The Quantum Coherence Induced by Geometric Curvature of Gauge Field in Non-equilibrium Quantum Dynamics

Haidong Feng[1] and Jin Wang[2]*

[1] State Key Laboratory of Electroanalytical Chemistry, Changchun Institute of Applied Chemistry, Chinese Academy of Sciences, Changchun, China, [2] Department of Chemistry, Physics, and Applied Mathematics, State University of New York at Stony Brook, Stony Brook, NY, United States

The study of non-equilibrium quantum dynamics has recently received attention. However, the nature and effects of non-equilibrium, such as detailed-balance breaking and the relationship to the underlying intrinsic geometry, is still unclear. In this study, we show that a gauge field will be induced by non-equilibrium in the coherence representation. Furthermore, we show that its internal geometrical curvature is directly related to the degree of detailed balance breaking. The non-equilibrium of the quantum system induces an intrinsic geometric curvature which can enhance the quantum coherence, leading to the possibility of a space time origin for non-local quantum correlations or the possibility of curved space time emergence from non-equilibrium quantum dynamics. We also uncovered that the internal curvature of the gauge field provides a bridge to connect the generalized quantum fluctuation dissipation theorem to the fluctuation theorem and time irreversibility of quantum dynamics. The quantum time irreversibility is due to the path dependent factor along any particular path in an internal curved space, which is analogous to the Wilson lines (or Wilson loops) in Abelian gauge theory. We also found that the steady state quantum coherence disappears when the non-trivial internal curvature vanishes for the quantum system coupled with environments. When the curvature is relatively small, indicating weak detailed balance breaking, the coherence increases as curvature increases. The internal curvature can provide a general and direct quantitative measure of the detail-balance breaking for any quantum/classical non-equilibrium systems, even without knowing the underlying steady state distribution or the steady state flux. Using an example of two harmonic oscillators, coupled to two environments with different temperatures, we explicitly show the dependence of the internal curvature and quantum coherence.

Keywords: gauge field, non-equilibrium, coherence, quantum, geometric curvature

1. INTRODUCTION

With the recent progress made in experimental and theoretical studies on quantum synchronization, energy/charge transports in molecular junctions, and quantum information devices [1–6], non-equilibrium quantum systems characterized by detailed-balance breaking have attracted more attention. However, it remains challenging to understand the fundamental nature and underlying mechanisms of the non-equilibrium quantum systems, for example, the effect on quantum coherence and entanglement [7–10].

For non-equilibrium systems at the classical level, it has been shown that the driving force for the dynamics of non-equilibrium systems can be decomposed into the gradient of a potential, quantified by the steady state probability distribution and a curl flux (current) quantified by the steady state probability flux [11]. The non-trivial curl flux provides a direct measure of the degree of the intrinsic non-equilibrium: the detailed balance breaking [12, 13]. Given a complex non-equilibrium system, the quantification of the curl flux requires the information of the resulting dynamics in the long run which is often challenging to obtain. In our recent work [14] on classical non-equilibrium systems, a connection between the curl flux or the detailed-balance breaking and an internal curvature of a gauge field [15] was uncovered. This leads to a new and geometric perspective of classical non-equilibrium physics: it is the internal curvature that leads to the detailed-balance breaking and the time irreversibility in non-equilibrium dynamics.

Therefore, it will be interesting to know how this concept can be extended to non-equilibrium quantum systems. In this work, using coherent phase space representation in quantum mechanics [16–19], we derive the gauge field and internal curvature to a generic class of non-equilibrium bosonic quantum systems coupled with the environments [9, 10]. The internal curvature of the gauge field, which is derived from the fundamental dynamics without requiring numerical or analytical steady state solution, provides a direct measure of detailed-balance breaking for non-equilibrium quantum systems. In addition, it provides a new and geometric view for the general nature and behaviors of non-equilibrium quantum systems, such as the fluctuation-dissipation theorem (FDT), the fluctuation theorem, and time irreversibility. In particular, the Wilson lines/loops of the gauge theory provide the direct measure of time irreversibility of non-equilibrium quantum systems. On the other hand, quantum coherence, which characterizes the quantum nature such as interferences, is also shown to be connected to the internal curvature. When the internal curvature is zero, there is no detailed-balance breaking and no steady quantum coherence. Vice versa, when the internal curvature is nonzero, detailed-balance breaking emerges with non-zero quantum coherence. From here, we can develop a new view in non-equilibrium quantum physics: the steady state coherence is associated with internal curvature quantified by the degree of the detailed-balance breaking. The non-equilibrium of the quantum system induces an intrinsic geometric curvature which can enhance the quantum coherence, leading to the possibility of the space time origin for the non-local quantum correlations or the possibility of curved space time emergence from non-equilibrium quantum dynamics.

As an explicit example, we study two harmonic oscillators coupled to two heat baths. For the first time, we explicitly calculate the internal curvature and associated gauge field for a specific non-equilibrium quantum system, which can be measured and examined in experiments. Furthermore, in this quantum system, we demonstrate the dependence of quantum coherence on the internal curvature. It is found that, when a quantum system is not very far away from the equilibrium, the quantum coherence increases with the internal curvature monotonically, which can also be tested in future experiments.

2. BOSE-HUBBARD MODEL AND QUANTUM MASTER EQUATION IN COHERENT SPACE

In this study, to quantify the gauge field and the curvature introduced by non-equilibrium quantum flux, we consider the Bose-Hubbard model on N sites with each site coupled to two environments with site-dependent coupling strengths [9, 10]. It is expected that the temperature difference of the two environments can lead to non-equilibrium quantum dynamics.

