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Coordinate-Free Approach for the Model Operator Associated With a Third-Order Dissipative Operator

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In this paper we investigate the spectral properties of a third-order differential operator generated by a formally-symmetric differential expression and maximal dissipative boundary conditions. In fact, using the boundary value space of the minimal operator we introduce maximal selfadjoint and maximal non-selfadjoint (dissipative, accumulative) extensions. Using Solomyak's method on characteristic function of the contractive operator associated with a maximal dissipative operator we obtain some results on the root vectors of the dissipative operator. Finally, we introduce the selfadjoint dilation of the maximal dissipative operator and incoming and outgoing eigenfunctions of the dilation.

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1. INTRODUCTION

A model operator may be regarded as an equivalent operator to another operator in a certain sense. Such an equivalent representation has been constructed by Szokefalvi-Nagy and Foiaş [1] for a contractive operator. The main idea for this construction is to obtain the unitary dilation of the contraction. In fact, if the following equality holds

$$T^n y = PU^n y,$$

where T is a contraction on the Hilbert space H and U is the operator on $\mathcal{H}$, $y \in H$, $n \geq 0$ and P is the orthogonal projection of $\mathcal{H}$ onto H, then U is called a dilation of T. U is called unitary provided that U is a unitary operator and in this case U is called unitary dilation of T. There exists a geometric meaning of the dilation space. This meaning has been given by Sarason [2]. Sarason showed that U is a dilation of T if and only if $\mathcal{H}$ has the representation

$$\mathcal{H} = G_* \oplus H \oplus G,$$

where $UG \subset G$ and $U^* G_* \subset G_*$. This representation is closely related with incoming and outgoing spaces in the scattering theory [3]. In the case that

$$\mathcal{H} = span\left\{U^n H, n \in \mathbb{Z}\right\}$$

then G and G_* are uniquely determined and U is called minimal. If U is unitary minimal dilation of T then one may consider the decomposition [4]

$$U = \begin{bmatrix} P_{G_*}U \mid G_* & 0 & 0 \\ D_{T^*}V_*^* & T & 0 \\ -VT^*V_*^* & VD_T & U \mid G \end{bmatrix},$$

where D_{T^*} and D_T are so called defect operators of T defined by

$$D_T = (I - TT^*)^{1/2}, D_{T^*} = (I - T^*T)^{1/2},$$

$\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$ are the defect spaces defined by

$$\mathfrak{D}_T = closD_T H, \mathfrak{D}_{T^*} = closD_{T^*}H,$$

V is a partial isometry with the initial space $\mathfrak{D}_T$ and final space $E = G \ominus UG$ and V_* is a partial isometry with initial space $\mathfrak{D}_{T^*}$ and the final space $E_* = G_* \ominus U^*G_*$.

Now consider the transformations

$$\begin{aligned} \pi : L^2(E) &\to \bigoplus_{n\in\mathbb{Z}} U^n(G \ominus UG), \\ \sum_n z^n e_n &\to \sum_n U^n \upsilon e_n, \end{aligned}$$

and

$$\begin{aligned} \pi_* : L^2(E_*) &\to \bigoplus_{n\in\mathbb{Z}} U^n(G_* \ominus U^*G_*), \\ \sum_n z^n e_n^* &\to \sum_n U^{n+1} \upsilon_* e_n^*, \end{aligned}$$

where υ and υ_* are the unitary mappings defined by

$$\upsilon : E \to G \ominus UG$$

and

$$\upsilon_* : E_* \to G_* \ominus U^*G_*.$$

In the literature the operators π and π_* are called functional embeddings. The function $\pi_*^*\pi$ acting from E into E_* is called the characteristic function of the contraction T. If υ and υ_* are chosen as the unitary identifications between $\mathfrak{D}_T$ and E, and, $\mathfrak{D}_{T^*}$ and E_*, respectively, then the characteristic function can be introduced as

$$\pi_*^*\pi = \Theta_T(\zeta)a = V_*(-T + \zeta D_{T^*}(I - \zeta T^*)^{-1}D_T)V^*a, a \in E.$$

Nagy and Foiaş introduced the characteristic function of a contraction as [1]

$$\Theta_T = -T + \zeta D_{T^*}(I - \zeta T^*)^{-1}D_T$$

and this can be obtained from the previous equation by choosing $E = \mathfrak{D}_T$ and $E_* = \mathfrak{D}_{T^*}$.

Solomyak [5] using the connection between dissipative operators and their Cayley transforms introduced an effective way to obtain the characteristic function of both dissipative operators and related contractions generated by Cayley transforms. By a dissipative operator it is meant an operator A with a dense domain $D(A)$ acting on a Hilbert space K satisfying

$$\mathrm{Im}(Ah, h) \geq 0, h \in D(A).$$

An immediate result on dissipative operators is that all eigenvalues lie in the closed upper half-plane. If a dissipative operator does not have a proper dissipative extension then A is called maximal dissipative. The Cayley transform of a dissipative operator

$$C(A) = (A - iI)(A + iI)^{-1}$$

is a contraction from $(A + iI)D(A)$ onto $(A - iI)D(A)$, i.e.,

$$\|C(A)\| \leq 1.$$

It is known that a dissipative operator is maximal if and only if $C(A)$ is a contraction such that domain of $C(A)$ is the Hilbert space K and 1 can not belong to the point spectrum of $C(A)$. Solomyak used these connections and boundary spaces associated with A to construct the characteristic function $S_A(\lambda)$ with the rule

$$S_A(\lambda) = P_*(A^* - \lambda I)^{-1}(A - \lambda I)P^{-1}, \tag{1.1}$$

where P and P_* are the natural projections. To be more precise we should note that for a maximal dissipative operator A the Hermitian part A_H of A is defined as the restriction of A to the following subspace

$$G_A = \left\{ f \in D(A) \cap D(A^*) : Af = A^*f \right\}.$$

The natural projection P is defined by

$$P : D(A) \to D(A)/G_A,$$

where $D(A)/G_A$ is the quotient space. Similarly P_* is defined by

$$P_* : D(A^*) \to D(A^*)/G_A.$$

On the quotient spaces the following inner products are defined

$$\langle P\varphi, P\psi \rangle = \frac{i}{2}\left((\varphi, P\psi) - (P\varphi, \psi)\right), \varphi, \psi \in D(A)$$

and

$$\langle P_*\varphi_*, P_*\psi_* \rangle = \frac{i}{2}\left(\left(\varphi_*, P_*\psi_*\right) - \left(P_*\varphi_*, \psi_*\right)\right), \varphi_*, \psi_* \in D(A^*).$$

Let $F(A)$ be the completion of the quotient space $D(A)/G_A$ with respect to the norm

$$\|P\varphi\|_F^2 = \mathrm{Im}(A\varphi, \varphi).$$

In a similar way one may define $F_*(A) := F(-A^*)$ and $F_*(A)$ is equipped with the norm

$$\left\|P_*\varphi_*\right\|_{F_*}^2 = -\mathrm{Im}(A^*\varphi_*, \varphi_*).$$

These spaces $F(A)$ and $F_*(A)$ are called boundary spaces. Solomyak showed for a maximal dissipative operator A and its Cayley transform $C(A)$ that there exist isometric isomorphisms

$$\rho : F(A) \to \mathfrak{D}_C, \rho_* : F_*(A) \to \mathfrak{D}_{C^*}$$

with the rules

$$\rho P(I - C) = \mathfrak{D}_C, \rho_* P_*(I - C^*) = \mathfrak{D}_{C^*}.$$

Then fixing arbitrary isometric isomorphisms $\Omega : E \rightarrow \mathfrak{D}_C$, $\Omega_* : E_* \rightarrow \mathfrak{D}_{C^*}$ the characteristic function Θ_C of the Cayley transform $C(A)$ can be introduced by

$$\Theta_C(\zeta) = \Omega_*^*(-C + \zeta D_{C^*}(I - \zeta C^*)D_C)\Omega.$$

Finally taking $\Omega = \rho$, $\Omega_* = \rho_*$, $E = F(A)$, $E_* = F_*(A)$ one obtains (1.1).

