Makayla applied her knowledge of the specific student (KCS) to thinking about how to help a student move on from touch points (KCT) because she saw the relationships between strategies (SCK).

Makayla: So I think, yeah, maybe we gotta move that person to maybe we should start with a smaller number for that kiddo. Okay. Maybe something as simple as just adding the six and the three with blocks.

The teachers then decided which type of blocks would be most useful. They knew where their students' thinking was developmentally speaking (KCS) and understood the mathematical advantages of different manipulatives (CCK) which helped them make another instructional decision (KCT).

Makayla: That would be easier than the linking cubes because who's gonna want to count out 163. I'm sure there are kids. I'll put it on here as an option.

Utilizing their curriculum—integrating KCC with other domains

As their PLC continued, they started evaluating the quality of tasks in their curriculum (KCC) using their mathematical knowledge (CCK) and their knowledge of their students' thinking (KCS). They then though about creating their own task that would invite students to build on a strategy they were already using (SCK) and the order in which those tasks would be presented (KCT).

Kate: Do we think this task is gonna be too simple for them? I bring it up because we don't have any making a 10 or making a hundred. Oh like those are friendly numbers to add six and nine, six and two or sorry, six and three mm-hmm. I would like it if there was a carryover, if you're doing standard algorithm, or something.

Heather: We could do this for like the very first problem and then move into maybe a second one. I like that idea. Great. Cause We'll just give 'em a starting point and some strategies and, and almost like a warmup. Yeah. That's a good idea.

Heather continued evaluating the task from the curriculum (KCC) by thinking through the way she would solve it (CCK) to see if it aligns with the kinds of thinking she hopes to engender which in and of itself requires a knowledge of student thinking (KCS) and a knowledge of instruction (KCT).

Heather: So we could use this one to get those strategies out and then see. Okay. Can you use those same strategies on one where you have to make a 10 or like a hundred? Because like this strategy, if you have to make a 10, you're not gonna be able to use that strategy.

Heather also saw the connections between observed and hoped for strategies (SCK). Now Makayla evaluates a task by using her knowledge of how to solve it (CCK) to determine if it would surface the thinking she wants (KCS) which happens to be thinking she has seen before (KCS).

Makayla: It'd be interesting to see how they would. Yeah. Yeah. Cause that's why I saw that like, okay, this is a good strategy to use for this problem. But if you're gonna have to make a 10, you're not going to want to use that.

Other Professional Learning Communities

Our analysis revealed numerous PLC conversations in which connections between and among MKT domains were being made and reflected upon. Indeed, all five PLCs we analyzed could have been used to demonstrate the nature of the connections the teachers were making, however, here are two brief conversations from two other PLCs that evidence the widespread integration of MKT domains in the minds of the teachers. Makayla used KCS to anticipate how her students would think about the mathematical topic, Kate used that information to suggests a task (KCT), and Heather selected another task from the curriculum to follow the first task (KCC).

Makayla: I know I'm gonna have some that are gonna struggle with that and this could be a good strategy.

Kate: Okay. So the first problem we have is this one, see animal stickers. Oh duh. If we don't have place value blocks, we can use the stickers. Oh then we can honestly just make copies of the tens and then singles and just leave 'em on a table. Yeah. Okay. Okay. I like that.

Heather: Um, the next one we should just do a problem similar to that, right, like this one here.

In the following conversation, Makayla uses her KCC to adjust a task from the curriculum, then KCT to propose an

10.3389/feduc.2022.1040805

additional task, a question to ask in the Explore, and ways to engage students in other students' thinking, then encourages the team to use CCK to create a representation associated with one of the tasks in their curriculum which enables them to anticipate the thinking the task might surface (KCS).

> Makayla: So my thought was to like, for your class, Kate was to do what Heather was saying, but maybe take out the number line at first and just draw the two rectangles and just like pose the question of like, how much do you see up here? How many wholes are there? Okay. How do we write that? And then, And then write it as a fraction as well. Cause I'm sure. It'll just say, oh, that's two.

Discussion and conclusion

The MKT model provides useful domains for describing, studying, and improving mathematics teaching. Another use of the MKT model inolves the mental connections or interactions between those domains. Our analysis revealed some very insightful connections the teachers were making between domains, both within the two model categories—subject matter knowledge and pedagogical content knowledge–and across categories. As connections were made, one domain of knowledge informed the use of another, knowledge in one domain was applied for use in another, or growth in two or more domains occurred simultaneously.

With the emphasis on student thinking inherent in the alternative PLC process, it is not surprising that KCS played a very prominent role in the teachers' reflections. It influenced other types of knowledge and in turn, was influenced by other types as well. When a teacher used her own math knowledge to anticipate student thinking, her CCK influenced her KCS. We also saw the reverse, that is, making sense of student thinking (KCS) enhanced the teachers' own math knowledge (CCK). Thus, KCS used in anticipating student thinking informed the teachers' knowledge of content and teaching (KCT) with regards to several aspects of lesson design and implementation—creating or evaluating a teacher-constructed task, determining the order in which tasks are launched in a lesson, developing questions during the explore stage, determining what student thinking will be shared and in what order in the discuss stage, as well as designing the ways students will engage with each others' thinking in that stage.

