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Invariant Subspace and Classification of Soliton Solutions of the Coupled Nonlinear Fokas-Liu System

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In this work, the coupled nonlinear Fokas-Liu system which is a special type of KdV equation is studied using the invariant subspace method (ISM). The method determines an invariant subspace and construct the exact solutions of the nonlinear partial differential equations (NPDEs) by reducing them to ordinary differential equations (ODEs). As a result of the calculations, polynomial and logarithmic function solutions of the equation are derived. Further more, the ansatz approached is utilized to derive the topological, non-topological and other singular soliton solutions of the system. Numerical simulation off the obtained results are shown.

Keywords: invariant subspace method, soliton, ansatz, coupled nonlinear Fokas-Liu system, numerical simulation

1. INTRODUCTION

As vastly known, NPDEs are commonly applied to describe a lot of relevant dynamic processes and phenomena in mechanics, biology, physics, chemistry, etc. [1]. The solutions of NPDEs may provide a significant information for scientists and engineers. The ISM, proposed in Galaktionov [2] and modified in Ma [3], is one of strongest techniques to derive the solutions of NPDEs. The technique involve several invariant subspaces which are defined as subspaces of solutions to linear ODEs have been utilized to solve special NPDEs [3]. In Shen et al. [4], Zhu and Qu [5], and Song et al. [6], the maximal dimensions of invariant subspaces for studying n system of NPDEs has been reported. On the other hand, the ansatz technique is a powerful technique used in deriving the soliton solutions of NPDEs. The approach is based upon substituting an ansatz directly into the equation. The method has been used to obtain the solutions of several NPDEs [7–10]. In the last few decades, several powerful integration approaches have utilized to study many equations [11–19].

In this paper, we aim to study Equation (3) using the ISM [4–6]. Then, we will classify the soliton solutions of the equation by utilizing the the powerful ansatz approach [7, 8].

2. MODEL DESCRIPTION

Fokas and Liu [20] introduced a system of integrable KdV system. The system in it's original form is given by

$$\begin{cases} u_t + \upsilon_x + (3\beta_1 + 2\beta_4)\beta_3 u u_x + (2 + \beta_4\beta_1)\beta_2(u\upsilon)_x \\ +\beta_1\beta_3\upsilon\upsilon_x + (\beta_1 + \beta_4)\beta_2 u_{xxx} + (1 + \beta_4\beta_1)\beta_2\upsilon_{xxx} = 0, \\ \\ \upsilon_t + u_x + (2 + 3\beta_1\beta_4)\beta_3\upsilon\upsilon_x + (\beta_1 + 2\beta_4)\beta_3(u\upsilon)_x \\ +\beta_1\beta_3\beta_4 u u_x + (\beta_1 + \beta_4)\beta_2\beta_4 u_{xxx} + (1 + \beta_1\beta_4)\beta_2\beta_4\upsilon_{xxx} \\ = 0. \end{cases} \quad (1)$$

Gurses and Karasu [12] further simplified Equation (1) by considering a linear transformation of the form

$$u = m_1 r + n_1 s, \quad \upsilon = m_2 r + n_2 s, \quad (2)$$

where m_1, m_2, n_1 and n_2 are arbitrary constants, s and r new dynamical variables, $q^i = (s, r)$. On properly choosing the constants, the coupled nonlinear Fokas-Liu system Equation (1) is reduced to a simpler form represented by:

$$\begin{cases} u_t = a u u_x + (\upsilon u)_x + b\upsilon_x, \\ \upsilon_t = c u_x + f u u_x + d\upsilon_x + 3\upsilon\upsilon_x + e\upsilon_{xxx}, \end{cases} \quad (3)$$

with transformation parameters given by Baskonus et al. [15]:

$$\begin{aligned} m_2 &= \frac{\beta_1 + \beta_4}{1 + \beta_1\beta_4} m_1, \quad n_2 \frac{\beta_4 n_1}{\delta\beta_3}, \\ n_1 &= -\frac{1}{\delta\beta_3}, \quad \delta = \beta_1(1 + \beta_4^2) + 2\beta_4. \end{aligned} \quad (4)$$

In Equation (3), u is the elevation of the water wave, υ is the surface velocity of water along x-direction [15]. The parameters a, b, c, e, f, d are constants. The only condition on the parameters a, b, c, e, f, d is given by $c = fb$. This guarantees the integrability of the above system.

3. THE INVARIANT SUBSPACE METHOD

Let us give a brief account of the ISM [6]

$$\begin{aligned} \overline{u}^1_t &= F^1(x, \overline{u}^1, \overline{u}^2, \ldots, \overline{u}^1_{k_1}, \overline{u}^2_{k_1}), \\ \overline{u}^2_t &= F^2(x, \overline{u}^1, \overline{u}^2, \ldots, \overline{u}^1_{k_2}, \overline{u}^2_{k_2}). \end{aligned} \quad (5)$$

The operator $F^1 \equiv F^1[\overline{u}^1, \overline{u}^2]$ and $F^2 \equiv F^1[\overline{u}^1, \overline{u}^2]$ are smooth functions with orders k_1 and k_2, namely

$$\left(F^1_{\overline{u}^1_{k_1}}\right)^2 + \left(F^1_{\overline{u}^2_{k_1}}\right)^2 \neq 0, \quad \left(F^2_{\overline{u}^1_{k_2}}\right)^2 + \left(F^2_{\overline{u}^2_{k_2}}\right)^2 \neq 0. \quad (6)$$

