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Fractional Approach for Equation Describing the Water Transport in Unsaturated Porous Media With Mittag-Leffler Kernel

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In this paper, we find the solution for a fractional Richards equation describing the water transport in unsaturated porous media using the *q-homotopy analysis transform method* (*q*-HATM). The proposed technique is to use graceful amalgamations of the Laplace transform technique with the q-homotopy analysis scheme as well as the fractional derivative that is defined with the Atangana-Baleanu (AB) operator. The fixed point hypothesis is considered in order to demonstrate the existence and uniqueness of the obtained solution for the proposed fractional order model. In order to validate and illustrate the efficiency of the future technique, we analyze the projected model in terms of fractional order. Meanwhile, the physical behavior of the *q*-HATM solutions are captured in terms of plots for diverse fractional order and the numerical simulation is also demonstrated. The achieved results illuminate that the future algorithm is easy to implement, highly methodical, effective, and very accurate in its analysis of the behavior of non-linear differential equations of fractional order that arise in the connected areas of science and engineering.

Keywords: Laplace transform, Atangana-Baleanu derivative, Richards equation, *q*-homotopy analysis method, fixed point theorem

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INTRODUCTION

Fractional calculus (FC) was originated in Newton's time, but, lately, it has fascinated and captured the attention of many scholars. For the last 30 years, the most intriguing leaps in scientific and engineering applications have been found within the framework of FC. The concept of the fractional derivative has been industrialized due to the complexities associated with a heterogeneous phenomenon. The fractional differential operators are capable of capturing the behavior of multifaceted media as they have diffusion processes. It has been a very essential tool, and many problems can be illustrated more conveniently and more accurately with differential equations having an arbitrary order. Due to the swift development of mathematical techniques that use computer software, many researchers started to work on generalized calculus to present their viewpoints while analyzing many complex phenomena.

Numerous pioneering directions are prescribed for the diverse definitions of fractional calculus by many senior researchers, and these have prearranged the foundation [1–6]. Calculus with fractional order is associated with practical ventures and is extensively employed within nanotechnology [7], optics [8], human diseases [9], chaos theory [10], and other areas [11–39]. The numerical as well as analytical solutions for these equations illustrate that these models have an important role in portraying the nature of non-linear problems within connected areas of science.

In order to illustrate the importance of the novel fractional order derivative and future scheme, we, in the present framework, consider the Richards equation, which plays a vital role in describing the nature of the porous medium as well as the penetration of unsaturated regions in the soil. In 1931, *Lorenzo A. Richards* was the first person to pioneer work on the unsaturated porous material in order to model water movement. Later, he derived an equation based on continuum mechanics, which govern the water flow in the soil [40]. In the proposed model for the momentum equation, the continuity equation is an amalgam with Darcy's law, and is defined in a one-dimensional form as follows, with soil water diffusivity symbolized by ρ and hydraulic conductivity by σ for unsaturated soil moisture content u

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial z}\left(\rho \frac{\partial u}{\partial z} - \sigma\right), \tag{1}$$

where z designates the elevation above a vertical datum. Recently, many authors employed numerical as well as analytical techniques in order to analyze and predict the suitable models for parameters in the equation and solve the governing equation of unsaturated flow in soils. Meanwhile, three models are generally applied, namely (i) the exponential model, (ii) the van Genuchten model, and (iii) the Brook-Coreysmodel (BCM). Among these models, BCM is extensively applied due to its well-defined configuration and because it is associated with the largest pore size. The following equations describe the complete wet ability of the BC model [41, 42]:

$$\begin{aligned} \sigma(u) &= \sigma_0 u^k, \\ \rho(u) &= \rho_0 (n+1) u^n, \end{aligned} \tag{2}$$

where $\sigma_0, k, \ \rho_0$, and n are constants denoting particle shape, pore-size distribution and many other soil properties. For $n = 0$ and $k = 2$, Equation (2) simplified it to the classic Burgers equation [43, 44], and some particular values signify the generalized Burgers equation, which is essential to describing the important physical phenomena. In the present study, we consider that BCM employed the RC equation. In this case, for the (n, 1) order, the RC equation coincides with the Burgers equation, and this is presented here [45, 46]:

$$u_t + a\left(u^n\right)_x + bu_{xx} = 0, \ \ a, \ b \neq 1, \ n \geq \ 1. \tag{3}$$

The analytical solution for the above equation is presented:

$$u(x,t) = \left(\frac{c}{2a}\left(1 + \tanh\left(\frac{c(n-1)}{2b}(x - ct)\right)\right)\right)^{\frac{1}{n-1}}. \tag{4}$$

In the present scenario, many important and non-linear models are methodically and effectively analyzed with the help of fractional calculus. There have been diverse definitions that have been suggested by many senior research scholars like, Riemann, Liouville, Caputo, and Fabrizio. However, these definitions have their own limitations. The Riemann-Liouville derivative is unable to explain the importance of the initial conditions; the Caputo derivative has overcome this shortcoming but cannot explain the singular kernel of the phenomena. In 2015, Caputo and Fabrizio solved the above issues [47], and many researchers consult this derivative in order to analyze and find the solution for diverse classes of non-linear complex problems. Some issues, however, were pointed out in the CF derivative; non-singular kernel and non-local properties are very essential in describing the physical behavior and nature of the non-linear problems. In 2016, Atangana and Baleanu introduced and natured a novel fractional derivative, namely the AB derivative. This novel derivative was defined with the aid of Mittag-Leffler functions [48]. This fractional derivative buried all the above-cited issues and helps us to understand the natural phenomena in the systematic and effective way.

In this framework, we consider the fractional RC equation of the form

$${}^{ABC}_{\ \ a}D_t^{\alpha} u(x,t) + a\left(u^n\right)_x + bu_{xx} = 0, \tag{5}$$

where α is fractional order of the system and defined with AB fractional operator, u is the water content with depth x. The fractional order is introduced in order to incorporate the memory effects and hereditary consequence in the system, and these properties aid us in capturing the essential physical properties of the complex problems.

Recently, many mathematicians and physicists have developed very effective and more accurate methods in order to find and analyze solutions for complex and non-linear problems that have arisen in science and technology. In connection with this is the homotopy analysis method (HAM) proposed by Chinese Mathematician *Liao Shijun* [49, 50]. HAM has been profitably and effectively applied to study the behavior of non-linear problems without perturbation or linearization. But, for computational work, HAM requires significant time and computer memory. To overcome this, there is a possibility of using an amalgamation of the considered method and well-known transformation techniques.

