



Non-equilibrium Annealed Damage Phenomena: A Path Integral Approach

Sergey G. Abaimov*

Center for Design, Manufacturing, and Materials, Skolkovo Institute of Science and Technology, Moscow, Russia

We investigate the applicability of the path integral of non-equilibrium statistical mechanics to non-equilibrium damage phenomena. As an example, a fiber-bundle model with a thermal noise and a fiber-bundle model with a decay of fibers are considered. Initially, we develop an analogy with the Gibbs formalism of non-equilibrium states. Later, we switch from the approach of non-equilibrium states to the approach of non-equilibrium paths. Behavior of path fluctuations in the system is described in terms of effective temperature parameters. An equation of path as an analog of the equation of state and a law of path-balance as an analog of the law of conservation of energy are developed. Also, a formalism of a free energy potential is developed. For fluctuations of paths in the system, the statistical distribution is found to be Gaussian. Also, we find the “true” order parameters linearizing the matrix of fluctuations. The last question we discuss is the applicability of the phase transition theory to non-equilibrium processes. From near-equilibrium processes to stationary processes (dissipative structures), and then to significantly non-equilibrium processes: Through these steps we generalize the concept of a non-equilibrium phase transition.

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INTRODUCTION

One of the most important aspects in the modern theory of damage phenomena is the possibility to describe damage occurrence on the base of the formalism of statistical physics. Many approaches have been developed [1–24], and surveys of developments can be found in [25–30]. However, the majority of these studies are devoted to equilibrium or near-equilibrium damage occurrence. In our study, we consider a non-equilibrium damage process (a cascade, an avalanche) far from the point of a quasi-static state.

Once the analogy between damage mechanics and statistical physics had been discovered, it immediately became clear that any damage phenomenon can be described as a phase transition, where damage growth represents the appearance of another phase and the point of material failure is the point of the phase transition. The best way to illustrate this similarity is to compare the phase diagrams in **Figure 1**. There are debates in the literature [3–13, 21, 22, 24, 29–35] about whether this phase transition is of the first order or of the continuous order. The reason for this uncertainty is that spinodal points are very similar to critical points in the sense that in their vicinity a system switches its behavior from the laws intrinsic only to this particular system to universal power-law

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*Correspondence:

Sergey G. Abaimov
s.abaimov@skoltech.ru

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scaling. However, there is no doubt that any structure under load is in the metastable state since there always exists a nucleus of the critical size. In other words, by breaking enough of the material with external forces, we can always cause material failure. This is a clear indication of metastability, because there is no critical nucleus (no matter how big it is) for stable states.

To discuss our approach, we involve the fiber-bundle model (further, FBM) as the most illustrative representative of models with damage. However, the applicability of the approach we develop lies beyond the one-dimensional formulation of the FBM and can be easily generalized to two or three-dimensional damage phenomena.

For all possible approaches, an analogy between statistical physics and damage phenomena should rely on mapping of one type of fluctuations on another. In this sense, damage exhibits more complex behavior than classical thermodynamic systems of statistical physics due to the presence of two different forms of disorder: quenched and annealed. Therefore, to perform the aforementioned mapping, these two forms of disorder are usually studied apart from one another.

The influence of quenched disorder on the model behavior is generally observed where the dynamical timescale of fracture is much faster than the timescale of fluctuations. In this case, “frozen” initial disorder in a model plays the crucial role. This type of disorder, employed in the FBM, has been studied by many authors [8, 16, 28, 29, 31–85]. In particular, in our previous studies [21, 22, 30], we introduced quenched disorder by means of the fiber strength variability and for the ensemble of constant strain we were able to map the fluctuations observed on thermodynamic fluctuations in statistical physics.

Another type of behavior is exhibited when a FBM is influenced by the annealed disorder. One of two approaches is generally employed: the disorder is introduced in the model formulation either by means of a phenomenological rate of fiber failures with time [29, 86–109] or by the direct input of a stochastic or thermal noise into the state of stress of a fiber [23, 110–118].

Annealed disorder generates fluctuations which are closely related to their thermodynamic analogs in statistical physics. Indeed, adding stochastic noise to a fiber load seems to be quite similar to the statistical fluctuations of, for example, gas pressure acting on a piston. However, as it was recently discovered [119–123], annealed fluctuations in damage phenomena exceed thermodynamic fluctuations by orders of magnitude. In particular, it was found that the effective temperature required to provide the annealed behavior observed in experiments is 10 times higher (about 3,000 K) than the real temperature during the tests. This discrepancy is explained by complex interactions of quenched and annealed types of disorder which are present simultaneously in the model [18, 113–118, 122, 124].

In our study, we continue our previous research [23, 125, 126] and investigate a FBM with annealed damage evolution. In Section Model, we introduce models which we will use to illustrate our approach. In Section The Classical Gibbs Approach of States, we develop an analog of the classical Gibbs approach for non-equilibrium states. However, this approach has its restrictions which we discuss in Section Restrictions of

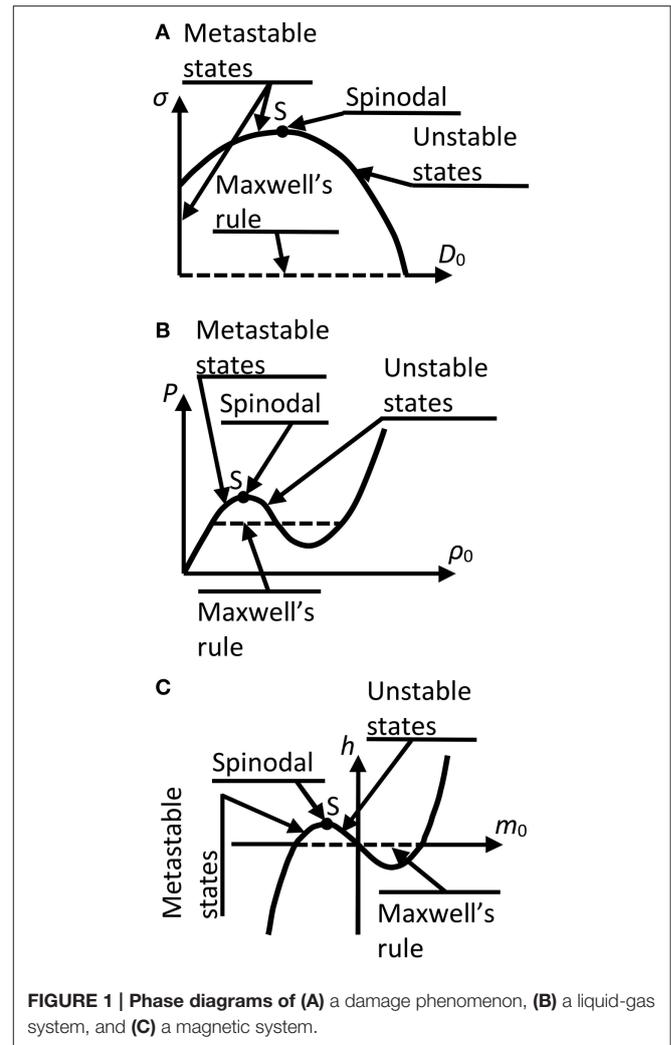


FIGURE 1 | Phase diagrams of (A) a damage phenomenon, (B) a liquid-gas system, and (C) a magnetic system.

the Classical Gibbs Approach of States. In Section Statistical Mechanics of the Path Integral Approach, we consider a path integral, when instead of non-equilibrium states of the system we investigate non-equilibrium paths. We map fluctuations of damage on the main concepts of statistical mechanics, such as temperature, entropy, free energy potential, and the law of conservation of energy. However, we find that these concepts are no longer associated with the energy characteristics of the states of the system but are defined over the space of all possible paths. Also, we investigate the behavior of fluctuations for this approach and find “actual” order parameters which diagonalize the matrix of the probability distribution. In the conclusions, we discuss how our theory leads to the description of non-equilibrium phase transitions.

MODEL

Damage is a complex phenomenon. The basic principles of damage are often completely disguised by the secondary side effects of its appearance. Therefore, to investigate the main

concepts of non-equilibrium behavior, it is reasonable to consider initially a simple model.

In this paper, as an illustration, we consider an annealed FBM. We assume that the number N of fibers in the model is constant and infinite in the thermodynamic limit. Intact fibers all carry the same strain ε_f which is identically equal to the strain ε of the total model: $\varepsilon_f \equiv \varepsilon$. The stress of each intact fiber is assumed to have a linear elastic dependence on the strain until a fiber failure: $\sigma_f = E\varepsilon_f$. In this paper, we consider constant strain $\varepsilon = \text{const}$ as an external boundary constraint of the model. The case of constant stress can be considered in a similar way (the redistribution of the load complicates the analytical solution but does not change the applicability of the path integral). The Young modulus E here is assumed to be the same for all fibers; however, the stress-strain dependence for the total model is non-linear because of fiber failures. We introduce an “intact” order parameter L as the fraction of intact fibers (unity minus the damage parameter). Everywhere further we assume that at the initial time $t = 0$ all fibers are intact: $L_{t=0} = 1$.

As a first modification of the FBM, we consider an annealed fiber-bundle model with noise [23, 110–118] (further, NFBM). Each fiber has thermal fluctuations of its energy characteristics. From statistical mechanics, we know that a piston, which works as a gas boundary and is supported by a spring, has oscillations due to the equipartition of energy. In particular, the elastic energy of the spring fluctuates as if a white Gaussian noise were added to the stress of the spring. Similarly, we assume that the stress of each fiber σ_f has an addition $\Delta\sigma_f$ of a white Gaussian noise with zero mean and standard deviation $\sqrt{k_B T}$. As it was discussed in Section Introduction, the temperature here may not correspond to the temperature during the experiment but may be several orders higher [119–123].