In the above and subsequent sections, we will apply the following notation

$$\overline{u}^q_0 = \overline{u}^q(x, t), \quad \overline{u}^q_j = \frac{\partial \overline{u}^q(x, t)}{\partial x^j}, \quad q = 1, 2; j = 1, 2, \ldots \quad (7)$$

Let $\mathcal{W}$ be a new linear subspace $\mathcal{W}^1_{n_1} \times \mathcal{W}^2_{n_2}$, where

$$\mathbb{W}^q_{n_q} = \mathcal{L}\{f_1(x)^q, \ldots, f_{n_q}(x)^q\} = \sum_{i=1}^{n_q} \lambda^q_j f_j(x)^q, \quad q = 1, 2 \quad (8)$$

and $f_1(x)^q, \ldots, f_{n_q}(x)^q$ are linearly independent. If the vector operator $F = (F^1, F^2)$ satisfies the condition

$$\mathbb{F} : \mathcal{W}^1_{n_1} \times \mathcal{W}^2_{n_2} \to \mathcal{W}^1_{n_1} \times \mathcal{W}^2_{n_2}, \quad (9)$$

i.e.,

$$F^q : \mathcal{W}^1_{n_1} \times \mathcal{W}^2_{n_2} \to \mathcal{W}^q_{n_q}, \quad q = 1, 2 \quad (10)$$

satisfies

$$\begin{aligned} &F^q\left[\sum_{j=1}^{n_1} \lambda^1_j f^1_j(x), \sum_{j=1}^{n_2} \lambda^2_j f_j(x)^2\right] \\ &= \sum_{j=1}^{n_q} \psi^q_j(\lambda^1_1, \ldots, \lambda^1_{n_1}, \lambda^2_1, \ldots, \lambda^2_{n_2}) f^q_j(x). \end{aligned} \quad (11)$$

Then the vector operator $\mathbb{F}$ admit an invariant subspace given by $\mathcal{W}$. If the subspace $\mathcal{W}$ is being admitted by the operator $\mathbb{F}$, then Equation (5) has a solution given by

$$\overline{u}^q = \sum_{j=1}^{n_q} \lambda^q_j(t) f_j(x)^q, \quad q = 1, 2, \quad (12)$$

with $\lambda^q_j(t)$ being functions of t satisfying the following ODEs

$$\frac{d\lambda^q_j(t)}{dt} = \psi^q_j(\lambda^1_1(t), \ldots, \lambda^1_{n_1}(t), \lambda^2_1(t), \ldots, \lambda^2_{n_2}(t)) \quad q = 1, 2. \quad (13)$$

Suppose $\mathcal{W}^q_{n_q} = \mathcal{L}\{f^q_1(x), \ldots, f^q_{n_q}(x)\}$ is generated by the solutions of the linear n_qth-order ODEs

$$\begin{aligned} \mathcal{L}^q[y] &= y_q^{(n_q)} + a^q_{n_q-1}(x) y^{(n_q-1)} + \cdots \\ &\quad + a^q_1(x) y'_q + a^q_0(x) y_q = 0, \quad q = 1, 2. \end{aligned} \quad (14)$$

Thus, the invariant conditions represented by

$$\mathcal{L}^q[F^q[\overline{u}^1, \overline{u}^2]]\big|_{[H_1]\cap[H_2]} = 0, \quad q = 1, 2 \quad (15)$$

one can denote by $[H_q]$ the equation $\mathcal{L}^q[\overline{u}^q] = 0$ and its differentials w.r.t x. Once one determined the maximal dimension, then the complete classification and exact solutions of the equation can be investigated. From Equation (15) representing the invariant condition, the estimation has been determined in Shen et al. [4].

Theorem 3.1. *Let $\mathbb{F} = (F^1, F^2)$ be a nonlinear vector and be coupled. We can assume without loss of generality ($k_1 \geq k_2$). If the operator $\mathbb{F}$ admits the invariant subspace $\mathcal{W}^1_{n_1} \times \mathcal{W}^2_{n_2}$ ($n_1 \geq n_2 > 0$), then there holds $n_1 - n_2 \leq k_2$, $n_1 \leq 2(k_1 + k_2) + 1$.*

In theorem 2.1, the operator $\mathbb{F}$ is couple meaning

$$\begin{aligned}\left(F^1_{\overline{u}^2_0}\right)^2+\left(F^1_{\overline{u}^2_1}\right)^2+\cdots+\left(F^1_{\overline{u}^2_{k_1}}\right)^2\neq 0,\\ \left(F^2_{\overline{u}^2_0}\right)^2+\left(F^2_{\overline{u}^2_1}\right)^2+\cdots+\left(F^2_{\overline{u}^2_{k_1}}\right)^2\neq 0.\end{aligned} \tag{16}$$

$\mathbb{F}$ represents a nonlinear vector, i.e., for certain $i_0, j_0, l_0 \in \{1,2\}$, $p_0, q_0 \in \{0,1,\ldots,k_{i_0}\}$, there holds

$$\frac{\partial F^{i_0}}{\partial \overline{u}^{j_0}_{p_0}\partial \overline{u}^{l_0}_{q_0}}\neq 0. \tag{17}$$

In the case of $k_1 = k_2$, the estimation of maximal dimension is given in Zhu and Qu [5]. Next, we consider the case $0 < n_1 < n_2$. We give the following results from Song et al. [6] in a more general form which we shall apply in the next section.