By using creation/annihilation operators, the free Hamiltonian of the system and the baths can be written as:

$$H_0 = \sum_{i=1}^{N} \epsilon_i a_i^\dagger a_i + \sum_{i \neq j} \Delta_{ij} a_i^\dagger a_j + \frac{g}{2} \sum_{i=1}^{N} a_i^\dagger a_i^\dagger a_i a_i \\ + \sum_{\nu=1}^{2} \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} b_{\mathbf{k}}^{(\nu),\dagger} b_{\mathbf{k}}^{(\nu)} \tag{1}$$

and the interactions between the sites and baths are given as

$$H_{int} = \sum_{i=1}^{N} \sum_{\nu=1}^{2} \sum_{\mathbf{k}} f_{\mathbf{k}}^{i(\nu)} (a_i^\dagger b_{\mathbf{k}}^{(\nu)} + a_i b_{\mathbf{k}}^{(\nu),\dagger}) \tag{2}$$

In what follows, for simplicity without the loss of generality, we will focus first on the case where the nonlinear self-coupling strength is small (set to zero): $g = 0$. Δ_{ij} are the hopping rates between different sites, which are real and symmetric. The creation/annihilation operators $a_i^\dagger / a_i$ creates/annihilates bosons on the i-th site, while $b_{\mathbf{k}}^{(\nu),\dagger} / b_{\mathbf{k}}^{(\nu)}$ create/annihilate bosons of mode k in the v-th environment.

In the interaction picture, the dynamics of the annihilator of the system is given as: $i \hbar \dot{a}_n = [a_n, H_0] = \epsilon_n a_n + \sum_{j \neq n} \Delta_{nj} a_j$. Following some standard procedures, after tracing out the environmental influences under certain approximations [18], quantum master equation (QME) in the interaction picture governs the time evolution density operator ρ_s of the reduced dynamics as:

$$\frac{d\rho_s}{dt} = \left[D_l^\nu \left(a_l \rho_s a_\nu^\dagger - a_\nu^\dagger a_l \rho_s \right) + \bar{D}_l^\nu \left(a_l^\dagger \rho_s a_\nu - a_\nu a_l^\dagger \rho_s \right) \right] \\ + h.c. \tag{3}$$

where the Einstein's summation rule is applied. The dissipation rates D_l^μ, $\bar{D}_l^\mu$ are given in the **Supplementary Materials**, which contains parameters of the form $n_\omega^T = 1/(e^{\hbar\omega/k_B T} - 1)$, i.e., the mean number of bosons with frequency ω in thermal equilibrium at temperature T. The non-equilibrium dynamics can emerge under different temperatures of the two environments while assuming identical coupling strengths to both environments: $f_{\mathbf{k}}^{i(1)} = f_{\mathbf{k}}^{i(2)}; i = 1, 2, ..., N$.

One conventional way to solve QME in Equation (3) was to write the density matrix as a supervector in Liouville space [20, 21]. However, this strategy gives rise to an infinite dimensional Fock space for bosons, which makes it difficult to manipulate.

Here we will consider the coherent representation, which was first developed by Glauber [18], with the establishment of a close analog to classical Fokker-Planck equations. The coherent state is the eigenstate of the annihilation operators and satisfies: $\hat{a}_j|\{\alpha_i\}\rangle = \alpha_j|\{\alpha_i\}\rangle$. The coherent state is introduced in a wide range of physical systems, from quantum harmonic oscillators, quantum optics, superconductivity, superfluid, to string theory. Coherent states mostly describe the classical-like states by displacing the ground-state wave packet from its origin, which minimizes the uncertainty relation. On the basis of Fock states, the coherent state can be written as:

$$|\alpha_i\rangle = e^{-\frac{|\alpha_i|^2}{2}} \sum_{n_i=0}^{\infty} \frac{\alpha_i^{n_i}}{\sqrt{n_i!}} |n_i\rangle \qquad (4)$$

with the probability of in state $|n_i\rangle$ following Poissonian distribution and the average boson number $\langle \hat{n}_i \rangle = |\alpha_i|^2$.

Then the density matrix can be expanded by the coherent states as: $\rho_s(t) = \int \prod_{m=1}^{N} d^2\alpha_m P(\{\alpha_i\}, \{\alpha_i^*\}, t)|\{\alpha_i\}\rangle\langle\{\alpha_i\}|$.

By introducing the short notation $\alpha \equiv \{\alpha_i\}$, the resulting PDE takes the form of the classical Fokker-Planck equation:

$$\frac{\partial}{\partial t} P(\alpha, \alpha^*, t) = \left[(D_l^\mu - \bar{D}_l^\mu) \left(\frac{\partial}{\partial \alpha_\mu} \alpha_l + \frac{\partial}{\partial \alpha_\mu^*} \alpha_l^* \right) \right.$$
$$\left. + \bar{D}_l^\mu \left(\frac{\partial^2}{\partial \alpha_\mu^* \partial \alpha_l} + \frac{\partial^2}{\partial \alpha_\mu \partial \alpha_l^*} \right) \right] P(\alpha, \alpha^*, t) \qquad (5)$$

with a quasiprobability distribution $P(\alpha, \alpha^*)$. The non-negative quasiprobability $P(\alpha, \alpha^*)$ acts like an ordinary probability distribution in Fokker-Planck equations with dependence on complex variables [22].

The driving force $(D_l^\mu - \bar{D}_l^\mu)\alpha_l$ represents the environmental influences from tracing out the baths. This driving force is linear in the coherent coordinate α_l, which will push the system to the coherent state with larger $|\alpha|$. Since $|\alpha|^2$ represents the average boson number and the large boson number represents a higher average energy of the system, this driving force, which is due to the coupling to the environment heat baths, will effectively drive the quantum system to a higher energy level.