The probability density function of the noise is

$$p(\Delta\sigma_f) = \frac{1}{\sqrt{2\pi k_B T}} e^{-\frac{\Delta\sigma_f^2}{2k_B T}} \quad (1)$$

and the cumulative distribution function is

$$P(\Delta\sigma_f) = \frac{1}{\sqrt{2\pi k_B T}} \int_{-\infty}^{\Delta\sigma_f} e^{-\frac{x^2}{2k_B T}} dx. \quad (2)$$

Each fiber has an *a priori* assigned strength threshold s which we choose to be the same for all fibers and not to change during the model evolution. A fiber can break only when its stress $\sigma_f = E\varepsilon + \Delta\sigma_f$ exceeds its strength s . We consider discrete time-steps dt of the model evolution. At each time-step, the probability of a fiber breaking is $1 - P(s - E\varepsilon)$ and the probability of a fiber staying intact is $P(s - E\varepsilon)$. Here we assume that there are no correlations of noise among adjacent fibers. Also, in spite of the fact that the time interval dt between the consecutive time-steps is assumed to be small, it is supposed to be much larger than the duration of any noise correlations. Therefore, we assume that there are no correlations of the noise in time either. Another assumption we utilize is that although the time interval between consecutive time-steps has zero limit, the thermodynamic limit

of infinite number of fibers $N \rightarrow +\infty$ is taken first. Therefore, although only the small fraction of fibers breaks at each time-step, the number of broken fibers is much higher than unity.

Although, we assume the noise to be Gaussian, this does not play any role in our model due to the fact that s , E , and ε are the same for all fibers as well as for all systems in the ensemble and are some constants. Therefore, the noise value $\Delta\sigma_{break} = s - E\varepsilon$ at which fibers break is fixed so that $P(s - E\varepsilon)$ and $1 - P(s - E\varepsilon)$ are two constants as well.

The second modification of the FBM which we consider (following the original terminology, a model with “breaking kinetics”) is an annealed decay fiber-bundle model [86–109; further, DFBM]. There is no thermal noise in this model. Instead, each (so far intact) fiber has *a priori* assigned probability $p_f dt$ of failing during the time interval dt and probability $(1 - p_f dt)$ of staying intact.

Here, p_f is the decay rate which is constant during model evolution. The duration of the time-step dt is considered to be fixed as well. Therefore, $p_f dt$ and $(1 - p_f dt)$ are some constants, as well as $1 - P(s - E\varepsilon)$ and $P(s - E\varepsilon)$ respectively, which represent fixed probabilities of a fiber breaking or staying intact during a time-step. In this sense, equating $p_f dt$ and $1 - P(s - E\varepsilon)$, we can map the DFBM on the NFBM. However, some care should be exercised when we say $p_f dt = 1 - P(s - E\varepsilon)$. Here, we consider dt to be the duration of the time interval during which the Gaussian noise loses its correlation with the value of the noise at the previous time-step and “switches” to a new value. An illustrative example is the output of the Gaussian random generator which provides a discrete set of uncorrelated i.i.d. numbers. In this sense, the duration of dt is not arbitrary but is dictated by the physical processes responsible for the correlations in the noise. Hence, we cannot say here that the considered probabilities are proportional to dt .

The issue can be illustrated with the aid of extreme statistics. If we observe a fiber during a long time interval τ consisting of $N \gg 1$ time-steps dt , the probability of the fiber staying intact during this time interval is $P^N(s - E\varepsilon) = e^{\frac{\tau}{dt} \ln P(s - E\varepsilon)}$, which is clearly not linear with time. Therefore, our mapping of the NFBM onto the DFBM is introduced only for one time-step. In this sense, we will consider only one of these models (the DFBM for the approach of states at Sections The Classical Gibbs Approach of States and Restrictions of the Classical Gibbs Approach of States, which is more convenient for this as it is a model with a well-known solution; the NFBM for the approach of paths at Section Statistical Mechanics of the Path Integral Approach since it is more illustrative of the generalization to more complex model formulations), assuming, however, that all results are valid for the other model as well.

Before going further, we should clarify that we do not consider the healing of fibers: Once a fiber has broken, it stays broken forever. However, our formalism easily allows healing to be added to the model formulation. We need only, in addition to $P(s - E\varepsilon)$ and $1 - P(s - E\varepsilon)$, to introduce the probability of a broken fiber healing. All formulae can be easily generalized for this case as well, but due to the complexity of the results already present for the simpler model, we leave it for further studies.

THE CLASSICAL GIBBS APPROACH OF STATES

The classical Gibbs approach is formulated in terms of states. To develop its analog for damage phenomena, we need to give some definitions: what we refer to as a non-equilibrium state (or a fluctuation). In more details, this approach is discussed in [30], Chapter 2.

Let us imagine a FBM where we observe separate fibers. As to a microstate $\{n\}$ we will refer to a particular microconfiguration when for each fiber of the model we prescribe whether it is intact or broken. For example, for a system with $N = 3$ fibers, one of the microstates is $\{xx|\}$ where by two symbols “x,” we denoted that the first two fibers of the model are broken, while the symbol “|” means that the third fiber is still intact. For a model with only three fibers, it is easy to list all the possible microstates: $\{|||\}$, $\{||x\}$, $\{|x|\}$, $\{x||\}$, $\{xx|\}$, $\{x|x|\}$, $\{xx|\}$, $\{xxx\}$.

What is the probability of observing a particular microstate in the ensemble of identical systems? For the DFMB, the solution is well-known as the solution of radioactive decay. The probability of a fiber failing at time t during time interval dt is $e^{-p_f t} p_f dt$ and the probability of a fiber failing before the time t is $\int_0^t e^{-p_f t} p_f dt = 1 - e^{-p_f t}$. Therefore, the probability of observing a microstate $\{n\}$ with L intact and $(1 - L)$ broken fibers at the time t is

$$w_{\{n\}}^{\text{ensemble}} = (e^{-p_f t})^{NL} (1 - e^{-p_f t})^{N(1-L)}. \tag{3}$$

We used here the superscript “ensemble” to emphasize that this probability is dictated by the ensemble considered.

We can rewrite the distribution (Equation 3) as

$$w_{\{n\}}^{\text{ensemble}} = \frac{1}{Z^{\text{ensemble}}(t)} e^{-NL/T^{\text{eff}}(t)} \tag{4}$$

where we denoted $T^{\text{eff}}(t) \equiv \ln^{-1}(e^{p_f t} - 1)$. Formula (4) could become identical to the Gibbs probability distribution if we called $T^{\text{eff}}(t)$ the effective temperature. Then $Z^{\text{ensemble}}(t) = (1 - e^{-p_f t})^{-N}$ plays the role of the partition function of the ensemble as we will prove later in Equations (9–11).

We see that the probability distribution of microstates is Gibbsian with the effective temperature changing with time; this allows us to call our ensemble as being “effectively” canonical. Similar to a thermodynamic system with the limited spectrum (like an Ising model), the effective temperature can be both positive and negative, choosing intact states for fibers in the beginning of the model evolution but switching this choice to broken once at later times.

As to a fluctuation $[L]$ (a macrostate $[L]$) we will refer to the union of all microstates corresponding to the given value of parameter L : $[L] \equiv \bigcup_{\{n\}:L(\{n\})=L} \{n\}$. For example, in the case

of a model with three fibers, we observe fluctuation $[L = 1/3]$ when we observe one of the three microstates: $\{xx|\}$, $\{x|x|\}$, or $\{xx|\}$. For the model with arbitrary N , the number $g_{[L]}$ of microstates, corresponding to the fluctuation $[L]$, is called the statistical weight of this fluctuation. It can be found as the

combinatorial number of outcomes when we distribute NL intact fibers and $N(1 - L)$ broken fibers among N fibers:

$$g_{[L]} = \frac{N!}{(NL)!(N(1 - L))!}. \tag{5}$$

To simplify this expression, we apply Stirling’s approximation

$$N! \approx \left(\frac{N}{e}\right)^N O(N^\alpha) \tag{6}$$

and in the limit $N \gg 1$ we discard the slow power-law dependence $O(N^\alpha)$ on N in comparison with the fast exponential multiplier $(N/e)^N$:

$$N! \approx_{\ln} \left(\frac{N}{e}\right)^N. \tag{7}$$

The special notation for the logarithmic accuracy, “ \approx_{\ln} ,” stands here to denote the accuracy to the order of an arbitrary multiplier with the power-law dependence on N (for detailed examples, see [30], Chapter 2).