4. APPLICATION TO THE COUPLED NONLINEAR FOKAS-LIU SYSTEM

In this section, we will construct the invariant subspace and solutions of Equation (3). Let us take an invariant subspace $\mathcal{W}_{2,2}=\mathcal{W}^1_2\times\mathcal{W}^2_2$ defined by

$$\begin{aligned}\mathcal{L}^1[y_1]=y_1^{''}+a_1y_1^{'}+a_0y_1=0,\\ \mathcal{L}^2[y_2]=y_2^{''}+b_1y_2^{'}+b_0y_2=0.\end{aligned} \tag{18}$$

where a_0, a_1, b_0, and b_1 are constants to be determined. The corresponding invariance conditions are given by

$$\begin{aligned}(D^2F+a_1DF+a_0F)\big|_{u\in\mathcal{W}^1_2,v\in\mathcal{W}^2_2}=0,\\ (D^2G+b_1DG+b_0G)\big|_{u\in\mathcal{W}^1_2,v\in\mathcal{W}^2_2}=0,\end{aligned} \tag{19}$$

where

$$\begin{cases}u_t=F=auu_x+(\upsilon u)_x+b\upsilon_x,\\ \upsilon_t=G=cu_x+fuu_x+d\upsilon_x+3\upsilon\upsilon_x+e\upsilon_{xxx}.\end{cases} \tag{20}$$

Substitute the expressions for F and G into the above equations, we obtain an overdetermined system of algebraic expressions which can be solved in general to obtain the invariant conditions given by

$$a_0=0, a_1=0, b_0=0, b_1=0, b=b, c=c, f=f. \tag{21}$$

Therefore, Equation (14) reduces to

$$\begin{aligned}\mathcal{L}^1[y_1]=y_1^{''}=0,\\ \mathcal{L}^2[y_2]=y_2^{''}=0.\end{aligned} \tag{22}$$

Thus, we get $\mathcal{W}^1_2 = span\{1,x\}$ and $\mathcal{W}^2_2 = span\{1,x\}$. This invariant subspace takes the exact solution of Equation (3) as

$$\begin{aligned}u(x,t)=\lambda_3(t)+x\lambda_4(t),\\ \upsilon(x,t)=\lambda_1(t)+x\lambda_2(t).\end{aligned} \tag{23}$$

where $\lambda_i(t),\quad i=1,2,3$ are unknown function to be determined. Putting Equation (23) into Equation (3), we acquire the following system of ODEs:

$$\begin{cases}-2\lambda_3(t)\lambda_1(t)+\lambda_3^{'}(t)=0,\\ -\lambda_3(t)^2f-3\lambda_1(t)^2+\lambda_1^{'}(t)=0,\\ -\lambda_4(t)\lambda_1(t)-\lambda_3(t)\lambda_2(t)-a\lambda_3(t)-b\lambda_1(t)+\lambda_4^{'}(t)=0,\\ -\lambda_4(t)\lambda_3(t)f-c\lambda_3(t)-3\lambda_1(t)\lambda_2(t)-d\lambda_1(t)+\lambda_2^{'}(t)=0.\end{cases} \tag{24}$$

Solving Equation (24), we acquire

$$\begin{aligned}\lambda_1(t)&=\frac{-1}{3t+c_3},\\ \lambda_2(t)&=\frac{-d(-3t+c_3)+3c_2}{-3t+c+3},\\ \lambda_3(t)&=0,\\ \lambda_4(t)&=-b+\frac{c_1}{(-3t+c_3)^{\frac{1}{3}}}.\end{aligned} \tag{25}$$

Subsequently, we obtain the following algebraic function solution

$$\begin{aligned}u(x,t)&=-bx+\frac{xc_1}{(-3t+c_3)^{\frac{1}{3}}},\\ \upsilon(x,t)&=\frac{-1}{3t+c_3}+\frac{x(-d(-3t+c_3)+3c_2)}{-3t+c_3}.\end{aligned} \tag{26}$$

where $c_i(i=1,\ldots,3)$ are arbitrary constants.

5. ANSATZ APPROACH

In this section, we will utilize the ansatz approach to derive the topological, non-topological and singular soliton solutions of Equation (3).

5.1. Non Topological Solitons

The non topological soliton solution of Equation (3) can be represented by the following ansatz:

$$u(x,t)=\sigma_1\mathrm{sech}^{p_1}\tau, v(x,t)=\sigma_2\mathrm{sech}^{p_2}\tau, \tag{27}$$

where $\tau=\eta(x-vt)$, σ_1, σ_2, p_1 and p_2 are to be determined later. η is the wave number of the soliton. Putting Equation (27) into Equation (3), we obtain

