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Set-Valued Weighted Value at Risk and Its Computation

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In this paper, we propose a new class of set-valued coherent risk measures called the set-valued weighted value at risk. Firstly, the "regulator" version is independent of other market scenarios. The second version, which is called the market extension, is related to different market scenarios. The proofs of the properties of both versions are given, and equivalent representations are provided that enable us to compute the values of both versions of set-valued weighted value at risk. Finally, we offer examples to illustrate various features of the theoretical constructions of the set-valued weighted value at risk.

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1. INTRODUCTION

Weighted value at risk for one-dimension random variables may be one of the most popular coherent risk measures (see [1]). Artzner et al. [2] initially introduced the first coherent risk measure by proposing four axioms. Cherny [1] showed that weighted value at risk possesses some desirable properties that are not shared by Expected Shortfall. For further details on Expected Shortfall, we refer the reader to Föllmer and Schied [3]. Weighted value at risk first appeared in Kusuoka [4]. Acerbi [5, 6] called it the spectral risk measure.

Jouini et al. [7] demonstrated that a set of set-valued risk measures are suitable for evaluating multivariate risks in market models with transaction costs/bid-ask spreads. Additional set-valued risk measures have since been introduced and studied (see [8–13], and the references therein).

Hamel et al. [11] introduced set-valued average value at risk, and reasons for using set-valued functions as risk measures have been further addressed from both financial and mathematical perspectives (see [14–17]), and the reference there in).

In this paper, we will extend the traditional weighted value at risk to a set-valued version for multivariate random variables. Therefore, we demonstrate their core properties and provide an alternative representation for computing their values. The first version is called "regulator weighted value at risk" since it does not take trading opportunities into account. The second version is called "the market extension" since it relates to a specific market scenario. These two versions are set-valued coherent risk measures. Then, we derive a benchmark when introducing set-valued weighted value at risk that can reflect the risk tolerance of the trader/regulator; see Remark 2.2 below. Finally, we offer examples to illustrate various features of the theoretical constructions of the set-valued weighted value at risk.

The remainder of this article is organized as follows. Section 2 introduces a primal and an equivalent representation of set-valued weighted value at risk, including the "regulator" and "the market extension" cases. The essential properties of both cases are then proven. In section 3, examples are given to illustrate the theoretical construction of the set-valued weighted value at risk.

2. SET-VALUED WEIGHTED VALUE AT RISK

2.1. The Regulator Case

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $d \geq 1$ be a positive integer. A multivariate random variable is an $\mathscr{F}$-measurable function $\mathbf{X}: \Omega \to \mathbf{R}^d$ for $d \geq 2$. Here, $d = 1$ represents a one-dimension random variate. Denote by $L_d^0 := L_d^0(\Omega, \mathscr{F}, P)$ the linear space of the equivalence classes (with respect to the probability P) of $\mathbf{R}^d$-valued random variables. An element $\mathbf{X} \in L_d^0$ has components $X_1, \cdots, X_d$ in $L^0 := L_1^0$. Denote by $(L_d^0)_+$ the set of $\mathbf{R}^d$-valued random variables with P almost surely non-negative components and by $L_d^1 := L_d^1(\Omega, \mathscr{F}, P)$ the linear space of all $\mathbf{X} = (X_1, \cdots, X_d) \in L_d^0$ with $\int_\Omega X_i dP < +\infty$, $1 \leq i \leq d$. We also define $E[\mathbf{X}] = (EX_1, \cdots, EX_d)^T$ for $\mathbf{X} \in L_d^1$, the transpose of row vector $(EX_1, \cdots, EX_d)$. Define $(L_d^1)_+ = L_d^1 \cap (L_d^0)_+$. If $d = 1$, we write L^0, L_+^0 and L_+^1 for L_1^0, $(L_d^0)_+$ and $(L_d^1)_+$, respectively. For $\alpha \in \mathbf{R}^d$, the symbol $\operatorname{diag}(\alpha)$ denotes the $d \times d$ matrix with the components of the vector α as entries on its main diagonal and zero entries elsewhere. x^+ stands for $\max(x, 0)$ for $x \in \mathbf{R}$ [see [18–21]] and the reference therein).

