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Hecke Groups, *Dessins d'Enfants*, and the Archimedean Solids

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Grothendieck's *dessins d'enfants* arise with ever-increasing frequency in many areas of twenty-first century mathematical physics. In this paper, we review the connections between *dessins* and the theory of *Hecke groups*. Focussing on the restricted class of highly symmetric *dessins* corresponding to the so-called *Archimedean solids*, we apply this theory in order to provide a means of computing representatives of the associated conjugacy classes of Hecke subgroups in each case. The aim of this paper is to demonstrate that *dessins* arising in mathematical physics can point to new and hitherto unexpected directions for further research. In addition, given the particular ubiquity of many of the *dessins* corresponding to the Archimedean solids, the hope is that the computational results of this paper will prove useful in the further study of these objects in mathematical physics contexts.

Keywords: Hecke groups, *dessins d'enfants*, Archimedean solids, Platonic solids, Belyi maps

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1. INTRODUCTION

Grothendieck's *dessins d'enfants*—bipartite graphs drawn on Riemann surfaces—arise with ever-increasing frequency in twenty-first century mathematical physics, appearing in e.g., the study of $\mathcal{N} = 2$ supersymmetric gauge theories [1–3], elliptically fibred Calabi-Yaus [4–6], toric conformal field theories [7, 8], topological strings [9, 10], and elsewhere. Given this growing ubiquity, it is valuable to study both the basic theory of *dessins*, and the applications of this theory to particularly significant cases. This paper aims to achieve both such goals, by first reviewing the mathematical results which connect *dessins d'enfants* to the theory of *Hecke groups*, before applying this work to the *dessins* for the so-called *Archimedean solids*. The hope is to provide a compendium of computational results to aid future research featuring these objects.

With the above in mind, let us begin our story by recalling that the *Platonic solids* are the convex polyhedra with equivalent faces composed of congruent convex regular polygons; these objects have been known and studied for millennia. These solids also appear in mathematical physics: for example, the symmetries of the Platonic solids are related to D-brane orbifold theories [11]. Now, more broadly, another well-known class of convex polyhedra are the *Archimedean solids*: the semi-regular convex polyhedra composed of two or more types of regular polygons meeting in identical vertices, with no requirement that faces be equivalent. There are three categories of such Archimedean solids: (I) the Platonic solids; (II) two infinite series solutions – the prisms and anti-prisms; and (III) 14 further exceptional cases.

In Magot and Zvonkin [12], a novel approach to the Archimedean solids was taken by interpreting the graphs of these solids as clean *dessins d'enfants*. By *clean*, we mean that all the nodes of one of the two possible colors of the bipartite graph have valency two [13, 14]. Now, the planar graph of any polytope can be interpreted as a clean *dessin* by inserting a black node into every edge of the graph, and coloring every vertex white. In this way, one can construct *dessins* for all the Archimedean solids. In each case, the underlying Riemann surface is the sphere $\mathbb{CP}^1$, and the *dessins* are drawn in a planar projection.

In a parallel vein, *trivalent* clean *dessins* can be associated to conjugacy classes of subgroups of the modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$. (Recall that the modular group $\Gamma \equiv \Gamma(1)$ is the group of linear fractional transformations $z \to \frac{az+b}{cz+d}$, with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. The presentation of Γ is $\langle x, y | x^2 = y^3 = I \rangle$.) To do so, we replace all n-valent vertices of the *dessin* with oriented n-gons. This constructs a *Schreier coset graph*, which displays the permutation action of generators x and y on each coset Gx_i of $\mathrm{PSL}(2, \mathbb{Z}) = \bigcup_i Gx_i$, $i = 1, \ldots, \mu$, where μ is the index of some subgroup G in $\mathrm{PSL}(2, \mathbb{Z})$ [15]. Generalizing, *any* clean *dessin* (not necessarily trivalent) can be associated with a conjugacy class of subgroups of a certain so-called *Hecke group* H_n, defined as having presentation $\langle x, y | x^2 = y^n = I \rangle$.

Once a Hecke subgroup G has been associated to the *dessin* in question, further results follow. For example, one can quotient the upper half plane by G to construct the surface $\mathcal{H}/G$. From this, one can construct a Belyi map (in a manner detailed in the main body of this paper), i.e., a holomorphic map to $\mathbb{P}^1$ *ramified* at only $\{0, 1, \infty\}$. To each Belyi map there corresponds a unique *dessin*; precisely the *dessin* with which we began! Hence, this chain of connections compactifies to a circle.

Returning now to the Archimedean solids, we see that, interpreting these as *dessins*, we should be able to explore the above circle of connections via explicit computations. Indeed, the Belyi maps associated to these *dessins* have already been computed in Magot and Zvonkin [12]; therefore, our task is to complete the circle by computing representatives of the conjugacy classes of Hecke subgroups associated to these objects. Not only will this provide an illustration of this aspect of the mathematical theory underlying *dessins*, but it will also provide a useful resource for further research in this area. Indeed, given that some *dessins* for the Archimedean solids have have already arisen in areas of mathematical physics (see for example [3, 4, 6]), it is not unreasonable to expect that such *dessins* will continue to manifest themselves in future research.

The structure of this paper is as follows. In Section 2 we present some technical details regarding Hecke groups and *dessins d'enfants*. We show that it is possible to interpret each clean *dessin* as the Schreier coset graph for a conjugacy class of subgroups of a certain Hecke group, and review the circle of connections mentioned above. In Section 3, we find the permutation representations for the conjugacy classes of subgroups of Hecke groups corresponding to every Archimedean solid, and provide an algorithm to compute explicit generating sets of matrices for representatives of these conjugacy classes in each case. In Section 4, we close with some conclusions, returning to the specific applications of this work in various subfields of mathematical physics.

2. *DESSINS D'ENFANTS* AND HECKE GROUPS

In this section, we review some essential details regarding both Hecke groups and clean *dessins d'enfants*. We begin by considering the modular group $\Gamma \cong H_3$, as although this is isomorphic to only one particular Hecke group, it is by far the most well-studied, and we shall draw upon the presented results at several points in the ensuing discussion. Subsequently, we discuss Hecke groups more generally, before moving on to consider clean *dessins* and their associated Belyi maps. Note that the connection between trivalent *dessins* and the modular group is discussed in Jones and Singerman [16, 17], while the relation between Hecke groups and maps is discussed in e.g., [18]. With these results in hand, we describe how every clean *dessin* is isomorphic to the Schreier coset graph for a conjugacy class of subgroups of a Hecke group, and how the circle of connections closes via a correspondence between such subgroups and the Belyi maps for the original *dessins*.