In this paper using the results of Solomyak we investigate some spectral properties of a regular third-order dissipative operator. We should note that such an investigation with the aid of Solomyak's approach has not been introduced for the third-order case. In fact, the literature has less works on odd-order operators than on even-order equations even if there exists some results in the literature [6–13]. The main reason is the confusion of imposing the boundary conditions because as Everitt says in [9] that it is impossible to impose separated boundary conditions for the solutions of a third-order equation. Consequently, this paper may give an idea to use Solomyak's method for the odd-order dissipative or accumulative operators.

2. MAXIMAL DISSIPATIVE OPERATOR

Throughout the paper we consider the following third-order differential expression

$$\ell(y) = \frac{1}{w}\left\{-i\left(q_0\left(q_0 y'\right)'\right)' - (p_0 y')' + i\left[q_1 y' + (q_1 y)'\right] + p_1 y\right\}$$
$$, x \in [a, b],$$

where $q_j, p_j, j = 0, 1, w$ are real-valued and continuous functions on $[a, b]$ and $q_0 > 0$ or $q_0 < 0$ and $w > 0$ on $[a, b]$.

The quasi-derivative $y^{[r]}$ of the function y is defined by

$$y^{[0]} = y,\ y^{[1]} = -\tfrac{1+i}{\sqrt{2}} q_0 y',\ y^{[2]} = i q_0 (q_0 y')' + p_0 y' - i q_1 y.$$

Let H denote the Hilbert space with the usual inner product

$$(y, z) = \int_a^b y \bar{z} w dx$$

and with the norm

$$\|y\|^2 = (y, y).$$

Now consider the subspace D of H which consists of the functions $y \in H$ such that $y^{[r]}$, $0 \leq r \leq 2$, is locally absolutely continuous on $[a, b]$ and $\ell(y) \in H$. The maximal operator L is defined on D by

$$Ly = \ell(y), y \in D, x \in [a, b].$$

For $y, z \in D$ following Lagrange's formula can be introduced

$$(Ly, z) - (y, Lz) = [y, z](b) - [y, z](a), \tag{2.1}$$

where

$$[y, z] := y\overline{z^{[2]}} - y^{[2]}\bar{z} + i y^{[1]}\overline{z^{[1]}}.$$

(2.1) particulary implies the meaning of $[y, z](a)$ and $[y, z](b)$ for $y, z \in D$.

Let D_0' be a set of D that consists of those functions $y \in D$ such that y has a compact support on $[a, b]$. The operator L_0' which is the restriction of L to D_0' is a densely defined symmetric operator and therefore it admits the closure. Let L_0 be the closure of L_0'. L_0 then becomes a densely defined, symmetric operator with domain D_0 that consists of the functions $y \in D$ satisfying

$$y^{[r]}(a) = y^{[r]}(b) = 0, 0 \leq r \leq 2.$$

Moreover one has $L_0^* = L$ [14, 15].

For the symmetric operators there exists a useful theory called deficiency indices theory to construct the extensions. In fact, let M be a symmetric operator on a Hilbert space B and R_λ denoted the range of $M - \lambda I$, where λ is a parameter and I is the identity operator in B. The deficiency spaces N_λ and $N_{\bar{\lambda}}$ are defined by Naimark [14]

$$N_\lambda = B \ominus R_\lambda, N_{\bar{\lambda}} = B \ominus R_{\bar{\lambda}}.$$

The deficiency indices (m, n) of the operator M are defined by

$$(m, n) = (\dim N_i, \dim N_{-i}).$$

Note that the deficiency indices of L_0 are $(3, 3)$.

To describe the extensions of a closed, symmetric operator with equal deficiency indices one may use the boundary value space. Boundary value space of the closed symmetric operator M is a triple (K, σ_1, σ_2) such that σ_1, σ_2 are linear mappings from $D(M^*)$ (domain of M^*) into K and following holds:

(*i*) for any $f, g \in D(M^*)$

$$(M^* f, g) - (f, M^* g) = (\sigma_1 f, \sigma_2 g)_K - (\sigma_2 f, \sigma_1 g)_K,$$

(*ii*) for and $F_1, F_2 \in K$, there exists a vector $f \in D(M^*)$ such that $\sigma_1 f = F_1$ and $\sigma_2 f = F$.

Now for $y \in D$ consider the following mappings

$$\sigma_1 y = \left(y^{[2]}(a), \frac{1}{2} y^{[1]}(a) + \frac{i}{2} y^{[1]}(b), y(b)\right)$$

and

$$\sigma_2 y = \left(y(a), i y^{[1]}(a) + y^{[1]}(b), y^{[2]}(b)\right).$$

Then we have the following Lemma.

Lemma 2.1. *For* $y, z \in D$

$$(\sigma_1 y, \sigma_2 z)_{\mathbb{C}^3} - (\sigma_2 y, \sigma_1 z)_{\mathbb{C}^3} = [y, z](b) - [y, z](a).$$

Proof: Let $y, z \in D$. Then

$$
\begin{aligned}
&(\sigma_1 y, \sigma_2 z)_{\mathbb{C}^3} - (\sigma_2 y, \sigma_1 z)_{\mathbb{C}^3} = \sigma_1 y(\sigma_2 z)^* - \sigma_2 y(\sigma_1 z)^* \\
&= y^{[2]}(a)\overline{z(a)} + \left(\tfrac{1}{2}y^{[1]}(a) + \tfrac{i}{2}y^{[1]}(b)\right)\left(-i\overline{z^{[1]}(a)} + \overline{z^{[1]}(b)}\right) \\
&+ y(b)\overline{z^{[2]}(b)} - \Big[y(a)\overline{z^{[2]}(a)} + \left(iy^{[1]}(a) + y^{[1]}(b)\right)\Big(\tfrac{1}{2}\overline{z^{[1]}(a)} \\
&- \tfrac{i}{2}\overline{z^{[1]}(b)}\Big) + y^{[2]}(b)\overline{z(b)}\Big] = [y, z](b) - [y, z](a).
\end{aligned}
$$

This completes the proof. □

One of our aim is to impose some suitable boundary conditions for the solution y of the equation

$$-i\left(q_0\left(q_0 y'\right)'\right)' - (p_0 y')' + i\left[q_1 y' + (q_1 y)'\right] + p_1 y = \lambda w y, x \in [a, b], \tag{2.2}$$

where λ is the spectral parameter and $y \in D$. We should note that the Equation (2.2) has a unique solution $\chi(x, \lambda)$ satisfying the initial conditions

$$\chi^{[r]}(c, \lambda) = l_r, 0 \leq r \leq 2,$$

where l_r is a complex number. This fact follows from the assumptions on the coefficients q_0, q_1, p_0, p_1, w, and following representation

$$Y' = A(x, \lambda)Y, \tag{2.3}$$

where

$$Y = \begin{bmatrix} y^{[0]} \\ y^{[1]} \\ y^{[2]} \end{bmatrix}, \ A = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{(1+i)q_0} & 0 \\ -\frac{1+i}{\sqrt{2}}\frac{q_1}{q_0} & i\frac{p_0}{q_0^2} & -\frac{\sqrt{2}}{(1+i)q_0} \\ p_1 - \lambda w & -\frac{1+i}{\sqrt{2}}\frac{q_1}{q_0} & 0 \end{bmatrix}.$$

Then the theory on ordinary differential equations may be applied to the first-order system (2.3), where the elements of A are integrable on each compact subintervals of $[a, b]$.