The diffusion term $\bar{D}_l^\mu$ represents the transitions between different energy levels of each site (for diagonal diffusion matrix elements) and between different sites (for off-diagonal diffusion matrix elements). In particular, the off-diagonal diffusion matrix

elements of $\bar{D}_l^\mu$ govern coupling between coherence and population dynamics for different coordinates α_μ and α_l in coherent space. Since the coherent state $|\alpha_i\rangle$ is the superpositions of Fock states $|n_i\rangle$ for sites i, The off-diagonal term of $\bar{D}_l^\mu$ naturally represents the coherence in Fock states $|n_i\rangle$ for site i, which is introduced by interaction terms $\sum_{i \neq j} \Delta_{ij} a_i^\dagger a_j$. When coupling between different sites $\Delta_{ij} \to 0$, the diffusion matrix will have no off-diagonal terms between different sites and the density matrix $P(\alpha, \alpha^*)$ in coherent space can be decomposed into the production of density matrices in sub-space of different coherent coordinates: $P(\alpha_i, \alpha_j, \alpha_i^*, \alpha_j^*) = P_i(\alpha_i, \alpha_j)P_j(\alpha_i^*, \alpha_j^*)$. On the other hand, with no interaction between different sites, obviously the coherence of density matrix ρ_s in Fock space $|n_i\rangle$ will vanish as well. Later, we will uncover the important link to the geometrical curvature introduced by non-equilibrium dynamics.

3. FORCE DECOMPOSITION, FLUX

In classical non-equilibrium dynamical systems, the dual description with both potential and flux has been identified and quantified to determine the global dynamics [11, 13, 23]. There, in the continuous space, the non-equilibrium dynamics are governed by Fokker-Planck equations with a driving force which can be decomposed into the gradient of a potential and a curl flux, quantifying the degree of the detailed-balance-breaking. Therefore, by observing the analogous form of the density matrix dynamical equation (5), a similar investigation can be applied for non-equilibrium quantum systems.

Defining the probability flux $\mathbf{J}$ as $J_\mu = \tilde{F}_\mu P - \tilde{D}_{\mu\nu}\partial_\nu P$, and quantum Fokker-Planck equation (5) can be written as a continuity equation $\partial_t P + \nabla \cdot \mathbf{J} = 0$ in coherent space. $\tilde{F}_\mu$ is the driving force in the complex coherent state space $x_\mu \equiv \{\alpha_i, \alpha_i^*\}$ with $i = 1, 2, ..., N$ representing N different sites, and $\tilde{D}_{\mu\nu}$ are symmetric diffusion coefficients:

$$\tilde{F}_a = -\sum_{l=1}^{N} \left[(\gamma_a^{(1)} + \gamma_a^{(2)})\delta_{al} \right] \alpha_l,$$
$$\tilde{D}_{al^*} = \frac{\bar{D}_l^a + \bar{D}_a^l}{2} \qquad (6)$$

Here, starred indexes indicate complex components corresponding to the coordinates $\{\alpha_i^*\}$. $\gamma_a^{(i)} = \pi \mathcal{N}(\bar{\nu})(f_{\bar{\nu}}^{a(i)})^2/\hbar^2$, $i = 1, 2$ is the damping coefficient depending on the site index and $\mathcal{N}(\bar{\nu})$ is the density of states.

The steady-state quasiprobability distribution satisfies $\partial_t P_{ss} = 0$ and the steady-state probability flux $\mathbf{J}_{ss}$ is given by

$$J_\mu^{ss} = \tilde{F}_\mu P_{ss} - \tilde{D}_{\mu\nu}\partial_\nu P_{ss} \qquad (7)$$

which implies that $\mathbf{J}_{ss}$ is a curl flux (a solenoidal vector field) satisfying $\nabla \cdot \mathbf{J}_{ss} = 0$. This does not necessarily mean that the flux $\mathbf{J}_{ss} = 0$. Instead, due to the detailed-balance-breaking, the divergence-free condition implies that the net non-zero coherent state space dependent flux is a rotational curl field in complex coherent state space. For general non-equilibrium

systems without detailed balance: $\mathbf{J}_{ss} \neq 0$, the force term $\tilde{F}_\mu$ can not be written as the gradient of a potential. From Equation (7), the driving force for non-equilibrium quantum dynamics can be decomposed into two parts in the complex coherent state space: a potential gradient term $-\tilde{D}_{\mu\nu}\frac{\partial}{\partial x_\nu}U$ where $U(\mathbf{x}) = -\ln P_{ss}$ and flux term $-J_\mu^{SS}/P^{SS} \equiv -v_\mu^{SS}$, where v_μ^{SS} represents a probabilistic velocity. In the next section, we can see that this potential-flux landscape provides new insights into the non-equilibrium quantum dynamics.

Analogous to the classical non-equilibrium systems, using Ito calculus, dynamically quantum Fokker-Planck equation (5) is equivalent to *Langevin* equations in the coherent state space:

$$\frac{dx_\mu}{dt} = \tilde{F}_\mu + B_{\mu\nu}\xi_\nu(t) \tag{8}$$

where $\xi_\mu(t)$ is the Gaussian distributed white stochastic force: $\langle \xi_\mu(t)\xi_\nu'(t') \rangle = \delta_{\mu\nu}\delta(t - t')$ and the diffusion coefficient is given as $\tilde{D}_{\mu\nu} = \frac{1}{2}(\mathbf{BB^T})_{\mu\nu}$.

4. GAUGE FIELD, CURVATURE

With the help of the driving force decomposition:

$$\tilde{F}_\mu = -\tilde{D}_{\mu\nu}\frac{\partial}{\partial x_\nu}U + v_\mu^{SS} \tag{9}$$

we can relate the non-equilibrium *Quantum Fokker-Planck equation* Equation(5) with *Abelian Gauge Theory* and internal curved space, as in *Quantum Electrodynamics (QED)*[15] and classic *Fokker-Planck equation*. Defining the covariant derivative $\nabla_\mu = \partial_\mu - \tilde{D}_{\mu\nu}^{-1}\tilde{F}_\nu = \partial_\mu + A_\mu$ and the *Abelian gauge field* $A_\mu = -\tilde{D}_{\mu\nu}^{-1}\tilde{F}_\nu$, *flux* can be rewritten as the form of covariant derivative: $\tilde{D}_{\mu\nu}\nabla_\nu P = -J_\mu$. As in *Abelian gauge theory*, the curvature of internal charge space is:

$$R_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = [\nabla_\mu, \nabla_\nu]. \tag{10}$$

where $[\cdot]$ indicates a commutator of two operators. According to Equation (7), for the detailed balance case: $\mathbf{J}^{SS} = 0$, $A_\mu = \partial_\mu \ln(P^{SS})$ is a pure gradient and the curvature is zero: $R_{\mu\nu} = 0$ which corresponds to a flat internal space. While for non-equilibrium cases, $\mathbf{A}$ cannot be written as a gradient and $R_{\mu\nu} \neq 0$ which corresponds to a curved internal space. On the other hand, $R_{\mu\nu} = 0$ also means that A_μ can be written as a pure gradient which can lead to a steady state solution P^{SS} to ensure $\mathbf{J}^{SS} = 0$. In other words, $\mathbf{J}^{SS} = 0$ and $R_{\mu\nu}$ provide equivalent measures of whether the detail balance is broken or not. Therefore, by checking the internal phase space curvature $R_{\mu\nu}$, we can know if the system is in detail balance or not without knowing the steady state solution or by solving the steady state flux. In addition, $R_{\mu\nu}$ is a gauge invariant tensor: for a gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \phi$, $R_{\mu\nu} \rightarrow R'_{\mu\nu} = R_{\mu\nu}$. Furthermore, the probabilistic velocity $\mathbf{v}$ and the flux $\mathbf{J}$ are also related to this internal curvature as:

$$\partial_\mu(\tilde{D}_{\nu\sigma}^{-1}v_\sigma) - \partial_\nu(\tilde{D}_{\mu\sigma}^{-1}v_\sigma) = R_{\mu\nu} \tag{11}$$

As we discussed in the previous section, the interactions between different sites introduce the coherence. When the coupling δ_{ij} between different sites vanish, the coherence between different sites will go to zero. Meanwhile, the diffusion matrix $\tilde{D}_{\mu\nu}$ and its inverse matrix $\tilde{D}_{\mu\nu}^{-1}$ will be diagonal. Then, following the linear driving force $\tilde{F}_\mu$ as in Equation (6), we have the gauge field $A_\mu \propto x_\mu$ and the curvature of the internal charge space $R_{\mu\nu} = 0$. In this way, we linked the coherence from the quantum systems to the internal curvature of coherent state space, which only become non-zero in non-equilibrium quantum dynamics.

Similar to *Abelian gauge theory*, we can define the *Wilson loop* or *Wilson line* along any specific path $\zeta(t) = \{\alpha(t), \alpha^*(t)\}$ as:

$$U_{\zeta(t)}(x, y) = e^{-\Delta s_m} = e^{-\int_{\zeta(t)} A_\mu d\zeta_\mu} \tag{12}$$

with

$$\Delta s_m(\zeta(t)) = -\int_{\zeta(t)} A_\mu d\zeta_\mu \tag{13}$$

$$= \int_{\zeta(t)} \tilde{D}_{\mu\nu}^{-1}\tilde{F}_\nu \dot{\zeta}_\mu dt$$

The above expression for Δs_m reminds the definition of the work done by the force or heat dissipation, which may be used as an analog to define quantum work or heat dissipation. Along a closed loop C, we have the *Wilson loop* $U(x, x) = e^{-\Delta s_m^C}$. Using the *Stokess theorem*, the phase factor can be written as:

$$\Delta s_m^C = -\oint_C A_\mu d\zeta_\mu = -\oint_C \tilde{D}_{\mu\nu}^{-1}v_\nu^{SS}d\zeta_\mu$$

$$= -\frac{1}{2}\int_\Sigma d\sigma_{\mu\nu}R_{\mu\nu} \tag{14}$$

where Σ is the surface of the closed loop C, $d\sigma_{ij}$ is the area element on this surface, and R_{ij} is the curvature due to the gauge field $\mathbf{A}$. Under the gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \phi$, the Wilson loop $U(x, y)$ or the exponential of the quantum work, or heat dissipation, transforms as: $U(x, y) \rightarrow e^{\phi(x)}U(x, y)e^{-\phi(y)}$, while R_{ij} and $U(x, x)$, or the exponential of quantum work/heat dissipation are gauge invariant. Therefore, the non-equilibrium quantum dynamics and thermodynamics relate to an internal curved coherent space. Here, the gauge field $\mathbf{A}$ can also be considered as a Berry connection and the curvature R_{ij} as a Berry curvature. The non-zero flux in quantum dynamics breaks the detailed balance, which leads to non-zero internal curvature in the coherent state space and a global topological non-trivial phase analogous to quantum mechanical Berry phase [11]. This can also lead to quantum work and heat dissipation, which is important for quantum thermodynamics. The phase factor of *Wilson line* Δs_m in Equation (13) plays an important role in the time irreversibility for non-equilibrium systems [14, 24–27] and generalized FDT for non-equilibrium dynamics [14].

5. FLUCTUATION-DISSIPATION THEOREM, FLUCTUATION THEOREM, AND IRREVERSIBILITY FOR NONEQUILIBRIUM QUANTUM SYSTEMS

Based on perturbation theories, Fluctuation-Dissipation Theorem (FDT) for classic equilibrium systems under detailed balance has been well studied [28–32]. Furthermore, such investigations are extended to classic nonequilibrium systems under detailed-balance breaking [14]. Since the operator master equation (Equation 5) has the form of the classical Fokker-Planck equation, the extension of FDT to non-equilibrium quantum systems is straight forward [19]: With a linear perturbation applied to the force: $\tilde{F}_\mu \to \tilde{F}'_\mu = \tilde{F}_\mu + h(t)\delta\tilde{F}_\mu$, we have

$$R^\Omega(t-t') \tag{15}$$
$$= \int [d^2\alpha]\Omega e^{\hat{L}(t-t')}\{\delta\tilde{F}_\mu[-\tilde{F}_\nu - v_\nu^{SS}]\tilde{D}_{\mu\nu}^{-1} - \partial_\mu\delta\tilde{F}_\mu\}P^{SS}$$
$$= -\langle\Omega(t)\partial_\mu\delta F_\mu(t')\rangle - \Big[\langle\Omega(t)\delta F_\mu(t')\tilde{F}_\nu(t')\tilde{D}_{\mu\nu}^{-1}(t')\rangle$$
$$+ \langle\Omega(t)\delta F_\mu(t')v_\nu^{SS}(t')\tilde{D}_{\mu\nu}^{-1}(t')\rangle\Big]$$

