The Propagators for δ and δ' Potentials With Time-Dependent Strengths

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We study the time-dependent Schrödinger equation with finite number of Dirac δ and δ' potentials with time dependent strengths in one dimension. We obtain the formal solution for generic time dependent strengths and then we study the particular cases for single delta potential and limiting cases for finitely many delta potentials. Finally, we investigate the solution of time dependent Schrödinger equation for δ' potential with particular forms of the strengths.

Keywords: propagator, delta potentials, delta prime potentials, Green's function, time dependent Schrödinger equation

1. INTRODUCTION

Dirac delta potentials in quantum mechanics have been used to model different physical systems almost since the beginning of quantum mechanics. Kronig Penney model [1] is the well-known example of these models. These potentials are the particular cases of much more general class of potentials, namely point interactions. In one dimension, one rigorous way of defining the point interaction at the origin is based on the self-adjoint extension of the free symmetric Hamiltonian defined on $\mathbb{R} \setminus \{0\}$. In this approach, the initially ill-defined formal δ and δ' function potentials appear naturally as two special cases of point interactions constructed from the self-adjoint extension theory. In general one has a 4 -parameter family of self-adjoint extension in one dimension. The monograph [2] includes a great deal of all the details and summarize the history of the literature about the δ interactions. The review article [3] and the book [4] are also good reference sources about the δ potentials from the physical point of view.

The δ' perturbation of free Hamiltonian $H_0 = -\frac{d^2}{dx^2}$ is defined as a limit of short range potentials in the distributional sense [5–7]. Although there are some controversial issues about δ' interactions (see e.g., [8–11]), they are also getting considerable amount of interest. The ambiguities about δ' interactions have been summarized in a very recent article [11], where the integral form of the Schrödinger equation for δ' potential has been studied based on the work of Kurasov [12]. We also adopt the distributional approach developed by Kurasov [12] for the functions having a discontinuity at the point of δ'. It is possible to overcome these ambiguities by considering different choices, as different type of δ' interactions [13]. Therefore, the different results on the spectrum of δ' potential obtained in [2] and in [8] using the Kurasov's approach -as a special case of $-a\delta(x)+b\delta'(x)$ potential- can be interpreted consistently. In other words, the Kurasov's approach corresponds to different self-adjoint extension of the free Hamiltonian H_0. These self-adjoint extensions are given by matching conditions at the origin (or at the point supporting the perturbation) and two of these matching conditions maybe identified as a δ' interaction and receive the names of non-local

and local δ' interaction, respectively [13]. Both approaches are used to see how the spectrum of the V-shaped potential changes when it is perturbed by $-a\delta(x) + b\delta'(x)$ perturbation in [9, 13]. The scattering, and resonant tunneling for δ' potential in one dimension is also a controversial issue because different results are obtained in the literature [8, 10], depending on whether the δ' interaction is the non-local or the local one. The results obtained in [14–16] for the non-local δ' potential, show that it is opaque for all energies of an incoming beam. However, other authors [17–20] claim that there are discrete energy values in the spectrum of δ' potential which lead to resonant tunneling. As an another physical application, δ' interactions are used to model Casimir effect in [21, 22].

The exact expression of the propagator for one dimensional single Dirac delta potentials have been found in different ways [23–25]. The generalization to two center case have been studied in Cacciari and Moretti [26]. The propagator for general four parameter family of point interactions have been given in Albeverio et al. [27]. Propagators for systems involving δ potentials are also studied from various points of view in references [28–34]. The propagator for derivatives of Dirac delta distribution for constant strengths has been recently studied in Lange [11]. The Cauchy problem for the non-local δ' potential with a time dependent strength has also been studied rigorously in detail [35]. Moreover, time dependent one dimensional point interactions have been studied in Campbell [36] and the exact solution to the initial value problem for Schrödinger equation has been given for some particular form of strength $\lambda(t)$ of the Dirac delta potential. Later on, the system has been investigated in Hmidi et al. [37] more rigorously and the regularity assumptions on $\lambda(t)$ is determined for which the initial value problem defines a unitary strongly continuous dynamical system on $L^2(\mathbb{R})$. Such time-dependent point interactions have been studied rigorously in order to model asymptotic complete ionization and suitable conditions on the function $\lambda(t)$ has been determined for ionization problem [38–41]. The higher dimensional generalizations of the problem have been studied in great detail and summarized in the thesis by Correggi [42]. Transmission properties of a monochromatic beam and wave packets by studying the scattering from the time-dependent δ potential are studied in Martinez and Reichl [43] and Kuhn et al. [32]. Utilizing δ potential with a time dependent coefficient in an infinite well, Baek et al. [44] showed that it is possible to split a wave function which may have applications in statistical mechanics and condensed matter physics. However, it may also lead to philosophical problems [44]. The time dependence of the Dirac delta potentials could also be expressed through the motion of its support [45]. As a physical application, a moving Dirac δ potential is used to describe particle displacement using a standard tunneling microscope [46].

The paper is organized as follows: In section 2, we obtain a formal expression of the propagator for a finite number of Dirac δ potentials with time dependent strengths and solve the time dependent Schrödinger equation for this system. In the following subsections, we investigate one δ potential with time dependent strength in more detail and find the propagator for N Dirac δ potential in the limit that centers are infinitely separated. In

section 3, we get an expression of the propagator for a finite number of δ' potentials with time dependent strengths and solve the time dependent Schrödinger equation for this potential. Finally, we elaborate on one δ' case.

2. THE PROPAGATOR FOR N DIRAC DELTA POTENTIALS WITH TIME DEPENDENT STRENGTHS

We begin with a one dimensional model in which a free Hamiltonian of the type $H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$ is perturbed with a time dependent potential

$$V(x, t) = \sum_{j=1}^{N} \lambda_j(t)\delta(x - x_j) . \qquad (1)$$

The initial value problem of the time-dependent Schrödinger equation for this potential is

$$i\frac{\partial}{\partial t}\psi(x, t) = \left[-\frac{d^2}{dx^2} + \sum_{j=1}^{N} \lambda_j(t)\delta(x - x_j) \right] \psi(x, t) , \qquad (2)$$

with the given sufficiently smooth function $\psi(x, 0)$. Here we have used the units such that $\hbar = 2m = 1$ for simplicity. It is well-known that the Laplace transform is a very useful tool to solve initial value problems so we first take the Laplace transformation of Equation (2) with respect to time variable t and get

$$\bar{\psi}_{xx}(x, s) - i\psi(x, 0) + is\bar{\psi}(x, s) = \sum_{j=1}^{N} \delta(x - x_j)\mathcal{L}\left\{\lambda_j(t)\psi(x_j, t)\right\} , \qquad (3)$$

where $\bar{\psi}(x, s) = \mathcal{L}\left\{\psi(x, t)\right\}$ and $\bar{\psi}_{xx}(x, s) = \mathcal{L}\left\{\frac{\partial^2}{\partial x^2}\psi(x, t)\right\}$.

