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Necessary and Sufficient Conditions for Expressing Quadratic Rational Bézier Curves

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Quadratic rational Bézier curve transformation is widely used in the field of computational geometry. In this paper, we offer several important characteristics of the quadratic rational Bézier curve. More precisely, on the basis of proving its monotonicity, the necessary and sufficient conditions for transforming a quadratic rational Bézier curve into a point, line segment, parabola, elliptic arc, circular arc, and hyperbola are proved, respectively.

Keywords: Bézier curve, quadratic rational, necessary and sufficient conditions, geometry, computer aided geometric design

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1. INTRODUCTION

Bézier curves have wide application in computer-aided geometric design, being used to provide precisely described points along a given curve [1]. Compared to other methods, such as the French curve, Bézier-based approaches are more computationally affordable and reliable. Additionally, the advantages of the Bézier curve in geometric design include its simple but clear mathematical function [2]. For instance, it is capable of incorporating both conic sections and parametric cubic curves as special cases [3]. As such, one can deal with two different curves simultaneously using one unique computational procedure. Some preliminary studies and applications of Bézier curves can be found in Lu et al. [4], Lee [5], and Han [6].

In this paper, to better understand the basic characteristics of Bézier curves, we conduct some fundamental research. In particular, we discuss the necessary and sufficient conditions for representing six different basic shapes, including a point, line segment, parabola, elliptic arc, circular arc, and hyperbola, using Bézier curves [7, 8]. These results play a fundamental role in the shape formulation and can help in facilitating any subsequent computer-based geometric design.

To begin with, we introduce the mathematical model of the quadratic rational Bézier curve [1].

Definition 1. *The quadratic rational Bézier curve is defined as follows:*

$$p(t) = \frac{(1-t)^2\omega_0 P_0 + 2t(1-t)\omega_1 P_1 + t^2\omega_2 P_2}{(1-t)^2\omega_0 + 2t(1-t)\omega_1 + t^2\omega_2}, \quad t \in [0,1], \tag{1}$$

where

$$t = \frac{\sqrt{\omega_0}\mu}{\sqrt{\omega_0}\mu + \sqrt{\omega_2}(1-\mu)}, \quad \mu \in [0,1], \tag{2}$$

and ω_0 and ω_2 are not zero values at the same time.

The monotonicity of Formula (1.2) is discussed below. Let $\mu_1 \in [0,1], \mu_2 \in [0,1]$, and $\mu_1 \leq \mu_2$. Accordingly, in the case of $\mu_1 = 0$, we have:

$$t_1 = \frac{\sqrt{\omega_0}\mu_1}{\sqrt{\omega_0}\mu_1 + \sqrt{\omega_2}(1-\mu_1)} = 0. \tag{3}$$

Note that $1 \geq \mu_2 > \mu_1 \geq 0$, and $t_2 = \frac{\sqrt{\omega_0}\mu_2}{\sqrt{\omega_0}\mu_2+\sqrt{\omega_2}(1-\mu_2)} \geq 0$; then it is easy to have $t_2 \geq t_1 = 0$.

In the case of $\mu_1 \neq 0$ and $\mu_2 \neq 0$, according to Formula (1.2), we have:

$$\begin{aligned}\frac{t_1}{t_2} &= \frac{\sqrt{\omega_0}\mu_1}{\sqrt{\omega_0}\mu_1 + \sqrt{\omega_2}(1-\mu_1)} \Big/ \frac{\sqrt{\omega_0}\mu_2}{\sqrt{\omega_0}\mu_2 + \sqrt{\omega_2}(1-\mu_2)} \\ &= (\sqrt{\omega_0} + \sqrt{\omega_2}(\frac{1}{\mu_2} - 1)) \,/\, (\sqrt{\omega_0} + \sqrt{\omega_2}(\frac{1}{\mu_1} - 1)) \leq 1.\end{aligned} \tag{4}$$

In other words, we have the conclusion that t is monotonically increasing [9–11]. Furthermore, if we apply linear transformation to Formula (1.1), it is easy to know

$$\begin{aligned}p(\mu) &= \frac{\omega_0\omega_2(1-\mu)^2P_0 + 2\sqrt{\omega_0}\sqrt{\omega_2}\omega_1\mu(1-\mu)P_1 + \omega_0\omega_2\mu^2P_2}{\omega_0\omega_2(1-\mu)^2 + 2\sqrt{\omega_0}\sqrt{\omega_2}\omega_1\mu(1-\mu) + \omega_0\omega_2\mu^2} \\ &= ((1-\mu)^2P_0 + 2\sqrt{\frac{\omega_1^2}{\omega_0\omega_2}}\mu(1-\mu)P_1 + \mu^2P_2)/((1-\mu)^2 \\ &+ 2\sqrt{\frac{\omega_1^2}{\omega_0\omega_2}}\mu(1-\mu) + \mu^2).\end{aligned} \tag{5}$$