In the present investigation, we analyzed the nature of the q-homotopy analysis transform method (q-HATM) solution for the FCDG equation by applying q-HATM. The future algorithm is the combination of q-HAM with LT [51]. The method of the considered scheme is merging two strong methods to solve linear and non-linear fractional differential equations both analytically as well as numerically. The future technique has many sturdy properties, including a non-local effect, straight forward solution procedure, and a promising large convergence region; moreover, it is free from any assumptions, discretization, and perturbation. Recently, due to its reliability and efficacy, the considered method has been exceptionally applied by many researchers to understand physical behavior in diverse classes of complex problems [52–60]. The novelty of the future method is that it aids a modest algorithm to evaluate the solution, and it is natured by the homotopy and axillary parameters, which provide the rapid convergence of the obtained solution for a non-linear portion of the given problem. Meanwhile, it has prodigious generality because it plausibly contains the results obtained by many algorithms like q-HAM, HPM, ADM and some other traditional techniques. The considered method can preserve great accuracy while decreasing the computational time and work in comparison with other methods.

The considered non-linear model recently caught the attention of researchers from different areas of science. Since RC equation plays a significant role in portraying several complex phenomena, many authors have found and analyzed the solution using analytical as well as numerical schemes; for instance, authors in [61] considered analytical techniques and found solutions for the considered model with arbitrary surface boundary conditions, and authors in [62] presented the compression approximation and infiltration of the RC equation with an analytical solution, authors in [45] applied the Adomian decomposition scheme, and authors in [46] applied HAM in order to find the approximated analytical solution. In this paper, we made an attempt to find the solution for the FRC equation using q-HATM.

PRELIMINARIES

Recently, many authors considered these derivatives to analyze a diverse class of models in comparison with classical order as well as other fractional derivatives, and they prove that the AB derivative is more effective while analyzing the nature and physical behavior of the models [63, 64]. Here, we define the basic notion of Atangana-Baleanu derivatives and integrals [48].

Definition 1. The fractional Atangana-Baleanu-Caputo derivative for a function $f \in H^1(a, b)\,(b > a,\ \alpha \in [0, 1])$ is presented:

$$ {}^{ABC}_{a} D^{\alpha}_{t}\left(f(t)\right) = \frac{\mathcal{B}[\alpha]}{1-\alpha}\int_a^t f'(\vartheta)\, E_\alpha\left[\alpha\frac{(t-\vartheta)^\alpha}{\alpha-1}\right] d\vartheta. \quad (6)$$

Definition 2. The AB derivative of fractional order for a function $f \in H^1(a, b),\ b > a,\ \alpha \in [0, 1]$ in the Riemann-Liouville sense is presented:

$$ {}^{ABR}_{a} D^{\alpha}_{t}\left(f(t)\right) = \frac{\mathcal{B}[\alpha]}{1-\alpha}\frac{d}{dt}\int_a^t f(\vartheta)\, E_\alpha\left[\alpha\frac{(t-\vartheta)^\alpha}{\alpha-1}\right] d\vartheta. \quad (7)$$

Definition 3. The fractional AB integral related to the non-local kernel is defined by

$$ {}^{AB}_{a} I^{\alpha}_{t}\left(f(t)\right) = \frac{1-\alpha}{\mathcal{B}[\alpha]} f(t) + \frac{\alpha}{\mathcal{B}[\alpha]\Gamma(\alpha)}\int_a^t f(\vartheta)(t-\vartheta)^{\alpha-1} d\vartheta. \quad (8)$$

Definition 4. The Laplace transform (LT) of AB derivative is defined by

$$ L\left[{}^{ABR}_{0} D^{\alpha}_{t}\left(f(t)\right)\right] = \frac{\mathcal{B}[\alpha]}{1-\alpha}\frac{s^\alpha L\left[f(t)\right] - s^{\alpha-1}f(0)}{s^\alpha + \left(\alpha/(1-\alpha)\right)}. \quad (9)$$

Theorem 1. The following Lipschitz conditions, respectively, hold true for both Riemann-Liouville and AB derivatives defined in Equations (6) and (7) [48],

$$ \left\| {}^{ABC}_{a}D^{\alpha}_{t} f_1(t) - {}^{ABC}_{a} D^{\alpha}_{t} f_2(t)\right\| < K_1 \left\| f_2(x) - f_2(x)\ \right\|, \quad (10)$$

and

$$ \left\| {}^{ABC}_{a}D^{\alpha}_{t} f_1(t) - {}^{ABC}_{a} D^{\alpha}_{t} f_2(t)\right\| < K_2 \left\| f_1(x) - f_2(x)\ \right\|. \quad (11)$$

Theorem 2. The time-fractional differential equation ${}^{ABC}_{a}D^{\alpha}_{t} f_1(t) = s(t)$ has a unique solution, which is defined as [48]

$$ f(t) = 1 - \frac{\alpha}{\mathcal{B}[\alpha]} s(t) + \frac{\mu}{\mathcal{B}[\alpha]\Gamma(\alpha)}\int_a^t s(\varsigma)(t-\varsigma)^{\alpha-1} d\varsigma. \quad (12)$$

FUNDAMENTAL IDEA OF THE PROPOSED SCHEME

Here, we consider the arbitrary order differential equation in order to demonstrate the basic solution procedure [65, 66]

$$ {}^{ABC}_{a}D^{\alpha}_{t} v(x,t) + \mathcal{R}\, v(x,t) + \mathcal{N} v(x,t) = f(x,t), \quad n-1 < \alpha \le n, \quad (13)$$

with the initial condition

$$ v(x,0) = g(x), \quad (14)$$

where ${}^{ABC}_{a}D^{\alpha}_{t} v(x,t)$ symbolize the AB derivative of $v(x,t)$. On using the LT on Equation (13), we have after simplification

$$ \mathcal{L}[v(x,t)] - \frac{g(x)}{s} + \frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)\{\mathcal{L}[\mathcal{R}v(x,t)] + \mathcal{L}[\mathcal{N}v(x,t)] - \mathcal{L}\left[f(x,t)\right]\} = 0. \quad (15)$$