Now the next Lemma can be introduced with the aid of Naimark's patching Lemma [14].

Lemma 2.2. *There existsy $\in D$ satisfying*

$$y^{[r]}(a) = \alpha_r, y^{[r]}(b) = \beta_r, 0 \leq r \leq 2,$$

where α_r, β_r are arbitrary complex numbers.

Now we may introduce the following.

Theorem 2.3. $(\mathbb{C}^3, \sigma_1, \sigma_2)$ *is a boundary value space for L_0.*

Proof: Since $L_0^* = L$ we obtain for $y, z \in D$ that

$$(L_0^* y, z) - (y, L_0^* z) = [y, z](b) - [y, z](a).$$

Therefore, Lemma 2.1 and Lemma 2.2 complete the proof. □

Let S be a contraction and N be a selfadjoint operator on $\mathbb{C}^3$. Then using the Theorem of Gorbachuks' [16], p. 156, the following abstract Theorem can be introduced.

Theorem 2.4. *Let $f \in D$. Then the conditions*

$$
\begin{aligned}
&(\sin N)\,\sigma_1 f - (\cos N)\sigma_2 f = 0, \\
&(S - I)\sigma_1 f + i(S + I)\sigma_2 f = 0, \\
&(S - I)\sigma_1 f - i(S + I)\sigma_2 f = 0,
\end{aligned}
$$

describe, respectively, the maximal selfadjoint, maximal dissipative, and maximal accumulative extensions of L_0.

Since we will investigate the spectral properties of the maximal dissipative extension of L_0 we shall introduce the following.

Corollary 2.5. *For $y \in D$ the maximal dissipative extension of L_0 is described by*

$$
\begin{aligned}
&y(a) + h_1 y^{[2]}(a) = 0, \operatorname{Im} h_1 \geq 0, \\
&iy^{[1]}(a) + y^{[1]}(b) + h_{2,*}\left(\tfrac{1}{2}y^{[1]}(a) + \tfrac{i}{2}y^{[1]}(b)\right) = 0, \operatorname{Im} h_{2,*} \geq 0, \\
&y^{[2]}(b) + h_3 y(b) = 0, \operatorname{Im} h_3 \geq 0.
\end{aligned}
$$

Corollary 2.6. *For $y \in D$ the conditions*

$$
\begin{aligned}
&y(a) + h_1 y^{[2]}(a) = 0, \operatorname{Im} h_1 = 0, \\
&(i + h_2)y^{[1]}(a) + (1 + ih_2)y^{[1]}(b) = 0, \operatorname{Im} h_2 > 0, h_2 \neq i, \\
&y^{[2]}(b) + h_3 y(b) = 0, \operatorname{Im} h_3 > 0,
\end{aligned}
\tag{2.4}
$$

where $h_2 = h_{2,*}/2$, describe the maximal dissipative extension of L_0.

Remark 2.7. *As may be seen in the next sections, the caseh$_2 =$ i may give rise to some complications. Therefore, we exclude this case.*

Now let $D(\mathcal{L})$ be a set consisting of all functions $y \in D$ satisfying the conditions (2.4). Let us define the operator $\mathcal{L}$ on $D(\mathcal{L})$ with the rule

$$\mathcal{L}y = \ell(y), y \in D(\mathcal{L}), x \in [a, b].$$

Then $\mathcal{L}$ is a maximal dissipative operator on H.

The adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ is given by

$$\mathcal{L}^* y = \ell(y), y \in D(\mathcal{L}^*), x \in [a, b],$$

where $D(\mathcal{L}^*)$ is the domain of $\mathcal{L}^*$ consisting of all functions $y \in D$ satisfying

$$
\begin{aligned}
&y(a) + h_1 y^{[2]}(a) = 0, \ \operatorname{Im} h_1 = 0, \\
&(-i + \overline{h}_2)y^{[1]}(a) + (1 - i\overline{h}_2)y^{[1]}(b) = 0, \ \operatorname{Im} h_2 > 0, h_2 \neq -i, \\
&y^{[2]}(b) + \overline{h}_3 y(b) = 0, \ \operatorname{Im} h_3 > 0.
\end{aligned}
$$

Theorem 2.8. *$\mathcal{L}$ is totally dissipative (simple) in H.*

Proof: This follows from choosing h_2 and h_3 with positive imaginary parts. Indeed, for $y \in D(\mathcal{L})$ one gets

$$\operatorname{Im}(\mathcal{L}y, y) = \frac{2 \operatorname{Im} h_2}{\left|1 + ih_2\right|^2}\left|y^{[1]}(a)\right|^2 + \operatorname{Im} h_3\left|y(b)\right|^2. \tag{2.5}$$

If $\mathcal{L}$ had a selfadjoint part $\mathcal{L}_s$ in $H_s \subset H$ then from (2.5) one would get

$$y(b) = y^{[1]}(b) = y^{[2]}(b) = 0$$

and therefore $y \equiv 0$. This completes the proof. □

3. CONTRACTIVE OPERATOR

There exists a connection between dissipative operator $\mathcal{L}$ and the contractive operator $\mathcal{C}$. This connection can be given by the following relation

$$\mathcal{C} = (\mathcal{L} - iI)(\mathcal{L} + iI)^{-1}.$$

Since $\mathcal{L}$ is maximal dissipative the domain of $\mathcal{C}$ is the whole Hilbert space H.

An important class of contractions on a Hilbert space consists of completely non-unitary (c.n.u.) contractions. A contraction C is said to be c.n.u. if there exists no non-zero reducing subspace H_0 such that $C \mid H_0$ is a unitary operator.

From the simplicity of $\mathcal{L}$ we have the following.

Theorem 3.1. *$\mathcal{C}$ is a c.n.u. contraction on$H$.*

Proof: Let $(\mathcal{L} + iI)^{-1}f = y$, where $y \in D(\mathcal{L})$ and $f \in H$. Then we get

$$\left\|(\mathcal{L} - iI)y\right\|^2 < \left\|(\mathcal{L} + iI)y\right\|^2 \tag{3.1}$$

because

$$\operatorname{Im}(\mathcal{L}y, y) > 0, y \in D(\mathcal{L}).$$

(3.1) implies that

$$\|\mathcal{C}\| < 1 \tag{3.2}$$

and this completes the proof. □

Now we define the defect operators of $\mathcal{C}$ as

$$D_{\mathcal{C}} = (I - \mathcal{C}^*\mathcal{C})^{1/2}, D_{\mathcal{C}^*} = (I - \mathcal{C}\mathcal{C}^*)^{1/2}$$

and the defect spaces of $\mathcal{C}$ as

$$\mathfrak{D}_{\mathcal{C}} = \overline{D_{\mathcal{C}}H}, \mathfrak{D}_{\mathcal{C}^*} = \overline{D_{\mathcal{C}^*}H}.$$

The numbers $\mathfrak{d}_C$ and $\mathfrak{d}_{C^*}$ defined by

$$\mathfrak{d}_C = \dim \mathfrak{D}_{\mathcal{C}}, \mathfrak{d}_{C^*} = \dim \mathfrak{D}_{\mathcal{C}^*}$$

are called the defect indices of $\mathcal{C}$.