The next definition offers an essential representation for set-valued weighted value at risk, which is an extension of the scalar case given by Cherny [1] to the set-valued case. It involves a linear subspace $\mathbf{M} \subseteq \mathbf{R}^d$, called the space of eligible assets, which we adopt from Hamel et al. [11]. We will also employ a benchmark level, which is one of the novelties of this article; see Remark 2.2 below. A natural choice for $\mathbf{M}$ is $\mathbf{M} = \mathbf{R}^m \times \{0\}^{d-m}$, $1 \leq m \leq d$, i.e., the first m of d assets are eligible as deposits (see [7, 11, 22]). We denote $\mathbf{M}_+ = \mathbf{M} \bigcap \mathbf{R}_+^d$, where $\mathbf{R}_+^d$ stands for the class of elements in $\mathbf{R}^d$ with non-negative components. We assume that $\mathbf{M}_+$ is non-trivial, i.e., $\mathbf{M}_+ \neq \{\mathbf{0}\}$.

Generally speaking, a scalar multivariate risk measure is any mapping from L_d^0 to $\mathbf{R}$. A set-valued risk measure is any mapping ρ from L_d^0 to a class of subsets of $\mathbf{R}^d$. $\rho(\mathbf{X})$ is interpreted as a set of acceptable margins of portfolio $\mathbf{X}$ (see [23–27]) and the reference therein).

Definition 2.1 Let $\theta \in (0, 1)$ and $\mu := (\mu_1, \cdots, \mu_d)$ be a probability on $[\theta, 1]^d$. For $\mathbf{X} \in L_d^0$, the set-valued weighted value at risk at $\mathbf{X}$ with respect to μ is defined as

$$WVaR_\mu(\mathbf{X}) := \left\{ \int_{[\theta,1]^d} \operatorname{diag}(\alpha)^{-1} E[\mathbf{Z}] \mu(d\alpha) - z; \right. \\ \left. \mathbf{Z} \in (L_d^1)_+, \mathbf{X} + \mathbf{Z} - z \in (L_d^0)_+, z \in \mathbf{R}^d \right\} \cap \mathbf{M}, \tag{2.1}$$

where $\int_{[\theta,1]^d} \operatorname{diag}(\alpha)^{-1} E[\mathbf{Z}] \mu(d\alpha) - z := \left[\int_{[\theta,1]} \frac{1}{\alpha_i} E[Z_i] \mu_i(d\alpha_i) - z_i \right]_{i=1}^d := \left(\int_{[\theta,1]} \frac{1}{\alpha_1} E[Z_1] \mu_1(d\alpha_1) - z_1, \cdots, \int_{[\theta,1]} \frac{1}{\alpha_1} E[Z_d] \mu_d(d\alpha_d) - z_d \right)^T$ for $\mathbf{Z} = (Z_1, \cdots, Z_d) \in (L_d^1)_+$ and $z = (z_1, \cdots, z_d) \in \mathbf{R}^d$.

Remark 2.1 If μ is a Dirac measure at some $\alpha \in (0, 1]^d$, that is, $\mu(\{\alpha\}) = 1$, then Definition 2.1 reverts to the definition of the set-valued regulator average value at risk of Hamel et al. [11] (Definition 2.1) because the benchmark level θ can be small enough. Moreover, in Example 3.2 below, we show that the $WVaR_\mu$ is better suited to the change in the market than the regulator average value at risk of Hamel et al. [11].

Remark 2.2 The financial interpretation of the benchmark level θ is as follows. Initially, it stems from the confidence level $1 - \alpha$ of value at risk. Given a confidence level $1 - \alpha \in (0, 1)$, the value at risk at $X \in L^0$ is defined as $\mathrm{VaR}_{1-\alpha}(X) := \inf\{t \in \mathbf{R};\ P(X > t) \leq \alpha\}$. From a practical perspective, in reality, the parameter $1 - \alpha$ can be very close to but cannot be 1. Thus, α can be very close to but cannot be zero, which motivates the introduction of the benchmark level θ, which reflects the risk tolerance of the investor/regulator in terms of probability. See Basel Committee [28–31] for the reasonability of the benchmark level. Therefore, the benchmark level θ can be very close to zero but cannot be exactly zero. Examples 3.1 and 3.2 below take this perspective into account.

