Aesthetical entanglements in mathematics education

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In this study, we develop a perspective on the diverse aesthetics historically associated with mathematics, inspired by Rancière's approach to aesthetics and politics. We call "Silencing Aesthetics" a dominant aesthetic that Rota has characterized as a "copout (…) intended to keep our formal description of mathematics as close as possible to the description of a mechanism". The challenge this study attempts to explore is how to question silencing aesthetics to make space for inclusive ones. Our efforts have focused on setting up and studying inclusive and pluralist "Studios", gathering craftworkers, anthropologists, mathematics educators, and mathematics enthusiasts. We include here a case study based on a conversation amongst basket weavers, anthropologists, and mathematics educators focused on the artisanal and mathematical nature of knots. We discuss the implications of aesthetical entanglements, such as those in our case study, for mathematics learning.

KEYWORDS

aesthetics, basketry, knots, mathematics, materiality, craft

Introduction

The literature on mathematical aesthetics is not easy to summarize or recapitulate (Sinclair et al., 2006, p. 1–17, 224–254), but we can succinctly identify certain themes that have been focal for it:

- Discussion about the nature of feelings evoked by mathematical work as well as those accompanying the perception of mathematical diagrams and symbols, which may include, amongst others:

  - Feelings of pleasure elicited by the senses of beauty, elegance, productivity, harmony, perfection, and the like.
  - Feelings of melancholy, detachment, coldness, austerity, and estrangement.

- The psychology of mathematical discovery and the roles of the unconscious.
- Differences and commonalities between mathematics and the arts.
- Criteria for what makes mathematical things beautiful or ugly, and the degree of uniformity of the corresponding aesthetic judgements amongst professional mathematicians.
- Mystical and uncanny dimensions of mathematics.
- Whether the appreciation of mathematical aesthetics is restricted to a small elite endowed with the "math gene" or is accessible to everyone.

Despite variations in conceptions of mathematical aesthetics, there is consensus amongst mathematics educators on questioning the aesthetic appreciation of mathematics developed by students in school settings. A number of mathematics education researchers have proposed approaches to address this issue. For example, Dreyfus and Eisenberg (1988) suggested that educators could foster explicit aesthetic evaluations of problem-solving. Others advocate a more radical approach that involves "pluralizing" mathematics education by expanding the range of perceptual, felt, and bodily aspects involved in mathematics learning (Brown, 1973; Papert, 1980; Nemirovsky et al., 1998; Sinclair, 2009; Sinclair and Pimm, 2010). In this study, we explore how creating entanglements, in the colloquial sense of the word, between mathematical and craft aesthetics might contribute in this direction.

Adding to the pioneering work of de Freitas and Sinclair (2014), this article will develop a perspective on mathematical aesthetics inspired by Rancière's (2000, 2006, 2010) approach to aesthetics and politics. We think that this perspective has the potential to illuminate new mutual implications between mathematical aesthetics and mathematical learning. Rancière (2000, p. 12) proposed that we all live in consensual worlds delineated by certain "distributions of the sensible", which are "systems of self-evident facts of perception based on the set horizons and modalities of what is visible and audible as well as what can be said, thought, made, or done". A distribution of the sensible is a demarcation of that which "truly" exists against a receding background of the unreal and meaningless, together with ways of behaving, perceiving, and valuing that are concordant with the embraced demarcation. These ways of behaving, perceiving, and valuing constitute the aesthetics of those who take part in a prevalent consensus. In times of political turmoil, the consensus validating a certain distribution of the sensible comes under scrutiny and questioning, making possible the opening of new spaces of legitimacy for some of those who had had no part in the prior consensus, which had been relegated to noise and meaninglessness. "Dissensus" (Rancière, 2004, p. 226) is the term used by Rancière for such political questioning. Dissensus is not equivalent to "disagreement" since disagreements are common within a consensual communitarian perspective consistent with a certain distribution of the sensible. Rather, dissensus refers to a struggle toward removing a veil of noise and meaninglessness that had hitherto rendered certain aspects of reality indiscernible.

The main examples chosen by Rancière in his writings on aesthetics were taken from art and literature. We surmise that Rancière's distrust of disciplinary traditions, including the ones permeating the natural, social, and mathematical sciences, prompted him to eschew elaboration on their aesthetical dimensions and moved him to rather focus on their hierarchical demarcations of "who can talk about what". Several artists and craftspeople we have been working with have found formal mathematics inaccessible or alienating, in school and even now. This is despite engaging in a study that is mathematically rich and powerful, where mathematics is conceived in its broadest sense. This silencing of their voice (by the discipline of a discipline) resonates with Rancière's acknowledgment of the political power of disciplinary enculturation in the cutting between "what is visible

and what is not, what can be heard and what cannot, what is noise and what is speech" (Rancière, 2004, p. 225).