Theorem 3.2. $\mathfrak{d}_C = \mathfrak{d}_{C^*} = 2$.

Proof: Consider the equation

$$D_{\mathcal{C}}^2 f = (\mathcal{L} + iI)y - (\mathcal{L}^* + iI)z,$$

where $f = (\mathcal{L} + iI)y, y \in D(\mathcal{L}), f \in H$ and $z \in D(\mathcal{L}^*)$. Then

$$z = (\mathcal{L}^* - iI)^{-1}(\mathcal{L} - iI)y$$

or

$$(\mathcal{L}^* - iI)z = (\mathcal{L} - iI)y. \tag{3.3}$$

Equation (3.3) implies that $\mathfrak{D}_{\mathcal{C}}$ is spanned by two independent solutions. In fact, let $\varphi(x, \lambda)$ and $\widetilde{\varphi}(x, \lambda)$ be two solutions of (2.2) satisfying

$$\varphi(a, \lambda) = -h_1, \varphi^{[2]}(a, \lambda) = 1, \varphi^{[1]}(a, \lambda) = c(\neq 0), \tag{3.4}$$

where c is a constant and

$$\widetilde{\varphi}(a, \lambda) = -h_1, \widetilde{\varphi}^{[2]}(a, \lambda) = 1, \varphi^{[1]}(a, \lambda) = 0.$$

(2.5) needs the solutions of (2.2) satisfying the condition

$$y(a) + h_1 y^{[2]}(a) = 0, h_1 \in \mathbb{R} \tag{3.5}$$

Clearly φ and $\widetilde{\varphi}$ satisfies (3.5) and φ can not be represented by a constant of $\widetilde{\varphi}$. If there exists any other solution $\psi(x, \lambda)$ of (2.2) satisfying

$$\psi(a, \lambda) = -h_1, \psi^{[2]}(a, \lambda) = 1, \psi^{[1]}(a, \lambda) = c_1,$$

where c_1 is another constant different from c then $\widetilde{\psi}(x, \lambda) := (c_1/c)\varphi(x, \lambda)$ becomes a solution of (2.2) satisfying (3.5) and $\psi(x, \lambda)$ may be introduced by $\varphi(x, \lambda)$.

Therefore,

$$D_{\mathcal{C}}^2 f = (\ell + iI)(y - z) = 2i(d_1\varphi + d_2\widetilde{\varphi}),$$

where d_1 and d_2 are constants and $\mathfrak{D}_{\mathcal{C}}$ is spanned by $\varphi(x, i)$ and $\widetilde{\varphi}(x, i)$.

With a similar argument one may see that

$$D_{\mathcal{C}^*}^2 f = (\ell - iI)(y - z) = -2i(d_1\varphi + d_2\widetilde{\varphi}),$$

and therefore $\mathfrak{D}_{\mathcal{C}^*}$ is spanned by $\varphi(x, -i)$ and $\widetilde{\varphi}(x, -i)$.

This completes the proof. □

Definition 3.3. [17] *The classes$C_{0.}$ and$C_{.0}$ are defined as*

$$\begin{aligned} C_{0.} &= \left\{T : \|T\| \leq 1, \lim_n \left\|T^n f\right\| = 0 \text{ for all} f\right\}, \\ C_{.0} &= \left\{T : \|T\| \leq 1, \lim_n \left\|T^{*n} f\right\| = 0 \text{ for all} f\right\}. \end{aligned}$$

C_{00} is defined by $C_{00} = C_{0.} \cap C_{.0}$.

Theorem 3.4. $\mathcal{C} \in C_{00}$.

Proof: This follows from (3.2), $\|\mathcal{C}\| = \|\mathcal{C}^*\|$ and the equalities

$$\left\|C^n f\right\| \leq \|C\|^n \left\|f\right\|,$$

and

$$\left\|C^{*n} f\right\| \leq \left\|C^*\right\|^n \left\|f\right\|.$$

□

The class C_0 consists of those c.n.u. contractions T for which there exists a non-zero function $u \in H^\infty$ (H^p denotes the Hardy class) such that $u(T) = 0$. Since C belongs to the class C_{00} with finite defect numbers this implies the following [1].

Theorem 3.5. $\mathcal{C} \in C_0$.

4. CHARACTERISTIC FUNCTION

We shall consider the inner product on the quotient space $D(\mathcal{L})/G_{\mathcal{L}}$ as follows

$$(Py, Pz) = \frac{i}{2}\left(\left(y, \mathcal{L}y\right) - \left(\mathcal{L}y, y\right)\right), y, z \in D(\mathcal{L}),$$

where P is the natural projection with $P : D(\mathcal{L}) \to D(\mathcal{L})/G_{\mathcal{L}}$. The completion of $D(\mathcal{L})/G_{\mathcal{L}}$ is denoted by $F(\mathcal{L})$ with respect to the corresponding norm. Similarly $F_*(\mathcal{L}) := F(-\mathcal{L}^*)$ and P_* is defined by $P_* : D(\mathcal{L}^*) \to D(\mathcal{L}^*)/G_{\mathcal{L}}$. One has

$$\left\|Py\right\|_F^2 = \operatorname{Im}(\mathcal{L}y, y), \ \left\|P_*y\right\|_{F_*}^2 = -\operatorname{Im}(\mathcal{L}^*y, y). \quad (4.1)$$

$F(\mathcal{L})$ and $F(\mathcal{L}^*)$ are the boundary spaces of $\mathcal{L}$. From (4.1) we get

$$\left\|Py\right\|_F^2 = \frac{2\operatorname{Im} h_2}{\left|1 + ih_2\right|^2}\left|y^{[1]}(a)\right|^2 + \operatorname{Im} h_3\left|y(b)\right|^2 \quad (4.2)$$

and

$$\|P_*z\|_{F_*}^2 = \frac{2\operatorname{Im} h_2}{\left|1 + ih_2\right|^2}\left|z^{[1]}(a)\right|^2 + \operatorname{Im} h_3\left|z(b)\right|^2. \quad (4.3)$$

From (4.2) and (4.3) we may set

$$Py = \begin{bmatrix} \frac{\sqrt{2\operatorname{Im} h_2}}{|1+ih_2|}y^{[1]}(a) \\ \sqrt{\operatorname{Im} h_3}y(b) \end{bmatrix}, \ P_*z = \begin{bmatrix} \frac{\sqrt{2\operatorname{Im} h_2}}{|1+ih_2|}z^{[1]}(a) \\ \sqrt{\operatorname{Im} h_3}z(b) \end{bmatrix}.$$

Setting $E = E_* = \mathbb{C}^2$ we define the following isometric isomorphisms

$$\begin{aligned} \Psi : E &\to F(\mathcal{L}), \\ c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &\to \Psi(c) = Py = \begin{bmatrix} \frac{\sqrt{2\operatorname{Im} h_2}}{|1+ih_2|}y^{[1]}(a) \\ \sqrt{\operatorname{Im} h_3}y(b) \end{bmatrix}, \end{aligned} \quad (4.4)$$

where $y \in D(\mathcal{L})$, $y^{[1]}(a) = c_2\left|1 + ih_2\right|\left(2\operatorname{Im} h_2\right)^{-1/2}$, $y(b) = c_1(\operatorname{Im} h_3)^{-1/2}$ and

$$\begin{aligned} \Psi_* : E_* &\to F_*(\mathcal{L}), \\ c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &\to \Psi_*(c) = P_*z = \begin{bmatrix} \frac{\sqrt{2\operatorname{Im} h_2}}{|1+ih_2|}z^{[1]}(a) \\ \sqrt{\operatorname{Im} h_3}z(b) \end{bmatrix}, \end{aligned} \quad (4.5)$$

where $z \in D(\mathcal{L}^*)$, $z^{[1]}(a) = c_2\left|1 + ih_2\right|\left(2\operatorname{Im} h_2\right)^{-1/2}$, $z(b) = c_1(\operatorname{Im} h_3)^{-1/2}$. Then we may introduce the characteristic function of $\mathcal{L}$.