$$\begin{cases}v\eta\mathrm{sech}^{1+p_1}(\tau)\sinh(\tau)p_1\rho_1+a\eta\mathrm{sech}^{1+2p_1}(\tau)\sinh_1^{p}(\tau)\rho_1^2+\\ b\eta\mathrm{sech}^{1+p_2}(\tau)\sinh(\tau)p_2\rho_2+\eta\mathrm{sech}^{1+p_1+p_2}(\tau)\sinh(\tau)p_1\rho_1\rho_2+\\ \eta\mathrm{sech}^{1+p_1+p_2}(\tau)\sinh(\tau)p_2\rho_1\rho_2=0,\\ \\ c\eta\mathrm{sech}^{1+p_1}(\tau)\sinh(\tau)p_1\rho_1+f\eta\mathrm{sech}^{1+2p_1}(\tau)\sinh(\tau)p_1\rho_1^2+\\ d\eta\mathrm{sech}^{1+p_2}(\tau)\sinh(\tau)p_2\rho_2+v\eta\mathrm{sech})^{1+p_2}(\tau)\sinh(\tau)p_2\rho_2-\\ 2e\eta^3\mathrm{sech}^{1+p_2}(\tau)\sinh(\tau)p_2\rho_2+2e\eta^3\mathrm{sech}^{3+p_2}(\tau)\sinh(\tau)^3p_2\rho_2-\\ 3e\eta^3\mathrm{sech}^{1+p_2}(\tau)\sinh(\tau)p_2^2\rho_2+3e\eta^3\mathrm{sech}^{3+p_2}(\tau)\sinh(\tau)^3p_2^2\rho_2+\\ e\eta^3\mathrm{sech}^{3+p_2}(\tau)\sinh(\tau)^3p_2^3\rho_2+3\eta\mathrm{sech}^{1+2p_2}(\tau)\sinh(\tau)p_2\rho_2^2\\ =0.\end{cases} \tag{28}$$

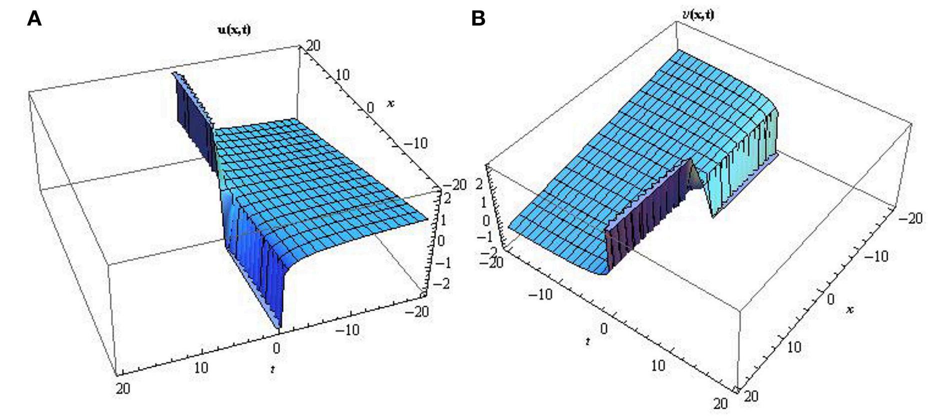

FIGURE 1 | Surface profile of the algebraic functions solutions **A** and **B** mathematically represented by Equation (26) by setting $c_1 = 0.5, c_2 = -0.2, c_3 = 0.2, d = 2$.

Upon equating the exponents in p_1 and p_2, we acquire

$$3 + p_2 = 1 + p_1 + p_2, \tag{29}$$

$$3 + p_2 = 1 + 2p_2, \tag{30}$$

thus, we obtain $p_1 = p_2 = 2$. Putting into Equation (28), we acquire

$$\begin{cases} 2v\eta\text{sech}(\tau)^2\rho_1\tanh(\tau) + 2a\eta\text{sech}^4(\tau)\rho_1^2\tanh(\tau)+ \\ 2b\eta\text{sech}^2(\tau)\rho_2\tanh(\tau) + 4\eta\text{sech}^4(\tau)\rho_1\rho_2\tanh(\tau) = 0, \\ \\ 2c\eta\text{sech}^2(\tau)\rho_1\tanh(\tau) + 2f\eta\text{sech}^4(\tau)\rho_1^2\tanh(\tau)+ \\ 2d\eta\text{sech}^2(\tau)\rho_2\tanh(\tau) + 2v\eta\text{sech}^2(\tau)\rho_2\tanh(\tau)+ \\ 8e\eta^3\text{sech}^2(\tau)\rho_2\tanh(\tau) - 24e\eta^3\text{sech}^4(\tau)\rho_2\tanh(\tau)+ \\ 6\eta\text{sech}^4(\tau)\rho_2^2\tanh(\tau) = 0. \end{cases} \tag{31}$$

After making some algebraic computations, we obtain the following soliton parameters:

$$v = \frac{ab}{2}, \quad \eta = \frac{1}{2}\sqrt{\frac{-a^2b + 4c - 2ad}{2ae}},$$
$$\rho_1 = \frac{3\left(a^2b - 4c + 2ad\right)}{3a^2 + 4f}, \quad \rho_2 = -\frac{a\rho_1}{2}. \tag{32}$$

The non-topological soliton solutions of Equation (3) are given by

$$\begin{cases} u(x,t) = \frac{3(a^2b-4c+2ad)}{3a^2+4f}\text{sech}^2\left[\frac{1}{2}\sqrt{\frac{-a^2b+4c-2ad}{2ae}}\left(-\frac{1}{2}abt + x\right)\right], \\ \\ v(x,t) = -\frac{3a(a^2b-4c+2ad)}{2(3a^2+4f)}\text{sech}^2\left[\frac{1}{2}\sqrt{\frac{-a^2b+4c-2a}{2ea}}\left(-\frac{1}{2}abt + x\right)\right]. \end{cases} \tag{33}$$