Applying Stirling’s approximation with the logarithmic accuracy to Equation (5), we find

$$g_{[L]} \approx_{\ln} L^{-NL} (1 - L)^{-N(1-L)}. \tag{8}$$

Let us now prove that $Z^{\text{ensemble}}(t) = (1 - e^{-p_f t})^{-N}$ is indeed the partition function of the ensemble. By definition, the partition function is the sum of exponential functions $e^{-NL/T^{\text{eff}}(t)}$ over all microstates:

$$Z^{\text{ensemble}}(t) \equiv \sum_{\{n\}} e^{-NL(\{n\})/T^{\text{eff}}(t)} \tag{9}$$

Grouping microstates by fluctuations $[L]$, since all microstates of the same fluctuation have equal values of L , we can write:

$$Z^{\text{ensemble}}(t) = \sum_{[L]} g_{[L]} e^{-NL/T^{\text{eff}}(t)}. \tag{10}$$

Substituting here $T^{\text{eff}}(t) \equiv \ln^{-1}(e^{p_f t} - 1)$ and the statistical weight from Equation (5), we easily prove the required statement:

$$\begin{aligned} Z^{\text{ensemble}}(t) &= \sum_{NL=0}^N \frac{N!}{(NL)!(N(1 - L))!} \left(\frac{1}{e^{p_f t} - 1}\right)^{NL} 1^{N(1-L)} \\ &= \left(\frac{1}{e^{p_f t} - 1} + 1\right)^N = (1 - e^{-p_f t})^{-N}. \end{aligned} \tag{11}$$

To find the probability $W_{[L]}^{\text{ensemble}}(t, L)$ of observing a fluctuation $[L]$ in the ensemble (to observe a system with the given fraction L of intact fibers), we multiply probability (Equations 3, 4) of the corresponding microstates by the number (Equation 8) of these microstates:

$$W_{[L]}^{\text{ensemble}} = g_{[L]} w_{\{n\} \in [L]}^{\text{ensemble}} \approx_{\ln} \left(\frac{e^{-p_f t}}{L}\right)^{NL} \left(\frac{1 - e^{-p_f t}}{1 - L}\right)^{N(1-L)}. \tag{12}$$

To attribute a microstate $\{n\}$ to a particular fluctuation $[L]$, we will write either $\{n\} \in [L]$ or $\{n\} : L_{\{n\}} = L$. There is no difference between these two notations, and we will utilize the one that will be more convenient at the moment.

The dependence $W_{[L]}^{\text{ensemble}}$ on NL has a very narrow maximum due to the “wrestling” of two exponentially fast dependences: $g_{[L]}$ and $w_{\{n\} \in [L]}^{\text{ensemble}}$. The absolute width of this maximum is of the order of \sqrt{N} while the relative width is of the order of $1/\sqrt{N}$. The most probable fluctuation (which will be observed most commonly) is determined by the maximum of this probability. Equating the derivative to zero, $\left. \frac{\partial W_{[L]}^{\text{ensemble}}}{\partial L} \right|_0 = 0$ or $\left. \frac{\partial \ln W_{[L]}^{\text{ensemble}}}{\partial L} \right|_0 = 0$, we find the equation of state:

$$L_0 = e^{-p_f t} \tag{13}$$

The probability distribution $W_{[L]}^{\text{ensemble}}$ is normalized by unity:

$$1 = \sum_{[L]} W_{[L]}^{\text{ensemble}} = \sum_{[L]} g_{[L]} w_{\{n\} \in [L]}^{\text{ensemble}}. \tag{14}$$

Let us multiply this equality by the partition function:

$$Z^{\text{ensemble}} \equiv \sum_{\{n\}} e^{-NL_{\{n\}}/T^{\text{eff}}(t)} = \sum_{[L]} g_{[L]} e^{-NL/T^{\text{eff}}(t)}, \tag{15}$$

where we have again substituted the sum over microstates $\{n\}$ by the sum over fluctuations $[L]$. The number of non-zero terms in this sum is of the order of \sqrt{N} (the width of the maximum of $W_{[L]}^{\text{ensemble}}$):

$$Z^{\text{ensemble}} = g_{[L_0]} e^{-NL_0/T^{\text{eff}}(t)} O(\sqrt{N}). \tag{16}$$

Applying the logarithmic approximation (neglecting slow power-law dependencies and keeping only fast exponential dependencies), we prove the classical result of statistical mechanics that a partition function equals its maximal term:

$$Z^{\text{ensemble}} \approx_{\ln} g_{[L_0]} e^{-NL_0/T^{\text{eff}}(t)}. \tag{17}$$

Dividing this equation by Z^{ensemble} , we find another important relationship:

$$w_{\{n\} : L_{\{n\}} = L_0}^{\text{ensemble}} \approx_{\ln} \frac{1}{g_{[L_0]}} \tag{18}$$

that the probability of microstates corresponding to the most probable fluctuation $[L_0]$ is the inversed number of these microstates (the principle of the equivalence between the canonical and microcanonical ensembles).

Equality (Equation 18) can be found without multiplying (Equation 14) by Z^{ensemble} . Only in this case, we arrive at the equation

$$1 = g_{[L_0]} w_{\{n\} : L_{\{n\}} = L_0}^{\text{ensemble}} O(\sqrt{N}) \approx_{\ln} g_{[L_0]} w_{\{n\} : L_{\{n\}} = L_0}^{\text{ensemble}} \tag{19}$$

which states that there are about \sqrt{N} fluctuations under the width of the maximum of $W_{[L]}^{\text{ensemble}}$ with the probability similar to the most probable fluctuation $[L_0]$. Thereby, the probability of $[L_0]$ is of the order of $1/\sqrt{N}$ which returns us to Equation (18).

Let us introduce the partial partition function (of a fluctuation)

$$Z_{[L]} \equiv \sum_{\{n\} \in [L]} e^{-NL_{\{n\}}/T^{\text{eff}}(t)} = g_{[L]} e^{-NL/T^{\text{eff}}(t)} \tag{20}$$

when we sum the exponential functions not over all microstates but only over microstates of the particular fluctuation. Then, probability (Equation 12) of observing the fluctuation is directly related to the partial partition function of this fluctuation:

$$W_{[L]}^{\text{ensemble}} = g_{[L]} w_{\{n\} \in [L]}^{\text{ensemble}} = \frac{Z_{[L]}}{Z^{\text{ensemble}}}. \tag{21}$$

Next, let us consider averaging in the ensemble:

$$\begin{aligned} \langle f \rangle^{\text{ensemble}} &\equiv \sum_{\{n\}} f_{\{n\}} w_{\{n\}}^{\text{ensemble}} = \sum_{[L]} f_{\{n\} : L_{\{n\}} = L} g_{[L]} w_{\{n\} : L_{\{n\}} = L}^{\text{ensemble}} \\ &= \sum_{[L]} f_{\{n\} : L_{\{n\}} = L} W_{[L]}^{\text{ensemble}}. \end{aligned} \tag{22}$$

As an example, we consider averaging of L :

$$\langle L \rangle^{\text{ensemble}} = \sum_{[L]} L W_{[L]}^{\text{ensemble}}. \tag{23}$$

Here the probability distribution $W_{[L]}^{\text{ensemble}}$ is the very fast dependence comparing to slow dependence L . Besides, $W_{[L]}^{\text{ensemble}}$ is normalized by unity. Therefore, it plays the role of a delta-function $\delta(L - L_0)$, and we immediately find:

$$\langle L \rangle^{\text{ensemble}} \approx L_0 \tag{24}$$

The definition $[L] \equiv \bigcup_{\{n\} : L_{\{n\}} = L} \{n\}$ of a fluctuation $[L]$ can be

presented in terms of non-equilibrium probability distributions which we developed in details in [30], Chapter 2 as the generalization of the Leontovich approach of non-equilibrium states [127, 128]. We observe a fluctuation $[L]$ when in the ensemble we observe a system with the given fraction L of intact fibers. It is like in statistical physics to observe fluctuations of gas density when, for example, the gas has gathered spontaneously all its molecules in the left half of its volume while there are no molecules in another (right) half of the volume. We call it a fluctuation (a macrostate). But to determine the properties of this state, it is not enough to observe it for a microsecond as a microstate. To study this state, we isolate the gas in the left half of the volume by a virtual boundary and observe for some time before releasing it. But what does this additional isolation mean? It means that the gas visits only the microstates corresponding to our fluctuation and does not visit other microstates.

The same purpose can be achieved by considering non-equilibrium probability distributions [30]. Distribution

(Equations 3, 4) is dictated by the ensemble boundary conditions similar to the pressure and temperature dictated by the boundary conditions for the gas. However, a system may disregard these orders and behave independently. Imagine students who are ordered to read one paper a week on average. One student works hard and may read five papers a week on average while another one does not study at all. Similar, a system may fluctuate violating orders from the ensemble. For the gas, it means that the system stays in the left half of its volume (have non-zero probabilities only for those microstates which do not have molecules in the right half). For our system with damage, it means that only those microstates have non-zero probabilities which possess the correct value of the intact fiber fraction L while probabilities of other microstates are zero:

$$w_{\{n\}} = \begin{cases} 1/g_{[L]}, \{n\} : L_{\{n\}} = L \\ 0, \{n\} : L_{\{n\}} \neq L \end{cases} \quad (25)$$

To find the entropy of the fluctuation, we apply Gibbs definition and arrive at Boltzmann's entropy:

$$S_{[L]} \equiv - \sum_{\{n\}} w_{\{n\}} \ln w_{\{n\}} = \ln g_{[L]}. \quad (26)$$

For the equilibrium probability distribution, this transforms into:

$$\begin{aligned} S^{\text{ensemble}} &\equiv - \sum_{\{n\}} w_{\{n\}}^{\text{ensemble}} \ln w_{\{n\}}^{\text{ensemble}} \\ &= - \sum_{[L]} g_{[L]} w_{\{n\}}^{\text{ensemble}} \ln w_{\{n\}}^{\text{ensemble}} \\ &= - \sum_{[L]} W_{[L]}^{\text{ensemble}} \ln w_{\{n\}}^{\text{ensemble}}, \end{aligned} \quad (27)$$

where we have substituted the sum over microstates $\{n\}$ by the sum over fluctuations $[L]$. In the right-hand side of Equation (27), the dependence $\ln w_{\{n\}}^{\text{ensemble}}$ is a slow, power-law dependence on NL while $W_{[L]}^{\text{ensemble}}$ is the exponentially fast dependence, acting like a delta-function due to its normalization $\sum_{[L]} W_{[L]}^{\text{ensemble}} = 1$. Therefore, utilizing Equation (18), we immediately find Boltzmann's expression for the equilibrium entropy as well:

$$S^{\text{ensemble}} = - \ln w_{\{n\}:L_{\{n\}}=L_0}^{\text{ensemble}} = \ln g_{[L_0]}. \quad (28)$$

The non-equilibrium free energy potential (the effective Helmholtz energy) of a fluctuation may be introduced as

$$F_{[L]} \equiv NL - T^{\text{eff}} S_{[L]} = -T^{\text{eff}} \ln Z_{[L]}. \quad (29)$$

For the equilibrium free energy we find

$$F^{\text{ensemble}} \equiv NL_0 - T^{\text{eff}} S^{\text{ensemble}} = -T^{\text{eff}} \ln Z^{\text{ensemble}}. \quad (30)$$

Instead of free energy, we can define the action of free energy on a fluctuation (this will remove the asymmetry of temperature; for details, see [30, Section 2.16]):

$$A_{[L]} \equiv NL/T^{\text{eff}} - S_{[L]} = - \ln Z_{[L]} = - \ln \left(Z^{\text{ensemble}} W_{[L]}^{\text{ensemble}} \right). \quad (31)$$

We see that the action of the free-energy is directly related to the probability distribution of fluctuations and its minimization is equivalent to the evolution from states of low probability to highly probable states.