Theorem 4.1. *The characteristic matrix-function $\Theta_{\mathcal{L}}$ of $\mathcal{L}$ is given by*

$$\Theta_{\mathcal{L}}(\lambda) = \begin{bmatrix} \frac{-i+\overline{h_2}}{i+h_2}\frac{(1+ih_2)\varphi^{[1]}(b)-(i+h_2)\varphi^{[1]}(a)}{(1-i\overline{h_2})\varphi^{[1]}(b)-(-i+\overline{h_2})\varphi^{[1]}(a)} & 0 \\ 0 & \frac{\varphi^{[2]}(b)+h_3\varphi(b)}{\varphi^{[2]}(b)+\overline{h_3}\varphi(b)} \end{bmatrix}, \operatorname{Im}\lambda > 0.$$

Proof: Consider the equation

$$\Theta_{\mathcal{L}}(\lambda)c = \Psi_*^* P_*(\mathcal{L}^* - \lambda I)^{-1}(\mathcal{L} - \lambda I)P^{-1}\Psi c. \quad (4.6)$$

(4.4) implies that $y \in P^{-1}\Psi c$ with $y \in D(\mathcal{L})$ and therefore

$$(\mathcal{L}^* - \lambda I)^{-1}(\mathcal{L} - \lambda I)y = z, \quad (4.7)$$

where $z \in D(\mathcal{L}^*)$ and

$$(\mathcal{L} - \lambda I)y = (\mathcal{L}^* - \lambda I)z. \quad (4.8)$$

Using (4.6) and (4.7) we obtain

$$\Psi_*^* P_* z = \begin{bmatrix} \frac{\sqrt{2\operatorname{Im} h_2}}{|1+ih_2|}z^{[1]}(a) \\ \sqrt{\operatorname{Im} h_3}z(b) \end{bmatrix}. \quad (4.9)$$

From (4.8) we should find a solution $u = z - y$ of the Equation (2.2) satisfying (3.5). Therefore, we may set $u = B(\lambda)\varphi(x, \lambda)$, where φ is the solution of (2.2) satisfying the conditions in (3.4). Consider the equation

$$(z - y)^{[2]}(b) = \frac{\varphi^{[2]}(b, \lambda)}{\varphi(b, \lambda)}(z - y)(b). \quad (4.10)$$

Since $y \in D(\mathcal{L})$ and $z \in D(\mathcal{L}^*)$ we get from (4.10)

$$z(b)(\varphi^{[2]}(b, \lambda) + \overline{h_3}\varphi^{[2]}(b, \lambda)) = y(b)(\varphi^{[2]}(b, \lambda) + h_3\varphi^{[2]}(b, \lambda)). \quad (4.11)$$

Similarly the equation

$$(z - y)^{[1]}(a) = \frac{\varphi^{[1]}(a, \lambda)}{\varphi^{[1]}(b, \lambda)}(z - y)^{[1]}(b)$$

gives

$$\begin{aligned} &z^{[1]}(a)\tfrac{-i+\overline{h_2}}{1-i\overline{h_2}}\left(\varphi^{[1]}(a, \lambda) - \tfrac{1-i\overline{h_2}}{-i+\overline{h_2}}\varphi^{[1]}(b, \lambda)\right) \\ &= y^{[1]}(a)\tfrac{i+h_2}{1+ih_2}\left(\tfrac{1+ih_2}{i+h_2}\varphi^{[1]}(b, \lambda) - \varphi^{[1]}(a, \lambda)\right). \end{aligned} \quad (4.12)$$

(4.6) and (4.9) show that

$$\Theta_{\mathcal{L}}(\lambda)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{z^{[1]}(a)}{y^{[1]}(a)} & 0 \\ 0 & \frac{z(b)}{y(b)} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \quad (4.13)$$

Consequently (4.11)–(4.13) complete the proof. □

Remind that a function $\Theta(\zeta)$ whose values are bounded operators from a Hilbert space $\mathbb{H}$ to a Hilbert space $\mathbb{H}_*$, both separable and which has a power series expansion

$$\Theta(\zeta) = \sum_{k=0}^{\infty}\zeta^k\Theta_k$$

whose coefficients are bounded operators from $\mathbb{H}$ to $\mathbb{H}_*$. Moreover assume that

$$\left\|\Theta(\zeta)\right\| \le const.$$

Such a function with the spaces $\mathbb{H}$ and $\mathbb{H}_*$ is called bounded analytic function. If *const* $= 1$ then it is called contractive analytic function. The contractive analytic function Θ is said to be inner if $\Theta(e^{it})$ is isometry from $\mathbb{H}$ into $\mathbb{H}_*$ for almost all t.

Since there exists a connection between the characteristic function $\Theta_{\mathcal{L}}$ of $\mathcal{L}$ and the characteristic function $\Theta_{\mathcal{C}}$ of $\mathcal{C}$ with the rule

$$\Theta_{\mathcal{L}}(\lambda) = \Theta_{\mathcal{C}}\left(\frac{\lambda - i}{\lambda + i}\right)$$

we have the following.

Corollary 4.2. *The characteristic function $\Theta_{\mathcal{C}}$ of C is given by*

$$\Theta_{\mathcal{C}}(\mu) = \Theta_{\mathcal{L}}(\lambda), \mu = \frac{\lambda - i}{\lambda + i}, \operatorname{Im}\lambda > 0.$$

Since $\mathcal{C}$ is a c.n.u. contraction belonging to the class $C_{.0}$ we have the following.

Theorem 4.3. $\Theta_{C}(\mu)$ *is inner.*

Corollary 4.4. $\det\Theta_{C}(\mu)$ *is inner.*

An operator $A \geq 0$ on a Hilbert space $\mathcal{H}$ is said to be of finite trace if A is compact and its eigenvalues is finite. This sum is called the trace of A [1, 18]. A contraction C on a Hilbert space H is called weak contraction if

(i) its spectrum does not fill the unit disc D,
(ii) $I - C^*C$ is of finite trace.

Since every C_0 contraction with finite-multiplicity is a weak contraction [18], p. 437, we may introduce the following.

Theorem 4.5. *$I - \mathcal{C}^*\mathcal{C}$ is of finite trace.*

The following Theorem is obtained from Nikolskiĭ [17], p. 134.

Theorem 4.6. *The followings are satisfied:*

(i) The root functions of $\mathcal{C}$ are complete in H,
(ii) The roots functions of $\mathcal{C}^*$ are complete in H,
(iii) $\det(\mathcal{C}^*\mathcal{C}) = \det\left|\Theta_{\mathcal{C}}(0)\right|^2 = \prod_{\mu} |\mu|^{2d(\mu)}$,

where μ belongs to the point spectrum of $\mathcal{C}$ and $d(\mu)$ is the rank of the Riesz projection at a point μ in the set of point spectrum.