The anchor of this study is a case study based on a recorded conversation between basket weavers and mathematics educators. We explore how the work of this kind of pluralist Studio—intermingling mathematics and craftwork—may contribute toward breaking through the silencing power often irradiated by prevalent elitist mathematical aesthetics. The case study, presented in the next section, is based on a conversation in one of the studio sessions hosted by the *Forces in Translation* project. This project[1] focuses on interactions between basket weaving, anthropology, and mathematics. It includes basket makers, anthropologists and mathematics educators, exploring, through in-person and online studio sessions, how different basket-weaving techniques and their cultural traditions interplay with the understanding of mathematical ideas, such as spatial relationships, surfaces, curvature, growth, and forces at play, such as tension, friction, and compression that hold together complex structures.

Methods

Throughout the *Forces in Translation* project, we have developed a methodology for interdisciplinary collaboration that we call Studio Practice. Following Ingold's (2017) differentiation between anthropology and ethnography, Studio Practice involves anthropological work because it is a time during which the role of all the participants, including the researchers, is not to learn *about* but to learn *with* others, materials, and tools. Members of a Studio Practice identify with a variety of personal backgrounds, ages, and life stories, primarily united by their senses of wonder, solidarity, and mutual respect. In addition, all members are presumed to be equally capable of making sense of new questions. Recordings of their activities, in the form of journal entries, videotaped interactions, collections of artifacts, and annotated stories, provide a basis for ensuing retrospective research.

With respect to retrospective research, we adopt microethnographic methods (Erickson, 1996, 2004; Goodwin, 2003; Streeck and Mehus, 2005). Microethnography encompasses a collection of techniques and means of analysis tracing the moment-by-moment bodily and situated activity of subjects engaged in events and interactions. These techniques include the preparation of multimodal transcriptions, bodily re-enactment of interactions recorded in the video, writing commentaries raising questions and issues fully grounded in the recorded events, and, to the extent possible, consulting with the participants in those events. Talk, gesture, facial expression, body posture, drawing of symbols, manipulation of tools, pointing, pace, and gaze are all instances of modalities to be traced. Whilst limited by the recording conditions of an online conversation, we pursued a microethnographic approach in this study.

This research included human participants and was reviewed and approved by the Health and Education Research Ethics and Governance Committee at Manchester Metropolitan University (EthOS Reference Number: 42821). All participants provided

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1 https://forcesintranslation.org/about/

FIGURE 1
Lark's head.

informed consent for participation in the Studio Practice and related research and for their images to be included in this study.

Crafting mathematics—A conversation between basketry and mathematics

The conversation selected for our case study interlaced knot tying as a craft and knot theory as a branch of mathematics. We conjecture that the transitions to and from knot tying and knot theory involve navigating across distinct distributions of the sensible and their aesthetics. For knot tyers, knots are to be made and valued according to their frictional strength, ease of unknotting, and other qualities critical to their use by fishermen, packers, etc. Within the consensual aesthetics of knot tyers, many knots studied in (mathematical) knot theory are not even knots, and vice versa. The two distributions of the sensible share some common entities, such as certain knots, e.g., the *trefoil* or *overhand,* but they are immersed in distinct dialects, patterns of expectations, bodily skills, and ways of making sense of them. The aesthetics of a distribution of the sensible are inseparable from the corresponding histories of communities and practices.

Learning knot tying or knot theory both entail the encounter with an unfamiliar aesthetic, which represents a political engagement in the sense of "bringing to reality" materials and events that had previously been invisible or occluded by irrelevant noise. Experiences with knot making of either kind, however, are not deterministic, in the sense that each of them can become sources for diverse aesthetical orientations, including the possibility of coming to appreciate the same knots and techniques as useless, inspiring, too easy, too difficult, powerful, very strange, being good at them, and so forth.

Results: case study in six scenes

A note on the formatting of the transcript for the following six scenes. Rounded double brackets are used to include references to related actions or materials. Aligned left square brackets on adjacent lines indicate simultaneous speech. The numbered speech segments are referred to as Turns in the commentary, e.g., Turn 5.

Scene 1

1. Geraldine: I suppose a hitch is not a knot, is it?
2. Stephanie: Yes it is ((showing a tied rope with a branch holding it, see Figure 1))
         ((Everybody laughs)).
3. Stephanie: Yeah.
4. Geraldine: It's not quite a knot, it's a hitch.
5. Stephanie: It's a Lark's Head [knot.
6. Geraldine: [Well, If you take it off.
7. Stephanie: Isn't it a Lark's [Head knot?
8. Geraldine: [if you take it ((the branch)) off, it's not a knot. I mean, it's one of those…not knots ((showing a rope forming a circle held by her hands, see Figure 2)).
9.   Ricardo: Unknot.
10. Geraldine: If you take it off the string, off the bar, and a knot is meant to be a knot that doesn't come undone ((showing a knot that stays as such when pulled apart, see Figure 3)).