Remark 2.3 In definition 2.1, the intersection with $\mathbf{M}$ has the following interpretation. To cancel the risk of portfolio $\mathbf{X}$, we would like to obtain a set of all margins when measuring the risk of portfolio $\mathbf{X}$. Intersecting with the set $\mathbf{M}$, $WVaR_\mu(\mathbf{X})$ shows both the valid margins and the aggregated margins, which aggregates the valid margins from the d-dimension to the m-dimension. The other $(d-m)$-dimension of $WVaR_\mu(\mathbf{X})$ should be zero. Aggregating the margin has plenty of financial explanations. For example, each element of the vector represents the amounts in a specific currency. Suppose that m different currencies should be taken into consideration. For the regulator, there is no need to ask for a d-dimensional margin. They could aggregate d elements of the margin into m elements that represent m different currencies. When considering the margin needed by a company with different departments, this idea is also reasonable. The decision-maker of a company may simply want to figure out the sum of the margins of different departments. More details can be found in Jouini et al. [7].

The next proposition provides another equivalent representation of $WVaR_\mu$ under the condition $\mathbf{M} = \mathbf{R}^m \times \{0\}^{d-m}$, which is easier to compute than (2.1).

Proposition 2.1 Let $\mathbf{M} = \mathbf{R}^m \times \{0\}^{d-m}$ (hence $\mathbf{M}_+ = \mathbf{R}_+^m \times \{0\}^{d-m}$). The set-valued weighted value at risk takes the following equivalent representation:

$$WVaR_\mu(\mathbf{X}) = \left(\left[\inf_{z_i \in \mathbf{R}} \left\{ \int_{[\theta,1]} \frac{1}{\alpha_i} E[(-X_i + z_i)^+] \mu_i(d\alpha_i) - z_i \right\} \right]_{i=1}^m \right. \\ \left. + \mathbf{R}_+^m \right) \times \{0\}^{d-m}$$

for $\mathbf{X} = (X_1, \cdots, X_d) \in L_d^0$.

Proof Considering a component of the portfolio, we know that the two conditions $Z_i \in L_+^1$ and $X_i + Z_i - z_i \in L_+^0$ are equivalent to $Z_i \geq (-X_i + z_i)^+$ for $1 \leq i \leq d$. Therefore, $\{\int_{[\theta,1]} \frac{1}{\alpha_i} E[Z_i]\mu_i(d\alpha_i) - z_i;\ Z_i \in L^1_+, X_i + Z_i - z_i \in L^0_+, z_i \in \mathbf{R}\}$ is equal to $\inf_{z_i\in\mathbf{R}}\{\int_{[\theta,1]} \frac{1}{\alpha_i} E[(-X_i + z_i)^+]\mu_i(d\alpha_i) - z_i\} + \mathbf{R}_+$. After intersecting with the set $\mathbf{M}$, we have that

$$WVaR_\mu(\mathbf{X}) = \left(\left[\inf_{z_i\in\mathbf{R}}\left\{\int_{[\theta,1]} \frac{1}{\alpha_i} E[(-X_i + z_i)^+]\mu_i(d\alpha_i) - z_i\right\}\right]_{i=1}^m + \mathbf{R}^m_+\right) \times \{0\}^{d-m}. \quad (1)$$

Proposition 2.1 is proved.

The next proposition will show that when $\mathbf{M} = \mathbf{R}^m \times \{0\}^{d-m}$, the set-valued weighted value at risk is exactly a set-valued coherent risk measure in the sense of Jouini et al. [7].

Proposition 2.2 Let $\mathbf{M} = \mathbf{R}^m \times \{0\}^{d-m}$. Then, the function $\mathbf{X} \longrightarrow WVaR_\mu(\mathbf{X})$ meets the listed properties:

(a) Positive homogeneity: for any $\mathbf{X} \in L^0_d$ and any $s > 0$, $WVaR_\mu(s\mathbf{X}) = sWVaR_\mu(\mathbf{X})$.
(b) Subadditivity: for any $\mathbf{X}^1, \mathbf{X}^2 \in L^0_d$, $WVaR_\mu(\mathbf{X}^1 + \mathbf{X}^2) \supseteq WVaR_\mu(\mathbf{X}^1) + WVaR_\mu(\mathbf{X}^2)$.
(c) $\mathbf{M}$-translation invariance: for any $\mathbf{X} \in L^0_d$ and any $u \in \mathbf{R}^m$, $WVaR_\mu(\mathbf{X} + \bar{u}) = WVaR_\mu(\mathbf{X}) - \bar{u}$, where $\bar{u} = u \times \{0\}^{d-m}$.
(d) Monotonicity with respect to $(L^0_d)_+$: for any $\mathbf{X}^1, \mathbf{X}^2 \in (L^0_d)_+$ with $\mathbf{X}^2 \geq \mathbf{X}^1$, which means that $\mathbf{X}^2 - \mathbf{X}^1 \in (L^0_d)_+$, we have $WVaR_\mu(\mathbf{X}^2) \supseteq WVaR_\mu(\mathbf{X}^1)$.
(e) It satisfies that $WVaR_\mu(\mathbf{X}) + \mathbf{M}_+ = WVaR_\mu(\mathbf{X})$ for $\mathbf{X} \in L^0_d$. Particularly, $WVaR_\mu(\mathbf{0})$ is a convex cone.

