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GENERAL COMMENTARY
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Response: Commentary: A Remark on the Fractional Integral Operators and the Image Formulas of Generalized Lommel-Wright Function

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A Commentary on

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In view of the commentary by Kiryakova [1] regarding our paper Agarwal et al. [2] published in "Frontiers in Physics" (2018), we reply to the commentary by proving the worth of our results and providing justification to all the points raised within commentary. Besides, our original contributions are better pointed out.

1. INTRODUCTION

We recall that numerous real world phenomena are closely related to the fractional order extensions of the Bessel function $J_{\varpi}(z)$ which in turn are generalized by the Lommel-Wright functions and their special cases $J_{\varpi}^{\varphi}(z), J_{\varpi,\varrho}^{\varphi}(z)$, where $\varphi > 0$ is a fractional parameter.

Substantial problems of mechanics, physics, astronomy, and various other engineering fields leads us to the generalized Bessel, Lommel, Struve, and Lommel-Wright functions.

In our paper Agarwal et al. [2], the image formulas for the generalized Lommel-Wright function involving the Saigo-Meada fractional integral operators, in term of the Fox-Wright function have been established. Following that certain theorems, with the outcomes achieved for the aforementioned functions in relation with the integral transforms like Beta transform, pathway transform, Laplace transform and Whittaker transform, have been proved. Interesting consequences of our results would involve the Saigo fractional integral operators $I_{0,x}^{\gamma,\tau,\eta}$ and $I_{x,\infty}^{\gamma,\tau,\eta}$ which can be deduced from the theorems in Agarwal et al. [2] by appropriately applying the relationships between these operators. Our paper contains results that are mathematically correct and novel in nature.

2. REPLY TO COMMENTARY

(a) As mentioned by Virginia Kiryakova in her commentary in the first paragraph [[1], p.3, Equation (8)] that Appell function F_3 can be expressed as Meijer G-function, it is to bring to the notice that almost all the special functions are expressible in the form of (G-function is a specific case of H-function itself) (see, for details [3, Equation 1.112], also refer [4]), namely

$$G^{m;n}_{p;q}\left[z\left|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right.\right]=H^{m;n}_{p;q}\left[z\left|\begin{matrix}(a_1,1),\ldots,(a_p,p)\\ (b_1,1),\ldots,(b_q,q)\end{matrix}\right.\right] \quad (1)$$

provided the existence conditions are satisfied.

Lot of work has been done on the various types of integrals and fractional operators involving H-function (see, e.g., [3, 5–8]). The limitations of H-function are due to the large number of parameters involved, its applications to the real world problems can be found only through its special cases. The large number of parameters interact each other and are not preferable from the experimental view point. In fact, numerical simulation is also difficult for the said generalized functions, namely, H-function and G-function.

(b) Paragraph 2 [[1], p. 3] highlights that the Marichev-Saigo-Maeda (MSM) fractional operator is actually composition of three Erdélyi-Kober (EK) operators. If it is so, we can raise the question on the novelty of the MSM operator. In our opinion, the composition of the operators is entirely different concept. These two operators are not linearly connected. In mathematics research, we know that even a different representation of the result/function is important. For example, a simple Beta function has so many representations and each one is important at its place. The fractional integral operators of Erdélyi-Kober are special cases of the operators of the fractional integral introduced by Marichev-Saigo-Maeda as can be observed from the following definitions:

For $\gamma,\gamma',\tau,\tau',\eta \in \mathbb{C}$ and $x > 0$, the generalized operators of Marichev-Saigo-Maeda are defined using the Appell's function in the kernel as follows:

$$\left(I^{\gamma,\gamma',\tau,\tau',\eta}_{0,\omega}h\right)(\omega) = \frac{\omega^{-\gamma}}{\Gamma\eta}\int_0^\omega(\omega-t)^{\eta-1}t^{-\gamma'}F_3 \left(\gamma,\gamma',\tau,\tau';\eta;1-\frac{t}{\omega},1-\frac{\omega}{t}\right)h(t)dt,\quad (\Re(\eta)>0), \quad (2)$$

and

$$\left(I^{\gamma,\gamma',\tau,\tau',\eta}_{\omega,\infty}h\right)(\omega) = \frac{\omega^{-\gamma}}{\Gamma\eta}\int_\omega^\infty(t-\omega)^{\eta-1}t^{-\gamma}F_3 \left(\gamma,\gamma',\tau,\tau';\eta;1-\frac{\omega}{t},1-\frac{t}{\omega}\right)h(t)dt,\quad (\Re(\eta)>0). \quad (3)$$

respectively.