Commentary on scene 1

According to Wikipedia, "a hitch knot is a type of knot used to secure a rope to an object or another rope. It is used in a variety

FIGURE 2
Unknot.

FIGURE 3
Knot that stays knotted.

of situations, including climbing, sailing, and securing loads."[2] In effect, the hitch knot shown by Stephanie in Figure 1, ties a rope to a branch. Stephanie names it "Lark's Head" (Turn 5), which is a type of hitch knot also called "Cow Hitch". Hitch and Lark's Head knots have definite places in the distribution of the sensible inhabited by knotters. However, Geraldine had suggested in Turn 1—the initial question of this scene—that it might not be a knot. Stephanie answered by showing it as an actual knot (Figure 1), which was funny because Stephanie picked up a hitch as if anyone would always have one of those around. However, in Turn 4, Geraldine reaffirms that a hitch "it's not quite a knot"; she makes the case for it by showing that if the branch is removed, it becomes one of those "not knots" (Turn 8). In Figure 2, Geraldine shows what a "not knot," called "unknot" by Ricardo (Turn 9), looks like: the rope tracing the shape of a closed figure with joined ends. Her

holding of the rope in Figure 2 emphasizes that in knot theory, knots—including the unknot—do not have loose ends, which can be achieved by fusing the two ends of a rope. Geraldine understands that if one pulls the ends of the rope without the branch, the hitch undoes itself instead of becoming tighter on its own. However, what does it mean that the hitch undoes itself? Could it mean that the rope can be stretched along a line segment? Perhaps, but she demonstrates the not-knot as a circular figure. A circle-like shape, free of crossings, is the standard way of displaying it in knot theory. Throughout this interaction, we recognize the entwining of two distributions of the sensible and their two corresponding aesthetics. One is the aesthetics of knot tyers, for whom the hitch is a knot and a not-knot is a stretchable rope without crossings; the other is the aesthetics of knot theorists, for whom the hitch is not a knot, as it can be converted into a closed planar figure free of crossings. These differing distributions of the sensible are differences in practices and values, not reducible to different definitions. For example,

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2 https://en.wikipedia.org/wiki/List_of_hitch_knots

filling squares of increasing area with unit squares to generate a number sequence—a classic task in school mathematics—differs from the work of tilers (those who lay tiles) who need to tile walls that may not be covered by square tiles without cutting some of them into smaller pieces; decisions about how to cut them entail criteria about symmetry, attachment strength, sharpness of the edges, continuity of graphic patterns on the titles, and more. All these are aesthetic considerations embedded in a distinct distribution of the sensible that a tiler inhabits. In sum, it is not about how squares are defined, but what kind of practices with squares matter.

Returning to our example of knots, to fit the knot theorist's distribution of the sensible, the branch in the hitch is to be removed or converted into being part of the rope, possibly becoming another knot. We can look at the same object, a hitch around a rod, and see what to do with it, depending on the aesthetics that we adopt. Each aesthetic has an inner coherence, which disallows comparisons between isolated parts. Unless it gets transformed properly, the branch does not exist in the knot theorist's distribution of the sensible, but it is very much part of the knot tyers' one because, crucially, knots secure one object to another one. In knot-tyer aesthetics, all knots can be undone if the ends are pulled properly, which is not the case in the aesthetics of knot theorists because their knots do not have ends. In Figure 3, Geraldine shows an overhand knot as an example of a knot that cannot be undone, which is what makes it a "real" knot; this is the case if the ends are joined, because otherwise, with loose ends, knots could be undone and cease to be knots for knot theorists. Joining the loose ends of an unknotted string leads to a circle/loop, the unknot. Given that its name suggests that it is not a knot, the unknot appears to have an odd role in knot theory. It is not unlike the dubious role that, for centuries, zero had among numbers. Zero doesn't really count anything (so is it a number?), but it serves as the identity or 'do nothing' number when it comes to addition. The unknot plays the same role when it comes to composing/adding knots. It serves as a base or boundary object, which helps complete the mathematical theory but may not be meaningful for knot tying. Scene 1 reflects a lived-in entanglement of distributions of the sensible respectively inhabited by knotters and knot theorists. This entanglement will be further articulated in Scene 2.