Let $\omega = \sqrt{\frac{\omega_1^2}{\omega_0\omega_2}}$ and substitute μ with t in the standard form of the quadratic rational Bézier curve. To this end, we have the simplified version of the quadratic rational Bézier curve, which is expressed as follows:

$$p(t) = \frac{(1-t)^2P_0 + 2\omega t(1-t)P_1 + t^2P_2}{(1-t)^2 + 2\omega t(1-t) + t^2}. \tag{6}$$

2. SUFFICIENT AND NECESSARY CONDITIONS FOR A QUADRATIC RATIONAL BÉZIER CURVE TO DEGENERATE INTO A POINT

Theorem 1. *A quadratic rational Bézier curve degenerates into a point if and only if three control points P_0, P_1, P_2 coincide.*

Proof: Assume that the quadratic rational Bézier curve degenerates to a point P_A. That is,

$$\begin{aligned}p(t) = \frac{(1-t)^2P_0 + 2\omega t(1-t)P_1 + t^2P_2}{(1-t)^2 + 2\omega t(1-t) + t^2} = P_A \Leftrightarrow \\ (1-t)^2(P_0 - P_A) + 2t(1-t)\omega(P_1 - P_A) + t^2(P_2 - P_A) = 0.\end{aligned} \tag{7}$$

As can be seen from Formula (7), when $t \in (0,1)$, we have $(1-t)^2 \neq 0, t^2 \neq 0, 2t(1-t) \neq 0$, so $P_0 = P_A, P_1 = P_A, P_2 = P_A$. That is, when the quadratic rational Bézier curve degenerates into a point, P_0, P_1, P_2 are the same point of P_A.

On the other hand, when three control points coincide (say, the same point P_A), we know that:

$$p(t) = \frac{(1-t)^2P_A + 2t(1-t)\omega P_A + t^2P_A}{(1-t)^2 + 2t(1-t)\omega + t^2} = P_A. \tag{8}$$

As can be seen from Formula (8), when three control points P_0, P_1, P_2 coincide, the quadratic rational Bézier curve degenerates into a point [12, 13].

Algorithm 1: To degenerate a Quadratic Rational Bézier Curve into a Point

Input: Control Points of Bezier Curve
Output: Points degenerated by Bezier Curve

1: *Input* Bézier Curve control points
2: *Set* coordinates of control points $P_1 = P_0$ and $P_2 = P_0$
3: *Output* coordinates of control points P_0, P_1, P_2
4: **if** the number of control points < 3 **then**
5: **goto** Step 1.
6: **end if**
7: Initializing the independent variable t in the standard formula of the quadratic rational Bezier curve to 0, *Set* $t = 0$
8: **for** $t = 0; t \leq 1; t += 0.00125$ **do**
9: Calculate the standard formula of the quadratic rational Bezier Curve.
10: $x = \frac{(1-t)^2x_0+2t(1-t)\omega x_1+t^2x_2}{(1-t)^2+2t(1-t)\omega+t^2}, y = \frac{(1-t)^2y_0+2t(1-t)\omega y_1+t^2y_2}{(1-t)^2+2t(1-t)\omega+t^2}$
11: **end for**
12: *Output* Bezier Curve.
13: *Clear* Bezier Curve, Bezier Curve control points. **goto** Step 1.
14: **return**

3. NECESSARY AND SUFFICIENT CONDITIONS FOR DEGRADATION OF A QUADRATIC RATIONAL BÉZIER CURVE INTO A LINEAR SECTION

Theorem 2. *The quadratic rational Bézier curve degenerates into a straight line segment if and only if the control points P_0, P_2 do not coincide, the weight factor $\omega = 0$, or the control point P_1 is on the line segment* [14–16].

Proof: First, we assume that one point is with two coordinates; alternatively, we have $P_0 = (x_0, y_0)$, $P_1 = (x_1, y_1)$, and $P_2 = (x_2, y_2)$. As such, for an arbitrary point $p(t) = (x, y)$, according to Formula (6) it is easy to have:

$$\begin{aligned}x &= \frac{(1-t)^2x_0 + 2t(1-t)\omega x_1 + t^2x_2}{(1-t)^2 + 2t(1-t)\omega + t^2}, \\ y &= \frac{(1-t)^2y_0 + 2t(1-t)\omega y_1 + t^2y_2}{(1-t)^2 + 2t(1-t)\omega + t^2},\end{aligned} \tag{9}$$