The operators $I^{\gamma,\gamma',\tau,\tau',\eta}_{0,\omega}(\cdot)$ and $I^{\gamma,\gamma',\tau,\tau',\eta}_{\omega,\infty}(\cdot)$, turn down to the Erdélyi-Kober fractional integral operators on placing $\tau=0$, $\gamma'=0$, namely:

$$\left(\mathcal{E}^{\gamma,\eta}_{0,\omega}h\right)(\omega)=\left(I^{\gamma,0,0,\tau',\eta}_{0,\omega}h\right)(\omega), \quad (4)$$

and

$$\left(\mathcal{K}^{\gamma,\eta}_{\omega,\infty}h\right)(\omega)=\left(I^{\gamma,0,0,\tau',\eta}_{\omega,\infty}h\right)(\omega). \quad (5)$$

The fractional integral operators of Erdélyi-Kober type are described as mentioned below [9]:

$$\left(\mathcal{E}^{\gamma,\eta}_{0,\omega}h\right)(\omega)=\frac{\omega^{-\gamma-\eta}}{\Gamma(\gamma)}\int_0^\omega(\omega-t)^{\gamma-1}t^\eta h(t)dt,\quad (\Re(\gamma)>0) \quad (6)$$

and

$$\left(\mathcal{K}^{\gamma,\eta}_{\omega,\infty}h\right)(\omega)=\frac{\omega^{\eta}}{\Gamma(\gamma)}\int_\omega^\infty(t-\omega)^{\gamma-1}t^{-\gamma-\eta}h(t)dt, \quad (\Re(\gamma)>0). \quad (7)$$

where $h(.)$ is so restricted that both the above integrals in (6) and (7) exist.

(c) Again, as per the last paragraph [1, p.3] in the commentary, the result in Agarwal et al. [2, Theorem 3.2] is obvious. This implies that majority of the research work done so far by many researchers worldwide is obvious because all the new results are derived from/based upon the earlier obtained results. For application purpose, a researcher (from a different domain like physics, engineering etc.) search for direct results, doesn't want to derive all the mathematics behind those results. Here is the Theorem 3.2 and its proof as in Agarwal et al. [2]:

Theorem 3.2. *Let $\gamma,\gamma',\tau,\tau',\eta,\rho,\varrho \in \mathbb{C}$, $\varphi>0$, $m\in\mathbb{N}$, $\Re(\rho)>0$, $\Re(s)>0$, $\sigma>1$ and $y>0$ be the parameters satisfying*

$$\Re(\wp)>0,\ \Re(\varpi)>-1,\ \Re(s)>0,$$
$$\Re(\rho+\varpi)>\max\{0,\Re(\gamma+\gamma'+\tau-\wp),\Re(\gamma'-\tau')\} \quad (8)$$

then the P_σ-transform formula is given by:

$$P_\sigma\left[z^{l-1}\left(I^{\gamma,\gamma',\tau,\tau',\wp}_{0+}t^{\rho-1}J^{\varphi,m}_{\varpi,\varrho}(tz)\right)(y):s\right]=\left(\Lambda(\sigma;s)\right)^{l+\varpi+2\varrho}\ \frac{y^{A-\gamma-\gamma'+\wp-1}}{2^{\varpi+2\varrho}}$$
$$\times\,{}_5\psi_{4+m}\left[\begin{matrix}(A,2),(A+\wp-\gamma-\gamma'-\tau,2),(A+\tau'-\gamma',2),(l+\varpi+2\varrho,2),(1,1)\\ (A+\tau',2),(A+\wp-\gamma-\gamma',2),(A+\wp-\gamma'-\tau,2),(\varpi+\varrho+1,\varphi),\underbrace{(\varrho+1,1)}_{m-times}\end{matrix}\left|\,-\frac{(\Lambda(\sigma;s)y)^2}{4}\right.\right] \quad (9)$$

such that $A=\rho+\varpi+2\varrho$ and $\Lambda(\sigma;s)=\left(\frac{\sigma-1}{\ln[1+(\sigma-1)s]}\right)$.