Scene 2

1. Ricardo: (...) can you think of the crossings in your hitch knot, even when you are with the branch in the middle?
2. Geraldine: Oh, with the branch, yeah. There's only two, aren't there?
3. Stephanie: Two, yeah. (...)
4. Geraldine: One, two ((counting two rope "contacts" in a knot similar to the one shown in Figure 1)). I think that's because it's not a knot.
5. Stephanie: But it is called a knot by knotters?
6. Geraldine: Yeah, but they don't call things non-knots either, do they?
7. Stephanie: And it is called the Lark's Head as well.
8. Geraldine: Yeah, they are called knots by the knotters, but not by mathematicians, I don't, I think.

Commentary on scene 2

Geraldine and Stephanie counted the times the rope touched itself for the Lark's Head as in Figure 1, which are two (Turns 2 and 3). It is tacitly understood that the rope touching the branch does not count as a crossing. This implicit assumption is a "natural" one from the aesthetics of knot theory, given that the branch is a spurious element in it, at times no more than background noise. In fact, the question of the number of crossings is essential for knot theorists but not for knot tyers, for whom the number of crossings is largely irrelevant. In this sense, Ricardo's question (Turn 1), just by asking about the number of crossings, foregrounded the aesthetics of knot theory, leading to Geraldine's additional argument (Turn 4) for the hitch not to be a knot. This inference, in all likelihood, is derived from a previous conversation situated in knot theory about the minimum number of crossings for a knot other than the unknot being three.

Stephanie shifts the conversation from "being a knot" to "being called a knot" (Turn 5): as opposed to mathematicians, knotters call it a knot. This is a crucial shift because it reframes the question in a manner relative to the communities of interest and their distributions of the sensible. Further elaborating on the differences between knot tyers and mathematicians, Geraldine says in Turn

FIGURE 4
Two linked half-hitches.

6 that knotters do not call things "non knots". This relativism removes hierarchical determinations, such as ascribing to either knotters or mathematicians the possession of a "true" criterion for what knots are: it is now possible, from the aesthetics of knot theory, for a knot to have at least three crossings and to include "not knots", but, at the same time, from the aesthetics of knot tying, knots may have fewer than three crossings and not-knots may be non-existent. What we attend to, what we choose to name, and what we are called to do are very much part of a distribution of the sensible that has been historically developed and presently adjusted by certain communities.

Scene 3

1. Ricardo: (...) Can you count the crossings without the branch in?
2. Geraldine: Well if you pull it off the branch and then it just comes apart.
3. Ricardo: Maybe there is something about the branch that somehow is playing a role of a crossing for some reason, I mean, it's something to wonder about.
4. Geraldine: It's just that lovely half hitch. I really like these half hitches, because this, this one... this one is two linked half hitches ((see Figure 4)). And it's the same as the looping. It's the looping. I mean, that's, but it's two half hitches.
5. Ricardo: But there are more than two crossings there.
6. Geraldine: One, two, three, four.
7. Ricardo: There is something that happens when you have a branch inside that it makes it not so easy to unknot it.

Commentary on scene 3

In Turn 1, Ricardo asks for counting crossings without the branch going through the hitch. In response, Geraldine expects the hitch to come apart (Turn 2). Ricardo wonders how to think of the branch as blocking the unknotting of the hitch (Turns 3, later reiterated in Turn 7). Whilst Ricardo was talking in Turn 3, Geraldine removed the branch from the hitch, and then she moved the rope in some way, which we cannot see because it was off-camera, such that she formed the knot shown in Figure 4: two half-hitches. We can think of a half-hitch as a rope going around and just touching itself. Two half-hitches can come one after another, or they can be made so that the second half-hitch goes through the first one. This is what we see in Figure 4: two half-hitches in which the second one goes through the first one as in a chain. She tells the rest of the group that they are "lovely" and that she "really likes them" (Turn 4). As a basket weaver, Geraldine uses looping techniques to create beautiful baskets of an extraordinary variety, and it turns out that two half-hitches are "the same as the looping." The immense creative value that looping has for Geraldine is also ascribed to two half-hitches; they are prominent in her aesthetic as a master of looping. We chose to highlight in this commentary the emotional investment that Geraldine expresses. This is not meant to imply that emotional values, so central to any aesthetic, are to be seen only in this scene. Anything said or shown in the six scenes is animated by emotional values. We elaborated on them here just because they are particularly salient and explicit. In Turn 5, Ricardo is puzzled by how the hitch became a loop with more than two crossings, four in fact (Turn 6). Looking at Figure 4, you notice that the left and right crossings are very clear, but in the middle, there is an area of contact that is not obviously one or two crossings. One needs to move the rope around there to disambiguate and notice that it is two of them, which is what Geraldine reported in Turn 6.

Scene 4

1. Ricardo: What are you showing, Mary?

FIGURE 5
Overhand knot.