On the other hand, a general form for a line function can be expressed as: $y = ax + b$, where a, and b is a constant [17]. Then, substituting x and y using Formula (12), we can get:

$$\begin{aligned}&\frac{(1-t)^2y_0 + 2t(1-t)\omega y_1 + t^2y_2}{(1-t)^2 + 2t(1-t)\omega + t^2} \\ &= a\,\frac{(1-t)^2x_0 + 2t(1-t)\omega x_1 + t^2x_2}{(1-t)^2 + 2t(1-t)\omega + t^2} + b,\end{aligned} \tag{10}$$

If we simplify the above formula, it is easy to know:

$$\begin{aligned}&(y_0 - ax_0 - b + y_2 - ax_2 - b - 2y_1\omega + 2ax_1\omega + 2b\omega)t^2 \\ &- 2(y_0 - ax_0 - b - y_1\omega + ax_1\omega + b\omega)t + y_0 - ax_0 - b = 0.\end{aligned} \tag{11}$$

First, we assume that one point is with two coordinates; alternatively, we have $P_0 = (x_0, y_0)$, $P_1 = (x_1, y_1)$, and $P_2 = (x_2, y_2)$. As such, for an arbitrary point $p(t) = (x, y)$, according to Formula (6) it is easy to have:

$$\begin{aligned}x &= \frac{(1-t)^2x_0 + 2t(1-t)\omega x_1 + t^2x_2}{(1-t)^2 + 2t(1-t)\omega + t^2}, \\ y &= \frac{(1-t)^2y_0 + 2t(1-t)\omega y_1 + t^2y_2}{(1-t)^2 + 2t(1-t)\omega + t^2},\end{aligned} \tag{12}$$

Now, control points P_0 and P_2 are the first and last points of the Bézier curve. As they are all on the Bézier curve, they will also be on the straight line [18–20]. Alternatively, we have:

$$y_0 = ax_0 + b, y_2 = ax_2 + b. \tag{13}$$

Therefore, Formula (11) is further simplified:

$$(y_1 - ax_1 - b)(\omega t - \omega t^2) = 0. \tag{14}$$

Next, Formula (14) is analyzed in the following aspects:

1. If the control point P_1 is also on the Bézier curve (or on the straight line), then $y_1 - ax_1 - b = 0$, and Formula (14) clearly holds.
2. If the control point P_1 is not on the Bézier curve (or not on the straight line), then $y_1 - ax_1 - b \neq 0$, and Formula (14) can be simplified as
$$-\omega t^2 + \omega t = 0. \tag{15}$$

Therefore, when $t \in [0, 1]$, in order to make Formula (15) hold, we have $\omega = 0$.

As such, it is proved that when the quadratic rational Bézier curve degenerates into a straight line segment, two conditions are met: (1) the weight factor $\omega = 0$, or (2) the control point P_1 is on the line segment with the control point P_0, P_2 as the end point. In the following, we discuss these two conditions separately.

1. According to Formula (6), when the weight factor $\omega = 0$, we have:

$$p(t) = \frac{(1-t)^2P_0 + 2\omega t(1-t)P_1 + t^2P_2}{(1-t)^2 + 2\omega t(1-t) + t^2} = \frac{(1-t)^2P_0 + t^2P_2}{(1-t)^2 + t^2}, \tag{16}$$

and

$$x = \frac{(1-t)^2x_0 + t^2x_2}{(1-t)^2 + t^2} = \frac{(1-t)^2x_0}{(1-t)^2 + t^2} + \frac{t^2x_2}{(1-t)^2 + t^2}. \tag{17}$$

$$y = \frac{(1-t)^2y_0 + t^2y_2}{(1-t)^2 + t^2} = \frac{(1-t)^2y_0}{(1-t)^2 + t^2} + \frac{t^2y_2}{(1-t)^2 + t^2}. \tag{18}$$

To simplify the calculation process, let us assume that:

$$\alpha = \frac{(1-t)^2}{(1-t)^2 + t^2}. \tag{19}$$

and

$$1 - \alpha = \frac{t^2}{(1-t)^2 + t^2}. \tag{20}$$

Now the following formula holds:

$$x = \alpha x_0 + (1-\alpha)x_2 \rightarrow x - x_2 = \alpha(x_0 - x_2). \tag{21}$$

$$y = \alpha y_0 + (1-\alpha)y_2 \rightarrow y - y_2 = \alpha(y_0 - y_2). \tag{22}$$

As the control points P_0, P_2 do not coincide, $x_0 \neq x_2, y_0 \neq y_2$,

$$\alpha = \frac{y - y_2}{y_0 - y_2} = \frac{x - x_2}{x_0 - x_2} \tag{23}$$

$$\frac{y}{y_0 - y_2} - \frac{x}{x_0 - x_2} = \frac{y_2}{y_0 - y_2} - \frac{x_2}{x_0 - x_2}, \tag{24}$$

where x_0, x_2, y_0, y_2 are constants. We assume that $\frac{1}{y_0 - y_2} = A$, $\frac{1}{x_0 - x_2} = B$, $\frac{y_2}{y_0 - y_2} - \frac{x_2}{x_0 - x_2} = C$ (that is, A,B,C are all constants). Accordingly, we know that $Ay - Bx = C$ is a line segment [21].