Likewise, the teachers' CCK enabled them to assess student thinking (KCS) in the course of teaching when they could see their own thinking in the thinking of their students. The teachers' KCS also enhanced their knowledge of content and curriculum (KCC) when they evaluated, adapted, and sequenced tasks obtained from their curriculum. The teachers' specialized and horizon content knowledge, i.e., knowledge of the connections between math topics within and across grades (SCK and HCK), empowered their KCS at times when anticipating student thinking, and vice versa, their anticipating of student thinking informed their HCK as they reflected on the use of invented strategies in future grades.

The teachers' use of their curriculum (KCC) was also enabled by their CCK when they used their own math knowledge to evaluate, select, and modify tasks in the curriculum. Their CCK also enabled them to anticipate how student thinking might change over time (KCS) and to design instruction to facilitate that change (KCT). Their SCK and HCK also facilitated the instructional decisions they made (KCT), particularly when surfaced previous learning revealed a lack of conceptual understanding (KCS).

Implications for the practice of mathematics teacher educators

Mathematical Knowledge for Teaching has become a useful model in informing the work of mathematics teacher educators in preservice and inservice settings (Chapman, 2017). Some of the more recent encouragements for its use in those settings also highlight the construction and value of connections across MKT domains. Based on Gess-Newsome and Lederman (1999) proposed integrative and transformative models for pedagogical content knowledge, Hurrell (2013) suggested conditions should be developed that encourage "creat[ing] connections to create a new knowledge" (p. 56). Indeed, he went so far as to recommend a revised MKT model that emphasizes connections.

Recognizing the roots of MKT in Shulman's pedagogical content knowledge (Shulman, 1986), Hurrell (2013) suggests that in order for teachers to "become more responsive when the opportunities for development of PCK in the teaching work place present themselves, teachers need to be given purposeful development opportunities to reflect upon their teaching. This may take the form of having opportunities to observe, analyze and reflect upon other teachers' teaching" (p. 62).

Beyond the analyses of the connections the teachers were making between domains of their MKT, we derived some generalizations about the alternative PLC process itself. The PLC seemed to be a environment where all participants felt safe enough to express their views openly which no doubt contributed to enhancing the MKT of all team members, but especially the two teachers new to teaching—contrary to the common problem associated with PLCs of stifling creativity and innovation (Miller, 2020). The safe environment was created by engaging teachers in inquiring about their own MKT in the service of making serious use of student thinking in their teaching, meaning, using inquiry with the teachers to help them learn to use inquiry with their students. The team had a positive experience with the PLC meetings because just as in inquiry-based teaching, the leader did not dictate what was to be done and how to do it, but rather, the team members were true collaborators. Instead of consistently discussing summative end-of-unit data, making the discussion of student thinking obtained through on-going formative assessment appeared to help the teachers feel less vulnerable in talking about data than they would otherwise, and therefore more open to change. In normal PLCs, teachers frequently compare themselves to each other on the basis of their students' summative unit assessment scores (Provini, 2013). This alternative process focuses more on student thinking and not on raw scores, (on formative not summative assessment), and finding out where students are in their thinking—not on evaluating them using traditional tests.

The mixture of personalities and years of experience seemed to augment the safe environment. The team leader (and lead author of this paper) was in her seventh year of teaching who has taken several mathematics education courses and earned an elementary mathematics license endorsement in doing so. She has also conducted professional development with the second author, and currently serves on a district mathematics committee of school-based leaders. She served in a "brokering" role (Wenger, 1998, p. 109) by coordinating the perspectives of her team thus helping the team members participate in the creation of a shared vision for mathematics teaching and learning. The other two members of the team, although disparate in age by about 10 years, were both in their first year of teaching and eager to learn. They were very amenable to engaging in the alternative PLC process as a means to improve their practice and learn from their leader. This collaboration helped to improve her practice as well, even though she has used inquiry for 7 years. Interestingly, she reports her improved practice has significantly reduced the need for Tier 2 (reteaching) instruction for students in her classroom. She also reports that these improvements in her mathematics practice has also helped her improve her literacy instruction.

In conclusion, when examining the webs of complex mathematical knowledge evident in the teachers' reflections, we saw evidence that their construction influenced changes in teacher perspectives on the nature of mathematics teaching and learning and produced an increased interest and ability in "making serious use of student thinking," (Ball, 2001, p. 11). Such an approach to conducting PLCs appears to possess some potential as a grassroots means of promoting mathematics education reform.

Data availability statement

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

Ethics statement

The studies involving human participants were reviewed and approved by the Brigham Young University Institutional Review Board. The participants provided their written informed consent to participate in this study. Written informed consent was also obtained from the participants for the publication of any potentially identifiable data presented in the manuscript.

Author contributions

MN and DB: conceptualize study, gather data, analyze data, and write report. BM and RN: conceptualize study and write report. QB and RR: conceptualize study. All authors contributed to the article and approved the submitted version.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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