The ensemble action of free energy is:

$$A^{\text{ensemble}} \equiv NL_0/T^{\text{eff}} - S^{\text{ensemble}} = - \ln Z^{\text{ensemble}}. \quad (32)$$

All other results, generally developed for the canonical ensemble, are applicable here as well (for details, see [30], Chapter 2). However, our purpose in this manuscript is to suggest possible approaches to solve an arbitrary system, not as simple as the DFBM. Therefore, we provided here the classical Gibbs approach of states only to the purpose to highlight differences with the proposed below path integral approach.

RESTRICTIONS OF THE CLASSICAL GIBBS APPROACH OF STATES

In the previous section, we were lucky that our system had an analytical solution. This was the consequence of the fact that we were able to find *a priori* the probability $\int_0^t e^{-p_f t} p_f dt = 1 - e^{-p_f t}$ of a fiber failing before time t . For the case of an arbitrary system, we generally know the distribution of probabilities only for one time-step. In other words, if at time t_i a system with probability $w_{\{n_i\}}^{\text{ensemble}}$ is in a microstate $\{n_i\}$ with NL_i intact fibers, we know that the probability of this system at the next time-step t_{i+1} being in a microstate $\{n_{i+1}\}$ with NL_{i+1} intact fibers is

$$w_{\{n_{i+1}\}}^{\text{ensemble}} = w_{\{n_i\}}^{\text{ensemble}} (1 - p_f t)^{NL_{i+1}} (p_f t)^{N(L_i - L_{i+1})}. \quad (33)$$

In fact, we utilized here the general Gibbs formula $\frac{dw_{\{i\}}^{\text{ensemble}}}{dt}(t) = F \left[w_{\{i\}}^{\text{ensemble}}(t'), t' \leq t \right]$ that the evolution of the distribution of probabilities of microstates is determined by some functional dependence on the history of the model. To obtain the distribution of probabilities of the microstates at time t we have to integrate this equation over all possible paths among all previous configurations [e.g., 129, 130]. We have to integrate all possible intermediate configurations; these configurations are interconnected by the tremendous number of different paths; each path can have its own probability or be prohibited (irreversible damage in the case of the FBM, when a broken fiber cannot become intact again). Therefore, the integration of combinatorial sums becomes cumbersome.

This approach corresponds to the classical Gibbs non-equilibrium statistical mechanics, when microstates of a system are identified with the system's microconfigurations and macrostates of a system are identified with the system's macroconfigurations. However, we know that the probabilities for Markov processes are associated not with the states but with the paths among these states (for the first-order Markov process, the probability of a path $A-B-C-D$ is $P(D|C)P(C|B)P(B|A)P(A)$ while to find the probability of the system arriving after three steps at state D we should have three sums involved:

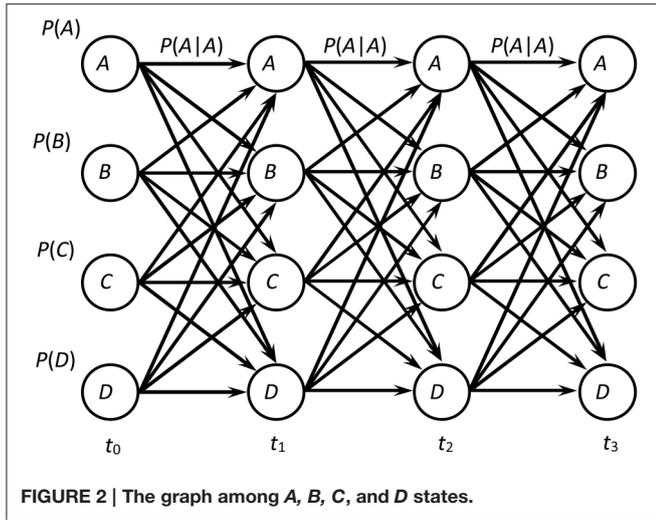


FIGURE 2 | The graph among A, B, C, and D states.

$\sum_{X_1, X_2, X_3} P(D|X_3)P(X_3|X_2)P(X_2|X_1)P(X_1)$, see Figure 2). Therefore, it is much easier to find a distribution of the probabilities only for a macrogroup of paths than for all the paths leading to a macrostate. In the Gibbs approach, the states of a system were chosen as bases, although we see that everything points to the fact that as bases we should choose not the states but the paths [125, 126, 131–142]. We will see how we can develop such an approach in the next section. The benefit of this approach will be that its bases will not include integrals of the states for the previous evolution of the system but, in contrast, will be this evolution itself.

STATISTICAL MECHANICS OF THE PATH INTEGRAL APPROACH

In the previous sections and in our previous publications [21, 22, 30], we identified microstates of a system with system’s microconfigurations and macrostates of a system with system’s macroconfigurations. However, this approach worked well only for equilibrium statistical mechanics. For non-equilibrium statistical mechanics, we should develop another approach [126].

As an illustration, we consider the case of the NFBM and assume that the process consists of ν time intervals of duration dt , from $t_0 = 0$ to $t_\nu = \nu \cdot dt$. For each time-step t_i as an order parameter we choose the value of the intact parameter L_i at this time-step. For the total process the order parameter is the total history of the intact parameter $L(t) \equiv L_0, \dots, L_\nu$. We assume that the process always starts from zero damage $L_0 = 1$, so the quantity L_0 will not be variable in the ensemble. We assume that broken fibers cannot become intact again; therefore, increments of the intact parameter $\Delta L_i \equiv L_i - L_{i-1}$ are always negative.

At each time-step t_i , a particular system in the ensemble has its own value of the intact parameter L_i and is in one of the microconfigurations $\{n_i\}$ corresponding to this intact parameter. The next possible microconfiguration for this system at the time-step t_{i+1} can only be a microconfiguration in

which all the broken fibers remain broken. This makes our system a Markov process of order 1. For the total process from $t_0 = 0$ to $t_\nu = \nu \cdot dt$, we construct all possible chains of microconfigurations. Each such chain, as a possible sequence of particular microconfigurations $\{n_0\}, \{n_1\}, \dots, \{n_\nu\}$, will be referred to as a micropath $\{n_0\} \rightarrow \{n_1\} \rightarrow \dots \rightarrow \{n_\nu\}$ (further, we will abbreviate this notation as $\{n_0\} \rightarrow \{n_\nu\}$). For example, for a system with $N = 3$ fibers, one of possible micropaths is $\{|||\} \rightarrow \{|x|\} \rightarrow \{xx|\} \rightarrow \{xxx\}$ where symbol “|” denotes an intact fiber while symbol “x” denotes a broken fiber.

Let us assume that for a micropath $\{n_0\} \rightarrow \{n_\nu\}$ the sequence of the configurations has the evolution of the intact parameter $L(t) \equiv L_0, \dots, L_\nu$. Then the probability of this micropath is

$$w_{\{n_0\} \rightarrow \{n_\nu\}}^{\text{ensemble}} = (1 - P)^{N|\Delta L_1|} P^{N(L_0 - |\Delta L_1|)} \dots (1 - P)^{N|\Delta L_\nu|} P^{N(L_\nu - |\Delta L_\nu|)} = \left(\frac{1 - P}{P}\right)^{N(1 - L_\nu)} P^{\sum_{i=1}^{\nu} L_{i-1}} \quad (34)$$

as the probability of $N|\Delta L_i|$ fibers failing and of $NL_i \equiv N(L_{i-1} - |\Delta L_i|)$ fibers staying intact, where $P \equiv P(s - E\epsilon)$ is constant. This probability $w_{\{n_0\} \rightarrow \{n_\nu\}}^{\text{ensemble}}$ is dictated by the ensemble and, in particular, by the value of P . This constraint is a model input and acts similarly to the temperature prescribed in the canonical ensemble. In the canonical ensemble, an external medium dictates the distribution of the probabilities for different paths but a system actually can be not in equilibrium with the dictated distribution and realizes some other probability distribution $w_{\{n_0\} \rightarrow \{n_\nu\}}$ for its paths. Therefore, we used here the abbreviation “ensemble” to emphasize that this probability distribution corresponds to the requirements stated regardless of possible fluctuations.