Equation (15) provides a general relation between the response functions and the correlation functions, which is a general extension of FDT to non-equilibrium quantum systems [19, 26, 27]. Here, $\langle...\rangle$ represents the average over the steady-state quasiprobability distribution P_{SS}. In particular, if the perturbation is independent on α, α^*: $\delta F_{\mu'} = \delta F_{\mu'\mu}$, we obtain

$$R_\mu^\Omega(t-t') = -\langle\Omega(t)\partial_\mu\ln[P^{SS}(\mathbf{x})]\rangle \tag{16}$$
$$= -\Big[\langle\Omega(t)\tilde{F}_\nu(t')\tilde{D}_{\mu\nu}^{-1}(t')\rangle + \langle\Omega(t)v_\nu^{SS}(t')\tilde{D}_{\mu\nu}^{-1}(t')\rangle\Big]$$

Equation (16) is a quantum generalization of FDT for non-equilibrium systems. The response of the system can be decomposed to two terms. The first term, which is present in FDT of equilibrium systems obeying the detailed balance is related to the equilibrium contribution due to the correlation of the variable with the driving force. The second term is directly related to the nontrivial non-zero flux which violates the detailed balance and measures the degree of non-equilibrium (how far away the system is from equilibrium).

For the quantum equilibrium system with detailed balance, we have $\mathbf{J}^{SS} = 0$ and time reversal invariant: $\langle\Omega(t)F_\nu(\mathbf{x}(t'))\rangle = \langle F_\nu(\mathbf{x}(t))\Omega(t')\rangle$. Using the *Langevin* equation (8) in coherent state space, $\langle F_\mu(\mathbf{x}(t))\Omega(t')\rangle = \langle[\dot{x}_\mu(t) - B_{\mu\nu}\xi_\nu(t)]\Omega(t')\rangle = \langle\dot{x}_\mu(t)\Omega(t')\rangle$, since random force will not correlate with Ω of previous time $(t > t')$: $\langle\xi_\mu(t)\Omega(t')\rangle = 0$. Then, we arrive at:

$$R_\mu^\Omega(t-t') = -\tilde{D}_{\mu\nu}^{-1}\Big[\frac{d}{dt}\langle x_\nu(t)\Omega(t')\rangle\Big] \tag{17}$$

Particularly, considering the operator $\Omega(\mathbf{x}) = x_\eta$, we have

$$R_\mu^{x_\eta}(t-t') = -\tilde{D}_{\mu\nu}^{-1}\Big[\frac{d}{dt}\langle x_\nu(t)x_\eta(t)\rangle\Big] \tag{18}$$

It can be considered as the FDT near quantum equilibrium, which is analogous to classic equilibrium FDT[32].

If the quantum system is in non-equilibrium without detailed balance, we have the flux $\mathbf{J} \neq 0$ or curvature $R_{\mu\nu} \neq 0$. Without detailed balance, the quantum system is time irreversible: $\langle\Omega(t)F_\eta(\mathbf{x}(t'))\rangle \neq \langle F_\eta(\mathbf{x}(t))\Omega(t')\rangle$. For general non-equilibrium quantum systems, due to the analogy to the classical case in the coherent state representation, we expect the quantum analog of the classical *Fluctuation theorem* [24–27, 33–37] to have the similar analytical form as:

$$\ln\frac{P^{SS}(\mathbf{x}')\tilde{P}(\mathbf{x},t|\mathbf{x}',t')}{P^{SS}(\mathbf{x})\tilde{P}(\mathbf{x}',t|\mathbf{x},t')} = \Delta s_m + \ln\frac{P^{SS}(\mathbf{x}')}{P^{SS}(\mathbf{x})} \tag{19}$$

with $\tilde{P}(\mathbf{x},t|\mathbf{x}',t')$ ($\tilde{P}(\mathbf{x}',t|\mathbf{x},t')$) indicating the transition probabilities of a forward (backward) path from $\mathbf{x}'$ at time t' to $\mathbf{x}$ at time t (from $\mathbf{x}$ at time t' to $\mathbf{x}'$ at time t). We define the commutator $\langle\Omega(t)F_i(\mathbf{x}(t'))\rangle - \langle F_i(\mathbf{x}(t))\Omega(t')\rangle = \int d\mathbf{x}d\mathbf{x}'\Omega(\mathbf{x})F_i(\mathbf{x}')A(\mathbf{x},\mathbf{x}',t-t')$ with