2.1. The Modular Group

To begin, let us recall some essential details regarding the modular group Γ. This is the group of linear fractional transformations $\mathbb{Z} \ni z \to \frac{az+b}{cz+d}$, with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. It is generated by the transformations T and S defined by:

$$T(z) = z + 1 \quad , \quad S(z) = -1/z \, . \tag{1}$$

The presentation of Γ is $\langle S, T | S^2 = (ST)^3 = I \rangle$, and we will later discuss the presentations of certain modular subgroups. The 2×2 matrices for S and T are as follows:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad , \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{2}$$

Letting $x = S$ and $y = ST$ denote the elements of order 2 and 3, respectively, we see that Γ is the free product of the cyclic groups $C_2 = \langle x | x^2 = I \rangle$ and $C_3 = \langle y | y^3 = I \rangle$. It follows that 2×2 matrices for x and y are:

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad y = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} . \tag{3}$$

With these details in hand, we can consider some important subgroups of Γ.

2.1.1. Congruence Modular Subgroups

The most significant subgroups of Γ are the *congruence* subgroups, defined by having the entries in the generating matrices S and T obeying some modular arithmetic. Some conjugacy classes of congruence subgroups of particular note are the following:

- Principal congruence subgroups:

$$\Gamma(m) := \left\{ A \in \mathrm{SL}(2; \mathbb{Z}) \; ; \; A \equiv \pm I \bmod m \right\} / \{\pm I\} \, ;$$

- Congruence subgroups of level m: subgroups of Γ containing $\Gamma(m)$ but not any $\Gamma(n)$ for $n < m$;
- Unipotent matrices:

$$\Gamma_1(m) := \left\{ A \in \mathrm{SL}(2; \mathbb{Z}) \; ; \; A \equiv \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \bmod m \right\} / \{\pm I\} \, ;$$

- Upper triangular matrices:

$$\Gamma_0(m) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \; ; \; c \equiv 0 \bmod m \right\} / \{\pm I\} .$$

- The congruence subgroups

$$\Gamma\left(m; \frac{m}{d}, \epsilon, \chi\right) := \left\{ \pm \begin{pmatrix} 1 + \frac{m}{\epsilon\chi}\alpha & d\beta \\ \frac{m}{\chi}\gamma & 1 + \frac{m}{\epsilon\chi}\delta \end{pmatrix} \; ; \; \gamma \equiv \alpha \bmod \chi \right\}$$

for certain choices of m, d, ϵ, χ (see [4, 6]).

We note here that:

$$\Gamma(m) \subseteq \Gamma_1(m) \subseteq \Gamma_0(m) \subseteq \Gamma . \tag{4}$$

In Section 3 of this paper, we shall remark on the connections between some specific conjugacy classes of congruence modular subgroups and the Archimedean solids.

2.2. Hecke Groups

We can now extend our discussion of the modular group $\Gamma \cong H_3$ to the more general Hecke groups H_n. The Hecke group H_n has presentation $\langle x, y | x^2 = y^n = I \rangle$, and is thus the free product of cyclic groups $C_2 = \langle x | x^2 = I \rangle$ and $C_n = \langle y | y^n = I \rangle$. Note that $\Gamma \cong H_3$ where Γ is the modular group; and that H_n is the triangle group $(2, n, \infty)$. H_n is generated by transformations T and S now defined by:

$$T(z) = z + \lambda_n \quad , \quad S(z) = -1/z , \tag{5}$$

where λ_n is some real number to be determined. The 2×2 matrices for these S and T are:

$$T = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix} \quad , \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{6}$$

Letting $x = S$ and $y = ST$ as in our discussion of Γ, we see that 2×2 matrices for x and y are:

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad y = \begin{pmatrix} 0 & -1 \\ 1 & \lambda_n \end{pmatrix} . \tag{7}$$

For H_n, we clearly have $(ST)^n = I$, thereby constraining λ_n for a given n [19]. Diagonalising y to compute y^n explicitly places a constraint which allows for a solution of λ_n; we find the following general expression, as well as some important values for small n:

$$\lambda_n = 2\cos(\pi/n) , \tag{8}$$

n	3	4	5	6
λ_n	1	$\sqrt{2}$	$\frac{1+\sqrt{5}}{2}$	$\sqrt{3}$

In particular, the λ_n are algebraic numbers.

2.2.1. Congruence Subgroups of Hecke Groups

By way of extension of the above discussion of subgroups of the modular group Γ, it is useful to consider congruence subgroups of Hecke groups. The Hecke groups are discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$; in fact, the matrix entries are in $\mathbb{Z}[\lambda_n]$, the extension of the ring of integers by the algebraic number λ_n. Note that:

$$H_n \subset \mathrm{PSL}(2, \mathbb{Z}[\lambda_n]) . \tag{9}$$

However, unlike the special case of the modular group, this inclusion is strict. With this point in mind, we can define the congruence subgroups of Hecke groups in the following way [19]. Let I be an ideal of $\mathbb{Z}[\lambda_n]$. We then define:

$$\mathrm{PSL}(2, \mathbb{Z}[\lambda_n], I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}[\lambda_n]) \; ; \; a-1, b, c, d-1 \in I \right\} . \tag{10}$$

By analogy, we also define:

$$\mathrm{PSL}^1(2, \mathbb{Z}[\lambda_n], I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}[\lambda_n]) \; ; \; a-1, c, d-1 \in I \right\} \tag{11}$$

$$\mathrm{PSL}^0(2, \mathbb{Z}[\lambda_n], I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}[\lambda_n]) \; ; \; c \in I \right\} . \tag{12}$$

Then we can define the congruence subgroups $H_n(m)$, $H_n^1(m)$, and $H_n^0(m)$ of H_n as follows:

$$H_n(I) = \mathrm{PSL}(2, \mathbb{Z}[\lambda_n], I) \cap H_n ; \tag{13}$$

$$H_n^1(I) = \mathrm{PSL}^1(2, \mathbb{Z}[\lambda_n], I) \cap H_n ; \tag{14}$$

$$H_n^0(I) = \mathrm{PSL}^0(2, \mathbb{Z}[\lambda_n], I) \cap H_n . \tag{15}$$

We have:

$$H_n(I) \subseteq H_n^1(I) \subseteq H_n^0(I) \subseteq H_n . \tag{16}$$

By analogy with our discussion of the modular group, we define congruence subgroups of level m of the Hecke group H_n as subgroups of H_n containing $H_n(m)$ but not any $H_n(p)$ for $p < m$ [20]. With these details regarding Hecke groups and their subgroups in hand, we can now consider their connections to clean *dessins d'enfants*.

2.3. *Dessins d'Enfants* and Belyi Maps

A *dessin d'enfant* in the sense of Grothendieck is an ordered pair $\langle X, \mathcal{D} \rangle$, where X is an oriented compact topological surface and $\mathcal{D} \subset X$ is a finite graph satisfying the following conditions [13]:

1. $\mathcal{D}$ is connected.
2. $\mathcal{D}$ is *bipartite*, i.e., consists of only black and white nodes, such that vertices connected by an edge have different colors.
3. $X \setminus \mathcal{D}$ is the union of finitely many topological discs, which we call the *faces*.