2. Mary: I've just sort of taken the knot that you were showing, the sort of overhand knot ((Figure 5)), and I was thinking if you had one end as a sort of standing still, not moving ((the horizontal end in her left hand in Figure 6)), but if you take the working end ((moving the working end back through the loop, as in Figure 7)), then it makes a complete turnabout. And if you actually sort of suspend it in space, then it only touches at three points, ((correcting herself)) two points ((Figure 8)), but if you pull it up and make it flat, it touches at three points ((Figure 9)). And also that bit ((referring to the horizontal section of the knot in Figure 10)) could be like Geraldine's rod.

3. Geraldine: [Yes.

4. Stephanie: [Yes.

5. Mary: That could be like a rod, that part. And then you've got another part coming around.

Commentary on scene 4

Mary begins by showing a knot that is a "sort of overhand knot". She holds it by pressing two points of contact for the rope, one in each hand (Figure 5). Then, she pulls one end of the rope, straightening the corresponding section of the rope with her left hand (Figure 6). Mary refers to this end of the rope as "standing still" or "not moving." Then, she alludes to the opposite end of the rope (pointing upwards in Figure 6) as "the working end." This distinction between the standing and working ends of a rope is very

FIGURE 6
Still vs. working end.

FIGURE 7
Retracing the working end.

FIGURE 8
Two-crossing view of the overhand knot.

FIGURE 9
Three-crossing view of the overhand knot with related knot diagram in the top left corner.

common amongst knot tyers as they clarify how to make a knot. Moreover, it is often the first thing that they set up, determining which end will be still and using the other one as the working end that will go through space. Mary passes the working end back under the loop to show how it goes from the beginning: "it makes a complete turnabout" (Figure 7). In Figure 8, Mary suspends the rope "in space," indicating how the rope only touches itself at two points. In contrast, "if you make it flat," that is, pulling the ends to press the whole knot closer to a plane, "it touches at three points" (Figure 9). If the two loose ends were joined, this representation would be close to a knot theorist's diagram for the overhand knot, shown in the top left corner of Figure 9. The bit of rope departing

from the still end, Mary adds, "could be like Geraldine's rod" (Figure 10), that is, like the branch that used to go through the hitch (Lark's Head knot in Figure 1), which is agreed upon by Stephanie and Geraldine (Turns 3 and 4). By showing the rod-like part next to the still end, Mary has transformed the branch across the hitch into a continuous part of the rope itself. Mary concludes with an overall description of her overhand knot, which starts with a rod-like part taking off from the still end and the rest of the rope fully "coming around" (Figure 10).

As part of her demonstration, Mary shows that the number of crossings of a knot depends on how one holds it, exhibiting that an overhand knot can feature two or three crossings. This

FIGURE 10
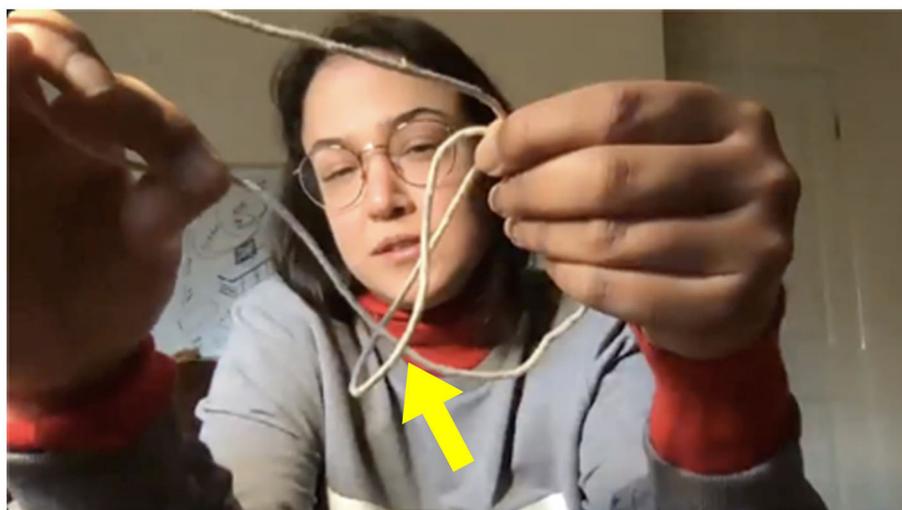Overhand knot with horizontal "rod" section.