Proof (a) For $\mathbf{X} = (X_1, \cdots, X_d) \in L^0_d$ and $s > 0$,

$$\begin{aligned}
&WVaR_\mu(s\mathbf{X}) = \\
&\left(\left[\inf_{z_i\in\mathbf{R}}\left(-z_i + \int_{[\theta,1]} \frac{1}{\alpha_i} E\left[(z_i - sX_i)^+\right]\mu_i(d\alpha_i)\right)\right]_{i=1}^m + \mathbf{R}^m_+\right) \times \{0\}^{d-m} \\
&= \left(\left[\inf_{z_i\in\mathbf{R}}\left(-z_i + s\int_{[\theta,1]} \frac{1}{\alpha_i} E\left[\left(\frac{z_i}{s} - X_i\right)^+\right]\mu_i(d\alpha_i)\right)\right]_{i=1}^m + \mathbf{R}^m_+\right) \times \{0\}^{d-m} \\
&= \left(\left[s\inf_{z_i\in\mathbf{R}}\left(-\frac{z_i}{s} + \int_{[\theta,1]} \frac{1}{\alpha_i} E\left[\left(\frac{z_i}{s} - X_i\right)^+\right]\mu_i(d\alpha_i)\right)\right]_{i=1}^m + \mathbf{R}^m_+\right) \times \{0\}^{d-m} \\
&= \left(\left[s\inf_{\frac{z_i}{s}\in\mathbf{R}}\left(-\frac{z_i}{s} + \int_{[\theta,1]} \frac{1}{\alpha_i} E\left[\left(\frac{z_i}{s} - X_i\right)^+\right]\mu_i(d\alpha_i)\right)\right]_{i=1}^m + \mathbf{R}^m_+\right) \times \{0\}^{d-m} \\
&= sWVaR_\mu(\mathbf{X}).
\end{aligned}$$

(b) For $\mathbf{X}^1 = (X^1_1, \cdots, X^1_d)$, $\mathbf{X}^2 = (X^2_1, \cdots, X^2_d) \in L^0_d$,

$$\begin{aligned}
&WVaR_\mu(\mathbf{X}^1 + \mathbf{X}^2) = \\
&\left(\left[\inf_{z_i\in\mathbf{R}}\left(-z_i + \int_{[\theta,1]} \frac{1}{\alpha_i} E\left[\left(z_i - X^1_i - X^2_i\right)^+\right]\mu_i(d\alpha_i)\right)\right]_{i=1}^m + \mathbf{R}^m_+\right) \times \{0\}^{d-m} \\
&\supseteq \left(\left[\inf_{z^1_i + z^2_i = z_i\in\mathbf{R}}\left(-z_i + \int_{[\theta,1]} \frac{1}{\alpha_i} E\left[\left(z^1_i - X^1_i\right)^+ + \left(z^2_i - X^2_i\right)^+\right]\mu_i(d\alpha_i)\right)\right]_{i=1}^m + \mathbf{R}^m_+\right) \times \{0\}^{d-m} \\
&= \left(\left[\inf_{z^1_i\in\mathbf{R}}\left(-z^1_i + \int_{[\theta,1]} \frac{1}{\alpha_i} E\left[\left(z^1_i - X^1_i\right)^+\right]\mu_i(d\alpha_i)\right)\right]_{i=1}^m + \mathbf{R}^m_+ + \left[\inf_{z^2_i\in\mathbf{R}}\left(-z^2_i + \int_{[\theta,1]} \frac{1}{\alpha_i} E\left[\left(z^2_i - X^2_i\right)^+\right]\mu_i(d\alpha_i)\right)\right]_{i=1}^m + \mathbf{R}^m_+\right) \times \{0\}^{d-m} \\
&= WVaR_\mu(\mathbf{X}^1) + WVaR_\mu(\mathbf{X}^2).
\end{aligned}$$