After this, we take the Fourier transformation of both sides of Equation (3) with respect to the coordinate variable x and get:

$$-k^2\hat{\bar{\psi}}(k, s) - i\hat{\psi}(k, 0) + is\hat{\bar{\psi}}(k, s) = \sum_{j=1}^{N} e^{-ikx_j}\mathcal{L}\left\{\lambda_j(t)\psi(x_j, t)\right\} , \qquad (4)$$

where $\hat{\bar{\psi}}(k, s) = \mathcal{F}(\bar{\psi}(x, s))$ denotes the Fourier transform of $\bar{\psi}(x, s)$ with respect to the variable x. This equation is an algebraic equation for the unknown wave function $\hat{\bar{\psi}}(k, s)$ and the solution is easily obtained as

$$\hat{\bar{\psi}}(k, s) = -\frac{1}{k^2 - is}\left[i\hat{\psi}(k, 0) + \sum_{j=1}^{N} e^{-ikx_j}\mathcal{L}\left\{\lambda_j(t)\psi(x_j, t)\right\} \right] . \qquad (5)$$

Now, we immediately find the inverse Fourier transform of the $\hat{\bar{\psi}}(k, s)$:

$$\bar{\psi}(x, s) = -i\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{ikx}}{k^2 - is}\hat{\psi}(k, 0)$$

$$- \sum_{j=1}^{N} \mathcal{L} \left\{ \lambda_j(t)\psi(x_j,t) \right\} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x-x_j)}}{k^2 - is} \ . \quad (6)$$

Using $\hat{\psi}(k,0) = \int_{-\infty}^{\infty} dx' \ e^{-ikx'} \psi(x',0)$, the Equation (6) becomes

$$\bar{\psi}(x,s) = -i \int_{-\infty}^{\infty} dx' \ \psi(x',0) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{k^2 - is}$$
$$- \sum_{j=1}^{N} \mathcal{L} \left\{ \lambda_j(t)\psi(x_j,t) \right\} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x-x_j)}}{k^2 - is} \ . \quad (7)$$

The integrals in Equation (7) are easily taken using residue theorem

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{k^2 - is} = i \frac{e^{i\sqrt{is}|x|}}{2\sqrt{is}} \ , \quad (8)$$

so we obtain

$$\bar{\psi}(x,s) = \frac{1}{2\sqrt{is}} \int_{-\infty}^{\infty} dx' \ e^{i\sqrt{is}|x-x'|} \psi(x',0)$$
$$- \frac{i}{2\sqrt{is}} \sum_{j=1}^{N} e^{i\sqrt{is}|x-x_j|} \mathcal{L} \left\{ \lambda_j(t)\psi(x_j,t) \right\} \ . \quad (9)$$

Now we need to take inverse Laplace transform to obtain the solution of the time dependent equation for the potential $\lambda(t)\delta(x)$. Using the Bromwich contour [47], one can easily find the inversion

$$\mathcal{L}^{-1} \left\{ \frac{e^{i\sqrt{is}|x|}}{\sqrt{is}} \right\} = \frac{1}{\sqrt{i\pi t}} \ \exp\left(\frac{ix^2}{4t} \right) \ , \quad (10)$$

and using the convolution theorem we get the formal solution as

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \ \frac{1}{\sqrt{4\pi it}} \ \exp\left[\frac{i(x-x')^2}{4t} \right] \psi(x',0)$$
$$- i \sum_{j=1}^{N} \int_{0}^{t} dt' \ \frac{\lambda_j(t')\psi(x_j,t')}{\sqrt{4\pi i(t-t')}} \ \exp\left[\frac{i(x-x_j)^2}{4(t-t')} \right] . \quad (11)$$

Although this is an explicit formal expression for $\psi(x,t)$, it is not completely expressed in terms of the initial condition $\psi(x,0)$ and includes the unknown factors $\psi(x_j,t)$. These can be found by simply inserting $x = x_j$ in the formal solution and then solving the resulting coupled Volterra type integral equations

$$\psi(x_j,t) = \int_{-\infty}^{\infty} dx' \ \frac{1}{\sqrt{4\pi it}} \ \exp\left[\frac{i(x_j-x')^2}{4t} \right] \psi(x',0)$$
$$- i \int_{0}^{t} dt' \ \frac{\lambda_j(t')\psi(x_j,t')}{\sqrt{4\pi i(t-t')}}$$
$$- i \sum_{\substack{k=1 \\ k \neq j}}^{N} \int_{0}^{t} dt' \ \frac{\lambda_k(t')\psi(x_k,t')}{\sqrt{4\pi i(t-t')}} \exp\left[\frac{i(x_j-x_k)^2}{4(t-t')} \right] , \quad (12)$$

where we split the term $k = j$ in the summation over k. Since this is not an easy problem for a generic function $\lambda(t)$, we will first investigate for particular cases, where λ is constant and λ is inversely proportional to t. All these results we present in the next subsection is previously obtained by Campbell [36] using a slightly different method, where only one integral transformation with the boundary conditions at the position of δ potential was used.

2.1. Single δ Potential With a Time Dependent Strength

As a particular case of (1), we consider a single delta potential with time-dependent strength

$$V(x,t) = \lambda(t)\delta(x) , \quad (13)$$

where $N = 1$ and $x_1 = 0$. We can formally obtain the solution of the time dependent Schrödinger equation for this case using Equation (11)

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \ \frac{1}{\sqrt{4\pi it}} \ \exp\left[\frac{i(x-x')^2}{4t} \right] \psi(x',0)$$
$$- i \int_{0}^{t} dt' \ \frac{\lambda(t')\psi(0,t')}{\sqrt{4\pi i(t-t')}} \exp\left[\frac{ix^2}{4(t-t')} \right] . \quad (14)$$

Actually this result can be directly obtained from the Duhamel's formula [48] for time-dependent Schrödinger equation associated with the Hamiltonian $H = H_0 + V$, where H_0 is self-adjoint free Hamiltonian and V is bounded (or relatively H_0-bounded with relative bound <1):

$$e^{-itH}|\psi_0\rangle = e^{-itH_0}|\psi_0\rangle +$$
$$(-i) \int_{0}^{t} dt' \ e^{-i(t-t')H_0} V e^{-it'H} |\psi_0\rangle \quad (15)$$

for every $|\psi_0\rangle = |\psi(t = 0)\rangle$. This shows that even if we formally take $V = \lambda(t')|0\rangle\langle0|$ which corresponds to our Dirac delta potential, one immediately sees that the Duhamel's formula is still formally valid for such singular interactions.