We can interpret any polytope as a *dessin* by inserting a black node into every edge, and coloring all vertices white. This process

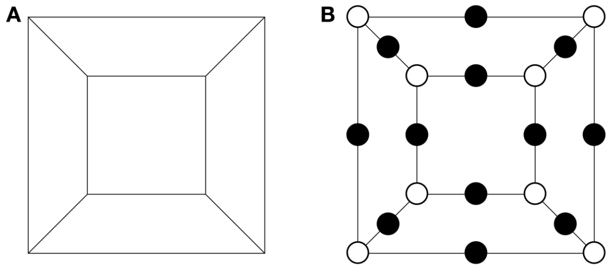

FIGURE 1 | Interpreting the planar graph of the cube as a clean *dessin*. **(A)** The planar graph for the cube. **(B)** The corresponding clean *dessin*.

of inserting into each edge a bivalent node of a certain color is standard in the study of *dessins d'enfants* and gives rise to so-called *clean dessins*, i.e., those for which all the nodes of one of the two possible colors have valency two. An example of this procedure for the cube is shown in **Figure 1**.

There is a one-to-one correspondence between *dessins d'enfants* and *Belyi maps* [13, 16, 17]. A Belyi map is a holomorphic map to $\mathbb{P}^1$ ramified at only $\{0, 1, \infty\}$, i.e., for which the only points $\tilde{x}$ where $\frac{\mathrm{d}}{\mathrm{d}x}\beta(x)|_{\tilde{x}} = 0$ are such that $\beta(\tilde{x}) \in \{0, 1, \infty\}$. We can associate a Belyi map $\beta(x)$ to a *dessin* via its *ramification indices*: the order of vanishing of the Taylor series for $\beta(x)$ at $\tilde{x}$ is the ramification index $r_{\beta(\tilde{x})\in\{0,1,\infty\}}(i)$ at that ith ramification point [4, 6]. To draw the *dessin* from the map, we mark one white node for the ith pre-image of 0, with $r_0(i)$ edges emanating therefrom; similarly, we mark one black node for the jth pre-image of 1, with $r_1(j)$ edges. We connect the nodes with the edges, joining only black with white, such that each face is a polygon with $2r_\infty(k)$ sides [6]. The converse direction (from *dessins* to Belyi maps) is detailed in e.g., [13], and involves using the *dessin* to construct a so-called *triangle decomposition* of the surface X in which it is embedded, before constructing a unique Belyi function from that decomposition. For more information on *dessins* and Belyi maps, the reader is referred to Girondo and Gonzalez-Diez [13], Schneps [21], Lochak and Schneps [22, 23].

2.4. Belyi Maps from Hecke Subgroups

Let G be a torsion-free subgroup of the Hecke group H_n of finite index (i.e., a subgroup that contains no element of finite order other than the identity). Then, the compact surface X_G is obtained from the surface $\mathcal{H}/G$ (where $\mathcal{H}$ is the upper half plane) by adding finitely many points on the boundary real line of $\mathcal{H}$, called *cusps* (in mathematical language: one for each G-orbit of boundary points at which the stabilizer is non-trivial). For example, choosing $n = 3$ and letting G be the full modular group Γ, there is a single cusp, equivalent to any rational number under G, namely, the point at infinity—this is the one-point compactification of the complex plane into the Riemann sphere. Then (returning to the general case), after carrying out this compactification, the holomorphic projection $\pi : \mathcal{H}/G \to \mathcal{H}/H_n \cong \mathbb{C}$ extends to a Belyi map $\beta : X_G \to \mathbb{P}^1$, with each cusp of X_G mapped by β to the single cusp $\{\infty\}$ of H_n which compactifies the plane $\mathcal{H}/H_n$ to $\mathbb{C} \cup \infty = \mathbb{P}^1$ [24]. In turn, this Belyi map will have a unique associated *dessin*, as discussed in the previous section.

2.5. Schreier Coset Graphs

As mentioned, the Hecke group H_n has presentation $\langle x, y | x^2 = y^n = I\rangle$, and is thus the free product of cyclic groups $C_2 = \langle x | x^2 = I\rangle$ and $C_n = \langle y | y^n = I\rangle$. Given the free product structure of H_n, we see that its Cayley graph is an infinite free n-valent tree, but with each node replaced by an oriented n-gon. Now, for any *finite index* subgroup G of H_n, we can quotient the Cayley graph to arrive at a finite graph by associating nodes to right cosets and edges between cosets which are related by action of a group element. In other words, this graph encodes the permutation representation of H_n acting on the right cosets of G. This is called a *Schreier coset graph*, sometimes also referred to in the literature as a *Schreier-Cayley coset graph*, or simply a *coset graph*.

It is useful consider the permutations induced by the respective actions of x and y on the cosets of each Hecke subgroup; denote these by σ_0 and σ_1, respectively. We can find a third permutation σ_∞ by imposing the following condition, thereby constructing a *permutation triple* [16, 17]:

$$\sigma_0 \cdot \sigma_1 \cdot \sigma_\infty = 1 \,. \tag{17}$$

The permutations σ_0, σ_1, and σ_∞ give the *permutation representations* of H_n on the right cosets of each subgroup in question. As elements of the symmetric group, σ_0 and σ_1 can easily be computed from the Schreier coset graphs by following the procedure elaborated in Sebbar [25], i.e., by noting that the doubly directed edges represent an element x of order 2, while the positively oriented triangles represent an element y of order n. Since the graphs are connected, the group generated by x and y is transitive on the vertices. Clearly, σ_0 and σ_1 tell us which vertex of the coset graph is sent to which, i.e., which coset of the Hecke subgroup in question is sent to which by the action of H_n on the right cosets of this subgroup.

Now, as detailed in Singerman and Wolfart [26], there is a direct connection between the Schreier coset graphs and the *dessins d'enfants* for each class of Hecke subgroups: the *dessins d'enfants* for a certain conjugacy class of subgroups of a Hecke group H_n can be constructed from the Schreier coset graphs by replacing each positively oriented n-gon with a white node, and inserting a black node into every edge. Conversely, the Schreier coset graphs can be constructed from the *dessins* by replacing each white node with a positively oriented n-gon, and removing the black node from every edge. Crucially, the *dessin* constructed in this way is the same *dessin* as that associated to the Belyi map for the surface X_G as discussed above, where G is the conjugacy class of Hecke subgroups under consideration [24, 26].

This last result means that our chain of connections between Hecke groups and *dessins d'enfants* has come full circle: for a given conjugacy class of subgroups G of a Hecke group H_n, there is an associated (compactified) surface X_G; in each case there exists a Belyi map $\beta : X_G \to \mathbb{P}^1$. Every such Belyi map has a unique associated *dessin*, which in turn can be translated into a Schreier coset graph for a certain conjugacy class of Hecke subgroups—precisely the G from which we began!