$$A(\mathbf{x},\mathbf{x}',t-t') \tag{20}$$
$$= P^{SS}(\mathbf{x}')P(\mathbf{x},t|\mathbf{x}',t') - P^{SS}(\mathbf{x})P(\mathbf{x}',t|\mathbf{x},t')$$
$$= P^{SS}(\mathbf{x}')\int D[\mathbf{x}]\tilde{P}(\mathbf{x},t|\mathbf{x}',t')\Big(1 - \frac{P^{SS}(\mathbf{x})}{P^{SS}(\mathbf{x}')}e^{-\Delta s_m}\Big)$$
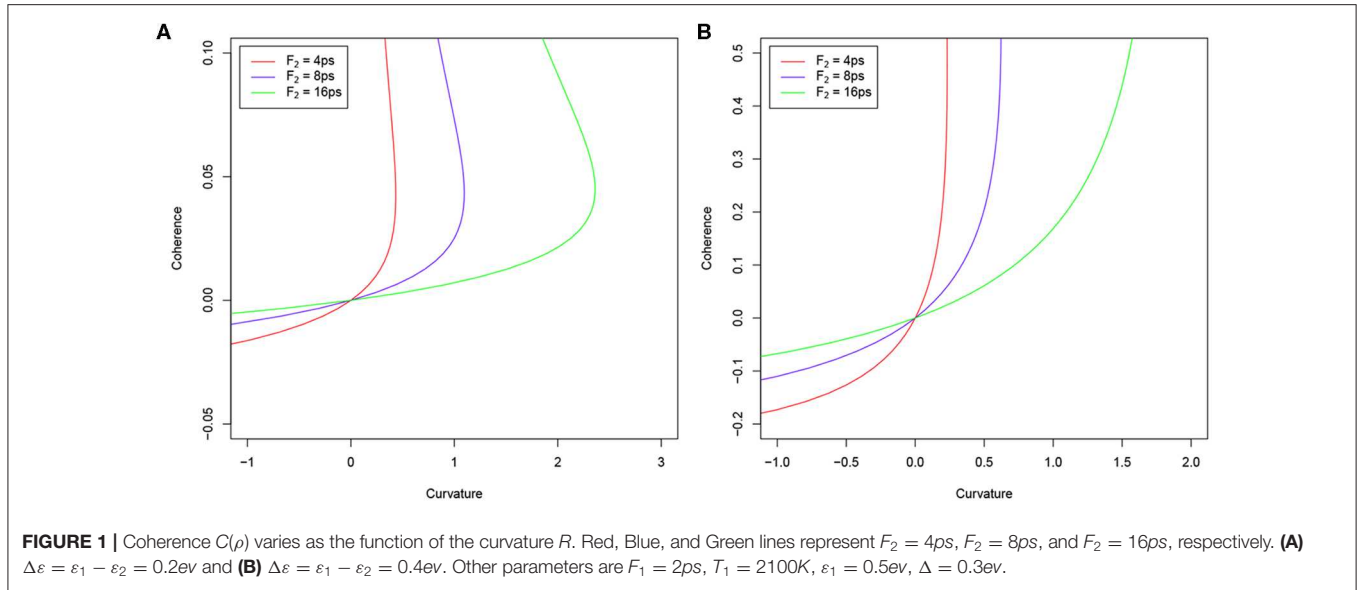
Here, $D[\mathbf{x}]$ is the path integral from $\mathbf{x}'(t')$ to $\mathbf{x}(t)$. Then, we can rewrite the response function, as

$$R_\mu^\Omega(t-t') = -\tilde{D}_{\mu\eta}^{-1}\Big[\frac{d}{dt}\langle x_\eta(t)\Omega(t')\rangle\Big]$$
$$- \tilde{D}_{\mu\eta}^{-1}\int d\mathbf{x}d\mathbf{x}'\Omega(\mathbf{x})F_\eta(\mathbf{x}')A(\mathbf{x},\mathbf{x}',t-t')$$
$$- \tilde{D}_{\mu\eta}^{-1}\langle\Omega(t)v_\eta^{SS}(t')\rangle \tag{21}$$

With the operator $\Omega(\mathbf{x}) = x_\nu$, the response function reads

$$R_\mu^\nu(t-t') = -\tilde{D}_{\mu\eta}^{-1}\Big[\frac{d}{dt}\langle x_\eta(t)x_\nu(t')\rangle\Big]$$
$$- \tilde{D}_{\mu\eta}^{-1}\int d\mathbf{x}d\mathbf{x}'x_\nu F_\eta(\mathbf{x}')A(\mathbf{x},\mathbf{x}',t-t')$$
$$- \tilde{D}_{\mu\eta}^{-1}\langle x_\nu(t)v_\eta^{SS}(t')\rangle \tag{22}$$

The first term is the same as the equilibrium case. The last two terms in Equation (22) are zero for equilibrium cases with detailed balance, which are related to the internal curvature due to the gauge field in space, as shown in Equations (11) and (14). In Equation (20), the path-dependent factor $e^{-\Delta s_m} = U_{\zeta(t)}(\mathbf{x},\mathbf{y})$ is defined as the *Wilson loop* or *Wilson line* due to the internal curvature $R_{\mu\nu}$, as Equation (12).

FIGURE 1 | Coherence $C(\rho)$ varies as the function of the curvature R. Red, Blue, and Green lines represent $F_2 = 4ps$, $F_2 = 8ps$, and $F_2 = 16ps$, respectively. **(A)** $\Delta\varepsilon = \varepsilon_1 - \varepsilon_2 = 0.2ev$ and **(B)** $\Delta\varepsilon = \varepsilon_1 - \varepsilon_2 = 0.4ev$. Other parameters are $F_1 = 2ps$, $T_1 = 2100K$, $\varepsilon_1 = 0.5ev$, $\Delta = 0.3ev$.

6. EXAMPLE

Now we illustrate the FDT for non-equilibrium quantum systems by studying an explicit example: a simple model describing energy transfer simulated by two harmonic oscillators coupled to two environments with different temperatures.

$$H_0 = \bar{\varepsilon}_1 a_1^\dagger a_1 + \bar{\varepsilon}_2 a_2^\dagger a_2 + \bar{\Delta}(a_1^\dagger a_2 + a_2^\dagger a_1)$$
$$+ \sum_{\nu=1}^{2} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} b_{\mathbf{k}}^{(\nu),\dagger} b_{\mathbf{k}}^{(\nu)} \quad (23)$$

and the interactions are

$$H_{int} = \sum_{i=1}^{2} \sum_{\nu=1}^{2} \sum_{\mathbf{k}} f_{\mathbf{k}}^{i(\nu)} (a_i^\dagger b_{\mathbf{k}}^{(\nu)} + a_i b_{\mathbf{k}}^{(\nu),\dagger}) \quad (24)$$