Algorithm 2: To Degenerate a Quadratic Rational Bézier Curve into a Linear section

Input: Control Points of Bezier Curve
Output: Linear section degenerated by Bezier Curve
1: *Set* $\omega = 0$
2: *Input* Bézier Curve control points P_0, P_1, P_2
3: **if** the number of control points < 3 **then**
4: **goto** Step 2.
5: **end if**
6: *Output* coordinates of control points P_0, P_1, P_2
7: *Output* line segment between control points P_0, P_1 and P_1, P_2
8: Initializing the independent variable t in the standard formula of the quadratic rational Bezier curve to 0, *Set* $t = 0$
9: **for** $t = 0; t \leq 1; t+ = 0.00125$ **do**
10: Calculate the standard formula of the quadratic rational Bezier Curve.
11: $x = \frac{(1-t)^2x_0 + 2t(1-t)\omega x_1 + t^2x_2}{(1-t)^2 + 2t(1-t)\omega + t^2}, y = \frac{(1-t)^2y_0 + 2t(1-t)\omega y_1 + t^2y_2}{(1-t)^2 + 2t(1-t)\omega + t^2}$
12: **end for**
13: *Output* Bezier Curve.
14: *Clear* Bezier Curve, Bezier Curve control points. **goto** Step 1.
15: **return**

2. Let the conditional control point P_1 be the end point (on the line segment with the control points P_0 and P_2) [22]; thus, it can be seen that:

$$P_1 = (1-v)P_0 + vP_2, \quad v \in [0,1], \tag{25}$$

The Formula (25) can be substituted with Formula (6) to have:

$$p(t) = \frac{(1-t)^2 + 2t(1-t)\omega(1-v)}{(1-t)^2 + 2t(1-t)\omega + t^2} P_0 + \frac{(2t(1-t)\omega v + t^2}{(1-t)^2 + 2t(1-t)\omega + t^2} P_2. \tag{26}$$

Then we set:

$$u = \frac{(2t(1-t)\omega v + t^2}{(1-t)^2 + 2t(1-t)\omega + t^2}. \tag{27}$$

Comparing Formula (26) with Formula (27), it is easy to find that

$$p(t) = (1-u)P_0 + uP_2. \tag{28}$$

In conclusion, Formula (28) is the parametric formula of the line segment. When the control point P_1 is on the line segment with the control point (P_0, P_2) as the end point, Formula (26) can be written as the parametric formula of the line segment of Formula (28). As such, it is proved that it degenerates into a line segment [23, 24].

Algorithm 3: To degenerate a Quadratic Rational Bézier Curve into a Linear section

Input: Control Points of Bezier Curve
Output: Linear section degenerated by Bezier Curve

1: *Set ωisaarbitraryvalue*
2: *Input* Bézier Curve control points, P_0, P_1, P_2
3: **if** the number of control points < 3 **then**
4: **goto** Step 2.
5: **else**
6: **if** the number of control points $= 3$ **then**
7: Set control point P_3 on line segment with the control points P_1 and P_2 as end points
8: **end if**
9: **end if**
10: *Output* coordinates of control points P_0, P_1, P_2
11: *Output* line segment between control points P_0, P_1 and P_1, P_2
12: Initializing the independent variable t in the standard formula of the quadratic rational Bezier curve to 0, *Set* $t = 0$
13: **for** $t = 0; t \le 1; t+ = 0.00125$ **do**
14: Calculate the standard formula of the quadratic rational Bezier Curve.
15: $x = \frac{(1-t)^2 x_0 + 2t(1-t)\omega x_1 + t^2 x_2}{(1-t)^2 + 2t(1-t)\omega + t^2}, y = \frac{(1-t)^2 y_0 + 2t(1-t)\omega y_1 + t^2 y_2}{(1-t)^2 + 2t(1-t)\omega + t^2}$
16: **end for**
17: *Output* Bezier Curve.
18: *Clear* Bezier Curve, Bezier Curve control points. **goto** Step 1.
19: **return**

4. NECESSARY AND SUFFICIENT CONDITIONS FOR A QUADRATIC RATIONAL BÉZIER CURVE TO REPRESENT A SECTION OF ARC

Theorem 3. *Quadratic rational Bézier curves can be used to represent an arc if and only if* $|P_0P_1| = |P_2P_1|$ *and* $0 \le \omega \le 1$ [25].