3. HECKE SUBGROUPS AND ARCHIMEDEAN SOLIDS

Having reviewed the theory connecting *dessins d'enfants* with Hecke subgroups, we can now apply this theory to the particular case of the Archimedean solids. In this section, we first give a precise definition of these geometrical entities. We then identify the conjugacy class of Hecke subgroups corresponding to each Archimedean solid, by giving the permutation representations σ_0 and σ_1 for each such class. Finally, we discuss some interesting aspects of these results.

3.1. Platonic and Archimedean Solids

The Platonic solids are the regular, convex polyhedra; they are the tetrahedron, cube, octahedron, dodecahedron and icosahedron. In order to introduce the wider class of *Archimedean solids*, consider planar graphs without loops, and with vertices of degree $k > 2$. Following Magot and Zvonkin [12], let us call the list of numbers $(f_1, f_2, \ldots, f_k)$, where the f_i are the number of edges of the adjacent faces taken in the counter-clockwise direction around the vertex, the *type* of that vertex. Two such lists are *equivalent* if one can be obtained from the other by (a) making a cyclic shift and (b) inverting the order of the f_i. A solid is called *Archimedean* if the types of all its vertices are equivalent [12][1].

We emphasize a subtlety here. One informal way of defining the Archimedean solids is as the semi-regular convex polyhedra composed of two or more types of regular polygons meeting in identical vertices, with no requirement that faces be equivalent. *Identical vertices* is usually taken to mean that for any two vertices, there must be an isometry of the entire solid that takes one vertex to the other. Sometimes, however, it is only required that the faces that meet at one vertex are related isometrically to the faces that meet at the other. On the former definition, the so-called *pseudorhombicuboctahedron*, also known as the *elongated square gyrobicupola*, is not considered an Archimedean solid; on the latter it is. The formal definition of the Archimedean solids above corresponds to the latter definition here; it is this latter definition which which we shall use.

The solids which satisfy the condition for being *Archimedean* are:

I The five Platonic solids;
II Two infinite series (the prisms and anti-prisms);
III Fourteen exceptional solutions[2].

All these solids are listed in **Table 1**. For completeness, we also give the symmetry groups of each of these solids in the column labeled "Sym," written in *Schönflies notation*. Here, we recognize the standard tetrahedral, octahedral and icosahedral groups:

$$T \simeq A_4\,, \quad O \simeq S_4\,, \quad I \simeq A_5\,, \tag{18}$$

as well as the dihedral group D_n. The subscripts d and h denote extra symmetries about a horizontal (h) or diagonal (d) plane.

3.2. Hecke Subgroups for the Archimedean Solids

As discussed, we can interpret all the Archimedean solids as clean *dessins d'enfants* by inserting a black node into every edge and coloring every vertex white. By the correspondences detailed in the previous section, we should then be able to associate a class of Hecke subgroups to every Archimedean solid. To achieve this, we begin by converting the *dessins* for these solids to Schreier coset graphs, reading off the associated permutations σ_0 and σ_1 for the conjugacy class of subgroups of the relevant Hecke group in each case; these results are tabulated in Appendix A. Next, we find a representative of the conjugacy class of subgroups of the relevant Hecke group in each case by inputting the permutations σ_0 and σ_1 into the GAP [31] algorithm presented in Appendix B. This yields a representative of the conjugacy classes of subgroups in each case.

To illustrate this procedure, consider the specific example of the octahedron. Inputting the permutations for the octahedron into the algorithm of Appendix B yields the following generators for a representative of the conjugacy class of subgroups of H_4 associated to this solid:

$$\left\{ \begin{pmatrix} -1 & 0 \\ 3\sqrt{2} & -1 \end{pmatrix}, \begin{pmatrix} 1 & -3\sqrt{2} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -7 & -3\sqrt{2} \\ 6\sqrt{2} & 5 \end{pmatrix}, \begin{pmatrix} -7 & -6\sqrt{2} \\ 3\sqrt{2} & 5 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 5 & -6\sqrt{2} \\ 3\sqrt{2} & -7 \end{pmatrix}, \begin{pmatrix} 5 & -3\sqrt{2} \\ 6\sqrt{2} & -7 \end{pmatrix}, \begin{pmatrix} 11 & -15\sqrt{2} \\ -15\sqrt{2} & 41 \end{pmatrix} \right\}. \tag{19}$$

Note that comparison with our results concerning congruence subgroups of Hecke groups in Section 2.2 reveals that the octahedron corresponds to the principal congruence subgroup $H_4(3)$. Repeating this procedure for all Archimedean solids, we find a number of results worthy of comment:

- While the octahedron corresponds to the conjugacy class of Hecke subgroups $H_4(3)$, no other Archimedean solid is readily associated with congruence Hecke subgroups in an analogous manner.
- From Section 2.2, the matrices of H_4 are of the following two types:

$$\begin{pmatrix} a & b\sqrt{2} \\ c\sqrt{2} & d \end{pmatrix} \quad , \quad \begin{pmatrix} a\sqrt{2} & b \\ c & d\sqrt{2} \end{pmatrix} \tag{20}$$

  The elements of the first type form a subgroup of index 2 in H_4; all the subgroups of H_4 corresponding to the Archimedean solids turn out to be subgroups of this subgroup.
- We have seen that, interpreted as *dessins*, the tetrahedron, cube and dodecahedron correspond to the conjugacy classes

[1] It is worth listing some classical sources on the Archimedean solids. Theses objects were first discussed by Archimedes, in a now lost work to which Pappus refers [27]. In the fifteenth century, the solids were rediscovered by Kepler [28]. Classical geometers who discuss the Archimedean solids include Sommerville [29] and Miller [27].

[2] An interesting aside: It is known that the five Platonic solids correspond to three symmetry groups, which in turn are related to the exceptional Lie algebras $E_{6,7,8}$, and furthermore to Arnold's simple surface singularities with no deformations [30]. It is curious that the next order generalization of the simple singularities, with exactly one deformation modulus, has 14 exceptional cases. We conjecture, therefore, some connection to the exceptional Archimedean solids.

TABLE 1 | The Archimedean solids.