Given the initial condition $\psi(x,0)$ and the function $\lambda(t)$, the function $\psi(0,t)$ can be determined by solving the following integral equation:

$$\psi(0,t) = \int_{-\infty}^{\infty} dx' \ \frac{1}{\sqrt{4\pi it}} \ \exp\left[\frac{ix'^2}{4t} \right] \psi(x',0)$$
$$- i \int_{0}^{t} dt' \ \frac{\lambda(t')\psi(0,t')}{\sqrt{4\pi i(t-t')}}. \quad (16)$$

However, this is in general hard to solve and one usually applies some approximation techniques, e.g., Dyson series [48]. Nevertheless, as shown in Campbell [36], there are cases where one can calculate the Green's function explicitly. We will show two such cases here explicitly. Although Green's functions for these cases are derived in Campbell [36], we repeat these results here for the sake of completeness. Instead of directly solving the above integral equation for particular cases, it is convenient to

start with the Laplace transformed wave function Equation (9) for a single δ potential centered at the origin:

$$\bar{\psi}(x,s) = \frac{1}{2\sqrt{is}} \int_{-\infty}^{\infty} dx'\, e^{i\sqrt{is}|x-x'|}\psi(x',0)$$
$$- \frac{i}{2\sqrt{is}} e^{i\sqrt{is}|x|}\mathcal{L}\left\{\lambda(t)\psi(0,t)\right\}.\quad (17)$$

The cases for constant λ and $\lambda \propto \frac{1}{t}$ are reviewed in **Appendices A** and **B** in detail.

2.2. The Propagator for N Dirac δ Potentials in a Limiting Case

When λ_j's are constant, the Laplace transformed wave function $\bar{\psi}(x,s)$ given by (9) becomes

$$\bar{\psi}(x,s) = \frac{1}{2\sqrt{is}} \int_{-\infty}^{\infty} dx'\, e^{i\sqrt{is}|x-x'|}\psi(x',0)$$
$$- \frac{i}{2\sqrt{is}} \sum_{k=1}^{N} \lambda_k\, e^{i\sqrt{is}|x-x_k|}\bar{\psi}(x_k,s).\quad (18)$$

The unknown functions $\bar{\psi}(x_k,s)$ can be found by evaluating the above expression at $x = x_j$:

$$\bar{\psi}(x_k,s) = \sum_{j=1}^{N} \left[\Phi^{-1}(s)\right]_{kj} \bar{\rho}(x_j,s),\quad (19)$$

where

$$\Phi_{kj}(s) = \begin{cases} 1 + \frac{i\lambda_j}{2\sqrt{is}} & \text{if } j = k \\ \frac{i\lambda_k}{2\sqrt{is}} e^{i\sqrt{is}|x_j-x_k|} & \text{if } j \neq k \end{cases},\quad (20)$$

and

$$\bar{\rho}(x_k,s) = \frac{1}{2\sqrt{is}} \int_{-\infty}^{\infty} dx'\, e^{i\sqrt{is}|x_k-x'|}\psi(x',0).\quad (21)$$

Substituting (19) into (18), we obtain

$$\bar{\psi}(x,s) = \bar{\rho}(x,s) - \frac{i}{2\sqrt{is}} \sum_{j=1}^{N}\sum_{k=1}^{N} \lambda_k\, e^{i\sqrt{is}|x-x_k|} \left[\Phi^{-1}(s)\right]_{kj} \bar{\rho}(x_j,s).$$
$$(22)$$

Although we have obtained the Laplace transformed wave function $\bar{\psi}(x,s)$, it is not explicitly given since one has to invert the matrix Φ and find the inverse Laplace transform of the resulting expression to get the final solution $\psi(x,t)$. In general, it is difficult to find the inverse Laplace transforms so one may apply some approximation schemes [49]. Moreover, one could use some numerical computations, but we will here simply show the limiting case, where the centers are infinitely far away from each other.

When all the centers are infinitely separated from each other, that is, $|x_j - x_k| \to \infty$, we expect that the off-diagonal elements of the matrix Φ given in (20) vanish, so that

$$\left[\Phi^{-1}(s)\right]_{kj} = \left(\frac{1}{1 + \frac{i\lambda_k}{2\sqrt{is}}}\right)\delta_{kj}.\quad (23)$$

Then, $\bar{\psi}(x,s)$ can be explicitly found as

$$\bar{\psi}(x,s) = \bar{\rho}(x,s) - i\sum_{j=1}^{N} e^{i\sqrt{is}|x-x_j|}\left(\frac{\lambda_j}{2\sqrt{is} + i\lambda_j}\right)\bar{\rho}(x_j,s),\quad (24)$$

from which the propagator $G(x,x',s)$ reads

$$G(x,x',s) = \frac{e^{i\sqrt{is}|x-x'|}}{2\sqrt{is}} - \sum_{j=1}^{N} e^{i\sqrt{is}|x-x_j|}\left(\frac{\lambda_j}{2\sqrt{s} + \sqrt{i}\lambda_j}\right)\frac{e^{i\sqrt{is}|x_j-x'|}}{2\sqrt{s}}.$$
$$(25)$$

Hence,

$$G(x,x',t) = \frac{1}{2\sqrt{i\pi t}} \exp\left[\frac{i(x-x')^2}{4t}\right]$$
$$- \sum_{j=1}^{N} \frac{\lambda_j}{4} \exp\left[\frac{\lambda_j}{2}(|x-x_j| + |x_j-x'|) + i\frac{\lambda_j^2}{4}t\right]\quad (26)$$
$$\text{erfc}\left[\frac{(|x-x_j| + |x_j-x'|)}{2\sqrt{it}} + \sqrt{it}\frac{\lambda_j}{2}\right].$$

This is actually the superposition of the individual propagators associated with single delta centers. This is expected since there is no correlation among the centers when they are far away from each other. Another limiting case is the case where all the centers coincide.