Proof: The proof follows from the fact $\det\Theta_{\mathcal{C}}$ is a Blashke product. So we shall prove this fact.

By Corollary 4.4 we may write

$$\frac{-i+\overline{h_2}}{i+h_2}\,\frac{(1+ih_2)\varphi^{[1]}(b)-(i+h_2)\varphi^{[1]}(a)}{(1-i\overline{h_2})\varphi^{[1]}(b)-(-i+\overline{h_2})\varphi^{[1]}(a)}\,\frac{\varphi^{[2]}(b)+h_3\varphi(b)}{\varphi^{[2]}(b)+\overline{h_3}\varphi(b)} = \mathbb{B}(\lambda)e^{i\lambda b}, \tag{4.14}$$

where $b > 0$, $\operatorname{Im}\lambda > 0$ and $\mathbb{B}(\lambda)$ is a Blashke product in the upper half-plane. Hence

$$\left|\frac{(1+ih_2)\varphi^{[1]}(b)-(i+h_2)\varphi^{[1]}(a)}{(1-i\overline{h_2})\varphi^{[1]}(b)-(-i+\overline{h_2})\varphi^{[1]}(a)}\,\frac{\varphi^{[2]}(b)+h_3\varphi(b)}{\varphi^{[2]}(b)+\overline{h_3}\varphi(b)}\right| \leq e^{-b\operatorname{Im}\lambda}. \tag{4.15}$$

For $\lambda_s = is$ we have from (4.15) that the following possibilities may occur:

(i) $\frac{\varphi^{[2]}(b)+h_3\varphi(b)}{\varphi^{[2]}(b)+\overline{h_3}\varphi(b)} \to 0$ as $s \to \infty$,

(ii) $\frac{(1+ih_2)\varphi^{[1]}(b)-(i+h_2)\varphi^{[1]}(a)}{(1-i\overline{h_2})\varphi^{[1]}(b)-(-i+\overline{h_2})\varphi^{[1]}(a)} \to 0$ as $s \to \infty$,

(iii) $\frac{\varphi^{[2]}(b)+h_3\varphi(b)}{\varphi^{[2]}(b)+\overline{h_3}\varphi(b)} \to 0$ and $\frac{(1+ih_2)\varphi^{[1]}(b)-(i+h_2)\varphi^{[1]}(a)}{(1-i\overline{h_2})\varphi^{[1]}(b)-(-i+\overline{h_2})\varphi^{[1]}(a)} \to 0$ as $s \to \infty$.

In fact (iii) is possible because in this case λ is an eigenvalue of $\mathcal{L}$ and this implies that λ_∞ is an eigenvalue of the operator $\mathcal{L}$ or equivalently 1 is an eigenvalue of the c.n.u. contraction $\mathcal{C}$. However the latter one is not possible. Therefore, this completes the proof. □

Definition 4.7. *Let all root functions of the operatorL span the Hilbert space$\mathcal{H}$. Such an operator is called complete operator. If everyL−invariant subspace is generated by root vectors of L belonging to the subspace then it is said L admits spectral synthesis.*

Since every complete operator in C_0 admits spectral synthesis [17], we obtain the following.

Theorem 4.8. *$\mathcal{C}$ admits spectral synthesis.*

Since the root functions of $\mathcal{L}$ span H then those of $\mathcal{C}$ must span H [19] (p. 42). Consequently we may introduce the following.

Theorem 4.9. *Root functions of $\mathcal{L}$ associated with the point spectrum of $\mathcal{L}$ in the open upper half-plane$\operatorname{Im}\lambda > 0$ span the Hilbert space H.*

5. DILATION OPERATOR AND ITS EIGENFUNCTIONS

In this section we investigate the properties of selfadjoint dilation of the operator $\mathcal{L}$ and eigenfunctions of selfadjoint dilation.

5.1. Selfadjoint Dilation of the Maximal Dissipative Operator

Following theorem gives the selfadjoint operator with free parameters [5].

Theorem 5.1.1. *The minimal selfadjoint dilation$\boldsymbol{\mathcal{L}}$ of the maximal dissipative operator$\mathcal{L}$ in the space*

$$H_{\mathcal{L}} = L^2(\mathbb{R}_-, E_*) \oplus H \oplus L^2(\mathbb{R}_+, E)$$

has the form

$$\boldsymbol{\mathcal{L}}\begin{bmatrix}\varphi_-\\ f\\ \varphi_+\end{bmatrix} = \begin{bmatrix} i\varphi_-' \\ i\left\{2\left(I-\mathcal{C}\right)^{-1}\left[f - \frac{i}{\sqrt{2}}D_{\mathcal{C}^*}\Omega_*\varphi_-(0)\right] - \rho\right\} \\ i\varphi_+' \end{bmatrix}$$

and the domain of $\boldsymbol{\mathcal{L}}$ *is given by the conditions*

$$\begin{aligned}&\varphi_-\in W_2^1(\mathbb{R}_-,E_*),\varphi_+\in W_2^1(\mathbb{R}_+,E),\\&f-\tfrac{i}{\sqrt{2}}D_{\mathcal{C}^*}\Omega_*\varphi_-(0)\in(I-\mathcal{C})\,H=D(\mathcal{L}),\\&\sqrt{2}iD_{\mathcal{C}}(I-\mathcal{C})^{-1}\left[f-\tfrac{i}{\sqrt{2}}D_{\mathcal{C}^*}\Omega_*\varphi_-(0)\right]=\mathcal{C}^*\Omega_*\varphi_-(0)+\Omega\varphi_+(0),\end{aligned}$$

where W_2^1 *denotes the Sobolev space.*

The isometries $\Omega: E\to D_{\mathcal{C}}$, $\Omega_*: E_*\to D_{\mathcal{C}^*}$ are the free parameters. In the case that $\dim\mathfrak{D}_{\mathcal{C}}<\infty$, $\dim\mathfrak{D}_{\mathcal{C}^*}<\infty$ then one may consider the boundary spaces $F(\mathcal{L})$ and $F_*(\mathcal{L})$ instead of $\mathfrak{D}_{\mathcal{C}}$ and $\mathfrak{D}_{\mathcal{C}^*}$. Then following Lemma gives a direct approach for the dilation [5].

Lemma 5.1.2. *The minimal selfadjoint dilation* $\boldsymbol{\mathcal{L}}$ *in the space* $H_{\mathcal{L}}$ *of the maximial dissipative operator* $\mathcal{L}$ *in* H *with finite defects has the form*

$$\boldsymbol{\mathcal{L}}\begin{bmatrix}\varphi_-\\ f\\ \varphi_+\end{bmatrix}=\begin{bmatrix}i\varphi_-'\\ \mathcal{L}\left(f-\frac{i}{\sqrt{2}}\left[\Psi_*\varphi_-(0)\right]\right)+\frac{i}{\sqrt{2}}\mathcal{L}^*\left[\Psi_*\varphi_-(0)\right]\\ i\varphi_+'\end{bmatrix}$$

where $\Psi: E\to F(\mathcal{L})$ *and* $\Psi_*: E_*\to F_*(\mathcal{L})$ *are the isometric isomorphisms and the domain of* $\boldsymbol{\mathcal{L}}$ *is given by the conditions*

$$\begin{aligned}&\varphi_-\in W_2^1(\mathbb{R}_-,E_*),\ \varphi_+\in W_2^1(\mathbb{R}_+,E),\\&f-\tfrac{i}{\sqrt{2}}\left[\Psi_*\varphi_-(0)\right]\in D(\mathcal{L}),\\&f-\tfrac{i}{\sqrt{2}}\left[\Psi_*\varphi_-(0)\right]+\tfrac{i}{\sqrt{2}}\left[\Psi\varphi_+(0)\right]\in G_{\mathcal{L}}.\end{aligned}$$

If $G_{\mathcal{L}}$ is dense in H one may consider $G_{\mathcal{L}}=D(\mathcal{L})\cap D(\mathcal{L}^*)$ and

$$\widetilde{\mathcal{L}}:=\begin{cases}\mathcal{L}\text{ on }D(\mathcal{L}),\\ \mathcal{L}^*\text{ on }D(\mathcal{L}^*).\end{cases}$$

The following Corollary now may be introduced [5].