5.2. Topological Solitons

The non topological soliton solution of Equation (3) can be represented by the following ansatz:

$$u(x,t) = \sigma_1\tanh^{p_1}\tau, v(x,t) = \sigma_2\tanh^{p_2}\tau, \tag{34}$$

where $\tau = \eta(x - vt)$. Putting Equation (34) into Equation (3), we obtain

$$\begin{cases} -v\eta\text{csch}(\tau)\text{sech}^{p_1}(\tau)\rho_1\tanh(\tau)^{p_1} - a\eta\text{csch}(\tau)\text{sech}(\tau)p_1\rho_1^2\tanh^{2p_1}(\tau)- \\ b\eta\text{csch}(\tau)\text{sech}^{p_2}(\tau)\rho_2\tanh^{p_2}(\tau) - \eta\text{csch}(\tau)\text{sech}^{p_1}(\tau)\rho_1\rho_2\tanh^{p_1+p_2}(\tau)- \\ \eta\text{csch}(\tau)\text{sech}(\tau)p_2\rho_1\rho_2\tanh)^{p_1+p_2}(\tau) = 0, \\ \\ -c\eta\text{csch}(\tau)\text{sech}(\tau)p_1\rho_1\tanh^{p_1}(\tau) - f\eta\text{csch}(\tau)\text{sech}(\tau)p_1\rho_1^2\tanh^{2p_1}(\tau)- \\ d\eta\text{csch}(\tau)\text{sech}(\tau)p_2\rho_2\tanh^{p_2}(\tau) - v\eta\text{csch}(\tau)\text{sech}(\tau)p_2\rho_2\tanh^{p_2}(\tau)- \\ 4e\eta^3\text{csch}(\tau)\text{sech}^3(\tau)p_2\rho_2\tanh^{p_2}(\tau) - 2e\eta^3\text{csch}^3(\tau)\text{sech}(\tau)^3p_2\rho_2\tanh^{p_2}(\tau)+ \\ 6e\eta^3\text{csch}(\tau)\text{sech}^3(\tau)p_2^2\rho_2\tanh^{p_2}(\tau) + 3e\eta^3\text{csch}^3(\tau)\text{sech}(\tau)^3p_2^2\rho_2\tanh^{p_2}(\tau)- \\ e\eta^3\text{csch}^3(\tau)\text{sech}^3(\tau)p_2^3\rho_2\tanh^{p_2}(\tau) - 3\eta\text{csch}(\tau)\text{sech}(\tau)p_2\rho_2^2\tanh^{2p_2}(\tau)- \\ 4e\eta^3\text{csch}(\tau)\text{sech}(\tau)p_2\rho_2\tanh^{2+p_2}(\tau) = 0 \end{cases} \tag{35}$$

Upon equating the exponents in p_1 and p_2, we acquire

$$2p_2 = 2 + p_2, \tag{36}$$

$$p_1 + p_2 = 1 + 2p_1, \tag{37}$$

thus, we obtain $p_1 = p_2 = 2$. Putting into Equation (38), we acquire

$$\begin{cases} 2v\eta\text{sech}^2(\tau)\rho_1\tanh(\tau) + 2a\eta\text{sech}^4(\tau)\rho_1^2\tanh(\tau)+ \\ 2b\eta\text{sech}^2(\tau)\rho_2\tanh(\tau) + 4\eta\text{sech}^4(\tau)\rho_1\rho_2\tanh(\tau) = 0, \\ \\ 2c\eta\text{sech}(\tau)^2\rho_1\tanh(\tau) + 2f\eta\text{sech}(\tau)^4\rho_1^2\tanh(\tau)+ \\ 2d\eta\text{sech}^2(\tau)\rho_2\tanh(\tau) + 2v\eta\text{sech}^2(\tau)\rho_2\tanh(\tau)+ \\ 8e\eta^3\text{sech}^2(\tau)\rho_2\tanh(\tau) - 24e\eta^3\text{sech}^4(\tau)\rho_2\tanh(\tau)+ \\ 6\eta\text{sech}^4(\tau)\rho_2^2\tanh(\tau) = 0. \end{cases} \tag{38}$$

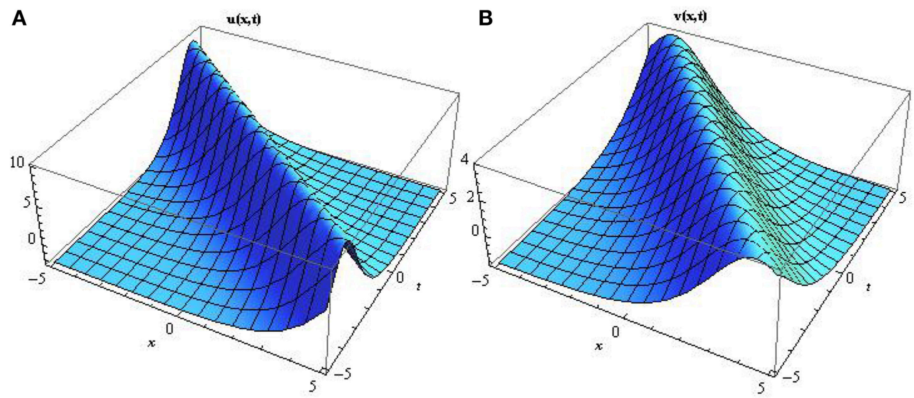

FIGURE 2 | Surface profile of the non topological soliton solutions **A** and **B** mathematically represented by Equation (33) describing several terminologies in the field of mathematical physics by setting $a_1 = 0.5, b_2 = -0.1, c_3 = 0.2, d = 2, e = 2, f = 0.4$.