(c) For $u = (u_1, \cdots, u_m) \in \mathbf{R}^m$,

$$\begin{aligned}
&WVaR_\mu(\mathbf{X} + \bar{u}) = \\
&\left(\left[\inf_{z_i\in\mathbf{R}}\left(-z_i + \int_{[\theta,1]} \frac{1}{\alpha_i} E\left[(z_i - \bar{u}_i - X_i)^+\right]\mu_i(d\alpha_i)\right)\right]_{i=1}^m + \mathbf{R}^m_+\right) \times \{0\}^{d-m} \\
&= \left(\left[\inf_{z_i\in\mathbf{R}}\left(-(z_i - \bar{u}_i) + \int_{[\theta,1]} \frac{1}{\alpha_i} E\left[(z_i - \bar{u}_i - X_i)^+\right]\mu_i(d\alpha_i) - \bar{u}_i\right)\right]_{i=1}^m + \mathbf{R}^m_+\right) \times \{0\}^{d-m} \\
&= WVaR_\mu(\mathbf{X}) - \bar{u}.
\end{aligned}$$

(d) Given $\mathbf{X}^1 = (X^1_1, \cdots, X^1_d)$, $\mathbf{X}^2 = (X^2_1, \cdots, X^2_d) \in L^0_d$ with $\mathbf{X}^2 - \mathbf{X}^1 \in (L^0_d)_+$, we have $(z_i - X^2_i)^+ \leq (z_i - X^1_i)^+$ for each $z_i \in \mathbf{R}$, $1 \leq i \leq d$. Hence,

$$\begin{aligned}
&\inf_{z_i\in\mathbf{R}}\left(-z_i + \int_{[\theta,1]} \frac{1}{\alpha_i} E\left[\left(z_i - X^2_i\right)^+\right]\mu_i(d\alpha_i)\right) \\
&\quad \leq \inf_{z_i\in\mathbf{R}}\left(-z_i + \int_{[\theta,1]} \frac{1}{\alpha_i} E\left[\left(z_i - X^1_i\right)^+\right]\mu_i(d\alpha_i)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left[\inf_{z_i\in\mathbf{R}}\left(-z_i + \int_{[\theta,1]} \frac{1}{\alpha_i} E\left[\left(z_i - X^2_i\right)^+\right]\mu_i(d\alpha_i)\right)\right]_{i=1}^m \\
&\quad \leq \left[\inf_{z_i\in\mathbf{R}}\left(-z_i + \int_{[\theta,1]} \frac{1}{\alpha_i} E\left[\left(z_i - X^1_i\right)^+\right]\mu_i(d\alpha_i)\right)\right]_{i=1}^m.
\end{aligned}$$

Consequently,

$$\left(\left[\inf_{z_i\in\mathbf{R}}(-z_i \quad +\int_{[\theta,1]}\frac{1}{\alpha_i}E\left[\left(z_i-X_i^2\right)^+\right]\mu_i(d\alpha_i)\right)\right]_{i=1}^m + \mathbf{R}_+^m\right)\times\{0\}^{d-m}$$

$$\supseteq\left(\left[\inf_{z_i\in\mathbf{R}}\left(-z_i+\int_{[\theta,1]}\frac{1}{\alpha_i}E\left[\left(z_i-X_i^1\right)^+\right]\mu_i(d\alpha_i)\right)\right]_{i=1}^m + \mathbf{R}_+^m\right)\times\{0\}^{d-m},$$

which implies that $WVaR_\mu(\mathbf{X}^2)\supseteq WVaR_\mu(\mathbf{X}^1)$.

(e) It is not difficult to verify that $WVaR_\mu(\mathbf{X})+\mathbf{M}_+ = WVaR_\mu(\mathbf{X})$ and that $WVaR_\mu(0)$ is a convex cone.

2.2. The Market Extension

The weighted value at risk from Definition 2.1 does not take into account the investment preferences of investors. Therefore, we define its market extension by replacing $(L_d^0)_+$ with a general closed convex cone K containing $(L_d^0)_+$ (see [7] or [8] for further motivation).

Definition 2.2 Let $\widetilde{K}$ be a closed convex cone that contains $(L_d^1)_+$ and K be a closed convex cone that contains $(L_d^0)_+$. The extended version of the set-valued weighted value at risk is defined as

$$WVaR_\mu^{ext}(\mathbf{X}):=\left\{\int_{[\theta,1]^d}\mathrm{diag}(\alpha)^{-1}E[\mathbf{Z}]\mu(d\alpha)-z;\ \mathbf{Z}\in\widetilde{K},\mathbf{X}+\mathbf{Z}-z\in K, z\in\mathbf{R}^d\right\}\cap\mathbf{M}.$$

In the proof of Proposition 2.1, through the same argument, we present the following proposition, which provides another equivalent representation of $WVaR_\mu^{ext}(\cdot)$.