Proof: Let us denote the left-hand side of the formula (9) as Ξ. Applying [[2], Equation 1.30] to [[2], Equation 3.2] we get,

$$\Xi=\int_0^\infty[1+(\sigma-1)s]^{-\frac{z}{\sigma-1}}z^{l-1}I^{\gamma,\gamma',\tau,\tau',\wp}_{0+}\left(t^{\rho-1}J^{\varphi,m}_{\varpi,\varrho}(tz)\right)(y)dz.$$

Here, applying the [[2], Equation 2.4] to the integral, we get

$$\Xi = \frac{y^{A-\gamma-\gamma'+\wp-1}}{2^{\varpi+2\varrho}} \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma(A+2p)\Gamma(A+\wp-\gamma-\gamma'-\tau+2p)}{\Gamma(A+\tau'+2p)\Gamma(A+\wp-\gamma-\gamma'+2p)\Gamma(A+\wp-\gamma'-\tau+2p)}$$
$$\frac{\Gamma(A+\tau'-\gamma'+2p)\Gamma(p+1)}{\Gamma(\varpi+\varrho+1+\varphi p)(\Gamma(\varrho+1+p))^m} \frac{(y)^{2p}}{4^p p} \times \int_0^\infty [1+(\sigma-1)s]^{-\frac{z}{\sigma-1}} z^{\varpi+2\varrho+2p+l-1} dz.$$

Using the result [[2], Equation 1.31] and swapping the order of the integration and the summation, we arrive at,

$$\Xi = \left(\Lambda(\sigma;s)\right)^{l+\varpi+2\varrho} \frac{y^{A-\gamma-\gamma'+\wp-1}}{2^{\varpi+2\varrho}} \sum_{p=0}^{\infty} \frac{\Gamma(A+2p)\Gamma(A+\wp-\gamma-\gamma'-\tau+2p)}{\Gamma(A+\tau'+2p)\Gamma(A+\wp-\gamma-\gamma'+2p)}$$
$$\times \frac{\Gamma(\varpi+2\varrho+2p+l)\Gamma(A+\tau'-\gamma'+2p)\Gamma(p+1)(-1)^p}{\Gamma(A+\wp-\gamma'-\tau+2p)\Gamma(\varpi+\varrho+1+\varphi p)(\Gamma(\varrho+1+p))^m} \frac{\{\Lambda(\sigma;s)y\}^{2p}}{4^p p}, \tag{10}$$

such that $A = \rho + \varpi + 2\varrho$ and $\Lambda(\sigma;s) = \left(\frac{\sigma-1}{\ln[1+(\sigma-1)s]}\right)$.

Taking into account, the definition [[2], Equation 1.2] we attain the required result (9). □

Thus, proof of the Theorem 3.2 clearly illustrates that its not as straightforward as claimed in Agarwal et al. [2]. It can be seen clearly that our work is new and the results are distinctive.

3. CONCLUSION

We proved that our results reported in Agarwal et al. [2] are correct, the claims of the commentary also support this "*as is the correctly evaluated…*".

We recall that the classification of fractional calculus operators has been a very hot topic during the last few years. In our opinion, the author of the commentary is considering the fractional calculus in the classical way, which has comparatively lesser number of applications as compared to the other new fractional calculus operators (see for example some of new related works on this topic, e.g., [10–12] and the references therein).

It is observed that, the commentary could have been placed in general, for the work on fractional calculus, as have been mentioned by the author of the commentary in the first paragraph of Kiryakova [[1], section 1], "The commented paper [1] is one example of a *long list of recently published works* devoted to evaluation of the images of classes of special functions…". However, only our paper has been pointed out within the comment. Therefore, we disagree with the aim of a particular investigation done in Kiryakova [1] on our correctly reported results.