As to the macropath $[L_0] \rightarrow [L_1] \rightarrow \dots \rightarrow [L_\nu]$ (further, we will abbreviate this notation as $[L_0] \rightarrow [L_\nu]$) we will refer to a subset of all micropaths $\{n_0\} \rightarrow \{n_\nu\}$ corresponding to the specified evolution of the intact parameter $L(t)$: $[L_0] \rightarrow [L_\nu] = \bigcup_{\{n_0\} \rightarrow \{n_\nu\}: L_{\{n_i\}} = L_i} \{n_0\} \rightarrow \{n_\nu\}$. For example, for the system with $N = 3$ fibers macropath $[1] \rightarrow [1/3] \rightarrow [1/3] \rightarrow [0]$ unites micropaths $\{|||\} \rightarrow \{|xx\} \rightarrow \{|xx\} \rightarrow \{xxx\}$, $\{|||\} \rightarrow \{x|x\} \rightarrow \{x|x\} \rightarrow \{xxx\}$, and $\{|||\} \rightarrow \{xx|\} \rightarrow \{xx|\} \rightarrow \{xxx\}$. In fact, macropath states that the system passes during its evolution the prescribed values of L but does not specify particular microstates.

The probabilities of micropaths corresponding to a macropath $[L_0] \rightarrow [L_\nu]$ are all equal one to another and are given by Equation (34); the number of these micropaths is

$$g_{[L_0] \rightarrow [L_\nu]} = \frac{(NL_0)!}{(N|\Delta L_1|)!(N(L_0 - |\Delta L_1|))!} \cdot \frac{(NL_1)!}{(N|\Delta L_2|)!(N(L_1 - |\Delta L_2|))!} \cdot \dots \cdot \frac{(NL_{\nu-1})!}{(N|\Delta L_\nu|)!(N(L_{\nu-1} - |\Delta L_\nu|))!} \quad (35)$$

as a combinatorial choice of $N|\Delta L_i|$ failed fibers among NL_{i-1} initial fibers. We can cancel $(NL_i)!$ in the numerators and

$(N(L_{i-1} - |\Delta L_i|)!)$ in the denominators to obtain

$$g_{[L_0] \rightarrow [L_\nu]} = \frac{(NL_0)!}{(NL_\nu)! \prod_{i=1}^{\nu} (N|\Delta L_i|)!} \approx_{\ln} (L_\nu)^{-NL_\nu} \prod_{i=1}^{\nu} |\Delta L_i|^{-N|\Delta L_i|}, \quad (36)$$

where the notation “ \approx_{\ln} ” means that in the thermodynamic limit $N \rightarrow +\infty$ all power-law multipliers are neglected in comparison with the exponential dependence on N .

The probability of the system having a macropath $[L_0] \rightarrow [L_\nu]$ (to move among macroconfigurations with the specified $L(t)$) is

$$W_{[L_0] \rightarrow [L_\nu]}^{\text{ensemble}} = \sum_{n=1}^{g_{[L_0] \rightarrow [L_\nu]}} w_{\{n_0\} \rightarrow \{n_\nu\}}^{\text{ensemble}} = g_{[L_0] \rightarrow [L_\nu]} w_{\{n_0\} \rightarrow \{n_\nu\}}^{\text{ensemble}}. \quad (37)$$

Maximization of $W_{[L_0] \rightarrow [L_\nu]}^{\text{ensemble}}$ over all the possible macropaths gives the most probable macropath $L^{(0)}(t)$:

$$W_{[L_0^{(0)}] \rightarrow [L_\nu^{(0)}]}^{\text{ensemble}} = \max_{L_1, \dots, L_\nu} W_{[L_0] \rightarrow [L_\nu]}^{\text{ensemble}}. \quad (38)$$

Dependence $L^{(0)}(t)$ is the dependence of L which we will observe generally in the ensemble.

The function $W_{[L_0] \rightarrow [L_\nu]}^{\text{ensemble}}$, given by Equation (37), is the product of $g_{[L_0] \rightarrow [L_\nu]}$ and $w_{\{n_0\} \rightarrow \{n_\nu\}}^{\text{ensemble}}$. Both these functions contain exponentially fast dependences on N (N is infinite in the thermodynamic limit). Therefore, in the space of all paths, the function $W_{[L_0] \rightarrow [L_\nu]}^{\text{ensemble}}$ has a very narrow maximum at the most probable macropath $[L_0^{(0)}] \rightarrow [L_\nu^{(0)}]$. As we will see below, the absolute width of this maximum is proportional to \sqrt{N} while the relative width is proportional to $1/\sqrt{N}$. Therefore, the number of different macropaths $[L_0] \rightarrow [L_\nu]$ under the width of this maximum has a power-law dependence on N while the number $g_{[L_0] \rightarrow [L_\nu]}$ of micropaths $\{n_0\} \rightarrow \{n_\nu\}$ corresponding to each of these macropaths $[L_0] \rightarrow [L_\nu]$ has the exponential dependence on N . Therefore, for the normalization of the function $W_{[L_0] \rightarrow [L_\nu]}^{\text{ensemble}}$ with the logarithmic accuracy we obtain

$$1 = \sum_{[L_0] \rightarrow [L_\nu]} W_{[L_0] \rightarrow [L_\nu]}^{\text{ensemble}} \approx_{\ln} W_{[L_0^{(0)}] \rightarrow [L_\nu^{(0)}]}^{\text{ensemble}} \equiv g_{[L_0^{(0)}] \rightarrow [L_\nu^{(0)}]} w_{\{n_0\} \rightarrow \{n_\nu\} \in [L_0^{(0)}] \rightarrow [L_\nu^{(0)}]}^{\text{ensemble}}. \quad (39)$$

This provides

$$g_{[L_0^{(0)}] \rightarrow [L_\nu^{(0)}]} \approx_{\ln} 1/w_{\{n_0\} \rightarrow \{n_\nu\} \in [L_0^{(0)}] \rightarrow [L_\nu^{(0)}]}^{\text{ensemble}}. \quad (40)$$

In other words, the probability of each micropath of the most probable macropath $[L_0^{(0)}] \rightarrow [L_\nu^{(0)}]$ has the order of the inverted number of these micropaths.

Next, let us consider averaging over the space of all possible paths:

$$\langle f \rangle^{\text{ensemble}}(t) \equiv \sum_{\{n_0\} \rightarrow \{n_\nu\}} w_{\{n_0\} \rightarrow \{n_\nu\}}^{\text{ensemble}} f_{\{n_0\} \rightarrow \{n_\nu\}}. \quad (41)$$

As an example, we again consider averaging of L :

$$\begin{aligned} \langle L \rangle^{\text{ensemble}}(t) &\equiv \sum_{\{n_0\} \rightarrow \{n_\nu\}} L_{\{n_0\} \rightarrow \{n_\nu\}} w_{\{n_0\} \rightarrow \{n_\nu\}}^{\text{ensemble}} \\ &\equiv \begin{cases} L_0, & t = t_0 \\ \sum_{\{n_0\} \rightarrow \{n_\nu\}} L_{\{n_0\} \rightarrow \{n_\nu\}} w_{\{n_0\} \rightarrow \{n_\nu\}}^{\text{ensemble}}, & t = t_i \end{cases} = \quad (42) \\ &= \begin{cases} L_0, & t = t_0 \\ \sum_{[L_0] \rightarrow [L_\nu]} g_{[L_0] \rightarrow [L_\nu]} L_t w_{\{n_0\} \rightarrow \{n_\nu\}}^{\text{ensemble}}, & t = t_i \end{cases} \end{aligned}$$

Of course, in the thermodynamic limit, $\langle L \rangle^{\text{ensemble}}(t)$ coincides with the most probable macropath: $\langle L \rangle^{\text{ensemble}}(t) \approx L^{(0)}(t)$.

Probability distribution (Equation 34) is the distribution dictated by the ensemble. However, we can also consider other distributions $w_{\{n_0\} \rightarrow \{n_\nu\}}$ which do not obey the ensemble requirements. Next, as a fluctuation we consider the macropath $[L_0] \rightarrow [L_\nu]$. The number $g_{[L_0] \rightarrow [L_\nu]}$ of micropaths corresponding to this macropath is given by Equation (36). Again, to observe this macropath, we need to “isolate” our system in this macropath. This is achieved by considering the probability distribution $w_{\{n_0\} \rightarrow \{n_\nu\}}$ such that only the micropaths belonging to the considered macropath $[L_0] \rightarrow [L_\nu]$ have non-zero probabilities:

$$w_{\{n_0\} \rightarrow \{n_\nu\}} = \begin{cases} 1/g_{[L_0] \rightarrow [L_\nu]}, & \{n_0\} \rightarrow \{n_\nu\} \in [L_0] \rightarrow [L_\nu] \\ 0, & \{n_0\} \rightarrow \{n_\nu\} \notin [L_0] \rightarrow [L_\nu] \end{cases}. \quad (43)$$

Here we assumed the equiprobability of all micropaths of the system isolated in the macropath. Actually, this may not always be true but depends solely on how we build fluctuations as an instrument of our investigation.

The entropy of the considered macropath is

$$S_{[L_0] \rightarrow [L_\nu]} \equiv - \sum_{\{n_0\} \rightarrow \{n_\nu\}} w_{\{n_0\} \rightarrow \{n_\nu\}} \ln w_{\{n_0\} \rightarrow \{n_\nu\}} = \ln g_{[L_0] \rightarrow [L_\nu]}. \quad (44)$$

We should emphasize here that the entropy introduced is the “dynamical” entropy of the distribution of probabilities for the paths and must not be confused with the classical Gibbs entropy associated with the distribution of probabilities for the states (configurations). In our notations, the classical Gibbs entropy would be $S_{\{n\}}(t) \equiv - \sum_{\{n\}} w_{\{n\}}(t) \ln w_{\{n\}}(t)$ as the average at time t over the probabilities $w_{\{n\}}(t)$ of microconfigurations at this time. On the contrary, the path entropy is associated with the probabilities $w_{\{n_0\} \rightarrow \{n_\nu\}}$ of micropaths of the total process and cannot be attributed to system’s characteristics at a particular time t .