Proposition 2.3 Let $\mathbf{M}=\mathbf{R}^m\times\{0\}^{d-m}$. $WVaR_\mu^{ext}$ has the following equivalent representation:

$$WVaR_\mu^{ext}(\mathbf{X})=\left(\left[\inf_{z_i\in\mathbf{R}}\left(\int_{[\theta,1]}\frac{1}{\alpha_i}E\left[(-X_i+z_i)^+\right]\mu_i(d\alpha_i)-z_i\right)\right]_{i=1}^m + C\right)\times\{0\}^{d-m}$$

where C is a closed convex cone that contains $\mathbf{R}_+^d$.

The next proposition will show that when $\mathbf{M}=\mathbf{R}^m\times\{0\}^{d-m}$, $WVaR_\mu^{ext}$ is exactly a set-valued coherent risk measure in the sense of Jouini et al. [7].

Proposition 2.4 Let $\mathbf{M}=\mathbf{R}^m\times\{0\}^{d-m}$. Then, the function $\mathbf{X}\longrightarrow WVaR_\mu^{ext}(\mathbf{X})$ satisfies the following properties:

(a) Positive Homogeneity: for each $\mathbf{X}\in L_d^0$ and each $s>0$, $WVaR_\mu^{ext}(s\mathbf{X})=sWVaR_\mu^{ext}(\mathbf{X})$.

(b) Subadditivity: for each $\mathbf{X}^1,\mathbf{X}^2\in L_d^0$, $WVaR_\mu^{ext}(\mathbf{X}^1+\mathbf{X}^2)\supseteq WVaR_\mu^{ext}(\mathbf{X}^1)+WVaR_\mu^{ext}(\mathbf{X}^2)$.

(c) $\mathbf{M}$-translation invariance: for each $\mathbf{X}\in L_d^0$ and each $u\in\mathbf{R}^m$, $WVaR_\mu^{ext}(\mathbf{X}+\bar{u})=WVaR_\mu^{ext}(\mathbf{X})-\bar{u}$, where $\bar{u}=u\times\{0\}^{d-m}$.

(d) Monotonicity with respect to K: for any $\mathbf{X}^1,\mathbf{X}^2\in K$ and $\mathbf{X}^2\succeq_K\mathbf{X}^1$, which means that $\mathbf{X}^2-\mathbf{X}^1\in K$, we have $WVaR_\mu^{ext}(\mathbf{X}^2)\supseteq WVaR_\mu^{ext}(\mathbf{X}^1)$.

(e) For each $\mathbf{X}\in L_d^0$, the set $WVaR_\mu^{ext}(\mathbf{X})\subset\mathbf{M}$ is convex and satisfies that $WVaR_\mu^{ext}(\mathbf{X})+C_M=WVaR_\mu^{ext}(\mathbf{X})$, where $C_M:=C\bigcap M$ and C is as in Proposition 2.3. In particular, $WVaR_\mu^{ext}(\mathbf{0})$ is a convex cone that satisfies $C_M\subseteq WVaR_\mu^{ext}(\mathbf{0})$ and $WVaR_\mu^{ext}(\mathbf{0})\bigcap-\mathbf{C_M}=\{\mathbf{0}\}$.

Proof: (a) For $\mathbf{X}\in L_d^0$ and $s>0$, we have

$$\begin{aligned}
WVaR_\mu^{ext}(s\mathbf{X}) &=\left\{\int_{[\theta,1]^d}\mathrm{diag}(\alpha)^{-1}E[\mathbf{Z}]\mu(d\alpha)-z;\ \mathbf{Z}\in\widetilde{K},s\mathbf{X}+\mathbf{Z}-z\in K,z\in\mathbf{R}^d\right\}\cap\mathbf{M}\\
&=\left\{\int_{[\theta,1]^d}\mathrm{diag}(\alpha)^{-1}E[\mathbf{Z}]\mu(d\alpha)-z;\ \frac{\mathbf{Z}}{s}\in\widetilde{K},s(\mathbf{X}+\frac{\mathbf{Z}}{s}-\frac{z}{s})\in K,\frac{z}{s}\in\mathbf{R}^d\right\}\cap\mathbf{M}\\
&=\left\{s\left(\int_{[\theta,1]^d}\mathrm{diag}(\alpha)^{-1}E\left[\frac{\mathbf{Z}}{s}\right]\mu(d\alpha)-\frac{z}{s}\right);\ \frac{\mathbf{Z}}{s}\in\widetilde{K},\mathbf{X}+\frac{\mathbf{Z}}{s}-\frac{z}{s}\in K,\frac{z}{s}\in\mathbf{R}^d\right\}\cap\mathbf{M}\\
&=sWVaR_\mu^{ext}(\mathbf{X}).
\end{aligned}$$