The non-linear operator is presented as

$$ \mathcal{N}\left[\varphi(x,t;q)\right] = \mathcal{L}\left[\varphi(x,t;q)\right] - \frac{g(x)}{s} + \frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)\left\{\mathcal{L}\left[\mathcal{R}\,\varphi(x,t;q)\right] + L\left[\mathcal{N}\varphi(x,t;q)\right] - L\left[f(x,t)\right]\right\}. \quad (16)$$

Here, $\varphi(x,t;q)$ is the real valued function with respect to x, t and $\left(q \in \left[0, \frac{1}{n}\right]\right)$. Now, we define a homotopy as follows

$$ \left(1-nq\right)\mathcal{L}\left[\varphi(x,\ t;q) - v_0(x,t)\right] = \hbar q \mathcal{N}\left[\varphi(x,\ t;q)\right], \quad (17)$$

where L is signifies LT, $q \in \left[0, \frac{1}{n}\right]$ $(n \ge 1)$ is the embedding parameter and $\hbar \neq 0$ is an auxiliary parameter. For $q = 0$ and $q = \frac{1}{n}$, the results given below are hold true

$$ \varphi(x,t;0) = v_0(x,t), \quad \varphi\left(x,t;\frac{1}{n}\right) = v(x,t). \quad (18)$$

Now, by intensifying q from 0 to $\frac{1}{n}$, then $\varphi(x,t;q)$ varies from $v_0(x,t)$ to $v(x,t)$. By using the Taylor theorem near to q, we define $\varphi(x,t;q)$ in series form and then we get

$$ \varphi(x,t;q) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t)\, q^m, \quad (19)$$

where

$$v_m(x,t) = \frac{1}{m!}\frac{\partial^m \varphi(x,\ t;q)}{\partial q^m}|_{q=0}. \quad (20)$$

The series (16) converges at $q = \frac{1}{n}$ for the proper chaise of $v_0(x,t)$, n and $\hbar$. Then

$$v(x,t) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t)\left(\frac{1}{n}\right)^m. \quad (21)$$

On m-times differentiating Equation (17) with q and lately dividing by $m!$ and then substituting $q = 0$, we get

$$\mathcal{L}[v_m(x,t) - k_m v_{m-1}(x,t)] = \hbar\Re_m\left(\vec{v}_{m-1}\right), \quad (22)$$

where the vectors are defined as

$$\vec{v}_m = \{v_0(x,t), v_1(x,t), \ldots, v_m(x,t)\}. \quad (23)$$

On employing the inverse LT on Equation (22), we have

$$v_m(x,t) = k_m v_{m-1}(x,t) + \hbar\mathcal{L}^{-1}\left[\Re_m\left(\vec{v}_{m-1}\right),\right. \quad (24)$$

where

$$\Re_m\left(\vec{v}_{m-1}\right) = L[v_{m-1}(x,t)] - \left(1 - \frac{k_m}{n}\right)\left(\frac{g(x)}{s} + \frac{1}{\mathcal{B}[\alpha]}\left(1 - \alpha + \frac{\alpha}{s^\alpha}\right)L\left[f(x,t)\right]\right) + \frac{1}{\mathcal{B}[\alpha]}\left(1 - \alpha + \frac{\alpha}{s^\alpha}\right)L\left[Rv_{m-1} + \mathcal{H}_{m-1}\right], \quad (25)$$

and

$$k_m = \begin{cases} 0, & m \le 1, \\ n, & m > 1. \end{cases} \quad (26)$$

In Equation (25), $\mathcal{H}_m$ signifies a homotopy polynomial and presented as follows

$$\mathcal{H}_m = \frac{1}{m!}\left[\frac{\partial^m \varphi\left(x,\ t;q\right)}{\partial q^m}\right]_{q=0} \text{ and } \varphi\left(x,\ t;q\right) = \varphi_0 + q\varphi_1 + q^2\varphi_2 + \ldots. \quad (27)$$

By the aid of Equations (24) and (25), one can get

$$v_m(x,t) = (k_m + \hbar)v_{m-1}(x,t) - \left(1 - \frac{k_m}{n}\right)\mathcal{L}^{-1}\left(\frac{g(x)}{s} + \frac{1}{\mathcal{B}[\alpha]}\left(1 - \alpha + \frac{\alpha}{s^\alpha}\right)L\left[f(x,t)\right]\right) + \hbar\mathcal{L}^{-1}\left\{\frac{1}{\mathcal{B}[\alpha]}\left(1 - \alpha + \frac{\alpha}{s^\alpha}\right)L\left[Rv_{m-1} + \mathcal{H}_{m-1}\right]\right\}. \quad (28)$$

Then, the terms of $v_m(x,t)$ we can obtain using the Equation (28). The q-HATM series solution is presented as

$$v(x,t) = \sum_{m=0}^{\infty} v_m(x,t). \quad (29)$$

SOLUTION FOR FRC EQUATION

In order to present the solution procedure and efficiency of the future scheme, in this segment we consider the DSW equation of fractional order with two distinct cases. Further, by the help of obtained results we made an attempt to capture the behavior of q-HATM solution for different fractional order. By the help of Equation (5) for the function of cubic water content and constant, we have

$${}^{ABC}_{a}D_t^\alpha u(x,t) + u^2 u_x - u_{xx} = 0, \qquad 0 < \alpha \le 1, \quad (30)$$

with initial conditions

$$u(x,0) = u_0(x,t). \quad (31)$$

Taking LT on Equation (29) and then using Equation (30), we get

$$L[u(x,t)] = \frac{1}{s}(u_0(x,t)) + \frac{1}{\mathcal{B}[\alpha]}\left(1 - \alpha + \frac{\alpha}{s^\alpha}\right)L\left\{u^2 u_x - u_{xx}\right\}. \quad (32)$$

The non-linear operator N is presented with the help of future algorithm as below

$$N\left[\varphi(x,t;q)\right] = L\left[\varphi(x,t;q)\right] - \frac{1}{s}(u_0(x,t)) + \frac{1}{\mathcal{B}[\alpha]}\left(1 - \alpha + \frac{\alpha}{s^\alpha}\right) L\left\{\varphi(x,t;q)\frac{\partial\varphi}{\partial x}(x,t;q) - \varphi(x,t;q)\right\}. \quad (33)$$