FIGURE 11
One crossing of a trefoil knot.

opens a huge topic: What is a crossing? How come that knot theory stipulates that an overhand knot with joining ends (i.e., a trefoil) has at least three crossings, whilst, at the same time, one can suspend it in space with only two? Or, as it will later transpire in the discussion, with only one or none? How is a crossing different from a "touching"? This isn't purely a semantic issue and brings to the fore how entanglements of different aesthetics can help shed light on and counter the prevalent silencing aesthetic in mathematics. The crossings in a mathematical knot diagram can be seen as arising when one flattens a (three-dimensional) knot onto a two-dimensional plane. In other words, knot diagrams are projections or flattenings. As an example, note the contrast in Figure 9, in which we see the knot held by Mary and the corresponding diagram

for the equivalent knot. However, these issues are often glossed over in the didactical use of diagrams. This sweeping under the carpet of essential features (under the prevailing mathematical aesthetics of brevity or assuming common understanding within the community) is brought to light as the participants try to clarify how touchings and crossings could be the same or different. This topic will be further elaborated on by Charlotte in Scene 5.

Scene 5

1. Charlotte: Yeah, well, I was interested in the idea of crossings and the relation between the three dimensions and
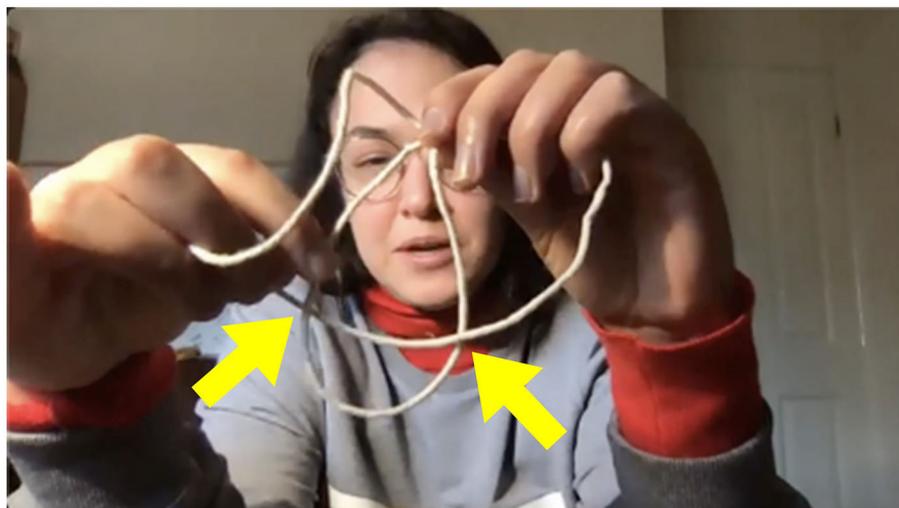
FIGURE 12
One crossing becomes two.

two dimensions, it's related to what Mary just said ((in Scene 4)). I was trying to making sense of the crossings in three dimension and then this one is here. ((see Figure 11, yellow arrow indicating a crossing she was probably referring to)) but then when once you flatten the knot, then it becomes two ((see Figure 12)). And yeah, because the knot is something, well it's three dimensional and then it kind of changes the nature in some way when you flatten it.

Commentary on scene 5

In Figure 11, Charlotte holds a trefoil in such a way that the string appears to touch itself at a point down below. However, as she turns the loop 90 degrees to "flatten" the knot, in Figure 12, that single touch in the lower section becomes two. She is operating all the time with three-dimensional materials, but she envisions a transformation from three dimensions to two, which coincides with what Mary had referred to in Scene 4 as "flattening," which seems to increase the number of crossings. In knot theory, this type of flattening is called projecting, which is more like creating a shadow of the knot on a flat surface. There is also the additional requirement that at most two points in the three-dimensional knot project or flatten onto a point in the diagram (or shadow) and that the strands of the knot that result in such crossing points genuinely cross each other (a transverse crossing) and are not just tangential touchings. In this operation, even a knot that does not touch itself anywhere will generate crossings in its shadow. However, some of the shadowed crossings can be thought of as artifacts that can be eliminated by re-arranging the rope in space. We witness in Scenes 4 and 5 the initial emergence of a new partition in a distribution of the real permeated by knot theory, which allows for the allocation of crossings as distinct from touchings.

The materials in use, such as ropes or diagrams on paper, fully participate in the realization of the corresponding aesthetics and are not just accessories.

Scene 6

1. Hilary: Well, I also agree with Charlotte about the three-dimensional sort of way that it changes because I make knots in willow for my frame baskets. So the frame of it is a knot ((Figure 13)), usually the hoop frame. It doesn't have to be made like that, but that's the way I do it. And so I have a knot in a hoop, in a sort of piece of willow ((Figure 14)). And then I have to find the place at which it balances. (...) So there's a point that I have to choose where everything is going to stay in the right place. That it's going to make a balanced basket. So I'm looking for that when I make the hoop, I'm looking for the point at which that is getting to stay upright, which is about there ((Figure 15)), where I can press down on it, and it's going to be, it sort of balances on itself.