Corollary 5.1.3. *The selfadjoint dilation* $\boldsymbol{\mathcal{L}}$ *of the maximal dissipative operator* $\mathcal{L}$ *with finite defects such that* $G_{\mathcal{L}}$ *is dense in* H *has the form*

$$\boldsymbol{\mathcal{L}}\begin{bmatrix}\varphi_-\\ f\\ \varphi_+\end{bmatrix}=\begin{bmatrix}i\varphi_-'\\ \widetilde{\mathcal{L}}f\\ i\varphi_+'\end{bmatrix},\ \widetilde{\mathcal{L}}=(\mathcal{L}\mid G_{\mathcal{L}})^*,$$

and the domain of $\boldsymbol{\mathcal{L}}$ *is given by the conditions*

$$\begin{aligned}&\varphi_-\in W_2^1(\mathbb{R}_-,E_*),\varphi_+\in W_2^1(\mathbb{R}_+,E),\\&f-\tfrac{i}{\sqrt{2}}\left[\Psi_*\varphi_-(0)\right]\in D(\mathcal{L}),\\&f+\tfrac{i}{\sqrt{2}}\left[\Psi\varphi_+(0)\right]\in D(\mathcal{L}^*).\end{aligned}$$

Now using Corollary 5.1.3 we may introduce the following.

Theorem 5.1.4. *The selfadjoint dilation* $\boldsymbol{\mathcal{L}}$ *of the maximal dissipative operator* $\mathcal{L}$ *in the space*

$$H_{\mathcal{L}}=L^2(\mathbb{R}_-;\mathbb{C}^2)\oplus H\oplus L^2(\mathbb{R}_+;\mathbb{C}^2)$$

is given by the rule

$$\boldsymbol{\mathcal{L}}\begin{bmatrix}\varphi_-\\ f\\ \varphi_+\end{bmatrix}=\begin{bmatrix}i\varphi_-'\\ \ell(f)\\ i\varphi_+'\end{bmatrix}$$

whose domain is given by the conditions

$$\begin{aligned}&(i+h_2)f^{[1]}(a)+(1+ih_2)f^{[1]}(b)=\tfrac{(1-|h_2|)(1+ih_2)}{\sqrt{\operatorname{Im}h_2}|1+ih_2|}\varphi_-^{(1)}(0),\\&f^{[2]}(b)+h_3f(b)=-\sqrt{2\operatorname{Im}h_3}\varphi_-^{(2)}(0),\\&(-i+\overline{h_2})f^{[1]}(a)+(1-i\overline{h_2})f^{[1]}(b)=\tfrac{(|h_2|-1)(1-i\overline{h_2})}{\sqrt{\operatorname{Im}h_2}|1+ih_2|}\varphi_+^{(1)}(0),\\&f^{[2]}(b)+\overline{h_3}f(b)=-\sqrt{2\operatorname{Im}h_3}\varphi_+^{(2)}(0),\end{aligned}$$

where

$$\varphi_\pm:=\begin{bmatrix}\varphi_\pm^{(1)}\\ \varphi_\pm^{(2)}\end{bmatrix}\in W_2^1(\mathbb{R}_\pm;\mathbb{C}^2).$$

Proof: Let $y\in D(\mathcal{L})$ with $y^{[1]}(a)=(2\operatorname{Im}h_2)^{-1/2}\left|1+ih_2\right|\varphi_+^{(1)}(0)$, $y(b)=(\operatorname{Im}h_3)^{-1/2}\varphi_+^{(2)}(0)$ and $z\in D(\mathcal{L}^*)$ with $z^{[1]}(a)=(2\operatorname{Im}h_2)^{-1/2}\left|1+ih_2\right|\varphi_-^{(1)}(0)$, $z(b)=(\operatorname{Im}h_3)^{-1/2}\varphi_-^{(2)}(0)$. Then $f-i2^{-1/2}\left[\Psi_*\varphi_-(0)\right]\in D(\mathcal{L})$ if and only if

$$f^{[2]}(b)-\frac{i}{\sqrt{2}}z^{[2]}(b)=-h_3\left(f(b)-\frac{i}{\sqrt{2}}z(b)\right)\tag{5.1}$$

and

$$(i+h_2)\left(f^{[1]}(a)-\frac{i}{\sqrt{2}}z^{[1]}(a)\right)=-(1+ih_2)\left(f^{[1]}(b)-\frac{i}{\sqrt{2}}z^{[1]}(b)\right).\tag{5.2}$$

(5.1) gives

$$f^{[2]}(b)+h_3f(b)=-2\sqrt{\operatorname{Im}h_3}\varphi_-^{(2)}(0)$$

and (5.2) implies

$$(i+h_2)f^{[1]}(a)+(1+ih_2)f^{[1]}(b)=\frac{(1-|h_2|)(1+ih_2)}{\sqrt{\operatorname{Im}h_2}\,|1+ih_2|}\varphi_-^{(1)}(0).$$

Similarly $f+i2^{-1/2}\left[\Psi\varphi_+(0)\right]\in D(\mathcal{L}^*)$ if and only if

$$f^{[2]}(b)+\frac{i}{\sqrt{2}}z^{[2]}(b)=-\overline{h_3}\left(f(b)+\frac{i}{\sqrt{2}}z(b)\right)\tag{5.3}$$

and

$$\begin{aligned}&(-i-\overline{h_2})\left(f^{[1]}(a)+\frac{i}{\sqrt{2}}z^{[1]}(a)\right)\\&=-(1-i\overline{h_2})\left(f^{[1]}(b)+\frac{i}{\sqrt{2}}z^{[1]}(b)\right).\end{aligned}\tag{5.4}$$

(5.3) shows that

$$f^{[2]}(b)+\overline{h_3}f(b)=-2\sqrt{\operatorname{Im}h_3}\varphi_+^{(2)}(0)$$

and (5.4) shows

$$(-i+\overline{h_2})f^{[1]}(a)+(1-i\overline{h_2})f^{[1]}(b)=\frac{-(1-\left|h_2\right|)(1+ih_2)}{\sqrt{\operatorname{Im}h_2}\left|1+ih_2\right|}\varphi_+^{(1)}(0).$$

Therefore the proof is completed. □

5.2. Eigenfunctions of the Dilation

As is pointed out in Solomyak [5] the generalized eigenfunctions of the dilation $\mathcal{L}$ may be introduced by incoming eigenfunctions

$$\begin{bmatrix} \Theta_{\mathcal{L}}(\lambda)\exp(-i\lambda r)c \\ \frac{i}{\sqrt{2}}\left(\left(\mathcal{L}^*-\overline{\lambda}I\right)^{-1}\left(\mathcal{L}-\overline{\lambda}I\right)-I\right)P^{-1}\Psi c \\ \exp(-i\lambda s)c \end{bmatrix}$$

and outgoing eigenfunctions

$$\begin{bmatrix} \exp(-i\lambda r)\widetilde{c} \\ -\frac{i}{\sqrt{2}}\left((\mathcal{L}-\lambda I)^{-1}(\mathcal{L}^*-\lambda I)-I\right)P_*^{-1}\Psi_*\widetilde{c} \\ \Theta_{\mathcal{L}}^*(\lambda)\exp(-i\lambda s)\widetilde{c} \end{bmatrix},$$

where $r\in\mathbb{R}_-$, $s\in\mathbb{R}_+$, $c\in E$, $\widetilde{c}\in E_*$ and $\lambda\in\mathbb{R}$.