The operator Fokker-Planck equation (in the interaction picture) in the coherent state space Equation (5) reads

$$\frac{\partial P(\alpha_\beta, \alpha_\beta^*)}{\partial t} = \left(\bar{F}_1 \frac{\partial}{\partial \alpha_1} \alpha_1 + \bar{F}_2 \frac{\partial}{\partial \alpha_2} \alpha_2 + c.c. \right) P(\alpha_\beta, \alpha_\beta^*)$$
$$+ \left[\bar{D}_1^1 \frac{\partial^2}{\partial\alpha_1^* \partial\alpha_1} + \bar{D}_2^2 \frac{\partial^2}{\partial\alpha_2^* \partial\alpha_2} \right.$$
$$\left. + 2\bar{D} \left(\frac{\partial^2}{\partial\alpha_1^* \partial\alpha_2} + \frac{\partial^2}{\partial\alpha_1 \partial\alpha_2^*} \right) \right] P(\alpha, \alpha^*)$$

with $\bar{F}_1 = \gamma_1^{(1)} + \gamma_1^{(2)}$, $\bar{F}_2 = \gamma_2^{(1)} + \gamma_2^{(2)}$ and

$$\bar{D}_1^1 = (\gamma_1^{(1)} n_{\omega_1}^{T_1} + \gamma_1^{(2)} n_{\omega_1}^{T_2}) \cos^2\theta + (\gamma_1^{(1)} n_{\omega_2}^{T_1} + \gamma_1^{(2)} n_{\omega_2}^{T_2}) \sin^2\theta,$$
$$\bar{D}_2^2 = (\gamma_2^{(1)} n_{\omega_1}^{T_1} + \gamma_2^{(2)} n_{\omega_1}^{T_2}) \sin^2\theta + (\gamma_2^{(1)} n_{\omega_2}^{T_1} + \gamma_2^{(2)} n_{\omega_2}^{T_2}) \cos^2\theta, \quad (25)$$
$$\bar{D} = \frac{(\gamma_1^{(1)} + \gamma_2^{(1)})(n_{\omega_2}^{T_1} - n_{\omega_1}^{T_1}) + (\gamma_1^{(2)} + \gamma_2^{(2)})(n_{\omega_2}^{T_2} - n_{\omega_1}^{T_2})}{2} \sin\theta\cos\theta$$

and $n_{\omega_i}^T = 1/(e^{\hbar\omega_i/k_B T} - 1)$, $\omega_1 = \frac{\bar{\varepsilon}_1}{\hbar}$, $\omega_2 = \frac{\bar{\varepsilon}_2}{\hbar}$, $\theta = \frac{1}{2} \tan^{-1}(\frac{2\bar{\Delta}}{\bar{\varepsilon}_1 - \bar{\varepsilon}_2})$.

Then, Abelian gauge field can be written as

$$A_\mu = -\tilde{D}_{\mu\nu}^{-1} \tilde{F}_\nu = - \begin{pmatrix} Y_1 \bar{F}_1 \alpha_1^* + Y_3 \bar{F}_2 \alpha_2^* \\ Y_1 \bar{F}_1 \alpha_1 + Y_3 \bar{F}_2 \alpha_2 \\ Y_3 \bar{F}_1 \alpha_1^* + Y_2 \bar{F}_2 \alpha_2^* \\ Y_3 \bar{F}_1 \alpha_1 + Y_2 \bar{F}_2 \alpha_2 \end{pmatrix} \quad (26)$$

where inverse matrix reads

$$\tilde{D}_{\mu\nu}^{-1} = \begin{pmatrix} 0 & Y_1 & 0 & Y_3 \\ Y_1 & 0 & Y_3 & 0 \\ 0 & Y_3 & 0 & Y2 \\ Y_3 & 0 & Y_2 & 0 \end{pmatrix} \quad (27)$$

with elements $Y_1 = \frac{\bar{D}_2^2}{\bar{D}_1^1 \bar{D}_2^2 - \bar{D}*\bar{D}}$, $Y_2 = \frac{\bar{D}_1^1}{\bar{D}_1^1 \bar{D}_2^2 - \bar{D}*\bar{D}}$, and $Y_3 = \frac{\bar{D}}{\bar{D}*\bar{D} - \bar{D}_1^1 \bar{D}_2^2}$. It easy to calculate the curvature of the internal charge space due to the non-trivial Abelian gauge field A_μ:

$$R_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & R \\ 0 & 0 & R & 0 \\ 0 & -R & 0 & 0 \\ -R & 0 & 0 & 0 \end{pmatrix} \quad (28)$$

with

$$R = Y_3(\bar{F}_1 - \bar{F}_2) \quad (29)$$

Therefore, without calculating steady state solution or the steady state flux, we can obtain the curvature $R_{\mu\nu}$. It is noticed that the internal curvature $R_{\mu\nu}$ is proportional to the $Y_3 \sim \bar{D}(\bar{F}_1 - \bar{F}_2)$. If internal curvature vanishes, the coherence vanishes $\bar{F}_1 neq \bar{F}_2$. The internal curvature can

promote the emergence of the steady state coherence. On the other hand, as we mentioned in the beginning of the article, when the coupling between different sites vanishes, it leads to vanishing coherence between different site $\bar{D} = 0$ or zero quantum correlations and the zero internal curvature $R_{\mu\nu} = 0$. The detailed-balance is more broken as the temperature difference between two baths increases, leading to higher coherence.