The deformation equation of m-th order by the help of q-HATM at $\mathcal{H}(x,t) = 1$, is given as follows

$$L[u_m(x,t) - k_m u_{m-1}(x,t)] = \hbar\Re_m\left[\vec{u}_{m-1}\right], \quad (34)$$

where

$$\Re_m\left[\vec{u}_{m-1}\right] = L[u_{m-1}(x,t)] - \left(1 - \frac{k_m}{n}\right)\left\{\frac{1}{s}(u_0(x,t))\right\} + \frac{1}{\mathcal{B}[\alpha]}\left(1 - \alpha + \frac{\alpha}{s^\alpha}\right) L\{\sum_{j=0}^{i}\sum_{i=0}^{m-1} u_j u_{i-j}\frac{\partial u_{m-1-i}}{\partial x} - \frac{\partial^2 u_{m-1}}{\partial x^2}\}. \quad (35)$$

On applying inverse LT on Equation (34), it reduces to

$$u_m(x,t) = k_m u_{m-1}(x,\ t) + \hbar L^{-1}\left\{\Re_m\left[\vec{u}_{m-1}\right]\right\}. \quad (36)$$

On simplifying the above equation systematically by using $u_0(x, t)$, we can evaluate the terms of the series solution

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m. \quad (37)$$

EXISTENCE OF SOLUTIONS FOR THE FUTURE MODEL

Here, we considered the fixed-point theorem in order to demonstrate the existence of the solution for the proposed model. Since the considered model cited in Equation (30) is non-local as well as complex, there are no particular algorithms or methods that exist to evaluate the exact solutions. However, under some particular conditions, the existence of the solution is assured. Now, Equation (30) is considered:

$$ {}_0^{ABC}D_t^{\alpha}\left[u(x,t)\right] = \mathcal{G}(x,t,\ u). \tag{38}$$

The foregoing system is transformed to the Volterra integral equation using the Theorem 2 as follows

$$\begin{aligned} u(x,t) - u(x,0) = {} & \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\mathcal{g}(x,t,u) \\ & + \frac{\alpha}{\mathcal{B}(\alpha)\,\Gamma(\alpha)}\int_0^t \mathcal{g}(x,\zeta,u)\,(t-\zeta)^{\alpha-1}\,d\zeta. \end{aligned} \tag{39}$$

Theorem 3. The kernel $\mathcal{g}$ satisfies the Lipschitz condition and contraction if the condition $0 \le \left(\delta\left(a^2+b^2+ab\right)-\delta^2\right) < 1$ holds.

Proof. In order to prove the required result, we consider the two functions u and u_1, then

$$\begin{aligned} \left\|\mathcal{g}(x,t,\ u) - \mathcal{g}(x,t,\ u_1)\right\| = {} & \left\|\left[u^2(x,t)\frac{\partial u(x,t)}{\partial x} - u^2(x,t_1)\frac{\partial u(x,t_1)}{\partial x}\right]\right. \\ & \left. - \left[\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x,t_1)}{\partial x^2}\right]\right\| \\ = {} & \left\|\left[\frac{1}{3}\frac{\partial}{\partial x}\left(u^3(x,t) - u^3(x,t_1)\right)\right]\right. \\ & \left. - \left[\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x,t_1)}{\partial x^2}\right]\right\| \\ \le {} & \left\|\delta\left(a^2+b^2+ab\right) - \delta^2\right\| \\ & \left\|u(x,t) - u(x,t_1)\right\| \\ \le {} & \left(\delta\left(a^2+b^2+ab\right) - \delta^2\right) \\ & \left\|u(x,t) - u(x,t_1)\right\|, \end{aligned}$$

where $a = \|u\|$ and $b = \|u_1\|$ (since u and u_1 are the bounded functions). Putting $\eta = \delta\left(a^2+b^2+ab\right) - \delta^2$ in the above inequality, then we have

$$\left\|\mathcal{g}(x,t,\ u) - \mathcal{g}(x,t,\ u_1)\right\| \le \eta\left\|u(x,t) - u(x,t_1)\right\|. \tag{40}$$

The Lipschitz condition is thus obtained for $\mathcal{G}$. Further, we can see that if $0 \le \left(\delta\left(a^2+b^2+ab\right)-\delta^2\right) < 1$, then it implies the contraction. The recursive form of Equation (36) is defined as

$$\begin{aligned} u_n(x,t) = {} & \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\mathcal{g}(x,t,u_{n-1}) \\ & + \frac{\alpha}{\mathcal{B}(\alpha)\,\Gamma(\alpha)}\int_0^t \mathcal{g}(x,\zeta,u_{n-1})\,(t-\zeta)^{\alpha-1}\,d\zeta. \end{aligned} \tag{41}$$

The associated initial condition is

$$u(x,0) = u_0(x,t). \tag{42}$$

The successive difference between the terms is presented as

$$\begin{aligned} \phi_n(x,t) = {} & u_n(x,t) - u_{n-1}(x,t) \\ = {} & \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\left(\mathcal{g}_1(x,t,u_{n-1}) - \mathcal{g}(x,t,u_{n-2})\right) \\ & + \frac{\alpha}{\mathcal{B}(\alpha)\,\Gamma(\alpha)}\int_0^t \mathcal{g}(x,\zeta,u_{n-1})\,(t-\zeta)^{\alpha-1}\,d\zeta \end{aligned} \tag{43}$$

Notice that

TABLE 1 | Numerical simulation presented for $u(x,\ t)$ of FR equation consider in Case 1 at $n=1,\ \hbar=-1$ and $\alpha=1$.

x	t	$\left\|u_{Exact} - u_{q-HATM}^{(3)}\right\|$	$\left\|u_{Exact} - u_{q-HATM}^{(4)}\right\|$	$\left\|u_{Exact} - u_{q-HATM}^{(5)}\right\|$
2.5	0.25	6.50636×10^{-7}	1.84782×10^{-8}	1.01181×10^{-10}
	0.50	5.35446×10^{-6}	2.97193×10^{-7}	3.15969×10^{-9}
	0.75	1.85802×10^{-5}	1.51192×10^{-6}	2.33755×10^{-8}
	1	4.52584×10^{-5}	4.80032×10^{-6}	9.57906×10^{-8}
5	0.25	3.86055×10^{-7}	8.87727×10^{-10}	5.42849×10^{-11}
	0.50	3.08044×10^{-6}	1.50958×10^{-8}	1.76075×10^{-9}
	0.75	1.03664×10^{-5}	8.10613×10^{-8}	1.35526×10^{-8}
	1	2.44931×10^{-5}	2.71248×10^{-7}	5.78869×10^{-8}
7.5	0.25	2.68674×10^{-7}	1.50440×10^{-9}	2.57157×10^{-12}
	0.50	2.16147×10^{-6}	2.41095×10^{-8}	8.01731×10^{-11}
	0.75	7.33582×10^{-6}	1.22240×10^{-7}	5.92186×10^{-10}
	1	1.74857×10^{-5}	3.86892×10^{-7}	2.42301×10^{-9}
10	0.25	1.26101×10^{-7}	8.43873×10^{-10}	4.16169×10^{-12}
	0.50	1.01562×10^{-6}	1.35692×10^{-8}	1.33829×10^{-10}
	0.75	3.45096×10^{-6}	6.90365×10^{-8}	1.01990×10^{-9}
	1	8.23569×10^{-6}	2.19278×10^{-7}	4.31168×10^{-9}