Proof: The equation of a circle passing through three collinear points $Q_i(x_i, y_i)$, $(i = 1, 2, 3)$, on a rectangular coordinate plane is:

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_0^2+y_0^2 & x_0 & y_0 & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{29}$$

Given three points that are not collinear, we have:

$$P_0(x_0, y_0, 0) = (-a, 0, 0), A(x_A, y_A, 0), P_2(x_2, y_2, 0) = (a, 0, 0). \tag{30}$$

The arc curve starts from point P_0 and passes through point A to point P_2. Now, let us find another control vertex P_1. To do so, P_0, A, P_2 are substituted into the three-point common-circle equation 29, and we get

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_0^2+y_0^2 & x_0 & y_0 & 1 \\ x_A^2+y_A^2 & x_A & y_A & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{31}$$

From Formula (30) to Formula (31), we can find that $x_0 = -a, y_0 = 0, x_2 = a, y_2 = 0$. Furthermore, by expanding the determinant 31 in the first row, we have

$$(x^2+y^2)\begin{vmatrix} -a & 0 & 1 \\ 0 & y_A & 1 \\ a & 0 & 1 \end{vmatrix} + (-1)x\begin{vmatrix} a^2 & 0 & 1 \\ y_A^2 & y_A & 1 \\ a^2 & 0 & 1 \end{vmatrix} + y\begin{vmatrix} a^2 & -a & 1 \\ y_A^2 & 0 & 1 \\ a^2 & a & 1 \end{vmatrix} + (-1)\begin{vmatrix} a^2 & -a & 0 \\ y_A^2 & 0 & y_A \\ a^2 & a & 0 \end{vmatrix} = 0. \tag{32}$$

Among them,

$$\begin{vmatrix} -a & 0 & 1 \\ 0 & y_A & 1 \\ a & 0 & 1 \end{vmatrix} = -2ay_A, \quad \begin{vmatrix} a^2 & 0 & 1 \\ y_A^2 & y_A & 1 \\ a^2 & 0 & 1 \end{vmatrix} = 0,$$
$$\begin{vmatrix} a^2 & -a & 1 \\ y_A^2 & 0 & 1 \\ a^2 & a & 1 \end{vmatrix} = -a^3 + ay_A^2 + ay_A^2 - a^3,$$
$$\begin{vmatrix} a^2 & -a & 0 \\ y_A^2 & 0 & y_A \\ a^2 & a & 0 \end{vmatrix} = -a^3 y_A - a^3 y_A. \tag{33}$$

Finally, the above formula can be simplified as follows:

$$(-2ay_A)(x^2+y^2)+y(-a^3+2ay_A^2-a^3)+2a^3y_A=0. \quad (34)$$

Because $y_A \neq 0$, it is easy to know

$$x^2+\left(y+\frac{a^2-y_A^2}{2y_A}\right)^2=a^2+\frac{(a^2-y_A^2)^2}{4y_A^2}. \quad (35)$$

On the other hand, as $x_A=0$, we can add x_A to have

$$x^2+\left(y+\frac{a^2-(x_A^2+y_A^2)}{2y_A}\right)^2=a^2+\frac{(a^2-(x_A^2+y_A^2))^2}{4y_A^2}. \quad (36)$$

Summarizing the above formula, the coordinates of the center of the circle O are:

$$x_O=0, \quad y_O=\frac{a^2-(x_A^2+y_A^2)}{2y_A}. \quad (37)$$

The radius of the circle is:

$$r=\sqrt{a^2+\frac{(a^2-(x_A^2+y_A^2))^2}{4y_A^2}}. \quad (38)$$

The vertical lines of OP_0 and OP_2 are made from points P_0 and P_2, respectively. According to the symmetry, if two vertical lines intersect with the Y axis at point P_1, then point P_1 is the control vertex of the arc curve. That is,

$$y_1=\frac{2a^2y_A}{a^2-(x_A^2+y_A^2)}. \quad (39)$$

Accordingly, the coordinates of point P_1 are:

$$x_1=0, \quad y_1=\frac{2a^2y_A}{a^2-(x_A^2+y_A^2)}. \quad (40)$$

From the definition of the Bézier Curve in Formula (1), we have:

$$\begin{aligned} x(t)&=\frac{(1-t)^2\omega_0x_0+2t(1-t)\omega_1x_1+t^2\omega_2x_2}{(1-t)^2\omega_0+2t(1-t)\omega_1+t^2\omega_2},\\ y(t)&=\frac{(1-t)^2\omega_0y_0+2t(1-t)\omega_1y_1+t^2\omega_2y_2}{(1-t)^2\omega_0+2t(1-t)\omega_1+t^2\omega_2}. \end{aligned} \quad (41)$$