(b) For $\mathbf{X}^1,\mathbf{X}^2\in L_d^0$,

$$\begin{aligned}
&WVaR_\mu^{ext}(\mathbf{X}^1)+WVaR_\mu^{ext}(\mathbf{X}^2)\\
&=\left\{\int_{[\theta,1]^d}\mathrm{diag}(\alpha)^{-1}E[\mathbf{Z}^1]\mu(d\alpha)-z^1+\int_{[\theta,1]}diag(\alpha)^{-1}E[\mathbf{Z}^2]\mu(d\alpha)-z^2;\ \mathbf{Z}^1,\mathbf{Z}^2\in\widetilde{K},\mathbf{X}^1+\mathbf{Z}^1-z^1\in K,\mathbf{X}^2+\mathbf{Z}^2-z^2\in K,z^1,z^2\in\mathbf{R}^d\right\}\cap\mathbf{M}\\
&\subseteq\left\{\int_{[\theta,1]^d}\mathrm{diag}(\alpha)^{-1}E[\mathbf{Z}^1+\mathbf{Z}^2]\mu(d\alpha)-(z^1+z^2);\ \mathbf{Z}^1+\mathbf{Z}^2\in\widetilde{K},\mathbf{X}^1+\mathbf{X}^2+\mathbf{Z}^1+\mathbf{Z}^2-(z^1+z^2)\in K,z^1+z^2\in\mathbf{R}^d\right\}\cap\mathbf{M}\\
&=\left\{\int_{[\theta,1]^d}\mathrm{diag}(\alpha)^{-1}E[\mathbf{Z}]\mu(d\alpha)-z;\ \mathbf{Z}\in\widetilde{K},\mathbf{X}^1+\mathbf{X}^2+\mathbf{Z}-z\in K,z\in\mathbf{R}^d\right\}\cap\mathbf{M}\\
&=WVaR_\mu^{ext}(\mathbf{X}^1+\mathbf{X}^2).
\end{aligned}$$

(c) It is straightforward.

(d) $WVaR^{ext}_{\mu}$ is K-monotone because for $\mathbf{Y} \in K$, we have $\mathbf{Y} + K \subseteq K$, and therefore, $WVaR^{ext}_{\mu}(\mathbf{X} - \mathbf{Y}) = \left\{\int_{[\theta,1]^d} \operatorname{diag}(\alpha)^{-1} E[\mathbf{Z}]\mu(d\alpha) - z;\ \mathbf{Z} \in \widetilde{K}, \mathbf{X} + \mathbf{Z} - z \in \mathbf{Y} + K, z \in \mathbf{R}^d\right\} \cap \mathbf{M} \subseteq WVaR^{ext}_{\mu}(\mathbf{X})$.

(e) It is straightforward. Proposition 2.4 is proved.

3. EXAMPLES

In this part, we give two examples of computing $WVaR_{\mu}$. In the rest of the paper, we will consider a finite financial market, that is, we assume that $(\Omega, \mathscr{F}, P)$ is a finite probability space. Namely, let $|\Omega| = N$, $\mathscr{F} = 2^{\Omega}$, $P = (p_1, p_2, ..., p_N)$ with $\sum_{n=1}^{N} p_n = 1$ and $P(\{\omega_n\}) = p_n$, $n = 1, 2, ..., N$. Here, N is a strictly positive number, and the probability measure P is given by N.

The first example is motivated by Hamel et al. [11] (Example 3.1).