TABLE 2 | Numerical simulation presented for $u(x,\ t)$ of FR equation consider in Case 2 at $n=1,\ \hbar=-1$ and $\alpha=1$.

x	t	$\left\|u_{Exact} - u_{q-HATM}^{(3)}\right\|$	$\left\|u_{Exact} - u_{q-HATM}^{(4)}\right\|$	$\left\|u_{Exact} - u_{q-HATM}^{(5)}\right\|$
2.5	0.25	3.32962×10^{-7}	5.38964×10^{-9}	1.86945×10^{-11}
	0.50	2.70710×10^{-6}	8.65185×10^{-8}	5.83456×10^{-10}
	0.75	9.28381×10^{-6}	4.39361×10^{-7}	4.31477×10^{-9}
	1	2.23573×10^{-5}	1.39264×10^{-6}	1.76784×10^{-8}
5	0.25	9.75190×10^{-8}	5.33186×10^{-10}	1.16072×10^{-11}
	0.50	7.75698×10^{-7}	8.72022×10^{-9}	3.74948×10^{-10}
	0.75	2.60229×10^{-6}	4.51222×10^{-8}	2.87429×10^{-9}
	1	6.12959×10^{-6}	1.45752×10^{-7}	1.22274×10^{-8}
7.5	0.25	9.03607×10^{-8}	2.66049×10^{-10}	2.09943×10^{-13}
	0.50	7.25010×10^{-7}	4.25260×10^{-9}	7.54358×10^{-12}
	0.75	2.45406×10^{-6}	2.15062×10^{-8}	6.08009×10^{-11}
	1	5.83395×10^{-6}	6.78929×10^{-8}	2.69468×10^{-10}
10	0.25	5.17685×10^{-8}	1.88526×10^{-10}	2.13869×10^{-10}
	0.50	4.15714×10^{-7}	3.07483×10^{-9}	3.36348×10^{-9}
	0.75	1.40835×10^{-6}	1.56930×10^{-8}	1.69010×10^{-8}
	1	3.35098×10^{-6}	4.98643×10^{-8}	5.31486×10^{-8}

$$u_n(x,t)=\sum_{i=1}^{n}\phi_i(x,t). \tag{44}$$

By using Equation (39) after applying the norm on the Equation (43), one can get

$$\|\phi_n(x,t)\| \le \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta\left\|\phi_{(n-1)}(x,t)\right\| + \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta\int_0^t\left\|\phi_{(n-1)}(x,\zeta)\ \right\|d\zeta. \tag{45}$$

We prove the following theorem by using the above result.

Theorem 4. The solution for the Equation (30) will exist, and if we have specific t_0, then

$$\frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta+\frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta<1.$$

Proof. Let us consider the bounded function $u(x,t)$ satisfying the Lipschitz condition. Then, by Equation (43), we have

$$\|\phi_i(x,t)\| \le \|u_n(x,0)\|\left[\frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta+\frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta\right]^n. \tag{46}$$

Therefore, the continuity as well as existence of the obtained solution is proved. Subsequently, in order to show the Equation (46) is a solution for the Equation (29), we consider

$$u(x,t)-u(x,0)=u_n(x,t)-\mathcal{K}_n(x,t). \tag{47}$$

In order to obtain require a result, we consider

$$\begin{aligned}\|\mathcal{K}_n(x,t)\| &= \|\frac{(1-\alpha)}{\mathcal{B}(\alpha)}\left(g(x,t,u)-\mathcal{G}\Leftarrow x,t,\ u_{n-1})\right)\\ &\quad+\frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^t(t-\zeta)^{\mu-1}\left(g(x,\zeta,u)-g(x,\zeta,u_{n-1})\right)d\zeta\|\\ &\le \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\|g(x,\zeta,u)-g(x,\zeta,u_{n-1})\|\\ &\quad+\frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^t\|g(x,\zeta,u)-g(x,\zeta,u_{n-1})\ \|d\zeta\\ &\le \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta_1\|u-u_{n-1}\|+\frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta_1\|u-u_{n-1}\|\,t. \end{aligned} \tag{48}$$

Similarly, at t_0 we can obtain

$$\|\mathcal{K}_n(x,t)\ \| \le \left(\frac{(1-\alpha)}{\mathcal{B}(\alpha)}+\frac{\alpha t_0}{\mathcal{B}(\alpha)\Gamma(\alpha)}\right)^{n+1}\eta^{n+1}M. \tag{49}$$

As n approaches to ∞, we can see that form Equation (49), $\|\mathcal{K}_n(x,t)\ \|$ tends to 0.