3. THE PROPAGATOR FOR N DIRAC δ' POTENTIALS WITH TIME DEPENDENT STRENGTHS

In this section we first obtain a formal solution of the time-dependent Schrödinger equation, where the potential term is chosen formally as

$$V(x,t) = \sum_{i=1}^{N} \lambda_i(t)\delta'(x - x_i).\quad (27)$$

The time dependent Schrödinger equation for this potential is

$$i\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{d^2}{dx^2} + \sum_{j=1}^{N} \lambda_j(t)\delta'(x - x_j)\right]\psi(x,t).\quad (28)$$

As we mentioned in the introduction we adopt the distributional approach given by [12] for the definition of $\delta'(x)$, for functions having discontinuity at the point of δ'

$$\delta'(x)f(x) = \langle f(0) \rangle \delta'(x) - \langle f_x(0) \rangle \delta(x) , \qquad (29)$$

where we define $\langle f(y) \rangle = \frac{f(y^+) + f(y^-)}{2}$ and $\langle f_x(y) \rangle = \frac{f_x(y^+) + f_x(y^-)}{2}$ and $f(y^\pm)$ denote the limits $\lim_{x \to y^\pm} f(x)$. Note that this definition reduces to the well-known property of $\delta'(x)$ [50]

$$\delta'(x)f(x) = f(0)\delta'(x) - f'(0)\delta(x) . \qquad (30)$$

for continuous functions. As in the previous section we proceed by taking the Laplace transform of all the terms in the Equation (28) for time variable t and find

$$\bar{\psi}_{xx}(x,s) - i\psi(x,0) + is\bar{\psi}(x,s)$$
$$= \sum_{j=1}^{N} \left[-\mathcal{L}\left\{ \lambda_j(t)\langle\psi(x_j)\rangle \right\} \delta'(x-x_j) \right. \qquad (31)$$
$$+ \mathcal{L}\left\{ \lambda_j(t)\langle\psi_x(x_j)\rangle \right\} \delta(x-x_j) \Big] .$$

Now, we take the Fourier transform with respect to the variable x and solve $\hat{\bar{\psi}}(k,s)$ to get:

$$\hat{\bar{\psi}}(k,s) = -\frac{1}{k^2 - is}\left(i\hat{\psi}(k,0) + \sum_{j=1}^{N} -ike^{-ikx_j}\mathcal{L}\left\{ \lambda_j(t)\langle\psi(x_j)\rangle \right\} \right.$$
$$\left. + e^{-ikx_j}\mathcal{L}\left\{ \lambda_j(t)\langle\psi_x(x_j)\rangle \right\} \right) . \qquad (32)$$

Before taking the inverse Fourier transform of this equation we write $\hat{\psi}(k,0) = \int_{-\infty}^{\infty} dx' e^{-ikx'} \psi(x',0)$ then take the inverse Fourier transform and get

$$\bar{\psi}(x,s) = \frac{1}{2\sqrt{is}} \int_{-\infty}^{\infty} dx' \, e^{i\sqrt{is}|x-x'|} \psi(x',0)$$

$$- \frac{1}{2}\sum_{j=1}^{N} \mathrm{sgn}(x-x_j) \, e^{i\sqrt{is}|x-x_j|} \, \mathcal{L}\left\{ \lambda_j(t)\langle\psi(x_j,t)\rangle \right\} \qquad (33)$$

$$- i\frac{1}{2\sqrt{is}}\sum_{j=1}^{N} e^{i\sqrt{is}|x-x_j|} \, \mathcal{L}\left\{ \lambda_j(t)\langle\psi_x(x_j,t)\rangle \right\} .$$

where $\mathrm{sgn}(x)$ denotes sign function $\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$. We have also used $\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikx}}{k^2 - is} = -\frac{i}{2\sqrt{is}} e^{i\sqrt{is}|x|}$ and $\int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{ike^{ikx}}{k^2 - is} = -\frac{1}{2\sqrt{is}} e^{i\sqrt{is}|x|} \mathrm{sgn}(x)$. Now using $\mathcal{L}\left\{ \frac{\exp\left[i\frac{x^2}{4t}\right]}{2\sqrt{i\pi t}} \right\} = \frac{e^{i\sqrt{is}|x|}}{2\sqrt{is}}$ and $\mathcal{L}\left\{ (-1)^{3/4} \frac{\exp\left[i\frac{x^2}{4t}\right]x}{2\sqrt{\pi}t^{3/2}} \right\} = e^{i\sqrt{is}|x|} \mathrm{sgn}(x)$ and convolution theorem

for Laplace transform we get the formal expression of the wave function

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \, \frac{1}{\sqrt{4\pi it}} \, e^{i\frac{(x-x')^2}{4t}} \psi(x',0)$$

$$- \int_0^t dt' \left\{ \sum_{j=1}^{N} \frac{(-1)^{3/4}}{4\sqrt{\pi}}\lambda_j(t')\langle\psi(x_j)\rangle \frac{\exp\left[i\frac{(x-x_j)^2}{4(t-t')}\right]}{(t-t')^{3/2}} (x-x_j) \right. \qquad (34)$$

$$\left. + \frac{i}{2\sqrt{i\pi}}\sum_{j=1}^{N} \lambda_j(t')\langle\psi_x(x_j)\rangle \frac{\exp\left[i\frac{(x-x_j)^2}{4(t-t')}\right]}{\sqrt{(t-t')}} \right\} .$$

Similar to the time dependent δ potential case, this formal equation contains unknown functions $\psi(x_j^\pm)$'s and $\psi_x(x_j^\pm)$'s. One can find equations for $\psi(x_j^\pm)$'s by inserting $x = x_j^\pm$ to the Equation (34). In order to find equations for $\psi_x(x_j)^\pm$'s, the derivative of the Equation (34) with respect to the variable x has to be calculated. By taking this derivative we get

$$\psi_x(x,t) = \frac{1}{4\sqrt{i\pi}t^{3/2}} \int_{-\infty}^{\infty} dx' \, i(x-x') \, e^{i\frac{(x-x')^2}{4t}} \psi(x',0)$$

$$- \int_0^t dt' \left\{ -\frac{1}{8}\sqrt{\frac{i}{\pi}}\sum_{j=1}^{N} \lambda_j(t')\langle\psi(x_j)\rangle i(x-x_j)^2 \frac{\exp\left[i\frac{(x-x_j)^2}{4(t-t')}\right]}{(t-t')^{5/2}} \right.$$

$$+ \frac{(-1)^{3/4}}{4\sqrt{\pi}}\sum_{j=1}^{N} \lambda_j(t')\langle\psi(x_j)\rangle \frac{\exp\left[i\frac{(x-x_j)^2}{4(t-t')}\right]}{(t-t')^{3/2}} \qquad (35)$$

$$\left. - \frac{1}{4\sqrt{i\pi}}\sum_{j=1}^{N} \lambda_j(t')\langle\psi_x(x_j)\rangle(x-x_j)\frac{\exp\left[i\frac{(x-x_j)^2}{4(t-t')}\right]}{(t-t')^{3/2}} \right\} .$$

Now putting $x = x_j^\pm$ in the Equation (35) one can get also integral equations for $\psi_x(x_j^\pm)$'s. Finally one has to solve the system of integral equations for $\psi(x_j^\pm)$'s and $\psi_x(x_j^\pm)$'s to get the complete solution.