If, instead of the probability distribution $w_{\{n_0\} \rightarrow \{n_\nu\}}$, we consider probability distribution (Equation 34) dictated by the

ensemble, for the ensemble path entropy we find:

$$\begin{aligned}
 S^{\text{ensemble}} &\equiv - \sum_{\{n_0\} \rightarrow \{n_v\}} w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}} \ln w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}} \\
 &= - \sum_{[L_0] \rightarrow [L_v]} g_{[L_0] \rightarrow [L_v]} w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}} \ln w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}} = \\
 &= - \sum_{[L_0] \rightarrow [L_v]} W_{[L_0] \rightarrow [L_v]}^{\text{ensemble}} \ln w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}}. \tag{45}
 \end{aligned}$$

Here, the function $\ln w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}}$ has a slow power-law dependence on N in comparison with the functions $g_{[L_0] \rightarrow [L_v]}$ and $w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}}$ which have exponentially fast dependences on N . Therefore, the dependence $W_{[L_0] \rightarrow [L_v]}^{\text{ensemble}}$ acts as a delta-function around the most probable macropath, and we find:

$$\begin{aligned}
 S^{\text{ensemble}} &\approx - \ln w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}} \Big|_{[L_0] \rightarrow [L_v]} \approx \ln g_{[L_0] \rightarrow [L_v]} \\
 &= S_{[L_0] \rightarrow [L_v]}^{\text{ensemble}}. \tag{46}
 \end{aligned}$$

In other words, the entropy of the total ensemble of paths equals the entropy of the most probable macropath.

We can rewrite the equilibrium distribution of probabilities, given by Equation (34), as

$$w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}} = \frac{1}{Z^{\text{ensemble}}} \exp \left(-NL_v/T_v - N \left(\sum_{i=1}^v L_{i-1} \right) / T_f \right) \tag{47}$$

where $Z^{\text{ensemble}} = \left(\frac{P}{1-P} \right)^N$, $T_v = \ln^{-1} \{ (1-P)/P \}$, and $T_f = -\ln^{-1} P$. It is easy to see that Z^{ensemble} is the path partition function of the system: $Z^{\text{ensemble}} = \sum_{\{n_0\} \rightarrow \{n_v\}} \exp \left\{ -NL_v/T_v - N \left(\sum_{i=1}^v L_{i-1} \right) / T_f \right\}$. T_v and T_f play the roles of effective temperatures. We see that when P is close to unity (the probability of a fiber breaking is small), the temperature T_f is infinite but positive.

The definition of the temperature T_v is, to some extent, ambiguous. Indeed, we can rewrite Equation (34) for probabilities as

$$\begin{aligned}
 w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}} &= (1-P)^{N|\Delta L_1|} P^{NL_1} \dots (1-P)^{N|\Delta L_v|} P^{NL_v} \\
 &= (1-P)^{N(1-L_v)} P^{N \sum_{i=1}^v L_i}. \tag{48}
 \end{aligned}$$

This gives $T_v = \ln^{-1}(1-P)$ which is different from the expression $T_v = \ln^{-1} \{ (1-P)/P \}$ above. However, the difference is $\ln(P)$ which is negligible in comparison with $\ln(1-P)$ in the limit $P \rightarrow 1$.

The temperature $T_f = -\ln^{-1} P$ of the system is complementary to the integral of the evolution of intact parameter: $N \sum_{i=1}^v L_{i-1} \propto N \int_0^{t_v} L(t) dt$. If at the final time-step t_v the total system fails, $L(t_v) = 0$, then the single order parameter left is the integral $N \int_0^{t_v} L(t) dt$.

Since our system has two effective temperatures, instead of a free-energy potential we should define the action of the free-energy potential to avoid the asymmetry of one temperature chosen in favor to another. The action of the free-energy for a macropath $[L_0] \rightarrow [L_v]$ can be defined as:

$$\begin{aligned}
 A_{[L_0] \rightarrow [L_v]} &\equiv NL_v/T_v + N \left(\sum_{i=1}^v L_{i-1} \right) / T_f - S_{[L_0] \rightarrow [L_v]} = \\
 &= - \ln \left\{ g_{[L_0] \rightarrow [L_v]} \exp \left(-NL_v/T_v - N \left(\sum_{i=1}^v L_{i-1} \right) / T_f \right) \right\} \\
 &= - \ln Z_{[L_0] \rightarrow [L_v]}, \tag{49}
 \end{aligned}$$

where $Z_{[L_0] \rightarrow [L_v]}$ is the partial path partition function [23, 125, 126] of this macropath:

$$\begin{aligned}
 Z_{[L_0] \rightarrow [L_v]} &= \sum_{\{n_0\} \rightarrow \{n_v\} \in [L_0] \rightarrow [L_v]} \exp \left\{ -NL_v/T_v - N \left(\sum_{i=1}^v L_{i-1} \right) / T_f \right\} \\
 &= Z^{\text{ensemble}} W_{[L_0] \rightarrow [L_v]}^{\text{ensemble}}. \tag{50}
 \end{aligned}$$

Therefore, for the action of the free-energy we obtain

$$A_{[L_0] \rightarrow [L_v]} = - \ln \left(Z^{\text{ensemble}} W_{[L_0] \rightarrow [L_v]}^{\text{ensemble}} \right). \tag{51}$$

Hence, the action of the free-energy is directly related to the probability distribution of the fluctuations and the principle of its minimization is equivalent to the evolution from paths of low probability to highly probable paths.

In Gibbs equilibrium statistical mechanics, an equilibrium state is found as a minimum of a free energy potential. For the microcanonical ensemble, the free energy potential is the negative entropy $-S \equiv \langle \ln w_{\{n\}} \rangle \equiv \sum_{\{n\}} w_{\{n\}} \ln w_{\{n\}}$; for the canonical ensemble, the free energy potential is the Helmholtz energy $F \equiv \langle H_{\{n\}} \rangle + T \langle \ln w_{\{n\}} \rangle \equiv \sum_{\{n\}} w_{\{n\}} [H_{\{n\}} + T \ln w_{\{n\}}]$. For the case of general ensemble in Gibbs equilibrium statistical mechanics, the principle of the minimization of the free energy potential always works because this potential is always proportional to the minus logarithm of the probability distribution of fluctuations. We see that the same principle is valid for the case of non-equilibrium mechanics as well, only now we have to construct the free energy potential not for the states but for the paths.

So, for the path microcanonical ensemble (for the system isolated in a macropath), the negative dynamical entropy

$$-S \equiv \langle \ln w_{\{n_0\} \rightarrow \{n_v\}} \rangle \equiv \sum_{\{n_0\} \rightarrow \{n_v\}} w_{\{n_0\} \rightarrow \{n_v\}} \ln w_{\{n_0\} \rightarrow \{n_v\}} \tag{52}$$

plays the role of the free energy potential. This immediately provides that all the micropaths corresponding to this macropath must be equiprobable. If we compare this result with our hypothesis (Equation 43), we see that we were right considering micropaths as equiprobable within the macropath.

For the path canonical ensemble, the role of the free energy potential is played by the action of the free-energy

$$\begin{aligned}
 A &\equiv \langle H_{\{n_0\} \rightarrow \{n_v\}} \rangle / T + \langle \ln w_{\{n_0\} \rightarrow \{n_v\}} \rangle \\
 &\equiv \sum_{\{n_0\} \rightarrow \{n_v\}} w_{\{n_0\} \rightarrow \{n_v\}} [H_{\{n_0\} \rightarrow \{n_v\}} / T + \ln w_{\{n_0\} \rightarrow \{n_v\}}]
 \end{aligned}
 \tag{53}$$

where $H_{\{n_0\} \rightarrow \{n_v\}}$ is supposed to be the dynamical Hamiltonian which appears in distribution (Equation 47) of the probabilities: $w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}} = \exp(-H_{\{n_0\} \rightarrow \{n_v\}} / T) / Z^{\text{ensemble}}$. This dynamical Hamiltonian happens to consist of two parts, each corresponding to its own temperature, $H_{\{n_0\} \rightarrow \{n_v\}} / T = NL_v / T_v + N \left(\sum_{i=1}^v L_{i-1} \right) / T_f$, and does not correspond to the classical Hamiltonian of the states because of being determined by the rules of the memory of the process:

$$\begin{aligned}
 A &\equiv \sum_{\{n_0\} \rightarrow \{n_v\}} w_{\{n_0\} \rightarrow \{n_v\}} \left[NL_v / T_v + N \left(\sum_{i=1}^v L_{i-1} \right) / T_f + \ln w_{\{n_0\} \rightarrow \{n_v\}} \right]
 \end{aligned}
 \tag{54}$$

For the most probable macropath $[L_0^{(0)}] \rightarrow [L_v^{(0)}]$, the action of the free-energy is $A_{[L_0^{(0)}] \rightarrow [L_v^{(0)}]} \approx -\ln Z_{[L_0^{(0)}] \rightarrow [L_v^{(0)}]}$ which coincides with the ensemble path action of the free-energy:

$$\begin{aligned}
 A^{\text{ensemble}} &\equiv \sum_{\{n_0\} \rightarrow \{n_v\}} \left\{ NL_v / T_v + N \left(\sum_{i=1}^v L_{i-1} \right) / T_f \right\} w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}} - S^{\text{ensemble}} = \\
 &= - \sum_{\{n_0\} \rightarrow \{n_v\}} \left\{ -\ln w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}} - NL_v / T_v - N \left(\sum_{i=1}^v L_{i-1} \right) / T_f \right\} w_{\{n_0\} \rightarrow \{n_v\}}^{\text{ensemble}} = -\ln Z^{\text{ensemble}} \approx \\
 &\approx -\ln Z_{[L_0^{(0)}] \rightarrow [L_v^{(0)}]} = A_{[L_0^{(0)}] \rightarrow [L_v^{(0)}]}.
 \end{aligned}
 \tag{55}$$