After making some algebraic computations, we obtain the following soliton parameters

$$v = \frac{ab}{2}, \quad \eta = \frac{1}{4}\sqrt{\frac{a^2b - 4c + 2ad}{ae}},$$
$$\rho_2 = -\frac{3\left(a^3b - 4ac + 2a^2d\right)}{4\left(3a^2 + 4f\right)}, \quad \rho_1 = \frac{3\left(a^3b - 4ac + 2a^2d\right)}{2a\left(3a^2 + 4f\right)}. \tag{39}$$

The topological soliton solutions of Equation (3) are given by

$$\begin{cases} u(x,t) = \frac{3(a^2b-4c+2ad)}{6a^2+8f}\tanh^2\left[\frac{1}{4}\sqrt{\frac{a^2b-4c+2ad}{ae}}\left(-\frac{1}{2}abt + x\right)\right], \\ v(x,t) = -\frac{3a(a^2b-4c+2ad)}{4(3a^2+4f)}\tanh^2\left[\frac{1}{4}\sqrt{\frac{a^2b-4c+2ad}{ae}}\left(-\frac{1}{2}abt + x\right)\right]. \end{cases} \tag{40}$$

5.3. Singular Soliton Solutions Type-I

The singular soliton solution type-I of Equation (3) can be represented by the following ansatz:

$$u(x,t) = \sigma_1 \mathrm{csch}^{p_1}\tau, v(x,t) = \sigma_2 \mathrm{csch}^{p_2}\tau, \tag{41}$$

where $\tau = \eta(x - vt)$. Inserting Equation (41) into Equation (3), we acquire

$$\begin{cases} v\eta\cosh(\tau)\mathrm{csch}^{1+p_1}(\tau)p_1\rho_1 + a\eta\cosh(\tau)\mathrm{csch}^{1+2p_1}(\tau)p_1\rho_1^2 + \\ b\eta\cosh(\tau)\mathrm{csch}^{1+p_2}(\tau)p_2\rho_2 + \eta\cosh(\tau)\,\mathrm{csch}^{1+p_1+p_2}(\tau)p_1\rho_1\rho_2 + \\ \eta\cosh(\tau)\mathrm{csch}^{1+p_1+p_2}(\tau)p_2\rho_1\rho_2 = 0, \\ \\ c\eta\cosh(\tau)\mathrm{csch}^{1+p_1}(\tau)p_1\rho_1 + f\eta\cosh(\tau)\mathrm{csch}^{1+2p_1}(\tau)p_1\rho_1^2 + \\ d\eta\cosh(\tau)\mathrm{csch}^{1+p_2}(\tau)p_2\rho_2 + v\eta\cosh(\tau)\mathrm{csch}^{3+p_2}(\tau)p_2\rho_2 - \\ 2e\eta^3\cosh(\tau)\mathrm{csch}^{1+p_2}(\tau)p_2\rho_2 + 2e\eta^3\cosh^3(\tau)\mathrm{csch}^{3+p_2}(\tau)p_2\rho_2 - \\ 3e\eta^3\cosh(\tau)\mathrm{csch}^{1+p_2}(\tau)p_2^2\rho_2 + 3e\eta^3\cosh^3(\tau)\mathrm{csch}(\tau)^{3+p_2}p_2^2\rho_2 + \\ e\eta^3\cosh^3(\tau)\mathrm{csch}^{3+p_2}(\tau)p_2^3\rho_2 + 3\eta\cosh(\tau)\mathrm{csch}^{1+2p_2}(\tau)p_2\rho_2^2 = 0. \end{cases} \tag{42}$$

Upon equating the exponents of p_1 and p_2 Equation (42), we acquire

$$3 + p_2 = 1 + p_1 + p_2, \tag{43}$$
$$3 + p_2 = 1 + 2p_2, \tag{44}$$

thus, we obtain $p_1 = p_2 = 2$. Putting into Equation (42), we obtain

$$\begin{cases} 2v\eta\coth(\tau)\mathrm{csch}^2(\tau)\rho_1 + 2a\eta\coth(\tau)\mathrm{csch}^4(\tau)\rho_1^2 + \\ 2b\eta\coth(\tau)\mathrm{csch}^2(\tau)\rho_2 + 4\eta\coth(\tau)\mathrm{csch}^4(\tau)\rho_1\rho_2 = 0, \\ \\ 2c\eta\coth(\tau)\mathrm{csch}^2(\tau)\rho_1 + 2f\eta\coth(\tau)\mathrm{csch}^4(\tau)\rho_1^2 + \\ 2d\eta\coth(\tau)\mathrm{csch}^2(\tau)\rho_2 + 2v\eta\coth(\tau)\mathrm{csch}^2(\tau)\rho_2 + \\ 8e\eta^3\coth(\tau)\mathrm{csch}^2(\tau)\rho_2 - 24e\eta^3\coth(\tau)\mathrm{csch}^4(\tau)\rho_2 + \\ 6\eta\coth(\tau)\mathrm{csch}^4(\tau)\rho_2^2 = 0. \end{cases} \tag{45}$$

After making some algebraic computations, we obtain the following soliton parameters

$$v = \frac{ab}{2}, \quad \eta = \frac{1}{2}\sqrt{\frac{-a^2b + 4c - 2ad}{2ae}},$$
$$\rho_1 = \frac{3\left(a^2b - 4c + 2ad\right)}{3a^2 + 4f}, \quad \rho_2 = -\frac{3a\left(a^2b - 4c + 2ad\right)}{2\left(3a^2 + 4f\right)}. \tag{46}$$