	Name	*V*	Vertex type	*F*	*E*	Sym	Hecke group
I	Tetrahedron	4	(3, 3, 3)	4	6	T_d	H_3
I	Cube	8	(4, 4, 4)	6	12	O_h	H_3
I	Octahedron	6	(3, 3, 3, 3)	8	12	O_h	H_4
I	Dodecahedron	20	(5, 5, 5)	12	30	I_h	H_3
I	Icosahedron	12	(3, 3, 3, 3, 3)	20	30	I_h	H_5
II	n-prism	$2n$	$(4, 4, n)$	$n+2$	$3n$	D_{nh}	H_3
II	n-antiprism	$2n$	$(3, 3, 3, n)$	$2n+2$	$4n$	D_{nd}	H_4
III	Truncated tetrahedron	12	(3, 6, 6)	8	18	T_d	H_3
III	Truncated cube	24	(3, 8, 8)	14	36	O_h	H_3
III	Truncated octahedron	24	(4, 6, 6)	14	36	O_h	H_3
III	Truncated isocahedron	60	(5, 6, 6)	32	90	I_h	H_3
III	Truncated dodecahedron	60	(3, 10, 10)	32	90	I_h	H_3
III	Truncated cuboctahedron	48	(4, 6, 8)	26	72	O_h	H_3
III	Truncated icosidodecahedron	120	(4, 6, 10)	62	180	I_h	H_3
III	Cuboctahedron	12	(3, 4, 3, 4)	14	24	O_h	H_4
III	Icosidodecahedron	30	(3, 5, 3, 5)	32	60	I_h	H_4
III	Rhombicuboctahedron	24	(3, 4, 4, 4)	26	48	O_h	H_4
III	Rhombicosidodecahedron	60	(3, 4, 5, 4)	62	120	I_h	H_4
III	Pseudorhombicuboctahedron	24	(3, 4, 4, 4)	26	48	D_{4d}	H_4
III	Snub cube	24	(3, 3, 3, 3, 4)	38	60	O	H_5
III	Snub dodecahedron	60	(3, 3, 3, 3, 5)	92	150	I	H_5

Type I are the five Platonic solids. Type II are the two infinite families: the prisms and anti-prisms. Type III are the 14 exceptional solids. We record the number of vertices V, faces F and edges E in each case. The vertex type *is the number of edges of the adjacent faces to the vertex, taken counter-clockwise; the fact that all vertices have the same vertex type is the definition of* Archimedean. *"Sym" denotes the symmetry group of the solid, given in Schönflies notation. We also indicate the associated Hecke group, determined by the valency of the vertices of each solid.*

of subgroups $\Gamma(3)$, $\Gamma(4)$, and $\Gamma(5)$, respectively [6]. In addition to these three arising as the *dessins* corresponding to certain congruence subgroups of the modular group, the *dessins* for the 33 conjugacy classes of genus zero, torsion-free congruence subgroups (discussed in detail in He et al. [6]) reveal that *other* Archimedean solids also correspond to certain conjugacy classes of congruence subgroups of the modular group. Specifically, the truncated tetrahedron is associated with the modular subgroup $\Gamma_0(2) \cap \Gamma(3)$, while the 8-prism is associated with the subgroup $\Gamma(8; 2, 1, 2)$. Given this, the question arises as to whether the other trivalent Archimedean solids can also be associated with congruence subgroups of the modular group. This can be checked in Sage [32] using the permutations σ_0 and σ_1 as input; doing so, we find that the answer to this question is in general *negative*. As a more specific result, we find that the truncated tetrahedron is the *only* trivalent, exceptional Archimedean solid which corresponds to a conjugacy class of congruence subgroups of the modular group.

4. CONCLUSIONS

In this paper, we have reviewed the connections between *dessins d'enfants* and Hecke subgroups. We have applied this theory to the case of the Archimedean solids, showing how, through interpretation as clean *dessins*, each of these geometric objects can be associated with a conjugacy class of subgroups of Hecke groups. In addition to opening up one more area of research into the Archimedean solids, this work is useful from the point of view of mathematical physics, as it can help to shed new light on cases where *dessins* naturally arise in physical contexts. To take two examples:

1. Certain $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions [33, 34] naturally give rise to trivalent *dessins* [3] (including Archimedean *dessins*); with the theory of this paper in mind, this suggests an otherwise overlooked connection between these gauge theories and modularity.
2. In He and McKay [4–6], it is shown that certain Calabi-Yaus give rise to clean, trivalent *dessins*, including many Archimedean *dessins*. Again, the theory of this paper therefore suggests a connection between the theory of Hecke groups and these Calabi-Yaus.

The general lessson is the following: given the growing ubiquity of *dessins* in mathematical physics, it is important to be clear and explicit on the connections of these objects to other areas of mathematics. Hence, it is important to illustrate the theory underlying these connections. It is particularly useful to lay out this theory explicitly in special cases such as that of the Archimedean solids, which we expect (on both inductive grounds and grounds of simplicity) to arise with greatest frequency in physical contexts.

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APPENDIX

A. PERMUTATIONS FOR THE ARCHIMEDEAN SOLIDS

In this appendix, we present the permutations σ_0 and σ_1 corresponding to the conjugacy classes of Hecke subgroups for each of the Archimedean solids. Although it is in principle easy to read these permutations off from the Schreier coset graphs in the manner detailed in Section 2.5, the procedure is extremely time-consuming, so it is worth presenting the results in full here. The method for computing explicit generators for a representative of each class of subgroups is detailed in the following appendix.

As a technical point, note that we have just presented the permutations σ_0 in each case. The permutations σ_1 can be written down by populating $2E/n$ n-tuples with numerals in ascending order from 1 to $2E$, where E is the number of edges of the solid, and H_n is the associated Hecke group as given in **Table 1**. So, for example, the permutations σ_1 for the octahedron are: (1, 2, 3, 4) (5, 6, 7, 8) (9, 10, 11, 12) (13, 14, 15, 16) (17, 18, 19, 20) (21, 22, 23, 24).

A.1. Platonic Solids

Interpreted as clean *dessins d'enfants*, the tetrahedron, cube, and dodecahedron correspond to the conjugacy classes of principal congruence subgroups $\Gamma(3)$, $\Gamma(4)$, and $\Gamma(5)$, respectively [6]. In addition, from **Table 1** we can see that the remaining two Platonic solids—the octahedron and the icosahedron—correspond to conjugacy classes of subgroups of H_4 and H_5, respectively[3]. The permutations σ_0 for these classes are:

Octahedron: (1, 5) (6, 11) (4, 12) (2, 13) (3, 23) (8, 14) (7, 18) (10, 19) (9, 22) (16, 24) (15, 17) (20, 21)

Icosahedron: (1, 6) (10, 11) (5, 15) (2, 21) (3, 16) (4, 43) (7, 22) (8, 27) (9, 33) (12, 34) (13, 37) (14, 42) (17, 25) (23, 26) (28, 32) (35, 36) (38, 41) (20, 44) (18, 46) (24, 47) (30, 48) (29, 52) (31, 53) (40, 54) (39, 57) (45, 58) (19, 59) (50, 60) (55, 56) (49, 51)

A.2. Prisms and Antiprisms

The prisms and antiprisms form infinite series of Archimedean solids. Since all prisms are trivalent, all correspond to a conjugacy class of subgroups of Γ; since all antiprisms are 4-valent, all correspond to a conjugacy class of subgroups of H_4. We can construct general expressions for the permutations σ_0 and σ_1 for the prisms and antiprisms. These expressions can be used to find the permutations σ_0 and σ_1 for any particular (anti)prism of interest.

[3]Of course, one should note that the octahedron and icosahedron *could* also be defined as subgroups of the modular group Γ, as their duals are trivalent.