3.1. Single δ' Potential With a Time Dependent Strength

Now we will elaborate more on the single δ' interaction which is described by the potential

$$V(x,t) = \lambda(t)\delta'(x) \qquad (36)$$

The formal solution of the time dependent Schrödinger equation for single δ' can be obtained from the above section for $N = 1$, $x_1 = 0$ and $\lambda_1 = \lambda$

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \, \frac{1}{\sqrt{4\pi it}} \, e^{i\frac{(x-x')^2}{4t}} \psi(x',0)$$

$$- \int_0^t dt' \left\{ \frac{(-1)^{3/4}}{4\sqrt{\pi}}\lambda(t')\langle\psi(0)\rangle \, x \, \frac{\exp\left[i\frac{x^2}{4(t-t')}\right]}{(t-t')^{3/2}} \right. \qquad (37)$$

$$\left. + \frac{i}{2\sqrt{i\pi}}\lambda(t')\langle\psi_x(0)\rangle \frac{\exp\left[i\frac{x^2}{4(t-t')}\right]}{\sqrt{(t-t')}} \right\} .$$

Similarly the expression for the $\psi_x(x, t)$ is obtained from the Equation (35)

$$
\begin{aligned}
\psi_x(x, t) = {} & \frac{1}{4\sqrt{i\pi}\, t^{3/2}} \int_{-\infty}^{\infty} dx'\; i(x - x')\, e^{i\frac{(x-x')^2}{4t}} \psi(x', 0) \\
& - \int_0^t dt' \left\{ -\frac{1}{8}\sqrt{\frac{i}{\pi}} \lambda(t')\langle\psi(0)\rangle\, x^2 \frac{\exp\left[i\frac{x^2}{4(t-t')} \right]}{(t - t')^{5/2}} \right. \\
& + \frac{(-1)^{3/4}}{4\sqrt{\pi}} \lambda(t')\langle\psi(0)\rangle \frac{\exp\left[i\frac{x^2}{4(t-t')} \right]}{(t - t')^{3/2}} \\
& \left. - \frac{1}{4\sqrt{i\pi}} \lambda(t')\langle\psi_x(0)\rangle\, x\, \frac{\exp\left[i\frac{x^2}{4(t-t')} \right]}{(t - t')^{3/2}} \right\}.
\end{aligned}
\tag{38}
$$

For a given $\lambda(t)$ one gets integral equations for $\psi(x, 0)$ and $\psi_x(x, 0)$ by inserting $x = 0$ in Equations (37) and (38), respectively:

$$
\begin{aligned}
\psi(0, t) = {} & \int_{-\infty}^{\infty} dx'\; \frac{1}{\sqrt{4\pi it}}\, e^{i\frac{x'^2}{4t}} \psi(x', 0) \\
& - \int_0^t dt' \left\{ \frac{(-1)^{3/4}}{4\sqrt{\pi}} \lambda(t')\langle\psi(0)\rangle \frac{1}{(t - t')^{3/2}} \right. \\
& \left. + \frac{i}{2\sqrt{i\pi}} \lambda(t')\langle\psi_x(0)\rangle \frac{1}{\sqrt{(t - t')}} \right\}.
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
\psi_x(0, t) = {} & \frac{1}{4\sqrt{i\pi}\, t^{3/2}} \int_{-\infty}^{\infty} dx'\; (-x')e^{i\frac{x'^2}{4t}} \psi(x', 0) \\
& - \frac{(-1)^{3/4}}{4\sqrt{\pi}} \int_0^t dt'\; \lambda(t')\langle\psi(0)\rangle \frac{1}{(t-t')^{3/2}}.
\end{aligned}
\tag{40}
$$

The Equations (39) and (40) constitute an equation system for $\psi(0, t)$ and $\psi_x(0, t)$. Solving this system one can determine $\psi(0, t)$ and $\psi_x(0, t)$ and insert them to Equation (37) to get the wave function for all times. When studying special cases the expression of the Laplace transform of the wave function is necessary. Therefore, utilizing Equation (33) we write the general formula of the Laplace transform of the wave function for a delta prime at $x_1 = 0$ and $\lambda_1(t) = \lambda(t)$:

$$
\begin{aligned}
\bar{\psi}(x, s) = {} & \frac{1}{2\sqrt{is}} \int_{-\infty}^{\infty} dx'\; e^{i\sqrt{is}|x-x'|} \psi(x', 0) \\
& - \frac{1}{2} \operatorname{sgn}(x)\, e^{i\sqrt{is}|x|} \mathcal{L}\left\{ \lambda(t)\langle\psi(0, t)\rangle \right\} \\
& - i\frac{1}{2\sqrt{is}} e^{i\sqrt{is}|x|} \mathcal{L}\left\{ \lambda(t)\langle\psi_x(0, t)\rangle \right\}.
\end{aligned}
\tag{41}
$$

Now will investigate some special cases.

3.1.1. Case 1: λ Is Constant

When the strength of the single δ' interaction is constant, we obtain from Equation (33)

$$
\begin{aligned}
\bar{\psi}(x, s) = {} & \frac{1}{2\sqrt{is}} \int_{-\infty}^{\infty} dx'\; e^{i\sqrt{is}|x-x'|} \psi(x', 0) \\
& - \frac{\lambda}{2} \operatorname{sgn}(x)e^{i\sqrt{is}|x|} \langle\bar{\psi}(\bar{0}, s)\rangle \\
& - i\frac{\lambda}{2\sqrt{is}} e^{i\sqrt{is}|x|} \langle\bar{\psi}_x(\bar{0}, s)\rangle,
\end{aligned}
\tag{42}
$$

where $\langle\bar{\psi}(0, s)\rangle = \mathcal{L}\left\{ \langle\psi(x, t)\rangle \right\}$ and $\langle\bar{\psi}_x(0, s)\rangle = \mathcal{L}\left\{ \langle\psi_x(x, t)\rangle \right\}$. We need also derivative of the Laplace transformed wave function $\bar{\psi}(x, s)$ with respect to x.