To simply Formula (41), we further introduce the Quadratic Bernstein Basis Function ($B_{i,2}(t)$), which can be expressed as follows:

$$B_{0,2}(t)=(1-t)^2, \quad B_{1,2}(t)=2t(1-t), \quad B_{2,2}(t)=t^2. \quad (42)$$

As such, Formula (41) can be rewritten by applying $B_{i,2}(t)$ in the following format:

$$\begin{aligned} x(t)&=\frac{-a\omega_0B_{0,2}(t)+a\omega_2B_{2,2}(t)}{\omega_0B_{0,2}(t)+\omega_1B_{1,2}(t)+\omega_2B_{2,2}(t)},\\ y(t)&=\frac{2t(1-t)\omega_1y_1}{\omega_0B_{0,2}(t)+\omega_1B_{1,2}(t)+\omega_2B_{2,2}(t)}. \end{aligned} \quad (43)$$

On the other hand, note that the standard equation of curve arc circle can be estimated as

$$x^2(t)+(y(t)+a\cot\theta)^2=a^2/\sin^2\theta. \quad (44)$$

Consequently, by substituting Formulas (43) into Equation (44), the following results are obtained:

$$\begin{aligned} &\left(\frac{-a\omega_0B_{0,2}(t)+a\omega_2B_{2,2}(t)}{\omega_0B_{0,2}(t)+\omega_1B_{1,2}(t)+\omega_2B_{2,2}(t)}\right)^2\\ &\quad+\left(\frac{\omega_1y_1B_{1,2}(t)}{\omega_0B_{0,2}(t)+\omega_1B_{1,2}(t)+\omega_2B_{2,2}(t)}+a\cot\theta\right)^2=\frac{a^2}{\sin^2\theta}, \end{aligned} \quad (45)$$

Note that

$$\frac{a^2}{\sin^2\theta}-a^2\cot^2\theta=a^2. \quad (46)$$

As such, Formula (45) can be further simplified as

$$\begin{aligned} &a^2\omega_0^2B_{0,2}^2(t)+a^2\omega_2^2B_{2,2}^2(t)-2a^2\omega_0\omega_2B_{0,2}(t)B_{2,2}(t)+\omega_1^2y_1^2B_{1,2}^2(t)\\ &\quad+(2\omega_1y_1B_{1,2}(t)a\cot\theta(\omega_0B_{0,2}(t)+\omega_1B_{1,2}(t)+\omega_2B_{2,2}(t))\\ &\quad=a^2(\omega_0B_{0,2}(t)+\omega_1B_{1,2}(t)+\omega_2B_{2,2}(t))^2 \end{aligned} \quad (47)$$

Furthermore, according to Formula (38) and Formula (40), we can have

$$y_1\cot\theta=\frac{2a^2y_A}{a^2-(x_A^2+y_A^2)}\times\frac{(a^2-(x_A^2+y_A^2))}{2ay_A}=a, \quad (48)$$

and then,

$$(y_1^2+a^2)\omega_1^2B_{1,2}^2(t)-4a^2\omega_0\omega_2B_{0,2}(t)B_{2,2}(t)=0. \quad (49)$$

Again, we consider the Quadratic Bernstein Basis Function, and then the above formula (in Formula 49) can be simplified as follows:

$$((y_1^2+a^2)\omega_1^2-a^2\omega_0\omega_2)(1-t)^2t^2=0. \quad (50)$$

Next, according to Formula (40), we know

$$(\omega_1^2\sec^2\theta-\omega_0\omega_2)(1-t)^2t^2=0, \quad (51)$$

and $t\in(0,1)$, $t^2(1-t)^2\neq 0$. It is thus easy to know

$$\omega_1^2=\omega_0\omega_2\cos^2\theta. \quad (52)$$

According to the standard form of the quadratic rational Bézier curve (see Formula 6), we can further estimate $\omega_0=\omega_2=1, \omega_1=\cos\theta$, and the value range of θ of the center angle of the semicircle should be $0\leq\theta\leq\pi/2$ [26].

In summary, the rational quadratic Bézier expressions of arc curves passing through points P_0, A, P_2 are as follows,

$$C(t)=\frac{(1-t)^2P_0+2\cos(\theta)t(1-t)P_1+t^2P_2}{(1-t)^2+2\cos(\theta)t(1-t)+t^2}. \quad (53)$$

Compared with the standard formula of a rational quadratic Bézier, the following results are obtained,

$$\omega = \cos(\theta), \tag{54}$$

where $0 \leq \theta \leq \pi/2$, $0 \leq \omega \leq 1$. Consequently, the necessary and sufficient conditions for a rational quadratic Bézier curve to represent a circular arc are expressed as follows:

$$|P_0P_1| = |P_2P_1| \quad and \quad 0 \leq \omega \leq 1. \tag{55}$$

Algorithm 4: For a Quadratic Rational Bézier Curve to Represent a section of an Arc