At the point of the maximum of $W_{[L_0] \rightarrow [L_v]}^{\text{ensemble}}$, we have

$$\left. \frac{\partial W_{[L_0] \rightarrow [L_v]}^{\text{ensemble}}}{\partial L_i} \right|_0 = 0 \text{ or } \left. \frac{\partial \ln W_{[L_0] \rightarrow [L_v]}^{\text{ensemble}}}{\partial L_i} \right|_0 = 0
 \tag{56}$$

(where $i \geq 1$ because we assume L_0 to be non-variable). For $\ln W_{[L_0] \rightarrow [L_v]}^{\text{ensemble}} = \ln g_{[L_0] \rightarrow [L_v]} + A^{\text{ensemble}} - NL_v / T_v - N \left(\sum_{i=1}^v L_{i-1} \right) / T_f$, we can write that

$$\frac{1}{T_f} = \left. \frac{\partial \ln g_{[L_0] \rightarrow [L_v]}}{\partial L_i} \right|_0 = \frac{\partial \ln g_{[L_0^{(0)}] \rightarrow [L_v^{(0)}]}}{N \partial L_i^{(0)}}, i = 1, \dots, v - 1
 \tag{57}$$

and

$$\frac{1}{T_v} = \left. \frac{\partial \ln g_{[L_0] \rightarrow [L_v]}}{\partial L_v} \right|_0 = \frac{\partial \ln g_{[L_0^{(0)}] \rightarrow [L_v^{(0)}]}}{N \partial L_v^{(0)}} \quad \frac{\partial^2 \ln W_{[L_0] \rightarrow [L_v]}^{\text{ensemble}}}{\partial L_i \partial L_j} \Big|_0 = -K_{ij}, i, j = 1, \dots, v,
 \tag{61}$$

at the most probable macropath. These equations could be used as definitions of the temperatures. As both the entropy of a macropath $S_{[L_0] \rightarrow [L_v]} = \ln g_{[L_0] \rightarrow [L_v]}$ and the ensemble entropy $S^{\text{ensemble}} \approx S_{[L_0^{(0)}] \rightarrow [L_v^{(0)}]} = \ln g_{[L_0^{(0)}] \rightarrow [L_v^{(0)}}$ have the same functional dependence on $L(t)$ and $L^{(0)}(t)$ respectively, we obtain

$$\begin{aligned}
 \frac{1}{T_f} &= \left. \frac{\partial S_{[L_0] \rightarrow [L_v]}}{\partial L_i} \right|_0 \approx \frac{\partial S^{\text{ensemble}}}{N \partial L_i^{(0)}}, i = 1, \dots, v - 1 \text{ and} \\
 \frac{1}{T_v} &= \left. \frac{\partial S_{[L_0] \rightarrow [L_v]}}{\partial L_v} \right|_0 \approx \frac{\partial S^{\text{ensemble}}}{N \partial L_v^{(0)}}.
 \end{aligned}
 \tag{58}$$

This resembles the law of conservation of energy $dE/T = dQ^{\leftarrow} / T = dS^{\text{ensemble}}$ in the canonical ensemble—an equation of “dynamical” path balance:

$$N \left(\sum_{i=2}^v dL_{i-1}^{(0)} \right) / T_f + NdL_v^{(0)} / T_v = dS^{\text{ensemble}}.
 \tag{59}$$

Differentials in this equation should be understood as the response of the most probable macropath to the change of the external boundary constraint P . This equation could be obtained directly by differentiating (Equation 44) as the logarithm of Equation (36).

To find the most probable macropath $[L_0^{(0)}] \rightarrow [L_v^{(0)}]$, we should find when the derivatives of the probability of macropaths (Equation 37) (or of the logarithm of this probability) with respect to L_i equal zero (Equation 56). This provides

$$L_i^{(0)} = P^i, |\Delta L_i^{(0)}| = (1 - P)P^{i-1}, i = 1, \dots, v.
 \tag{60}$$

as an analog of the equation of state in equilibrium statistical mechanics. Being consistent, we should probably call it the equation of path.

To investigate the behavior of fluctuations, we should find the second derivatives of $\ln W_{[L_0] \rightarrow [L_v]}^{\text{ensemble}}$:

where $K_{i,j}$ is the symmetric matrix of covariance, non-zero elements of which are

$$K_{i,i} = \frac{N}{1-P} \cdot \left\{ \frac{1+P}{P^i} \right\}, K_{i,i+1} = K_{i+1,i} = \frac{N}{1-P} \cdot \left\{ -\frac{1}{P^i} \right\},$$

$$K_{v,v} = \frac{N}{1-P} \cdot \left\{ \frac{1}{P^v} \right\}. \tag{62}$$

For the probability of fluctuations around the most probable path, we obtain

$$\ln W_{[L_0] \rightarrow [L_v]}^{\text{ensemble}} = \ln W_{[L_0^{(0)}] \rightarrow [L_v^{(0)}]}^{\text{ensemble}} - \frac{1}{2} \sum_{i,j=1}^v (L_i - L_i^{(0)}) K_{i,j} (L_j - L_j^{(0)}) + \dots \text{ or}$$

$$W_{[L_0] \rightarrow [L_v]}^{\text{ensemble}} \propto \exp \left\{ -\frac{1}{2} \sum_{i,j=1}^v (L_i - L_i^{(0)}) K_{i,j} (L_j - L_j^{(0)}) \right\}. \tag{63}$$

The quadratic form $K_{i,j}$ is positively defined. This can be proved directly. However, we see that it is easy to diagonalize the form $K_{i,j}$. Indeed, transforming to new coordinates

$$\xi_i = (L_i - L_i^{(0)}) - \frac{1-P^i}{1-P^{i+1}} (L_{i+1} - L_{i+1}^{(0)}), i = 1, \dots, v-1;$$

$$\xi_v = (L_v - L_v^{(0)}), \tag{64}$$

we obtain that the new diagonal quadratic form is

$$\tilde{K}_{i,i} = \frac{N}{1-P} \cdot \left\{ \frac{1-P^{i+1}}{(1-P^i)P^i} \right\}, \tilde{K}_{v,v} = \frac{N}{1-P} \cdot \left\{ \frac{1-P}{(1-P^v)P^v} \right\}, \tag{65}$$

which is positively defined. Also, we see that the true, ‘‘diagonal’’ order parameters are not L_i but the quantities given by Equation (64). For these quantities, we have

$$\xi_i = (L_i - L_i^{(0)}) - \frac{1-L_i^{(0)}}{1-L_{i+1}^{(0)}} (L_{i+1} - L_{i+1}^{(0)}), i = 1, \dots, v-1;$$

$$\xi_v = (L_v - L_v^{(0)}). \tag{66}$$

In the limit $P \rightarrow 1$ (the probability of a fiber breaking at a time-step is much less than unity), we obtain

$$\xi_i = (L_i - L_i^{(0)}) - (L_{i+1} - L_{i+1}^{(0)}), i = 1, \dots, v-1;$$

$$\xi_v = (L_v - L_v^{(0)}) \tag{67}$$

as it could be expected. Indeed, the process is the first order Markov process and fluctuations $(L_{i+1} - L_{i+1}^{(0)})$ at the time-step t_{i+1} depend on what the system really was at the previous time-step t_i and do not depend on what it were supposed to be. But at the previous time-step the system realized fluctuations $(L_i - L_i^{(0)})$. Therefore, the new fluctuations depend on the

previous fluctuations as if it were a reference state, from which the new move starts. In other words, if the system had some fluctuation $(L_i - L_i^{(0)})$ at the previous time-step, this fluctuation is equivalent to the most probable macropath, but taken at some other, different time. Therefore, the system forgets that this was in fact a fluctuation and refers to it as to a new most probable path. Therefore, all the new fluctuations $(L_{i+1} - L_{i+1}^{(0)})$ are realized already relatively to the new, shifted, most probable path. Similarly to how the non-equilibrium statistical mechanics required us to move from the states to the paths, for the order parameters we also have to move from the quantities to the changes of the quantities. This is again the requirement of non-equilibrium statistical mechanics. Indeed, any non-equilibrium state is a source of some most probable path. Any fluctuation does shift a system out from this macropath but simultaneously initiates a new most probable macropath. The system forgets its previous most probable path and refers to the fluctuation as if it were the initiation of the new most probable path. Therefore, new fluctuations will be spread around the old fluctuation, not around the first, forgotten, most probable path.

Fluctuations (Equation 63) are Gaussian and relative fluctuations are proportional to $1/\sqrt{N}$. Therefore, the maximum of the function $W_{[L_0] \rightarrow [L_v]}^{\text{ensemble}}$ is indeed very narrow in the thermodynamic limit.