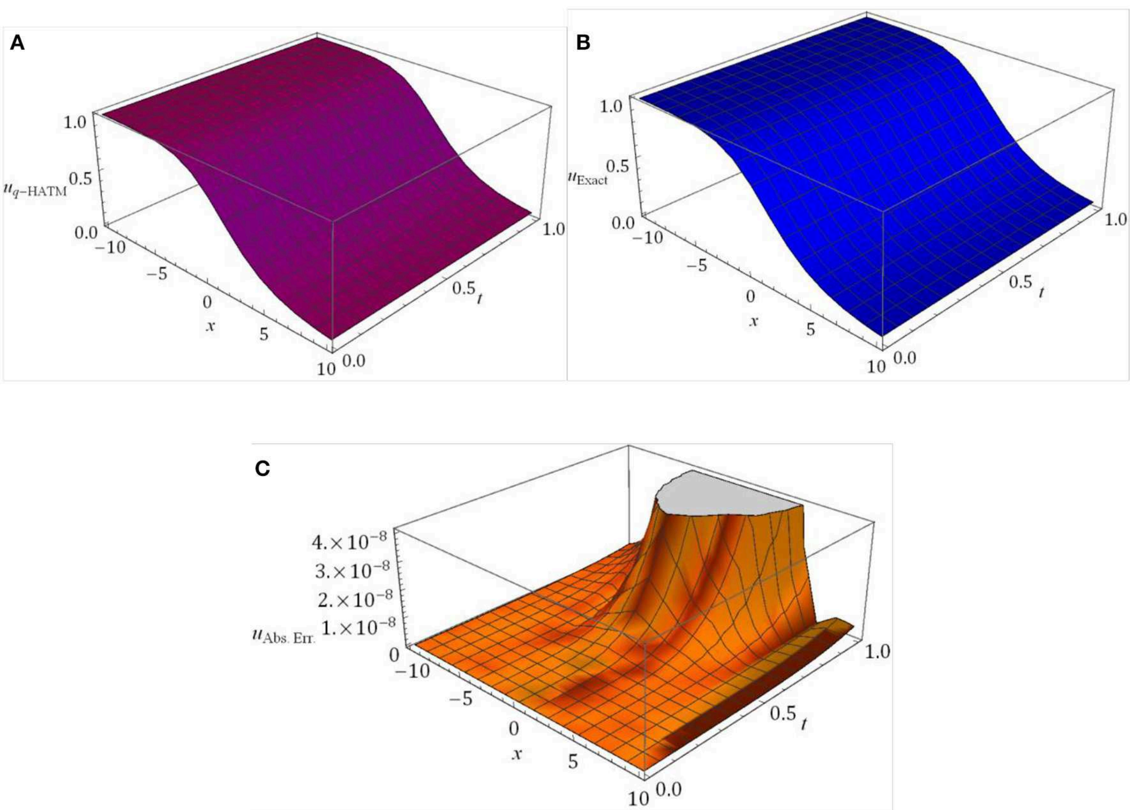

FIGURE 1 | Surfaces of **(A)** u_{q-HATM}, **(B)** u_{Exact} **(C)** $u_{Abs.\ Err.}=\left|u_{Exact}-u_{q-HATM}\right|$ for FR equation considered in Case 1 at $\hbar=-1$, $n=1$ and $\alpha=1$.

Next, it is a necessity to demonstrate uniqueness for the solution of the considered model. Suppose $u^*(x,t)$ is the other solution, then we have

$$u(x,t) - u^*(x,t) = \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\left(g(x,t,u) - g\left(x,t,\ u^*\right)\right) + \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^t \left(g(x,\zeta,u) - g\left(x,\zeta,u^*\right)\right)d\zeta. \tag{50}$$

On applying norm, the Equation (50) simplifies to

$$\begin{aligned}\left\|u(x,t) - u^*(x,t)\right\| &= \left\|\frac{(1-\alpha)}{\mathcal{B}(\alpha)}\left(g(x,t,u) - g\left(x,,t,\ u^*\right)\right) + \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^t \left(g(x,\zeta,u) - g\left(x,\zeta,u^*\right)\right)d\zeta\right\| \\ &\leq \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta\left\|u(x,t) - u^*(x,t)\right\| + \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta t\left\|u(x,t) - u^*(x,t)\ \right\|.\end{aligned} \tag{51}$$

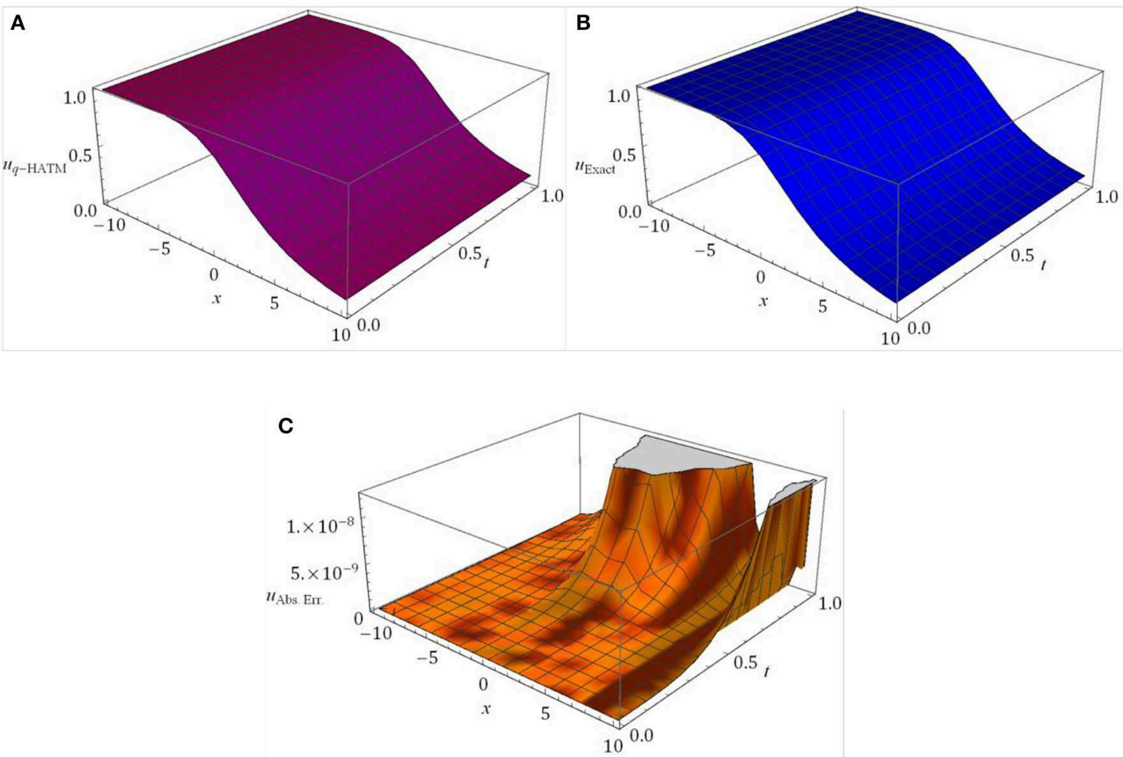

FIGURE 2 | Surfaces of **(A)** u_{q-HATM}, **(B)** u_{Exact} **(C)** $u_{Abs.\ Err.} = \left|u_{Exact} - u_{q-HATM}\right|$ for FR equation considered in Case 2 at $\hbar = -1,\ n = 1$ and $\alpha = 1$.