The singular soliton solutions type-I of Equation (3) are given by

$$\begin{cases} u(x,t) = \frac{3(a^2b-4c+2ad)}{3a^2+4f}\mathrm{csch}^2\left[\frac{1}{4}\sqrt{\frac{4c-a(ab+2d)}{2ae}}(-abt + 2x)\right], \\ v(x,t) = -\frac{3a(a^2b-4c+2ad)}{2(3a^2+4f)}\mathrm{csch}^2\left[\frac{1}{4}\sqrt{\frac{4c-a(ab+2d)}{2ae}}(-abt + 2x)\right]. \end{cases} \tag{47}$$

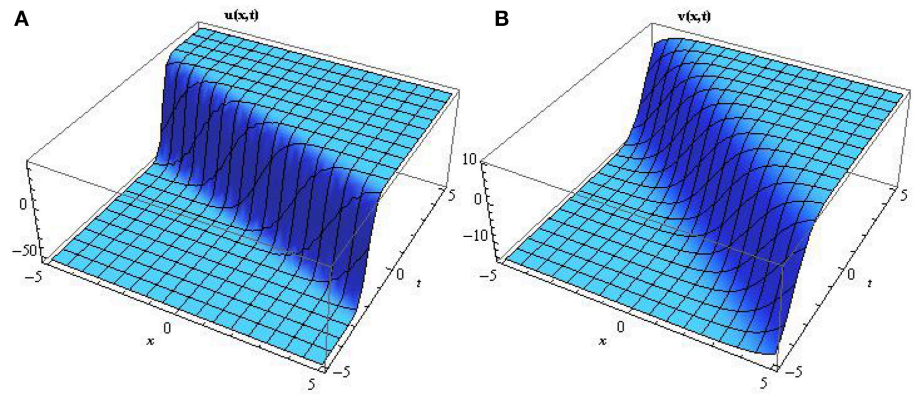

FIGURE 3 | Surface profile of the topological soliton solutions **A** and **B** mathematically represented by Equation (40) describing several terminologies in the field of mathematical physics by setting $a_1 = 0.4, b_2 = -0.2, c_3 = 0.1, d = 2, e = 2, f = 0.4$.

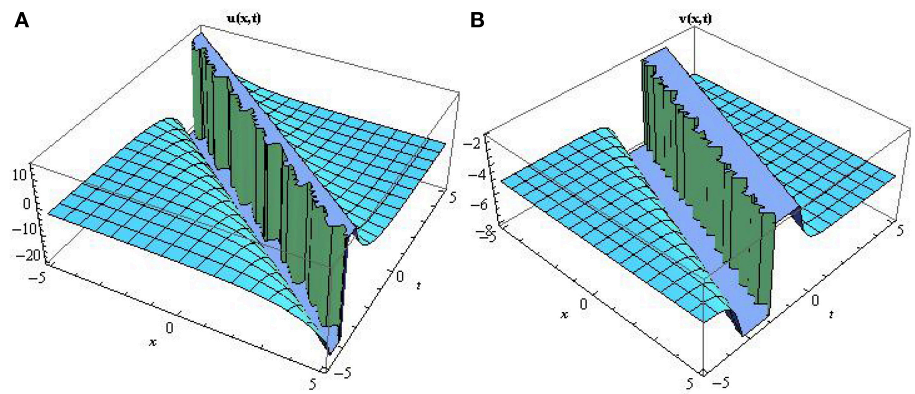

FIGURE 4 | Surface profile of the singular soliton solutions type-I **A** and **B** mathematically represented by Equation (47) describing several terminologies in the field of mathematical physics by setting $a_1 = 0.6, b_2 = -0.7, c_3 = 0.2, d = 1, e = 2, f = 0.4$.

5.4. Singular Soliton Type-II

The singular soliton solutions type-II of Equation (3) can be represented by the following ansatz:

$$u(x,t) = \sigma_1 \coth^{p_1}\tau, v(x,t) = \sigma_2 \coth^{p_2}\tau, \tag{48}$$

where $\tau = \eta(x - vt)$. Putting Equation (48) into Equation (3), we obtain

$$\begin{cases} v\eta \coth^{p_1}(\tau)\mathrm{csch}(\tau)\mathrm{sech}(\tau)p_1\rho_1 + a\eta \coth^{2p_1}(\tau)\mathrm{csch}(\tau)\mathrm{sech}(\tau)p_1\rho_1^2 + \\ b\eta \coth^{p_2}(\tau)\mathrm{csch}(\tau)\mathrm{sech}(\tau)p_2\rho_2 + \eta \coth^{p_1+p_2}(\tau)\mathrm{csch}(\tau)\mathrm{sech}(\tau)p_1\rho_1\rho_2 + \\ \eta \coth^{p_1+p_2}(\tau)\mathrm{csch}(\tau)\mathrm{sech}(\tau)p_2\rho_1\rho_2 = 0, \\ \\ c\eta \coth^{p_1}(\tau)\mathrm{csch}(\tau)\mathrm{sech}(\tau)p_1\rho_1 + f\eta \coth^{2p_1}(\tau)\mathrm{csch}(\tau)\mathrm{sech}(\tau)p_1\rho_1^2 + \\ d\eta \coth^{p_2}(\tau)\mathrm{csch}(\tau)\mathrm{sech}(\tau)p_2\rho_2 + v\eta \coth^{p_2}(\tau)\mathrm{csch}(\tau)\mathrm{sech}(\tau)p_2\rho_2 + \\ 4e\eta^3 \coth^{2+p_2}(\tau)\mathrm{csch}(\tau)\mathrm{sech}(\tau)p_2\rho_2 - 4e\eta^3 \coth^{p_2}(\tau)\mathrm{csch}^3(\tau)\mathrm{sech}(\tau)p_2\rho_2 + \\ e\eta^3 \coth^{p_2}(\tau)\mathrm{csch}^3(\tau)\mathrm{sech}^3(\tau)p_2\rho_2 + 6e\eta^3 \coth^{p_2}(\tau)\mathrm{csch}(\tau)^3\mathrm{sech}(\tau)p_2^2\rho_2 - \\ 3e\eta^3 \coth^{p_2}(\tau)\mathrm{csch}^3(\tau)\mathrm{sech}^3(\tau)p_2^2\rho_2 + e\eta^3 \coth^{p_2}(\tau)\mathrm{csch}^3(\tau)\mathrm{sech}^3(\tau)p_2^3\rho_2 + \\ 3\eta \coth^{2p_2}(\tau)\mathrm{csch}(\tau)\mathrm{sech}(\tau)p_2\rho_2^2 = 0. \end{cases} \tag{49}$$