Prisms:

Call the n-gon faced prism the n-prism. The n-prism has the following permutations σ_0 and σ_1:

$$\begin{aligned}\sigma_0\colon\ & (3n-2,3)\,(3n-1,6n-2)\,(6n-1,3n+3)\\ &\cdot\prod_{i=0}^{n-2}(3i+1,3i+6)\,(3i+2,3n+3i+1)\\ &\quad(3n+3i+2,3n+3i+6)\\ \sigma_1\colon\ &\prod_{i=0}^{n+2}(3i+1,3i+2,3i+3)\ . \end{aligned}\tag{A1}$$

Antiprisms:

Call the n-gon faced anti-prism the n-antiprism. The n-antiprism has the following permutations σ_0 and σ_1:

$$\begin{aligned}\sigma_0\colon\ & (4n-3,4)\,(4n-2,8n-2)\,(3,8n-1)\,(4n+1,8n)\\ &\cdot\prod_{i=0}^{n-2}(4i+1,4i+8)\,(4i+2,4n+4i+2)\\ &\quad(4i+7,4n+4i+3)\,(4n+4i+4,4n+4i+5)\\ \sigma_1\colon\ &\prod_{i=0}^{n+2}(4i+1,4i+2,4i+3,4i+4)\ . \end{aligned}\tag{A2}$$

A.3. Exceptional Archimedean Solids

In addition to the Platonic solids and the prisms and antiprisms, there remain 14 exceptional Archimedean solids, as given in **Table 1**. Here we give the permutations σ_0 for each corresponding class of Hecke subgroups.

Truncated tetrahedron: (1, 32) (3, 4) (2, 7) (5, 9) (6, 24) (8, 10) (11, 13) (12, 18) (15, 16) (17, 19) (14, 28) (29, 31) (33, 34) (30, 36) (26, 35) (20, 25) (23, 27) (21, 22)

Truncated cube: (3, 4) (2, 7) (5, 9) (60, 61) (63, 55) (57, 58) (20, 22) (21, 27) (24, 25) (11, 13) (12, 18) (15, 16) (39, 45) (43, 42) (38, 40) (30, 31) (29, 35) (32, 34) (66, 72) (65, 68) (67, 70) (48, 49) (47, 53) (50, 52) (1, 59) (62, 64) (69, 54) (51, 6) (41, 46) (8, 10) (56, 23) (36, 71) (33, 37) (17, 44) (14, 19) (26, 28)

Truncated octahedron: (1, 11) (2, 4) (5, 8) (9, 10) (12, 13) (3, 36) (6, 56) (7, 65) (35, 26) (32, 34) (29, 31) (27, 28) (25, 17) (16, 14) (15, 24) (21, 22) (18, 19) (23, 68) (70, 69) (66, 67) (62, 64) (63, 72) (59, 61) (57, 58) (51, 60) (50, 52) (54, 55) (49, 48) (45, 71) (20, 42) (30, 37) (39, 40) (38, 47) (44, 46) (41, 43) (33, 53)

Truncated icosahedron: (1, 14) (11, 13) (8, 10) (5, 7) (2, 4) (3, 16) (15, 28) (12, 26) (9, 22) (6, 20) (19, 35) (32, 34) (17, 31) (18, 58) (56, 60) (29, 55) (30, 53) (50, 52) (27, 49) (25, 47) (45, 46) (44, 23) (24, 41) (38, 40) (21, 37) (36, 73) (33, 64) (59, 61) (57, 110) (54, 107) (51, 98) (48, 95) (43, 86) (42, 83) (39, 77) (71, 75) (70, 68) (65, 67) (66, 62) (63, 116) (115, 113) (111, 112) (108, 109) (106, 104) (101, 103) (100, 99) (96, 97) (92, 94) (89, 91) (87, 88) (84, 85) (80, 82) (119, 79) (78, 118) (74, 76) (120, 122) (72, 164) (69, 158) (117, 155) (114, 149) (146, 105) (140, 102) (93, 137) (131, 90) (128, 81)

(163, 161) (159, 160) (156, 157) (152, 154) (150, 151) (147, 148) (143, 145) (141, 142) (138, 139) (134, 136) (132, 133) (129, 130) (125, 127) (123, 124) (121, 165) (162, 166) (153, 179) (144, 176) (135, 173) (126, 170) (167, 169) (171, 172) (174, 175) (177, 178) (168, 180)

Truncated dodecahedron: (1, 29) (3, 32) (30, 31) (27, 44) (23, 25) (24, 43) (21, 41) (18, 40) (17, 19) (11, 13) (15, 39) (12, 38) (5, 7) (9, 35) (6, 34) (2, 4) (26, 28) (22, 20) (14, 16) (8, 10) (36, 62) (33, 46) (45, 107) (42, 92) (37, 77) (61, 59) (60, 66) (63, 64) (47, 49) (51, 120) (48, 119) (108, 109) (111, 105) (104, 106) (90, 96) (93, 94) (89, 91) (74, 76) (78, 79) (75, 81) (58, 56) (50, 52) (55, 53) (54, 122) (57, 123) (121, 136) (117, 118) (112, 110) (116, 113) (115, 135) (114, 134) (133, 149) (101, 103) (95, 97) (98, 100) (102, 131) (99, 130) (132, 146) (86, 88) (80, 82) (83, 85) (87, 128) (84, 127) (129, 142) (71, 73) (65, 67) (68, 70) (72, 126) (69, 125) (124, 139) (137, 154) (156, 153) (152, 138) (151, 180) (179, 150) (148, 176) (177, 178) (174, 175) (173, 147) (170, 145) (171, 172) (168, 169) (165, 166) (164, 144) (143, 167) (162, 163) (159, 160) (141, 158) (140, 161) (155, 157)

Truncated cuboctahedron: (1, 23) (24, 20) (21, 17) (18, 14) (15, 12) (10, 9) (7, 6) (4, 3) (2, 25) (22, 29) (19, 32) (16, 35) (13, 38) (11, 41) (8, 44) (5, 46) (26, 28) (30, 55) (56, 58) (31, 59) (33, 34) (36, 62) (63, 64) (65, 37) (39, 40) (42, 67) (68, 70) (43, 71) (45, 48) (47, 49) (50, 52) (27, 53) (54, 77) (51, 73) (76, 74) (78, 79) (75, 119) (57, 86) (60, 89) (88, 87) (85, 83) (90, 91) (61, 98) (66, 101) (100, 99) (95, 97) (102, 103) (72, 113) (69, 110) (111, 112) (114, 115) (107, 109) (80, 82) (84, 128) (126, 127) (81, 125) (129, 130) (132, 133) (96, 134) (92, 94) (93, 131) (135, 136) (137, 105) (104, 106) (108, 140) (138, 139) (141, 142) (117, 143) (123, 144) (120, 121) (116, 118) (122, 124)