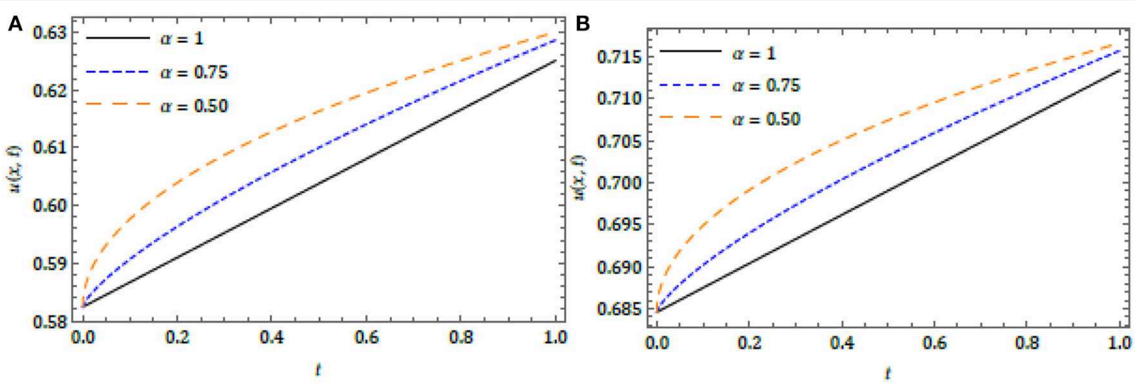

FIGURE 3 | Nature of the$q$-HATM solution for **(A)** Case 1 and **(B)** Case 2 with distinct α at $\hbar = -1,\ n = 1$ and $x = 1$.

On simplification

$$\left\|u(x,t)-u^{*}(x,t)\right\|\left(1-\frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta-\frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta t\right)\leq 0. \tag{52}$$

From the above condition, it is clear that $u(x,t)=u^{*}(x,t)$, if

$$\left(1-\frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta-\frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta t\right)\geq 0. \tag{53}$$

Hence, Equation (53) proves our essential result.

Theorem 5. Suppose $u_n(x,t)$ and $u(x,t)$ are defined in the Banach space $(\mathfrak{B}[0,T],\|\cdot\|)$. The series solution defined in Equation (29) converges to the solution of the Equation (13), if $0<\lambda<1$.

Proof: Consider the sequence $\{\mathcal{S}_n\}$, which is the partial sum of the Equation (29), and we have to prove $\{\mathcal{S}_n\}$ is the Cauchy sequence in $(\mathfrak{B}[0,T],\|\cdot\|)$. Now consider

$$\begin{aligned}&\|\mathcal{S}_{n+1}(x,t)-\mathcal{S}_n(x,t)\|=\|u_{n+1}(x,t)\|\\&\leq\lambda\|u_n(x,t)\|\\&\leq\lambda^2\|u_{n-1}(x,t)\|\leq\ldots\leq\lambda^{n+1}\|u_0(x,t)\|.\end{aligned}$$

Now, we have for every $n,\ m\in N\ (m\leq n)$

$$\begin{aligned}\|\mathcal{S}_n-\mathcal{S}_m\|&=\|(\mathcal{S}_n-\mathcal{S}_{n-1})+(\mathcal{S}_{n-1}-\mathcal{S}_{n-2})\\&+\ldots+(\mathcal{S}_{m+1}-\mathcal{S}_m)\|\\&\leq\|\mathcal{S}_n-\mathcal{S}_{n-1}\|+\|\mathcal{S}_{n-1}-\mathcal{S}_{n-2}\|+\ldots+\|\mathcal{S}_{m+1}-\mathcal{S}_m\|\\&\leq\left(\lambda^n+\lambda^{n-1}+\ldots+\lambda^{m+1}\right)\|u_0\|\\&\leq\lambda^{m+1}\left(\lambda^{n-m-1}+\lambda^{n-m-2}+\ldots+\lambda+1\right)\|u_0\|\\&\leq\lambda^{m+1}\left(\frac{1-\lambda^{n-m}}{1-\lambda}\right)\|u_0\|.\end{aligned} \tag{54}$$

But $0<\lambda<1$, therefore $\|\mathcal{S}_n-\mathcal{S}_m\|=0$. Hence, $\{\mathcal{S}_n\}$ is the Cauchy sequence. This proves the required result.

NUMERICAL RESULTS AND DISCUSSION

In the present investigation, we have found the solution for equation describing the water transport in unsaturated porous media using q-HATM with the help of Mittag-Leffler law. Here, we consider two distinct cases to present the effectiveness of the proposed method.

Case 1: In this case, we consider the conductivity term as a function of cubic water content and constant $\sigma=\frac{u^3}{3}cm/h$ and $\rho=1\ cm^{2/h}$. At $a=c=\frac{1}{3}$, $n=3$ and $b=-1$, Equation (4) becomes