$$
\begin{aligned}
\bar{\psi}_x(x, s) = {} & \frac{i}{2} \int_{-\infty}^{\infty} dx'\; \operatorname{sgn}(x - x')\, e^{i\sqrt{is}|x-x'|} \psi(x', 0) \\
& - \frac{i\lambda\sqrt{is}}{2} \operatorname{sgn}^2(x)e^{i\sqrt{is}|x|} \langle\bar{\psi}(\bar{0}, s)\rangle \\
& - \frac{\lambda}{2} \frac{d\operatorname{sgn}(x)}{dx} e^{i\sqrt{is}|x|} \langle\bar{\psi}(\bar{0}, s)\rangle + \frac{\lambda\operatorname{sgn}(x)e^{i\sqrt{is}|x|}}{2} \langle\bar{\psi}_x(\bar{0}, s)\rangle,
\end{aligned}
\tag{43}
$$

where we have used $\frac{d|x|}{dx} = \operatorname{sgn}(x)$. Now we find wave function and its derivative at $0^{\pm}$ by choosing $x = 0^{\pm}$ in Equations (42) and (43):

$$
\begin{aligned}
\bar{\psi}(0^{\pm}, s) = {} & \frac{1}{2\sqrt{is}} \int_{-\infty}^{\infty} dx'\; e^{i\sqrt{is}|x'|} \psi(x', 0) \mp \frac{\lambda}{2} \langle\bar{\psi}(\bar{0}, s)\rangle \\
& - \frac{i\lambda}{2\sqrt{is}} \langle\bar{\psi}_x(\bar{0}, s)\rangle,
\end{aligned}
\tag{44}
$$

where we take $\operatorname{sgn}(0^{\pm}) = \pm 1$. In order to find $\bar{\psi}(x, s)$ we need to calculate $\langle\bar{\psi}(0, s)\rangle$ and $\langle\bar{\psi}_x(0, s)\rangle$. From Equation (44) we get

$$
\langle\bar{\psi}(0, s)\rangle + \frac{i\lambda}{2\sqrt{is}} \langle\bar{\psi}_x(0, s)\rangle = \frac{1}{2\sqrt{is}} \int_{-\infty}^{\infty} dx'\; e^{i\sqrt{is}|x'|} \psi(x', 0).
\tag{45}
$$

In order to find another equation for $\langle\bar{\psi}(0, s)\rangle$ and $\langle\bar{\psi}_x(0, s)\rangle$ we calculate $\bar{\psi}_x(0^{\pm}, s)$

$$
\begin{aligned}
\bar{\psi}_x(0^{\pm}, s) = {} & \frac{i}{2} \int_{-\infty}^{\infty} dx'\; \operatorname{sgn}(-x')\, e^{i\sqrt{is}|x'|} \psi(x', 0) \\
& - \frac{i\lambda\sqrt{is}}{2} \langle\bar{\psi}(0, s)\rangle \mp \frac{\lambda}{2} \langle\bar{\psi}_x(0, s)\rangle.
\end{aligned}
\tag{46}
$$

where we take $\frac{d\operatorname{sgn}(x)}{dx}\big|_{x=0^{\pm}} = 0$. Using this equation we get

$$
\frac{i\lambda\sqrt{is}}{2} \langle\bar{\psi}(0, s)\rangle + \langle\bar{\psi}_x(0, s)\rangle = \frac{i}{2} \int_{-\infty}^{\infty} dx'\; \operatorname{sgn}(-x')\, e^{i\sqrt{is}|x'|} \psi(x', 0).
\tag{47}
$$

Solving Equations (45) and (47) we get

$$
\langle\bar{\psi}(0, s)\rangle = \frac{1}{2\sqrt{is}} \left(1 + \frac{\lambda^2}{4} \right)^{-1}
$$

$$\int_{-\infty}^{\infty} dx' \left(1 - \frac{\lambda}{2} \operatorname{sgn}(x')\right) e^{i\sqrt{is}|x'|} \psi(x',0), \quad (48)$$

and

$$\langle \bar{\psi}_x(0,s) \rangle = \int_{-\infty}^{\infty} dx' \left[-\frac{i}{2} \operatorname{sgn}(x') - \frac{i\lambda}{4} \left(1 + \frac{\lambda^2}{4}\right)^{-1} \right.$$
$$\left. \left(1 - \frac{\lambda}{2} \operatorname{sgn}(x')\right) \right] e^{i\sqrt{is}|x'|} \psi(x',0), \quad (49)$$

respectively. Substituting the expressions in Equations (48) and (49) into Equation (42) we get

$$\bar{\psi}(x,s) = \int_{-\infty}^{\infty} dx' \left\{ \frac{e^{i\sqrt{is}|x-x'|}}{2\sqrt{is}} + \left(1 + \frac{\lambda^2}{4}\right)^{-1} \left[-\frac{\lambda}{4} \operatorname{sgn}(x') \right. \right.$$
$$\left. \left. -\frac{\lambda^2}{8} - \frac{\lambda}{4} \operatorname{sgn}(x) + \frac{\lambda^2}{8} \operatorname{sgn}(x) \operatorname{sgn}(x') \right] \frac{e^{i\sqrt{is}(|x|+|x'|)}}{\sqrt{is}} \right\} \psi(x',0). \quad (50)$$

The factor that multiplies $\psi(x',0)$ in the integral of the equation above is the Green's function:

$$\bar{G}(x,x',s) = \left\{ \frac{e^{i\sqrt{is}|x-x'|}}{2\sqrt{is}} + \left(1 + \frac{\lambda^2}{4}\right)^{-1} \left[-\frac{\lambda}{4} \operatorname{sgn}(x') - \frac{\lambda^2}{8} \right. \right.$$
$$\left. \left. -\frac{\lambda}{4} \operatorname{sgn}(x) + \frac{\lambda^2}{8} \operatorname{sgn}(x) \operatorname{sgn}(x') \right] \frac{e^{i\sqrt{is}(|x|+|x'|)}}{\sqrt{is}} \right\}. \quad (51)$$

The Green's function in terms of s variable can be converted using Equation (10) to get

$$G(x,x',t) = \frac{1}{\sqrt{4\pi it}} \left\{ e^{i\frac{(x-x')^2}{4t}} + \left(1 + \frac{\lambda^2}{4}\right)^{-1} \left[-\frac{\lambda}{2} \operatorname{sgn}(x') \right. \right.$$
$$\left. \left. -\frac{\lambda^2}{4} - \frac{\lambda}{2} \operatorname{sgn}(x) + \frac{\lambda^2}{4} \operatorname{sgn}(x) \operatorname{sgn}(x') \right] e^{i\frac{(|x|+|x'|)^2}{4t}} \right\}. \quad (52)$$

So, we can write the wave function for all times as:

$$\psi(x,t) = \frac{1}{\sqrt{4\pi it}} \int_{-\infty}^{\infty} dx' \left\{ e^{i\frac{(x-x')^2}{4t}} + \left(1 + \frac{\lambda^2}{4}\right)^{-1} \right.$$
$$\left[-\frac{\lambda}{2} \operatorname{sgn}(x') - \frac{\lambda^2}{4} - \frac{\lambda}{2} \operatorname{sgn}(x) + \frac{\lambda^2}{4} \operatorname{sgn}(x) \operatorname{sgn}(x') \right] \quad (53)$$
$$\left. e^{i\frac{(|x|+|x'|)^2}{4t}} \right\} \psi(x',0).$$

This result is completely consistent with the one given in Lange [11] except the convention $\lambda = -c$.