Input: Control Points of Bezier Curve
Output: A section of Arc Represented by Bezier Curve
1: *Set* $-1 < \omega < 1, and\ \omega \neq 0$
2: *Input* Bézier Curve control points, P_0, P_1, P_2
3: **if** the number of control points < 3 **then**
4: **goto** Step 2.
5: **else**
6: **if** the number of control points $= 3$ **then**
7: Set $|P_0P_1| = |P_1P_2|$:
8: $P_2(y) = P_0(y)$
9: $P_1(x) = \frac{P_0(x)+P_2(x)}{2}$
10: $P_1(y) = P_0(y) - \frac{P_2(x)-P_0(x)}{2} \times \tan(\frac{\Pi}{3})$
11: **end if**
12: **end if**
13: *Output* coordinates of control points P_0, P_1, P_2
14: *Output* line segment between control points P_0, P_1 and P_1, P_2
15: Initializing the independent variable t in the standard formula of the quadratic rational Bezier curve to 0, *Set* $t = 0$
16: **for** $t = 0; t \leq 1; t+ = 0.00125$ **do**
17: Calculate the standard formula of the quadratic rational Bezier Curve.
18: $x = \frac{(1-t)^2x_0+2t(1-t)\omega x_1+t^2x_2}{(1-t)^2+2t(1-t)\omega+t^2}, y = \frac{(1-t)^2y_0+2t(1-t)\omega y_1+t^2y_2}{(1-t)^2+2t(1-t)\omega+t^2}$
19: **end for**
20: *Output* Bezier Curve.
21: *Clear* Bezier Curve, Bezier Curve control points. **goto** Step 1.
22: **return**

5. NECESSARY AND SUFFICIENT CONDITIONS FOR QUADRATIC RATIONAL BÉZIER CURVES TO REPRESENT A PARABOLA, ELLIPTIC ARC AND HYPERBOLA

Theorem 4. *Quadratic rational Bézier curve represents a parabola, elliptic arc, and hyperbola if and only if* $\omega = \pm 1$, $-1 < \omega < 1$, *and* $\omega < -1$ *or* $\omega > 1$, *respectively* [27].

Proof: According to the second order Bernstein basis function of Formula (42), Bézier curve from Formula (1) is written as follows,

$$p(t) = \frac{\omega_0 B_{0,2}(t)P_0}{\sum_{j=0}^{2} B_{j,2}(t)\omega_j} + \frac{\omega_1 B_{1,2}(t)P_1}{\sum_{j=0}^{2} B_{j,2}(t)\omega_j} + \frac{\omega_2 B_{2,2}(t)P_2}{\sum_{j=0}^{2} B_{j,2}(t)\omega_j} = \sum_{i=0}^{2} R_{i,2}(t)P_i, \tag{56}$$

where

$$R_{i,2}(t) = \frac{\omega_i B_{i,2}(t)}{\sum_{j=0}^{2} B_{j,2}(t)\omega_j}. \tag{57}$$

Next, we introduce the Local Oblique Coordinate System P_1, S, T, so that $S = P_0 - P_1$, $T = P_2 - P_1$. Since point $P(t)$ is within $\delta P_0P_1P_2$ for arbitrary $t \in [0, 1]$, $P(t)$ can be rewritten as

$$\begin{aligned} P(t) &= P_1 + u(t)S + v(t)T \\ &= P_1 + u(t)(P_0 - P_1) + v(t)(P_2 - P_1) \\ &= u(t)P_0 + [1 - u(t) - v(t)]P_1 + v(t)P_2. \end{aligned} \tag{58}$$

Comparing the coefficients from both Formula (56) and Formula (58), we know that

$$\begin{aligned} R_{0,2}(t) &= u(t), \\ R_{1,2}(t) &= 1 - u(t) - v(t), \\ R_{2,2}(t) &= v(t). \end{aligned} \tag{59}$$

Let $k = \omega_0\omega_2/\omega_1^2$, where k is the shape-invariant factor of a conic, so

$$u(t)v(t) = R_{0,2}(t)\dot{R}_{2,2}(t) = \frac{1}{4}k[1 - u(t) - v(t)]^2. \tag{60}$$