$$^{ABC}_{\ \ a}D_t^{a}u(x,t)+u^2u_x-u_{xx}=0,\qquad 0<\alpha\leq 1, \tag{55}$$

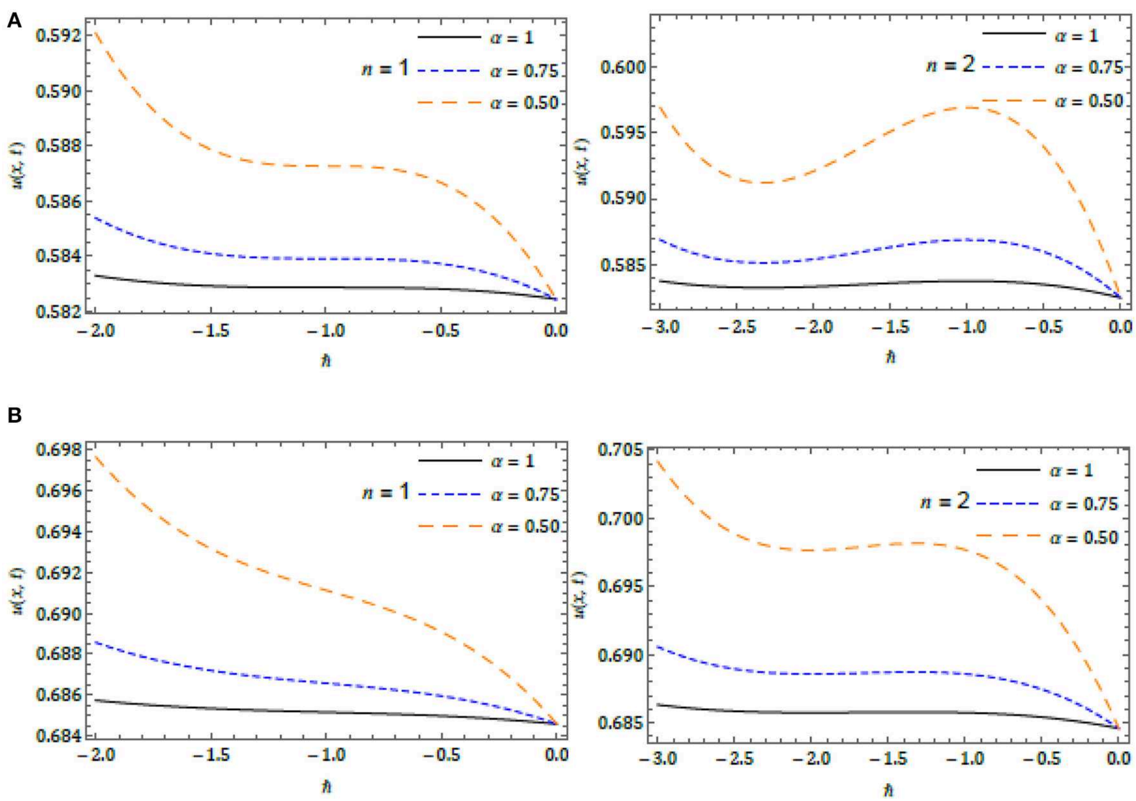

FIGURE 4 | $\hbar$-curves for **(A)** Case 1 **(B)** Case 2 with distinct α at $x=1$ and $t=0.01$ with $n=1$ and 2.

with initial condition

$$u(x,0) = \sqrt{\frac{1}{2}\left(1+\tanh\left(-\frac{x}{3}\right)\right)}. \qquad (56)$$

Case 2: In this segment, we consider the conductivity term as a function of quadric water content and constant $\sigma = \frac{u^4}{4} cm/h$ and $\rho = 1\ cm^{2/h}$. At $a = c = \frac{1}{1}$, $n = 1$ and $b = -1$, the Equation (4) becomes

$${}_{a}^{ABC}D_t^{\alpha} u(x,t) + u^3 u_x - u_{xx} = 0, \qquad 0 < \alpha \leq 1, \qquad (57)$$

with initial condition

$$u(x,0) = \sqrt[3]{\frac{1}{2}\left(1+\tanh\left(-\frac{3x}{8}\right)\right)}. \qquad (58)$$

Here, we demonstrate the numerical simulation for the considered non-linear. In **Tables 1** and **2**, the error analysis has been validated. From the tables we can see that the proposed scheme is more accurate, and we confirm that the iterations increase the q-HATM solutions so that they get closer to the analytical solution.

The surfaces of the obtained solution and the exact solution in comparison with absolute error have been captured, respectively, in **Figures 1** and **2** for Case 1 and Case 2. The behavior of the obtained solution for different orders is presented in **Figure 3** for both the cases in terms of 2D plots. In order to analyze the variations of the obtained solution for the FRC equation cited in Case 1 and Case 2 with respect to the homotopy parameter ($\hbar$), and the ($\hbar$) curves are drawn for diverse μ and presented in **Figure 4** with distinct n. In the plots, the horizontal line signifies the convergence region of the q-HATM solution and these curves aid us to adjust and handle the convergence province of the solution. For an appropriate value of $\hbar$, the achieved solution quickly tends to the exact solution. The small deviation in the physical behavior of the complex models stimulates the enormous new results to analyze and understand the nature in a better and systematic manner. Moreover, from all the plots we can see that the proposed method is more accurate and very effective in its analysis of the considered non-linear fractional order equations.

Since every non-linear differential equation does not have an exact solution we look for an approximated analytical solution thorugh which we can prove the exactness or accuracy of the proposed scheme, as opposed to an exact solution. As we mentioned earlier, the q-HATM is a modified algorithm of HAM, and it thus does not require perturbation, dissertation, linearization, or any assumptions. More importantly, the future method generalizes many traditional techniques, such as HAM, HPM, FRDTM, and others, because these are a special case of q-HATM ($n = 1$, $\hbar = 1$). In connection with this, we capture the physical behavior of q-HATM solution to illustrate the accuracy. Further, we noticed that the considered non-linear phenomenon is highly dependent on a fractional operator. In order to illustrate the computational level and computational cost, the numerical simulation has been presented. From the table, it shows that as a number of series terms increases the solution converges to an analytical solution.

CONCLUSION

In this paper, the q-HATM is applied profitably to find the solution for an arbitrary order RC equation describing the water transport in the unsaturated porous media. Since AB derivatives and integrals having fractional order are defined with the help of generalized Mittag-Leffler function as the non-local kernel and non-singular, the present investigation illuminates the effectiveness of the considered derivative. The existence and uniqueness of the obtained solution is demonstrated by the fixed point hypothesis. The results obtained by the future scheme are more stimulating as compared to results available in the literature. Further, the proposed algorithm finds the solution of the non-linear problem without considering any discretization, perturbation or transformations.

The behavior of the obtained series solution has been captured in terms of 2D and 3D plots for distinct fractional order. These plots show that the q-HATM solution is more accurate and also conformed with the help of numerical simulation, and this is cited in the tables. Further, we confirm that, as the order of the solution increases, the obtained solutions converge to the exact solution. The present investigation illuminates how the considered complex non-linear phenomena noticeably depend on the time history and the time instant, which can be proficiently analyzed by applying the concept of calculus to fractional order. The present investigation helps the researchers to study the behavior of non-linear problems, and this gives very interesting and useful consequences. The proposed derivative provides non-singular kernel and non-local properties; these properties are very essential in describing the physical behavior and nature of the non-linear problems, and hence researchers can consider the AB derivative to solve many non-linear complex problems. Lastly, we can conclude the projected method is extremely methodical, effective and very accurate, and that it can be applied to the analysis of the diverse classes of non-linear problems that exist in science and technology.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

All the authors have worked equally in this manuscript. All the authors have read and approved the final manuscript.

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