In addition, the steady state of the two quantum oscillators under the two baths is exactly solvable, which has the form of:

$$P_{SS}(\alpha, \alpha^*) = \frac{1}{A} e^{-[A_1|\alpha_1|^2 + A_2|\alpha_2|^2 + BRe(\alpha_1\alpha*_2)]} \quad (30)$$

with

$$A_1 = \frac{(\bar{F}_1 + \bar{F}_2)\bar{D}_2^2}{\bar{D}_1^1\bar{D}_2^2\frac{\bar{F}_1+\bar{F}_2}{\bar{F}_1} - 4\frac{\bar{F}_2}{\bar{F}_1+\bar{F}_2}\bar{D}*\bar{D}} \quad (31)$$

$$A_2 = \frac{(\bar{F}_1 + \bar{F}_2)\bar{D}_1^1}{\bar{D}_1^1\bar{D}_2^2\frac{\bar{F}_1+\bar{F}_2}{\bar{F}_2} - 4\frac{\bar{F}_1}{\bar{F}_1+\bar{F}_2}\bar{D}*\bar{D}} \quad (32)$$

$$B = \frac{2\bar{D}}{\bar{D}_1^1\bar{D}_2^2\frac{\bar{F}_1+\bar{F}_2}{\bar{F}_1\bar{F}_2} - 4\frac{1}{\bar{F}_1+\bar{F}_2}\bar{D}*\bar{D}} \quad (33)$$

With a given density matrix, the coherence

$$C(\rho) = Tr(\rho_s a_1^\dagger a_2) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\langle n_1-1, n_2|\rho_s|n_1, n_2-1\rangle \quad (34)$$

can be used to quantify the non-local correlations between the vibrational modes of spatially separated sites, from the combination of off-diagonal elements of the density matrix in the Fock space. In our model, the coherence can be written as:

$$C(\rho) = \frac{2\bar{D}}{\bar{F}_1 + \bar{F}_2} \quad (35)$$

When the coupling between different sites $\bar{D} = 0$, we have $B = 0$, which leads to $C(\rho) = 0$ and the curvature $R = 0$. This means, when the coupling between different sites vanishes, the coherence between spatially separated sites vanishes at the steady state. In **Figures 1A,B**, we plot the coherence $C(\rho)$ vs curvature R with different sets of parameters. One observation is that all curves cross the same points: when the coherence $C(\rho)$ is zero, the curvature R is 0. The coherence $C(\rho)$ depends on the curvature R in a nonlinear and non-monotonic way. In the near-equilibrium region where $R \approx 0$, the quantum coherence $C(\rho)$ increases with the internal curvature R.

Therefore, quantum coherence is naturally connected to the internal curvature of the gauge field. Without quantum coherence, the gauge field is trivial associated with flat internal space. On the other hand, when the internal space is not flat and

the gauge field is non-trivial, the detailed-balance is broken with non-local quantum coherence. This provides a new fundamental view in quantum physics: the non-local coherence can emerge from the non-equilibrium detailed-balance breaking which can be measured by an internal curvature of the gauge field in phase space.

The steady state curl quantum flux is of the form:

$$v_{\alpha_1}^{SS} = \frac{J_{\alpha_1}^{SS}}{P_{SS}} = (v_{\alpha_1^*}^{SS})^* \quad (36)$$

$$= \frac{\bar{D}*\bar{D}(\frac{1}{2} - \frac{\bar{F}_1}{\bar{F}_1+\bar{F}_2})A_1}{(\bar{F}_1 + \bar{F}_2)\bar{D}_2^2}\alpha_1 + (\frac{\bar{F}_1}{\bar{F}_1+\bar{F}_2} - \frac{1}{2})\bar{D}A_2\alpha_2$$

$$v_{\alpha_2}^{SS} = \frac{J_{\alpha_2}^{SS}}{P_{SS}} = (v_{\alpha_2^*}^{SS})^* \quad (37)$$

$$= \frac{\bar{D}*\bar{D}(\frac{1}{2} - \frac{\bar{F}_2}{\bar{F}_1+\bar{F}_2})A_2}{(\bar{F}_1 + \bar{F}_2)\bar{D}_1^1}\alpha_2 + (\frac{\bar{F}_2}{\bar{F}_1+\bar{F}_2} - \frac{1}{2})\bar{D}A_1\alpha_1$$

Here, from steady state curl quantum flux, we have the same observations as from the internal curvature $R_{\mu\nu}$ discussed above.

On the other hand, from the coherent space $\{\alpha, \alpha^*\}$ to the Fock space, the coherence between α_i and α_j is equivalent to the coherence or coupling between eigenstate of different sites: $|n_i\rangle$ and $|n_j\rangle$, since the coherence is introduced by the interaction term $\sum_{i\neq j}\Delta_{ij}a_i^\dagger a_j$. If the coupling between different sites vanishes, we have $\bar{D} \to 0$ and no interactions between sites, which leads to the curvature $R_{\mu\nu} \to 0$ and steady curl quantum flux $v_\mu^{SS} \to 0$. Therefore, there will be no quantum correlations regardless of the environmental conditions.

7. CONCLUSION

In this study, we have uncovered that non-equilibrium quantum dynamics gives rise to an intrinsic geometric curvature which can enhance quantum coherence. The non-equilibrium can be characterized by the curvature. This may help reveal an intrinsic connection between the space time geometry/topology and the quantum nature. On the one hand, curved space time may emerge from the non-equilibrium quantum dynamics, on the other hand, the intrinsic underlying curved space time may provide a possible channel (space time shortcut) or physical origin for the non-local quantum correlations such as coherence and entanglement. Furthermore, we illustrated that intrinsic curvature could lead to new fluctuation-dissipation theorem for non-equilibrium quantum systems. Therefore, the curved space time geometry/topology from non-equilibrium can give rise to new types of fluctuations in addition to the original spontaneous fluctuations. The non-equilibrium response is now linked not only to the spontaneous fluctuations around the equilibrium but also to the non-equilibrium fluctuations that originated from the curved geometry/topology or non-equilibrium. We have also shown that the curved geometry/topology characterized by the intrinsic curvature

from non-equilibrium can generate quantum work and heat dissipation important for quantum thermodynamics. We believe our approach and results in the current study are general and can be applied to further study many interesting non-equilibrium quantum systems.

AUTHOR CONTRIBUTIONS

HF and JW have contributed to the organization, research performance, and writing of this article.

SUPPLEMENTARY MATERIAL

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fphy.2020.00129/full#supplementary-material

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