Example 3.1 Suppose that the elements of a portfolio are $d = 2$ and $\mathbf{M} = \mathbf{R}^2$ (hence all the initial portfolios are eligible). In a binary model with $N = 2$ and $P = (0.4, 0.6)$, the potential income is given by

$$\mathbf{X}(\omega_1) = (12, -20)^T, \quad \mathbf{X}(\omega_2) = (4, -6)^T.$$

We set the benchmark level $\theta = 0.01$ and let $\mu_1 = \mu_2 := \nu$. If ν is set to be uniformly distributed on $[\theta,\ 1]$, that is, for Borel measurable set $A \subset [\theta,\ 1]$,

$$\nu(A) := \int_A f(x)dx,$$

where $f(x) = \frac{1}{1-\theta}$ for $\theta \leq x \leq 1$. By a simple calculation, we have that

$$WVaR_{\mu}(\mathbf{X}) = (-4,\ 20)^T + \mathbf{R}^2_+.$$

If we let ν be a (discrete) probability law with $\nu(\{0.01\}) = \nu(\{0.02\}) = 0.5$, then calculation shows that

$$WVaR_{\mu}(\mathbf{X}) = (-4,\ 20)^T + \mathbf{R}^2_+$$

again. For the first and second assets, the margins that the manager/regulator needs for compensating the risk are at least 4 units and −20 units, respectively.

In the above example, the value of $WVaR_{\mu}(\mathbf{X})$ is equal to that of $AV@R^{reg}_{\alpha}(\mathbf{X})$, the set-valued regulator average value at risk (see [11], Definition 2.1 and Example 3.1), where $\alpha = (0.01,\ 0.02)^T$. The next example will show that the values of $WVaR_{\mu}(\mathbf{X})$ and $AV@R^{reg}_{\alpha}(\mathbf{X})$ are not necessarily the same and that $WVaR_{\mu}(\mathbf{X})$ is better suited to a market featuring extreme events than is $AV@R^{reg}_{\alpha}(\mathbf{X})$.

Example 3.2 Let all the input parameters and the potential incomes of $\mathbf{X}$ be as in Example 3.1 except for the probability law P and the probability measure μ. Here, we set $P = (0.99,\ 0.01)$. If ν is set to be uniformly distributed on $[\theta,\ 1]$, then

$$WVaR_{\mu}(\mathbf{X}) = (-11.628,\ 20)^T + \mathbf{R}^2_+.$$

If ν is again a (discrete) probability law with $\nu(\{0.01\}) = \nu(\{0.02\}) = 0.5$, then,

$$WVaR_{\mu}(\mathbf{X}) = (-6,\ 20)^T + \mathbf{R}^2_+.$$

In contrast to the above example, the probability measure μ concerning the confidence levels does affect the risk measure because the minimal margin to cancel the risk for a manager/regulator covers the worst case only for the second asset, which is −20 units.

On the other hand,

$$AV@R^{reg}_{\alpha}(\mathbf{X}) = (-4,\ 20)^T + \mathbf{R}^2_+,$$

where $\alpha = (0.01,\ 0.02)^T$, which is the same as in Example 3.1.

From the above two examples, we observe that when all the input parameters remain the same except for the change in the (binary) probability law P from (0.4, 0.6) to (0.99, 0.01), the minimal risk-compensating portfolio of $WVaR_{\mu}(\mathbf{X})$ changes from (−4, 20) to (−11.628, 20) and (−6, 20), respectively, whereas the minimal risk-compensating portfolio of $AV@R^{reg}_{\alpha}(\mathbf{X})$ remains unchanged, which is (−4, 20). Thus, we conclude that $WVaR_{\mu}(\mathbf{X})$ can reflect the change in the market, that is, the change in the (binary) probability P, whereas $AV@R^{reg}_{\alpha}(\mathbf{X})$ cannot. In the case of $P = (0.99,\ 0.01)$, the event with probability 0.01 could be regarded as an extreme event compared with the other event with probability 0.99. Therefore, we conclude that $WVaR_{\mu}$ is better suited to a market featuring extreme events than is $AV@R^{reg}_{\alpha}$.

4. CONCLUSIONS

In this paper, we proposed two new classes of set-valued coherent risk measures: the "regulator" version and "market" version. Their essential properties are discussed, and equivalent representations are given. Moreover, the coherency of the set-valued weighted value at risk is characterized. These newly introduced set-valued risk measures complement the study of set-valued risk measures. Examples are also presented that show that set-valued weighted value at risk is better suited to a market featuring extreme events than is $AV@R^{reg}_{\alpha}$.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary materials, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

YD, YH, and YF: conceptualization, formal analysis, writing-original draft preparation, writing-review and editing, and funding acquisition. All authors contributed to the article and approved the submitted version.

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