It is easy to demonstrate the benefits of the path integral approach relatively to the state approach. For example, let us calculate the probability that at a given time t_v , the model has a given value L_v of the intact parameter. For the formalism of states we would have to sum the probabilities of all possible intermediate configurations and all the possible paths for v time-steps

$$\Pr(t_v, L_v) \propto \int_{L_v}^1 \frac{dL_{v-1}}{1/N} \frac{(NL_{v-1})!(1-P)^{N(L_{v-1}-L_v)} P^{NL_v}}{(N(L_{v-1}-L_v))!(NL_v)!}$$

$$\int_{L_{v-1}}^1 \frac{dL_{v-2}}{1/N} \frac{(NL_{v-2})!(1-P)^{N(L_{v-2}-L_{v-1})} P^{NL_{v-1}}}{(N(L_{v-2}-L_{v-1}))!(NL_{v-1})!} \times \dots \times$$

$$\times \int_{L_2}^1 \frac{dL_1}{1/N} \frac{(NL_1)!(1-P)^{N(L_1-L_2)} P^{NL_2}}{(N(L_1-L_2))!(NL_2)!} \frac{(NL_0)!(1-P)^{N(L_0-L_1)} P^{NL_1}}{(N(L_0-L_1))!(NL_1)!}. \tag{68}$$

For the path integral, we do not need to perform these integrations. Instead, we already know that the system follows its equilibrium macropath $[L_0^{(0)}] \rightarrow [L_v^{(0)}]$ with the Gaussian fluctuations of the order of $1/\sqrt{N}$ around it. For fluctuations of $\xi_v = (L_v - L_v^{(0)})$ we have

$$\Pr(t_v, L_v) \propto \exp \left\{ -\frac{1}{2} \tilde{K}_{v,v} (L_v - L_v^{(0)})^2 \right\}. \tag{69}$$

Therefore, the probability that at a given time t_v the model has a given value of L_v has the Gaussian distribution around the equilibrium path $L_v^{(0)}$ at this time.

CONCLUSIONS

In this paper, we have developed a formalism of non-equilibrium statistical mechanics for non-equilibrium damage phenomena. Far from the state of equilibrium, we switched from states to paths to base our theory on the most basic concepts which directly determine the probability ensembles. We developed non-equilibrium statistical mechanics for the path ensembles and found the equation of path, the path balance equation, the expression for the dynamical entropy and the free energy potential. Also we showed that the ensemble of systems can be described in terms of the effective temperatures. Although, we used the fiber-bundle model to illustrate all the concepts developed, we believe that our results have general applicability for other, less simplified damage phenomena. Another important result of this study is that we generalized the Gibbs principle of the minimization of the free energy potential for path ensembles. Only in this case, instead of characteristics of the states we had to move to the dynamical characteristics in the space of all possible paths.

The importance of the path integral approach relies on the fact that it is the most fundamental approach for processes far from equilibrium, the importance of which is difficult to underestimate for damage phenomena. Instead of following the evolution of a system among the states, we observe the paths followed by the system which is much easier and simplifies analytical formulae significantly [e.g., compare Equation (69) with Equation (68)]. Imagine a city during a rush hour. To analyze the traffic, we do not count all the cars on all the streets; instead, we inquire about a traffic jam along one particular route and either follow it or choose another one. The same holds for the path approach: To study the states we need to know how all the microstates are interconnected with one another through the whole system phase space; to study the paths we compare one with another and choose one which is more probable.

However, characteristics of the path formalism are different from the classical approach of states. The mathematics at the base of the formalism is quite similar but the results it provides differ very much from our intuitive thoughts influenced for more than a century by the brilliance of the Gibbs mechanics. Let us take, for example, the entropy. For the approach of the states, we utilize famous formula $S \equiv - \sum_{\{\dots\}} w_{\{\dots\}} \ln w_{\{\dots\}}$ which

is applicable for any system (both thermal and not) for any distribution of probabilities of microstates. The universality of this formula is such that considering an arbitrary phenomenon we may immediately conclude that one state differs from another by how it is disordered, which is quantified by the entropy. The whole formalism of statistical physics can be developed based on only this one formula (see, e.g., [30], Chapter 2).

But when we come to the path integral approach, everything should be rethought anew. The formula is the same, $S \equiv - \sum_{\{\dots\}} w_{\{\dots\}} \ln w_{\{\dots\}}$, but the probabilities in it are no longer related to states but to paths. And all the results require new interpretation. What, for example, does it mean that the entropy

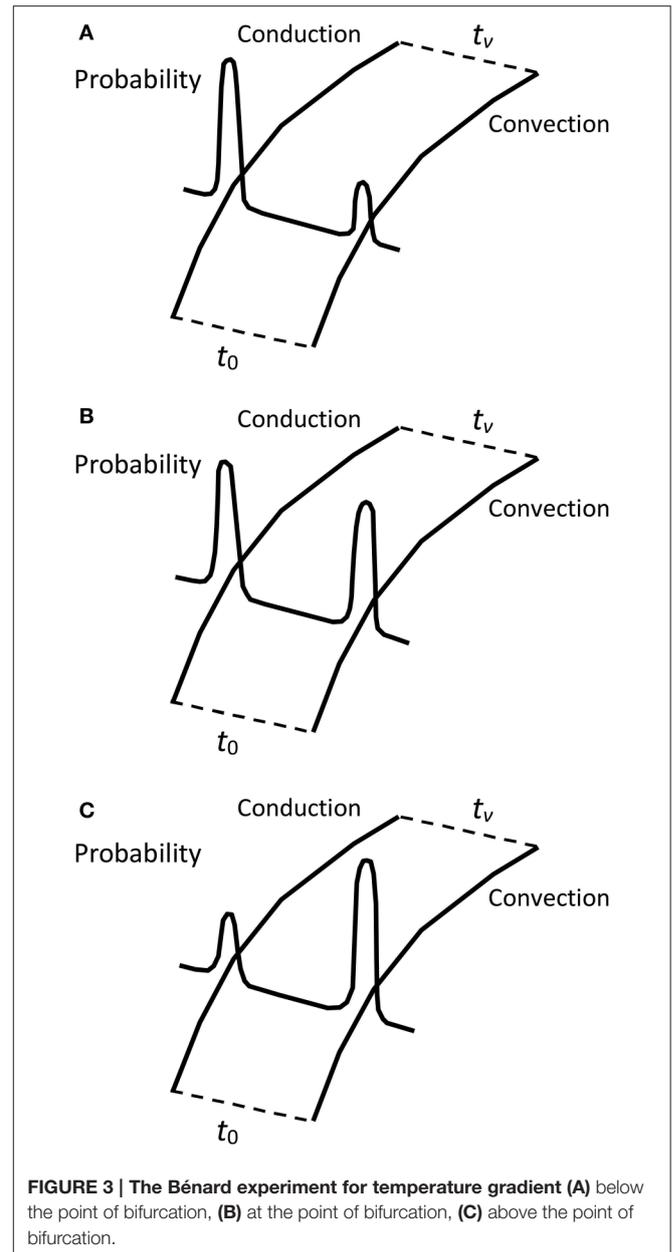


FIGURE 3 | The Bénard experiment for temperature gradient (A) below the point of bifurcation, (B) at the point of bifurcation, (C) above the point of bifurcation.

of one macropath is higher than of another? Is it related to a more “disordered path?”

The answers still usually appear to be absent in the literature. Let us consider, for example, the situation where after the free energy action minimization, we reach the conclusion that there is not one but two most probable macropaths for the non-equilibrium process. In other words, comparing probabilities for all possible macropaths, we find that the probability distribution has two maxima instead of one. These maxima have equal heights but correspond to two different macropaths.

Then, the system is allowed to choose between these two paths (breaking some symmetry as an Ising model in zero field breaks its symmetry choosing one of two phases). Does

this mean that there is a “non-equilibrium phase transition” from one macropath to another? In other words, if there are two most probable macropaths with equal probabilities, should we generalize the concept of phase transitions here as well, considering these paths as phases?

To illustrate this concept, let us first consider the near-equilibrium processes. The simplest example is the system of domains of opposite phases (magnetic domains in an Ising model with long-range interactions in a zero field below a critical temperature; domains with different values of L in damage phenomena with long-range load-redistribution). If the system has to eliminate the domain structure to move to an equilibrium state (when we turn on the magnetic field in the Ising model or apply loading in the system with damage), the situation when there are several equally probable macropaths to destroy domains is quite possible. The system can follow one of these paths but then due to fluctuations of a size larger than critical, it may switch its evolution to another path.

A more illustrative example appears when we consider a system that fluctuates not around an equilibrium state but around a stationary process. Let us consider Bénard cells as a classic example of dissipative structures. A thin layer of fluid is located between two plates, where the lower plate is hotter than the upper. Depending on the temperature gradient, the system chooses to transfer heat by means of either heat conduction or convection in the form of Bénard cells. Exactly at the bifurcation point, both scenarios have equal probabilities and correspond to the two most probable macropaths. Therefore, the case where the free energy action minimization returns several macropaths with equal probabilities (instead of one) is typical for the bifurcation points in dissipative structures.

What if there are two maxima of the probability distribution corresponding to two different macropaths, but the probability of one of them is lower than the probability of another? Does the first maximum correspond to a metastable phase? To illustrate

this, let us return to the Bénard cells (**Figure 3**). If initially the temperature gradient was low, the heat was transferred by conduction. If we consider two macropaths, one for conduction, the other for convection, the probability of conduction is higher, and the system follows the conduction macropath. When we increase the temperature gradient, the probability of convection increases, reaches the value of the probability of conduction at the point of bifurcation, and even exceeds this value if we continue to increase the temperature gradient. Therefore, conduction becomes metastable, and the system that follows the conduction macropath, switches to the convection macropath.

We have considered near equilibrium processes and stationary dissipative structures. But non-equilibrium processes are in no way limited to these two extremes. We may consider a significantly non-equilibrium process when even the local equilibrium is not reached and we do not have temperature or pressure defined even locally. The formalism developed in our study remains applicable for this general case as well, leading to the description of non-equilibrium phase transitions even for such complex processes far from equilibrium.

In statistical physics, many questions about the path integral approach are still opened. And damage phenomena are probably a crucial step for these studies due to the intrinsic property of damage growth as being a non-equilibrium process. Besides, damage evolution is often more illustrative than thermodynamical models of statistical physics can provide. We hope that the application of non-equilibrium statistical physics to damage phenomena will be beneficial for the further development of statistical physics itself as it has already taken place, for example, with complex systems.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and approved it for publication.

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