Therefore we may introduce the following.

Theorem 5.2.1. *The incoming and outgoing eigenfunction of* $\mathcal{L}$ *can be introduced by*

$$\begin{bmatrix} \Theta_{\mathcal{L}}(\lambda)\exp(-i\lambda r)c \\ \begin{bmatrix} \frac{-\sqrt{2\operatorname{Im}h_3}}{\varphi^{[2]}(b,\overline{\lambda})+\overline{h_3}\varphi(b,\overline{\lambda})} & 0 \\ 0 & \left(\frac{1+ih_2}{i+h_2}\frac{\varphi^{[1]}(b,\overline{\lambda})}{\varphi^{[1]}(a,\overline{\lambda})}-1-\frac{1}{\varphi^{[1]}(a,\overline{\lambda})}\right)\frac{\left|1+ih_2\right|}{\sqrt{2\operatorname{Im}h_2}} \end{bmatrix} \\ \exp(-i\lambda s)\widetilde{c} \end{bmatrix},$$

$$\begin{bmatrix} \exp(-i\lambda r)c \\ \begin{bmatrix} \frac{-\sqrt{2\operatorname{Im}h_3}}{\varphi^{[2]}(b,\overline{\lambda})+\overline{h_3}\varphi(b,\overline{\lambda})} & 0 \\ 0 & \left(\frac{1+ih_2}{i+h_2}\frac{\varphi^{[1]}(b,\overline{\lambda})}{\varphi^{[1]}(a,\overline{\lambda})}-1-\frac{1}{\varphi^{[1]}(a,\overline{\lambda})}\right) \end{bmatrix} \\ \Theta_{\mathcal{L}}^*(\lambda)\exp(-i\lambda s)\widetilde{c} \end{bmatrix}$$

or

$$\begin{bmatrix} \Theta_{\mathcal{L}}(\lambda)\exp(-i\lambda r)c \\ \begin{bmatrix} \frac{\sqrt{2\operatorname{Im}h_3}}{\varphi^{[2]}(b,\lambda)+h_3\varphi(b,\lambda)} & 0 \\ 0 & -\left(\frac{1-i\overline{h_2}}{-i+\overline{h_2}}\frac{\varphi^{[1]}(b,\lambda)}{\varphi^{[1]}(a,\lambda)}-1-\frac{1}{\varphi^{[1]}(a,\lambda)}\right) \end{bmatrix} \\ \exp(-i\lambda s)\widetilde{c} \end{bmatrix},$$

$$\begin{bmatrix} \exp(-i\lambda r)c \\ \begin{bmatrix} \frac{\sqrt{2\operatorname{Im}h_3}}{\varphi^{[2]}(b,\lambda)+h_3\varphi(b,\lambda)}\frac{\left|1+ih_2\right|}{\sqrt{2\operatorname{Im}h_2}} & 0 \\ 0 & -\left(\frac{1-i\overline{h_2}}{-i+\overline{h_2}}\frac{\varphi^{[1]}(b,\lambda)}{\varphi^{[1]}(a,\lambda)}-1-\frac{1}{\varphi^{[1]}(a,\lambda)}\right) \end{bmatrix} \\ \Theta_{\mathcal{L}}^*(\lambda)\exp(-i\lambda s)\widetilde{c} \end{bmatrix},$$

where $r\in\mathbb{R}_-$, $s\in\mathbb{R}_+$, $\lambda\in\mathbb{R}$.

Proof: Consider the equation

$$\left(\left(\mathcal{L}^*-\overline{\lambda}I\right)^{-1}\left(\mathcal{L}-\overline{\lambda}I\right)-I\right)P^{-1}\Psi c=B(\overline{\lambda})\varphi(x,\overline{\lambda})c, \qquad (5.5)$$

where $z-y=B(\overline{\lambda})\varphi(x,\overline{\lambda})$, $z\in D(\mathcal{L}^*)$ and $y\in D(\mathcal{L})$.

One gets

$$\left(z-y\right)(b)=\frac{2i\operatorname{Im}h_3\varphi(b,\overline{\lambda})}{\varphi^{[2]}(b,\overline{\lambda})+\overline{h_3}\varphi(b,\overline{\lambda})}y(b)$$

or

$$B(\overline{\lambda})\varphi(b,\overline{\lambda})=\frac{i2\sqrt{\operatorname{Im}h_3}}{\varphi^{[2]}(b,\overline{\lambda})+\overline{h_3}\varphi(b,\overline{\lambda})}\varphi(b,\overline{\lambda})c_1$$

and

$$\left(z-y\right)^{[1]}(a)=\left(\frac{1+ih_2}{i+h_2}\varphi^{[1]}(b,\overline{\lambda})-\varphi^{[1]}(a,\overline{\lambda})-1\right)y^{[1]}(a)$$

or

$$B(\overline{\lambda})\varphi^{[1]}(a,\overline{\lambda})=\left(\frac{1+ih_2}{i+h_2}\varphi^{[1]}(b,\overline{\lambda})-\varphi^{[1]}(a,\overline{\lambda})-1\right)\frac{\left|1+ih_2\right|}{\sqrt{2\operatorname{Im}h_2}}c_2.$$

Therefore the left-hand side of (5.5) can be introduced as

$$B(\overline{\lambda})\begin{bmatrix} \frac{i2\sqrt{\operatorname{Im}h_3}}{\varphi^{[2]}(b,\overline{\lambda})+\overline{h_3}\varphi(b,\overline{\lambda})} & 0 \\ 0 & \left(\frac{1+ih_2}{i+h_2}\varphi^{[1]}(b,\overline{\lambda})-\varphi^{[1]}(a,\overline{\lambda})-1\right)\frac{\left|1+ih_2\right|}{\sqrt{2\operatorname{Im}h_2}} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Now consider the equation

$$\left((\mathcal{L}-\lambda I)^{-1}\left(\mathcal{L}^*-\lambda I\right)-I\right)P_*^{-1}\Psi_*c=B(\lambda)\varphi(x,\lambda),$$

where $y-z=B(\lambda)\varphi(x,\lambda)$, $y\in D(\mathcal{L})$ and $z\in D(\mathcal{L}^*)$. A similar argument completes the proof. □

6. CONCLUSION AND REMARKS

This paper provides a new method to analyze the spectral properties of some third-order dissipative boundary value problems and it seems that such a method has not been introduced previously for third-order case. This method is very effective and can be applied for other odd-order dissipative operators generated by suitable odd-order differential equation and boundary conditions.

Finally we should note that the differential expression ℓ can also be handled as the following

$$\ell(y)=\frac{1}{w}\left\{-i\left[(ry')''+(ry'')'\right]-(p_0y')'+i\left[q_1y'+(q_1y)'\right]+p_1y\right\},$$

where r is a suitable function. Then with some modifications a similar boundary value problem as (2.2), (2.4) can be analyzed.

DATA AVAILABILITY

All datasets generated for this study are included in the manuscript and the supplementary files.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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