3.1.2. Case 2: $\lambda(t) = \alpha/t$

The general expression (41) for the Laplace transform of the wave function in this particular case $\lambda(t) = \frac{\alpha}{t}$, where α is a constant, becomes

$$\bar{\psi}(x,s) = \frac{1}{2\sqrt{is}} \int_{-\infty}^{\infty} dx' \, e^{i\sqrt{is}|x-x'|} \psi(x',0) - \frac{1}{2} \operatorname{sgn}(x) \, e^{i\sqrt{is}|x|}$$
$$(54)$$
$$\mathcal{L}\left\{ \frac{\alpha}{t} \langle \psi(0,t) \rangle \right\} - i \frac{1}{2\sqrt{is}} e^{i\sqrt{is}|x|} \mathcal{L}\left\{ \frac{\alpha}{t} \langle \psi_x(0,t) \rangle \right\}.$$

Using the identity given in Equation (B.2) in **Appendix B** this equation becomes

$$\bar{\psi}(x,s) = \frac{1}{2\sqrt{is}} \int_{-\infty}^{\infty} dx' \, e^{i\sqrt{is}|x-x'|} \psi(x',0) - \frac{1}{2} \operatorname{sgn}(x) \, e^{i\sqrt{is}|x|}$$
$$(55)$$
$$\alpha \int_s^{\infty} ds' \langle \bar{\psi}(0,s') \rangle - i \frac{1}{2\sqrt{is}} e^{i\sqrt{is}|x|} \alpha \int_s^{\infty} ds' \langle \bar{\psi}_x(0,s') \rangle.$$

In order to calculate this expression we need $\psi(0^{\pm},s)$ and $\psi_x(0^{\pm},s)$. The first of this is easily calculated by choosing $x = 0^{\pm}$ in Equation (55)

$$\bar{\psi}(0^{\pm},s) = \frac{1}{2\sqrt{is}} \int_{-\infty}^{\infty} dx' \, e^{i\sqrt{is}|x'|} \psi(x',0)$$
$$\mp \frac{\alpha}{2} \int_s^{\infty} ds' \langle \bar{\psi}(0,s') \rangle - \frac{i\alpha}{2\sqrt{is}} \int_s^{\infty} ds' \langle \bar{\psi}_x(0,s') \rangle. \quad (56)$$

In order to find $\psi_x(0^{\pm},s)$ we take the derivative of the Equation (54) with respect to x variable and get

$$\bar{\psi}_x(x,s) = \frac{i}{2} \int_{-\infty}^{\infty} dx' \, \operatorname{sgn}(x-x') e^{i\sqrt{is}|x-x'|} \psi(x',0)$$
$$- \frac{i\sqrt{is}\alpha}{2} \operatorname{sgn}^2(x) \, e^{i\sqrt{is}|x|} \int_s^{\infty} ds' \langle \bar{\psi}(0,s') \rangle$$
$$- 2e^{i\sqrt{is}|x|} \alpha \frac{d \operatorname{sgn}(x)}{dx} \int_s^{\infty} ds' \langle \bar{\psi}(0,s') \rangle \quad (57)$$
$$+ \frac{\alpha}{2} \operatorname{sgn}(x) e^{i\sqrt{is}|x|} \int_s^{\infty} ds' \langle \bar{\psi}_x(0,s') \rangle.$$

From this equation we get

$$\bar{\psi}_x(0^{\pm},s) = -\frac{i}{2} \int_{-\infty}^{\infty} dx' \, \operatorname{sgn}(x') e^{i\sqrt{is}|x'|} \psi(x',0)$$
$$- \frac{i\sqrt{is}\alpha}{2} \int_s^{\infty} ds' \langle \bar{\psi}(0,s') \rangle \pm \frac{\alpha}{2} \int_s^{\infty} ds' \langle \bar{\psi}_x(0,s') \rangle. \quad (58)$$

We denote $u_1(s) = \int_s^{\infty} ds' \langle \bar{\psi}(0,s') \rangle$ and $u_2(s) = \int_s^{\infty} ds' \langle \bar{\psi}_x(0,s') \rangle$. We obtain from Equations (56) and (58)

$$\langle \bar{\psi}(0,s) \rangle = \frac{I_0(s)}{2\sqrt{is}} - \frac{i\alpha}{2\sqrt{is}} u_1(s) \quad (59)$$

and

$$\langle \bar{\psi}_x(0,s) \rangle = -i \frac{I_1(s)}{2} - \frac{i\sqrt{is}\alpha}{2} u_2(s). \quad (60)$$

Here $I_0(s) = \int_{-\infty}^{\infty} dx' \, e^{i\sqrt{is}|x'|} \psi(x',0)$ and $I_1(s) = \int_{-\infty}^{\infty} dx' \, \operatorname{sgn} x' e^{i\sqrt{is}|x'|} \psi(x',0)$. Note that $\langle \bar{\psi}(0,s) \rangle = -\frac{du_1(s)}{ds}$ and $\langle \bar{\psi}_x(0,s) \rangle = -\frac{du_2(s)}{ds}$. Inserting these equalities into Equations (59) and (60) we obtain two coupled differential equations:

$$\frac{du_1(s)}{ds} - \frac{i\alpha}{2\sqrt{is}} u_2(s) = -\frac{I_0(s)}{2\sqrt{is}} \quad (61)$$

and

$$\frac{du_2(s)}{ds} - \frac{i\sqrt{is}\,\alpha}{2}\,u_1(s) = i\frac{I_1(s)}{2\sqrt{is}}\,. \tag{62}$$