We recall that with the increased number of indices in the multi-index Mittag-Leffler function and multi-parameter E-K integral [13], the domain gets narrowed and hence it is not always suitable for describing the dynamics of the real world phenomenon. Besides, these indices interact each other, reducing the practical usage of the above said functions. In our opinions, the construction suggested by Kiryakova [13] will face problems while it will be applied to the real world problems mainly because:

- contains many indices and they interact each other and from experimental view point the researchers will not prefer to work with many indices;
- the laws of Nature are simple and do not require non-physical operators to describe them. So far, we were unable to find any numerical implementation suggested by the generalization in Kiryakova [13, 14].

Besides, in our opinions, the conclusion of Kiryakova [1] lack in clarifying its full objectives. For example, the meaning of the third point, namely:

"If so (usually it is the case), apply a general result like [1]".

is not clear because of ambiguity in the phrase "usually it is the case". Therefore, in accordance with our view, the conclusion part of Kiryakova [1] has failed to reach its aim.

AUTHOR CONTRIBUTIONS

The original paper "A Remark on the Fractional Integral Operators and the Image Formulas of Generalized Lommel-Wright Function" was authored by the four authors. In preparing the response to the commentary, all the authors have contributed with lots of discussions among them. RA and SJ prepared the first draft of the response on the basis of points provided by RPA and DB. The successive discussions among the authors lead to the final form of the response to the commentary.

REFERENCES

1. Kiryakova V. Commentary: A remark on the fractional integral operators and the image formulas of generalized Lommel-Wright function. *Front. Phys.* (2019) **7**:145. doi: 10.3389/fphy.2019.00145
2. Agarwal R, Jain S, Agarwal Ravi P, Baleanu D. A remark on the fractional integral operators and the image formulas of generalized Lommel-Wright function. *Front. Phys.* (2018) **6**:79. doi: 10.3389/fphy.2018.00079
3. Mathai AM, Saxena RK, Haubold HJ. *The H-Function: Theory and Applications*. New York, NY: Springer-Verlag (2010).
4. Mathai AM, Saxena RK. *Generalized Hypergeometric Functions With Applications in Statistics and Physical Sciences*. New York, NY: Springer-Verlag (1973).

5. Kumar D, Agarwal P, Purohit SD. Generalized fractional integration of the $\bar{H}$ -function involving general class of polynomials. *Walailak J Sci Tech*. (2014) **11**:1019–30. doi: 10.14456/WJST.2014.57
6. Srivastava HM, Gupta KC, Goyal SP. *The H-Functions of One and Two Variables With Applications*. New Delhi; Madras: South Asian Publishers (1982).
7. Saxena RK, Saigo M. Generalized fractional calculus of the H-function associated with the Appell function F3. *J Fract Calc*. (2001) **19**:89–104.
8. Srivastava HM, Singh NP. The integration of certain product of the multivariable H-function with a general class of polynomials. *Rend Circ Mat Palermo*. (1983) **32**:157–87.
9. Kober H. On fractional integrals and derivatives. *Quart J Math Oxford Ser*. (1940) **11**:193–212.
10. Zhao D, Luo M. Representations of acting processes and memory effects: general fractional derivative and its application to theory of heat conduction with finite wave speeds. *Appl Math Comput*. (2019) **346**:531–44. doi: 10.1016/j.amc.2018.10.037
11. Hilfer R, Luchko Y. Desiderata for fractional derivatives and integrals. *Mathematics*. (2019) **7**:149. doi: 10.3390/math7020149
12. Baleanu D, Fernandez A. On fractional operators and their classifications. *Mathematics*. (2019) **7**:830. doi: 10.339/math7090830
13. Kiryakova V. Fractional calculus operators of special functions? The result is well predictable! *Chaos Solitons Fract*. (2017) **102**:2–15. doi: 10.1016/j.chaos.2017.03.006
14. Kiryakova V. Use of fractional calculus to evaluate some improper integrals of special functions. *AIP Conf. Proc.* (2017) **1910**:050012. doi: 10.1063/1.5013994

Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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