Truncated icosidodecahedron: (1, 29) (26, 28) (23, 25) (20, 22) (17, 19) (15, 16) (11, 14) (8, 10) (5, 7) (2, 4) (3, 31) (30, 59) (27, 56) (24, 52) (21, 50) (18, 47) (13, 44) (12, 41) (9, 38) (6, 34) (33, 60) (58, 89) (88, 86) (85, 57) (53, 55) (83, 54) (80, 82) (79, 51) (48, 49) (46, 77) (75, 76) (45, 74) (42, 43) (40, 71) (68, 70) (39, 67) (35, 37) (36, 65) (64, 62) (61, 32) (90, 119) (87, 116) (84, 113) (81, 109) (78, 107) (73, 104) (72, 101) (69, 98) (66, 95) (63, 91) (93, 121) (123, 179) (178, 176) (175, 174) (173, 120) (118, 117) (115, 170) (169, 167) (166, 164) (163, 162) (161, 114) (112, 110) (111, 158) (155, 157) (152, 154) (149, 151) (108, 148) (105, 106) (103, 146) (143, 145) (140, 142) (138, 139) (102, 137) (99, 100) (97, 134) (131, 133) (128, 130) (125, 127) (96, 124) (92, 94) (122, 181) (180, 269) (177, 260) (257, 172) (171, 254) (168, 251) (165, 242) (160, 239) (159, 236) (156, 233) (153, 224) (150, 221) (147, 218) (144, 215) (141, 206) (136, 203) (135, 200) (132, 197) (129, 188) (126, 185) (184, 182) (183, 270) (268, 266) (265, 262) (264, 261) (259, 258) (256, 255) (252, 253) (248, 250) (247, 245) (244, 243) (241, 240) (237, 238) (234, 235) (230, 232) (227, 229) (226, 225) (222, 223) (219, 220) (216, 217) (214, 212) (209, 211) (207, 208) (204, 205) (202, 201) (199, 198) (196, 194) (191, 193) (189, 190) (186, 187) (267, 271) (263, 323) (249, 314) (246, 311) (231, 302) (228, 299) (290, 213) (210, 287) (195, 278) (192, 274) (276, 329) (328, 326) (325, 272) (273, 324) (322, 320) (319, 317) (316, 315) (313, 312) (308, 310) (307, 305) (304, 303) (300, 301) (296, 298) (293, 295) (291, 292) (288, 289) (284, 286) (281, 283) (279, 280) (275, 277) (330, 332) (327, 359) (321, 356) (353, 318) (350, 309) (347, 306) (344, 297) (294, 341) (285, 338) (282, 335) (333, 334) (331, 360) (358, 357) (355, 354) (351, 352) (348, 349) (345, 346) (342, 343) (339, 340) (337, 336)

Cuboctahedron: (1, 10) (11, 14) (15, 8) (4, 5) (2, 21) (3, 17) (20, 6) (7, 31) (30, 16) (27, 13) (22, 9) (12, 26) (25, 39) (28, 42) (43, 29) (32, 46) (47, 19) (18, 33) (24, 34) (23, 38) (35, 37) (40, 41) (44, 45) (36, 48)

Icosidodecahedron: (1, 9) (10, 17) (18, 26) (27, 34) (35, 4) (3, 37) (2, 5) (6, 12) (11, 13) (14, 20) (19, 21) (22, 25) (28, 29) (30, 33) (36, 40) (38, 41) (44, 79) (39, 78) (8, 45) (7, 49) (46, 52) (16, 54) (55, 57) (15, 58) (24, 63) (64, 65) (23, 66) (32, 70) (31, 74) (71, 73) (75, 77) (80, 110) (76, 109) (42, 48) (47, 82) (81, 43) (50, 53) (56, 86) (51, 85) (60, 93) (59, 62) (61, 94) (67, 69) (72, 102) (101, 68) (89, 96) (95, 100) (90, 99) (98, 107) (103, 108) (97, 104) (105, 112) (111, 116) (106, 115) (84, 113) (83, 117) (114, 120) (88, 118) (87, 92) (91, 119)

Rhombicuboctahedron: (1, 5) (6, 9) (10, 13) (4, 14) (2, 17) (3, 30) (20, 31) (8, 33) (7, 38) (34, 37) (12, 54) (11, 59) (55, 58) (15, 70) (67, 69) (16, 66) (18, 36) (35, 48) (21, 47) (19, 24) (40, 41) (39, 53) (50, 56) (42, 49) (57, 64) (61, 79) (68, 80) (65, 60) (72, 76) (28, 73) (25, 32) (29, 71) (22, 82) (27, 81) (23, 26) (44, 45) (46, 86) (43, 87) (51, 63) (62, 91) (52, 90) (75, 77) (74, 95) (78, 94) (84, 96) (83, 85) (88, 89) (92, 93)

Rhombicosidodecahedron: (1, 19) (4, 5) (8, 12) (11, 16) (15, 20) (2, 21) (3, 28) (6, 29) (7, 36) (9, 40) (10, 44) (13, 47) (14, 52) (17, 56) (18, 237) (240, 22) (24, 25) (27, 30) (32, 33) (35, 37) (39, 41) (43, 48) (46, 49) (51, 53) (55, 238) (54, 116) (57, 239) (23, 68) (26, 72) (31, 80) (34, 84) (38, 92) (42, 96) (45, 104) (50, 108) (115, 58) (60, 61) (64, 65) (67, 69) (71, 73) (76, 77) (79, 81) (83, 85) (88, 89) (91, 93) (95, 97) (100, 101) (103, 105) (107, 109) (112, 113) (114, 117) (175, 118) (62, 124) (63, 128) (66, 132) (70, 131) (74, 136) (75, 140) (78, 144) (82, 143) (86, 148) (87, 152) (90, 156) (94, 155) (98, 160) (99, 164) (102, 168) (106, 167) (110, 172) (111, 176) (119, 121) (123, 125) (127, 129) (130, 133) (135, 137) (139, 141) (142, 145) (147, 149) (151, 153) (154, 157) (159, 161) (163, 165) (166, 169) (171, 173) (59, 120) (122, 184) (126, 188) (134, 192) (138, 196) (146, 200) (150, 204) (158, 208) (162, 212) (170, 213) (174, 177) (183, 185) (187, 189) (191, 193) (195, 197) (199, 201) (203, 205) (207, 209) (211, 214) (178, 216) (180, 181) (182, 224) (186, 223) (190, 228) (194, 227) (198, 232) (202, 231) (206, 236) (210, 235) (215, 218) (179, 217) (222, 225) (226, 229) (230, 233) (234, 219) (220, 221)

Pseudorhombicuboctahedron: (1, 5) (6, 10) (11, 14) (4, 15) (2, 17) (3, 46) (20, 47) (8, 21) (7, 26) (22, 25) (9, 30) (12, 35) (31, 34) (13, 38) (16, 43) (39, 42) (18, 24) (23, 53) (50, 56) (19, 49) (28, 58) (59, 61) (32, 62) (27, 29) (33, 65) (66, 70) (40, 71) (36, 37) (44, 45) (48, 80) (76, 79) (41, 75) (52, 77) (51, 81) (84, 96) (78, 95) (55, 82) (83, 88) (60, 85) (54, 57) (86, 64) (63, 68) (67, 91) (87, 90) (89, 93) (73, 94) (72, 74) (69, 92)