Although the coefficients of the unknown functions are not constants in these coupled equations, by taking the derivative of the Equations (61) and (62) with respect to the variable s one can uncouple these equations and get second order differential equations for $u_1(s)$ and $u_2(s)$:

$$\frac{d^2u_1(s)}{ds^2} + \frac{1}{2s}\frac{du_1(s)}{ds} + \frac{\alpha^2}{4}u_1(s) = \frac{\alpha}{4\sqrt{is}}I_1(s) - \frac{1}{2\sqrt{is}}\frac{dI_0(s)}{ds} \tag{63}$$

and

$$\frac{d^2u_2(s)}{ds^2} - \frac{1}{2s}\frac{du_2(s)}{ds} + \frac{\alpha^2}{4}u_2(s) = \frac{1}{4is}I_1(s) - \frac{i\alpha}{4}I_0(s) + \frac{i}{2}\frac{dI_1(s)}{ds}\,. \tag{64}$$

The solutions of these equations are elementary and easily obtained after some algebra as

$$u_1(s) = -\frac{\pi}{4\sqrt{2}}\int_{-\infty}^{\infty}dx'\,\psi(x',0)\left\{\int_0^s ds'\,\frac{s^{1/4}s'^{3/4}}{\sqrt{is'}}g_1(s,s')\right.$$
$$\left.\left[\alpha\,\mathrm{sgn}(x') + \frac{|x'|}{\sqrt{is'}}\right]e^{i\sqrt{is'}|x'|}\right\} \tag{65}$$

where

$$g_1(s,s') = J_{\frac{1}{4}}\left(\frac{\alpha s'}{2}\right)J_{-\frac{1}{4}}\left(\frac{\alpha s}{2}\right) - J_{\frac{1}{4}}\left(\frac{\alpha s}{2}\right)J_{-\frac{1}{4}}\left(\frac{\alpha s'}{2}\right)\,. \tag{66}$$

Here and in the following expressions $J_a(x)$, stands for the first kind of Bessel's function. The solution of the differential equation in Equation (64) is

$$u_2(s) = -\frac{\pi}{4\sqrt{2}}\int_{-\infty}^{\infty}dx'\,\psi(x',0)\left\{\int_0^s ds'\,s^{3/4}s'^{1/4}g_2(s,s')\right.$$
$$\left.\left[\frac{\mathrm{sgn}(x')}{is'} - i\alpha - \frac{ix'}{\sqrt{is'}}\right]e^{i\sqrt{is'}|x'|}\right\} \tag{67}$$

where

$$g_2(s,s') = J_{\frac{3}{4}}\left(\frac{\alpha s'}{2}\right)J_{-\frac{3}{4}}\left(\frac{\alpha s}{2}\right) - J_{\frac{3}{4}}\left(\frac{\alpha s}{2}\right)J_{-\frac{3}{4}}\left(\frac{\alpha s'}{2}\right)\,. \tag{68}$$

Note that

$$\mathcal{L}\left\{\frac{1}{t}\langle\psi(0,t)\rangle\right\} = u_1(s) = \int_s^{\infty}ds'\,\langle\bar\psi(0,s')\rangle \tag{69}$$

and

$$\mathcal{L}\left\{\frac{1}{t}\langle\psi_x(0,t)\rangle\right\} = u_2(s) = \int_s^{\infty}ds'\,\langle\bar\psi_x(0,s')\rangle\,. \tag{70}$$

Thus using Equations (69) and (70) in (41) we get

$$\bar\psi(x,s) = \int_{-\infty}^{\infty}dx'\,\psi(x',0)\left\{\frac{e^{i\sqrt{is}|x-x'|}}{2\sqrt{is}}e^{i\sqrt{is}|x-x'|}\right.$$
$$\left.+\frac{\mathrm{sgn}(x)\,\pi\,\alpha\,s^{1/4}e^{i\sqrt{is}|x|}}{8\sqrt{2i}}\int_s^{\infty}ds'\,s'^{1/4}g_1(s,s')\left[\alpha\,\mathrm{sgn}(x') + \frac{|x'|}{\sqrt{is'}}\right]\right. \tag{71}$$

$$e^{i\sqrt{is'}|x'|} + \frac{\sqrt{i}\,\pi\,\alpha\,s^{1/4}e^{i\sqrt{is}|x|}}{8\sqrt{2}}\int_s^{\infty}ds'\,s'^{1/4}g_2(s,s')$$
$$\left.\left[\frac{\mathrm{sgn}(x')}{is'} - i\alpha - \frac{ix'}{\sqrt{is'}}\right]e^{i\sqrt{is'}|x'|}\right\}\,.$$

Thus the Green's function in Laplace transformed space for the Schrödinger equation with a potential $V(x,t) = \frac{\alpha}{t}\delta'(x)$ is

$$G(x,x',s) = \left\{\frac{e^{i\sqrt{is}|x-x'|}}{2\sqrt{is}}e^{i\sqrt{is}|x-x'|} + \frac{\mathrm{sgn}(x)\,\pi\,\alpha\,s^{1/4}e^{i\sqrt{is}|x|}}{8\sqrt{2i}}\right.$$
$$\int_s^{\infty}ds'\,s'^{1/4}g_1(s,s')\left[\alpha\,\mathrm{sgn}(x') + \frac{|x'|}{\sqrt{is'}}\right]e^{i\sqrt{is'}|x'|}$$
$$+\frac{\sqrt{i}\,\pi\,\alpha\,s^{1/4}e^{i\sqrt{is}|x|}}{8\sqrt{2}}\int_s^{\infty}ds'\,s'^{1/4}g_2(s,s') \tag{72}$$
$$\left.\left[\frac{\mathrm{sgn}(x')}{is'} - i\alpha - \frac{ix'}{\sqrt{is'}}\right]e^{i\sqrt{is'}|x'|}\right\}\,.$$

This Green's function cannot be converted in terms of elementary functions but it is possible to use numerical methods to obtain Green's function in the position-time space.

4. CONCLUSION

In this work, we have studied some analytically solvable time-dependent point interactions. First, we have obtained a formal expression of the propagator for finite number of Dirac δ potentials with time dependent strengths and solved the time dependent Schrödinger equation for this system. Then we have investigated one δ potential with various time dependent strengths in more detail and found the propagator for N Dirac δ potential in the limit that centers are infinitely separated. Furthermore, we have found an expression of the propagator for finite number of δ' potentials with time dependent strengths and solved the time dependent Schrödinger equation for this potential. We believe that these results obtained are useful in models of ionization problems, where the particle is initially bound to the time dependent δ or δ' potentials. Such type of models have been studied (see e.g., [42, 51]) and the results obtained are compared with the experiment [52].

DATA AVAILABILITY STATEMENT

All datasets analyzed for this study are included in the article/**Supplementary Material**.

AUTHOR CONTRIBUTIONS

FUNDING

SUPPLEMENTARY MATERIAL

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.