Upon equating the exponents in p_1 and p_2, we acquire

$$2 + p_2 = 2p_2, \tag{50}$$

$$p_1 + p_2 = 2p_1, \tag{51}$$

thus, we obtain $p_1 = p_2 = 2$. Putting into Equation (49), we acquire

$$\begin{cases} 2v\eta\coth(\tau)\mathrm{csch}^2(\tau)\rho_1 + 2a\eta \coth^3(\tau)\mathrm{csch}^2(\tau)\rho_1^2 + \\ 2b\eta \coth(\tau)\mathrm{csch}^2(\tau)\rho_2 + 4\eta \coth^3(\tau)\mathrm{csch}^2(\tau)\rho_1\rho_2 = 0, \\ \\ 2c\eta \coth(\tau)\mathrm{csch}(\tau)^2\rho_1 + 2f\eta \coth^3(\tau)\mathrm{csch}^2(\tau)\rho_1^2 + \\ 2d\eta \coth(\tau)\mathrm{csch}^2(\tau)\rho_2 + 2v\eta \coth(\tau)\mathrm{csch}^2(\tau)\rho_2 + \\ 16e\eta^3 \coth(\tau)\mathrm{csch}^2(\tau)\rho_2 - 8e\eta^3 \coth^3(\tau)\mathrm{csch}^2(\tau)\rho_2 + \\ 6\eta \coth^3(\tau)\mathrm{csch}^2(\tau)\rho_2^2 = 0. \end{cases} \tag{52}$$

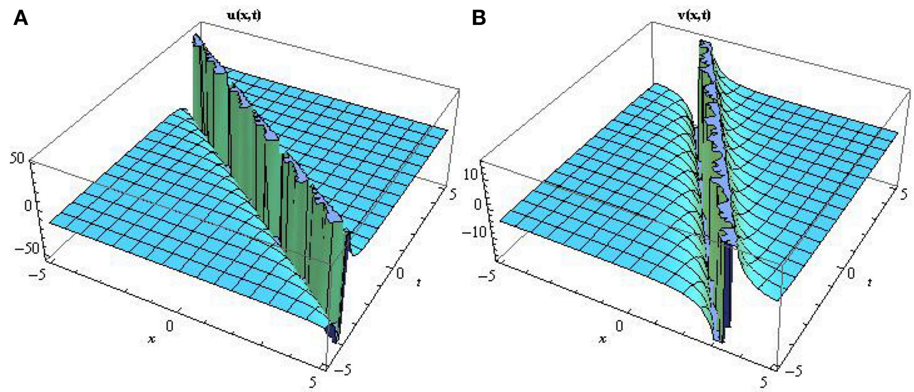

FIGURE 5 | Surface profile of the singular soliton solutions type-I I **A** and **B** mathematically represented by Equation (54) describing several terminologies in the field of mathematical physics by setting $a_1 = 0.2, b_2 = 0.8, c_3 = 0.2, d = 1, e = 2, f = 0.5$.

After making some algebraic computations, we acquire the following soliton parameters

$$\nu = \frac{ab}{2}, \ \eta = \frac{1}{4}\sqrt{\frac{-a^2b + 4c - 2ad}{ae}},$$
$$\rho_2 = \frac{-a^3b + 4ac - 2a^2d}{4\left(3a^2 + 4f\right)}, \ \rho_1 = -\frac{-a^3b + 4ac - 2a^2d}{2a\left(3a^2 + 4f\right)}. \quad (53)$$

The singular soliton solutions type-II of Equation (3) are given by

$$\begin{cases} u(x,t) = \frac{(a^2b-4c+2ad)}{6a^2+8f}\coth^2\left[\frac{1}{8}\sqrt{\frac{4c-a(ab+2d)}{ae}}(-abt+2x)\right], \\ v(x,t) = -\frac{a(a^2b-4c+2ad)}{4(3a^2+4f)}\coth^2\left[\frac{1}{8}\sqrt{\frac{4c-a(ab+2d)}{ae}}(-abt+2x)\right]. \end{cases} \quad (54)$$

6. CONCLUSION

In this work, we obtained the invariant subspaces and soliton solutions the coupled nonlinear Fokas-Liu system. The ISM and the ansatz approach were the methods employed to study the equation. New forms of algebraic solutions, topological, non-topological and singular soliton solutions have been reported. These solutions have a lot of application in mathematical physics and have not been reported in previous time in the literature. Some figures showing the physical description and numerical results of the acquired solutions. This has been shown in **Figures 1–5**.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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