Formula (60) is an implicit equation of a quadratic curve in the local oblique coordinate system P_1, S, T. The expansion of Formula (60) further indicates that:

$$ku^2(t) + (2k-4)u(t)v(t) + kv^2(t) - 2ku(t) - 2kv(t) + k = 0. \tag{61}$$

In the Cartesian coordinate system, the image of a binary quadratic equation can represent a conic curve, and all conic curves can be derived in the aforementioned way [1]. The equation has the following forms [28]:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad A, B, C \text{ are not all zero}, \tag{62}$$

where A, B, C, D, E, F are polynomial coefficients. If the following conditions are satisfied,

$$B^2 - 4AC < 0, \tag{63}$$

then Formula (62) represents an ellipse; furthermore, under the same condition, if the conic degenerates (that is, $A = C, B = 0$), the equation represents a circle. Additionally, if the following conditions are satisfied,

$$B^2 - 4AC = 0, \tag{64}$$

Algorithm 5: For Quadratic Rational Bézier Curves to Represent a Parabola, Elliptic Arc and Hyperbola

Input: Control Points of Bezier Curve
Output: A section of a Parabola, Elliptic Arc or Hyperbola Represented by a Bezier Curve

1: **if** Quadratic Rational Bézier Curves to Represent a Parabola **then**
2: *Set* $\omega = 1$ *or* $\omega = -1$
3: **end if**
4: **if** Quadratic Rational Bézier Curves to Represent an Elliptic Arc **then**
5: *Set* $-1 < \omega < 1$, *and* $\omega \neq 0$
6: **end if**
7: **if** Quadratic Rational Bézier Curves to Represent a Hyperbola **then**
8: *Set* $\omega < -1$ *or* $\omega > 1$
9: **end if**
10: *Input* Bézier Curve control points P_0, P_1, P_2
11: **if** the number of control points < 3 **then**
12: **goto** Step 2.
13: **else**
14: **if** the number of control points $= 3$, and Quadratic Rational Bézier Curves to Represent an Elliptic Arc **then**
15: Set $|P_0P_1| \neq |P_1P_2|$
16: **end if**
17: **end if**
18: *Output* coordinates of control points P_0, P_1,P_2
19: *Output* line segment between control points P_0, P_1 and P_1, P_2
20: Initializing the independent variable t in the standard formula of the quadratic rational Bezier curve to 0, *Set* $t = 0$
21: **for** $t = 0; t \leq 1; t+ = 0.00125$ **do**
22: Calculate the standard formula of the quadratic rational Bezier Curve.
23: $x = \frac{(1-t)^2 x_0 + 2t(1-t)\omega x_1 + t^2 x_2}{(1-t)^2 + 2t(1-t)\omega + t^2}, y = \frac{(1-t)^2 y_0 + 2t(1-t)\omega y_1 + t^2 y_2}{(1-t)^2 + 2t(1-t)\omega + t^2}$
24: **end for**
25: *Output* Bezier Curve.
26: *Clear* Bezier Curve, Bezier Curve control points. **goto** Step 1.
27: **return**

then Formula (62) represents a parabola [29]. Finally, if the following conditions are satisfied,

$$B^2 - 4AC > 0 \tag{65}$$

then Formula (62) represents an hyperbola. The coefficients from Formula (61) and Formula (62) can be obtained as follows: $A = k, B = k - 2, C = k, D = -2k, E = -2k, F = k$. As such, we can get:

$$B^2 - 4AC = 1 - k. \tag{66}$$

We then provide the discussion and judgment of Formula (66). That is, from the condition of Formula (63), if the curve is an ellipse, then in Formula (66) we have $B^2 - 4AC = 1 - k < 0$. Therefore, when $k > 1$, the curve is an ellipse. From the condition of (64), if the curve is a parabola, then $B^2 - 4AC = 1 - k = 0$ (again see Formula 66). Therefore, when $k = 1$, the curve is a parabola. From the condition of 65, if the curve is a hyperbola, then $B^2 - 4AC = 1 - k > 0$, so when $k < 1$, the curve is a hyperbola.

Note that $k = \omega_0\omega_2/\omega_1^2$. In summary, under the standard form of the quadratic rational Bézier curve, we have $\omega_0 = \omega_2 = 1$, and $\omega = \omega_1$. Consequently, we prove that when $-1 < \omega < 1$, the quadratic rational Bézier curve is a ellipse; when $\omega = \pm 1$, the quadratic rational Bézier curve is a parabola; when $\omega < -1, or\ \omega > 1$, the quadratic rational Bézier curve is a hyperbola.

6. CONCLUSION

In this paper, we discuss the necessary and sufficient conditions for utilizing quadratic rational Bézier curves to represent different shapes, such as a point, line segment, parabola, elliptic arc, circular arc, and hyperbola. These results can be further used to facilitate other computer-aided geometric designs.

DATA AVAILABILITY STATEMENT

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

AUTHOR CONTRIBUTIONS

CY: conceptualization, methodology, software, validation, investigation, visualization, and writing original draft. JY: software, writing - review & editing, and supervision. YL: software, visualization, and writing - original draft. XG: writing - review & editing, validation, and visualization.

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