Commentary on scene 6

Charlotte's reflections in Scene 5 prompted Hilary to think of her experiences balancing the hoop of a frame basket. Unlike other types of baskets, which take shape gradually as they are woven, frame baskets are made on the basis of a "skeleton" or "framework" that is shaped from the beginning, orienting and determining the basket's overall shape (Jensen, 1991).

Hilary is a skilled basket weaver, with willow being one of her preferred materials to work with. In Figure 13, Hilary shows the skeleton of a frame basket with two hoops, one vertical and one

FIGURE 13
Knot as part of basket hoop frame.

FIGURE 14
Forming a knot in willow for a hoop frame.

horizontal. The upper side of the vertical hoop, also called the hoop frame, is going to be the handle of the basket. Hilary had made the ring of the hoop frame by tying a piece of willow as an overhand knot. She clarifies that there are other ways of making it, but this is how she does it. In Figure 14, she shows a hoop frame before the horizontal ring is attached to it, just the overhand knot. In making the skeleton of a rib basket, it is important to ensure that it is "balanced" at every step. For example, in Figure 15, Hilary shows that a point can be found at which the skeleton stays upright. This demonstration reveals how critical sensing the materiality of a willow-made knot is, to ascertain conditions enabling the weaving

of a balanced basket. For basket weavers, balanced and unbalanced baskets are part of the distribution of the sensible they inhabit. Knot theorists make use of symmetry (a form of balance) in their knot diagrams, which can aid efficient analysis and classification. In Hilary's case, the balance is integral to the use and materiality of the basket. Could we say that this emphasis on materiality marks a radical difference between the aesthetics of basket weaving and the aesthetics of knot theory? We think it does not. Even the very planar knot diagrams, customary in knot theory, preserve three-dimensional features as they show points of over and underpassing. Such crucial under/over reflects material impenetrability. In fact,

FIGURE 15
Part of the knot serving as a balance point.

knot theory's distribution of the sensible does not involve the melding of two filaments or strings into one, in other words, actual flattening.

Discussion

> Knots seem to refuse to be seen from one particular point of view or perspective (de Freitas and McCarthy, 2014, p. 45).

We have set up and studied inclusive and pluralist "Studios", gathering craftworkers, anthropologists, mathematics educators, and mathematics enthusiasts. Many of these Studios were open to the public and family groups as well. The six scenes of our case study, which are part of a much longer conversation, reflect the messy work of a group of participants with diverse backgrounds and ages, as they navigate and comingle various aesthetics, sharing the notion that no single aesthetic has ultimate and dominant value. Scenes 1 and 2 reflect that there are different distributions of the sensible consensually adopted by, in this case, knotters and knot theorists, such that that which exists for them is perspectival, interrelated, and sensible. Scene 3 suggests that materials and techniques allocated in a communitarian distribution of the sensible have multiple and intense powers of emotional attachment. Scenes 4 and 5 evoke how grappling with the idea of "crossing" is a matter of degrees and nuance, dependent on careful attention to materials and diagrams. Scene 6 highlights the multiple lives of knots and their crossings as they basket-weave materials and patterns.

The conversation that we have portrayed entangles different aesthetics, including ones from knot tying, knot theory, and basket weaving. Other aesthetics surrounding knots have been studied by anthropologists:

> The knot is ascribed more than functional value in the Pacific as it becomes the object of meditative thought and holds together through binding not two things but two concepts: that of the visible, and that of the invisible whose momentary entanglement facilitates temporal concepts of genealogy and remembrance (Küchler, 2003, p. 207).

In this discussion, we want to elaborate on the significance of aesthetical entanglements in mathematics learning because they may help question a prevalent and widespread aesthetic of mathematics that works by silencing those who fail to appreciate it, a community that includes the majority of students. This aesthetic is not an honest one either. The actual practice of mathematicians is much more similar to that of craftspeople or students, though that is occluded in much of how the subject is presented and taught. The mathematician Rota (1997) has described two interrelated aspects of what we call "Silencing Aesthetics": (1) sudden light in the darkness and (2) attributions of mathematical beauty to cover up the messiness of mathematical enlightenment. They are two sides of the same impetus. Regarding the former, he writes:

> All the effort that went in understanding the proof of a beautiful theorem, all the background material that is needed if the statement is to make any sense, all the difficulties we met in following an intricate sequence of logical inferences, all these features disappear once we become aware of the beauty of a mathematical theorem (…) [all that will remain] is the image of a flash of light of insight, of a sudden light in the darkness (ibid, p. 179).