Snub cube: (1, 18) (13, 17) (8, 12) (2, 7) (5, 22) (4, 60) (3, 53) (21, 19) (20, 26) (16, 35) (34, 14) (40, 15) (45, 11) (9, 44) (10, 49) (6, 54) (52, 56) (59, 23) (25, 27) (30, 31) (33, 36) (39, 41) (43, 50) (48, 55) (51, 73) (57, 72) (58, 67) (24, 61) (65, 28) (29, 97) (32, 93) (37, 92) (38, 87) (42, 84) (46, 83) (47, 78) (71, 68) (66, 62) (64, 98) (96, 94) (88, 91) (85, 86) (79, 82) (74, 77) (75, 108) (69, 107) (70, 103) (63, 102) (99, 101) (100, 119) (118, 95) (89, 117) (90, 114) (81, 113) (80, 112) (76, 109) (104, 106) (105, 120) (115, 116) (110, 111)

Snub dodecahedron: (1, 7) (8, 12) (13, 16) (17, 21) (5, 22) (2, 37) (3, 32) (4, 27) (6, 38) (10, 42) (9, 47) (11, 48) (15, 52) (14, 57) (20, 58) (19, 62) (18, 67) (25, 68) (24, 72) (26, 23) (28, 31) (33, 36) (39, 41) (43, 46) (49, 51) (53, 56) (59, 61) (63, 66) (69, 71) (30, 73) (29, 76) (35, 87) (34, 92) (40, 93) (45, 102) (44, 107) (50, 108) (55, 117) (54, 122) (60, 123) (65, 132) (64, 137) (70, 138) (75, 147) (74, 80) (77, 81) (82, 86) (88, 91) (94, 96) (97, 101) (103, 106) (109, 111) (112, 116) (118, 121) (124, 126) (127, 131) (133, 136) (139, 141) (142, 146) (79, 148) (149, 182) (78, 183) (85, 184) (84, 187) (83, 192) (90, 193) (89, 197) (95, 198) (100, 199) (99, 202) (98, 206) (105, 207) (104, 212) (110, 213) (115, 214) (114, 217) (113, 223) (120, 224) (119, 152) (125, 153) (130, 154) (129, 157) (128, 162) (135, 163) (134, 167) (140, 168) (145, 169) (144, 172) (143, 177) (150, 178) (200, 201) (203, 210) (208, 211) (215, 216) (218, 222) (225, 151) (155, 156) (158, 161) (164, 166) (170, 171) (173, 176) (179, 181) (185, 186) (188, 191) (194, 196) (195, 258) (205, 262) (204, 267) (209, 268) (220, 272) (219, 230) (221, 226) (160, 232) (159, 237) (165, 238) (175, 242) (174, 247) (180, 248) (190, 252) (189, 257) (259, 261) (263, 266) (269, 271) (273, 229) (227, 231) (233, 236) (239, 241) (243, 246) (249, 251) (253, 256) (260, 293) (265, 294) (264, 297) (270, 298) (275, 299) (274, 280) (228, 276) (235, 277) (234, 282) (240, 283) (284, 245) (287, 244) (288, 250) (289, 255) (292, 254) (295, 296) (300, 279) (278, 281) (285, 286) (290, 291)

B. ALGORITHM FOR COMPUTING GENERATORS FOR HECKE SUBGROUPS

We can find the generators for a representative of all the conjugacy classes of subgroups of interest using GAP [6, 31]. First, we use the permutation data σ_0, σ_1 obtained from each of the Schreier coset graphs (in turn obtained from each of the *dessins*) to find the group homomorphism by images between the relevant Hecke group and a representative of the conjugacy class of subgroups of interest. We then use this to define the representative in question. Finally, we use the GAP command `IsomorphismFpGroup(G)`, which returns an isomorphism from the given representative to a finitely presented group isomorphic to that representative. This function first chooses a set of generators of the representative and then computes a presentation in terms of these generators.

To give an example, consider the clean *dessin* for the octahedron. We can find a set of generators as 2×2 matrices for a representative of the associated class of subgroups by implementing the following code in GAP:

```
gap> f:=FreeGroup("x","y");
<free group on the generators [ x, y ]>
gap> H4:=f/[f.1^2,f.2^4];
<fp group on the generators [ x, y ]>
gap>hom:=GroupHomomorphismByImages(H4,Group(
(1,5)(6,11)(4,12)(2,13)(3,23)(8,14)(7,18)(10,19)(9,22)(16,24)(15,17)(20,21),
(1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)(17,18,19,20)(21,22,23,24)),
GeneratorsOfGroup(H4),
[(1,5)(6,11)(4,12)(2,13)(3,23)(8,14)(7,18)(10,19)(9,22)(16,24)(15,17)(20,21),
(1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)(17,18,19,20)(21,22,23,24)]]);
[ x, y ] ->
[ (1,5)(2,13)(3,23)(4,12)(6,11)(7,18)(8,14)(9,22)(10,19)(15,17)(16,24)(20,21),
(1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)(17,18,19,20)(21,22,23,24) ]
gap> octahedron_group:=PreImage(hom,Stabilizer(Image(hom),1));
Group(<fp, no generators known>)
gap> iso:=IsomorphismFpGroup(octahedron_group);
[ <[ [ 1, 1 ] ]|y*x*y*x^-1*y*x^-1>,
<[ [ 2, 1 ] ]|y^-1*x*y^-1*x^-1*y^-1*x^-1>,
<[ [ 3, 1 ] ]|y^2*x*y*x^-1*y*x^-1*y^ -1>,
<[ [ 4, 1 ] ]|y^-1*x*y*x*y*x^-1*y^-2>,
<[ [ 5, 1 ] ]|x*y^2*x*y*x^-1*y*x^-1*y^-1*x^-1>,
<[ [ 6, 1 ] ]|x*y^-1*x*y*x*y*x^-1*y^-2*x^-1>,
<[ [ 7, 1 ] ]|y*x*y^-1*x*y*x*y^-1*x^-1*y*x^-1*y^-1*x^-1> ]
-> [ F1, F2, F3, F4, F5, F6, F7 ]
```

We define the relevant Hecke group (here H_4) in the third line. Then, the only input in each case is the permutation data σ_0, σ_1. Once the output has been obtained, we see the generators (here seven: [$F1$, $F2$, $F3$, $F4$, $F5$, $F6$, $F7$]) in the final line, as functions of the x and y for the Hecke group in question. Now, returning to the explicit matrices for these x and y presented in Section 2.2, the only thing left to do is to multiply together the matrices and their inverses as indicated. This will produce the generators for each representative, as required.