Rota views veiling the messiness inherent in enlightenment, which is regularly hidden when mathematicians formally present their work, as motivated by powerful aesthetical and political forces:

> The term "mathematical beauty", together with the light-bulb mistake, are tricks that mathematicians have devised to avoid facing up to the messy phenomenon of enlightenment. The comfortable one-shot but misleading idea of mathematical beauty saves us from having to deal with the messy situation of a concept having degrees. All talk of mathematical beauty is a copout from confronting the logic of enlightenment, a copout that is intended to keep our formal description of mathematics as close as possible to the description of a mechanism. This copout is a step in a cherished activity of mathematicians, that of building a perfect world immune from the messiness of the ordinary world, a world where what we think should be true turns out to be forever true, a world that is to be kept free from the disappointments, the ambiguities, the failures of that other world in which we are forced to live (ibid, p. 182).

Numerous efforts have been made to *unveil* the messiness of mathematical enlightenment. There is a considerable literature in mathematics education research dedicated to this, most notably from philosophy, *Proofs and Refutations* (Lakatos, 1976), and from cognitive linguistics, *Where Mathematics Comes From* (Lakoff and Núñez, 2000). However, these untidy accounts of mathematical work are often marginalized as belonging either to historical epochs prior to the development of contemporary mathematics or to students who are still far from mastering the subject. The underlying message for many students, who seldom experience flashes of light of mathematical insight whilst often sensing unapproved messiness in their understandings, is that they lack the abilities necessary to enjoy mathematics and appreciate its alleged aesthetics.

This study attempts to explore how we could question silencing aesthetics to make space for inclusive ones. Given the dominant cultural images of mathematics, it is a complex issue. Aesthetical entanglements such as the one reflected in our case study, in which knot theory and knot tying aesthetics are compared on an equal footing, each equally legitimate despite their radical differences, may help in grasping an aesthetic of knot theory that is neither mechanistic nor flawless. The episode was not, for the most part, a matter of dissensus; none of the participants were prone to argue that knots used by fishermen should be considered knots by knot theorists or that the latter should discriminate knots according to their frictional strength. These would be dissensual claims to the extent that they attempted to subvert the distribution of the sensible consensually adopted by each of the communities. We can recognize many points of contact between knot theory and knot tying, such as the trefoil being an overhand knot with its ends joined. These points of contact constitute rich counterpoints to be explored. There is nothing absolute about defining knots, say, as not having loose ends. The issue can be highlighted by noticing that without friction, knots with loose ends can always be unscrambled onto a linear rope, which would make knot theory pointless.

Regarding classroom implications, as we touched on earlier, there is a small strand of literature on mathematics classroom dynamics favoring or countering what we have called "silencing aesthetics," as well as exploring ways of nurturing inclusive aesthetics (Brown, 1973; Dreyfus and Eisenberg, 1988; Sinclair, 2008, 2009; Sinclair and Pimm, 2010). The thematic lines introduced in this study, on taking a Rancièrian perspective on aesthetics and intermingling mathematics and craftwork, could be of significance on this front by facilitating the entanglement of diverse aesthetics in the classroom and questioning silencing aesthetics. This is a complex agenda because attending to more inclusive aesthetics is likely to demand slower and less predictable courses of action, which are inherent in striving to ground learners' conceptions in their own tangible experiences. This creates tensions with respect to constraints of time, "covering" extensive curricula, and standardized assessments. These tensions make the work in informal learning, such as the Studio Practices reported in this study or our past study with 10-year-old children in an afterschool programme (Nemirovsky, 2018), not easily transferable to goal- and assessment-driven classroom mathematics. The entanglements we propose could result in slower and more deliberative mathematics, with opportunities for reflecting on different aesthetic values. There are examples of curricular materials that could support such practice. For example, the *Math in a Cultural Context* materials for elementary school, developed in southwestern Alaska, bring together classroom mathematics and indigenous Yup'ik cultural knowledge of Yup'ik cosmology and everyday practices, such as garment-making (Lipka et al., 2013, 2015).

With an emphasis on the entanglement of diverse aesthetics, this study contributes to a vision that interlaces mathematics learning and craftwork, which has become a small but growing movement in recent years (Belcastro and Yackel, 2011; Gresalfi and Chapman, 2017; Nemirovsky, 2018; Taimina, 2018; O'Brien, 2022; Peppler et al., 2022). More broadly, the overall idea is to liberate space and time for the artisan, the practitioner, the child, and the lay citizen to partake in the sharing of the sensible and to become "a deliberative citizen" (Rancière, 2000, p. 12) in mathematics and beyond.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Ethics statement

The studies involving humans were approved by the Health and Education Research Ethics and Governance Committee at Manchester Metropolitan University. The studies were conducted in accordance with the local legislation and institutional requirements. The participants provided their written informed consent to participate in this study. Written informed consent was obtained from the individual(s) for the publication of any potentially identifiable images or data included in this article.

Author contributions

RN: Writing – original draft. VK: Writing – original draft. CM: Writing – original draft.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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