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Some Effective Numerical Techniques for Chaotic Systems Involving Fractal-Fractional Derivatives With Different Laws

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Chaotic systems are dynamical systems that are highly sensitive to initial conditions. Such systems are used to model many real-world phenomena in science and engineering. The main purpose of this paper is to present several efficient numerical treatments for chaotic systems involving fractal-fractional operators. Several numerical examples test the performance of the proposed methods. Simulations with different values of the fractional and fractal parameters are also conducted. It is demonstrated that the fractal-fractional derivative enables one to capture all the useful information from the history of the phenomena under consideration. The numerical schemes can also be implemented for other chaotic systems with fractal-fractional operators.

Keywords: chaotic attractors, computational efficiency, fractal-fractional operators, thomas attractor, Newton's method, product integration rule

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1. INTRODUCTION

In recent decades numerical methods have been recognized as powerful mathematical techniques for solving nonlinear equations that model real-world problems [1–5]. Numerical techniques have been used to solve different classes of differential equations, including those of arbitrary order. Due to the outstanding contribution of nonlinear models to human understanding of many phenomena and the prediction of the future behavior of systems, many researchers are devoting their attention to developing new and reliable numerical techniques that could be applied to more complex cases. Chaotic behaviors are among the natural phenomena that have attracted the attention of many researchers, who aim to replicate and predict those behaviors. To achieve this, differential operators are often used as mathematical tools to construct the underlying models. Recently a new class of differential operators was introduced, which are convolutions of fractal derivatives and fractional kernels with different forms such as power law, exponential decay and the Mittag-Leffler function [6]. These differential operators are able to represent complexities that cannot be described with classical fractional differentiation and integration. One of the strengths of these operators is their two orders, where one is considered a fractional order and the other is the fractal dimension [7–9]. With these efficient operators, a new class of nonlinear differential and integral equations can be constructed, and existing chaotic models can be extended. Nevertheless, to verify the effectiveness of such fractal-fractional operators in modeling chaotic attractors, one needs to solve the models numerically as their exact solutions cannot be easily obtained by existing analytical methods. So far, a few numerical methods have been used to discretize such models, and some numerical results for chaotic attractors have been obtained. A different numerical scheme was suggested very recently [10–27] and was found to be efficient for solving nonlinear differential equations. Some articles have examined the analysis of errors and the determination of error bands in the context of the possible deficit differential equations [28–30]. Since the operators defined in this paper are novel in the field, the numerical methods associated with them are also very limited. One of the main motivations for this article was to introduce methods for solving fractal-fractional problems that have not been considered before. By using the proposed methods, approximate solutions to these problems can be determined more easily and with higher accuracy. In addition, the methods can be applied to real-world problems. In this work, we present applications of such numerical schemes in solving chaotic models that involve the new class of differential operators. We consider some well-known chaotic models with fractal-fractional differential operators, so that we can compare our results with those in the literature. The article is organized as follows. In section 2, we give a brief overview of some basic definitions of fractional differential calculus. Two efficient and effective numerical methods for determining approximate solutions to fractal-fractional problems are presented in section 3. The first is for the Caputo derivative and the second is for the Atangana-Baleanu-Caputo derivative. The kernels used in these two definitions are singular and non-singular, respectively. Several numerical simulations for chaotic systems are described in section 4. The results obtained are accurate, interesting, and meaningful. Finally, we present our overall conclusions.

2. PRELIMINARY DEFINITIONS

In this section, we give a brief review of some existing definitions of fractal-fractional operators. These operators result from the combination of two important concepts: fractional differentiation and fractal derivatives. Most of the definitions given here are taken from Atangana [6] and Atangana and Qureshi [31].

Definition 1. If $g(t)$ is a differentiable function on a finite open interval, then we define the fractal-fractional derivative of $g(t)$ in the Caputo sense as

$$ {}_0^{\text{FF-C}}\mathbb{D}_t^{\rho,\tau} g(t) = \frac{1}{\Gamma(k-\rho-1)} \int_0^t \frac{dg(\omega)}{dt^\tau}(t-\omega)^{k-\rho-1}\,d\omega, \quad k-1<\rho\le k,\ 0<k-1<\tau\le k, \tag{1} $$

where

$$ \frac{dg(\omega)}{dt^\tau} = \lim_{t\to y}\frac{g(t)-g(\omega)}{t^\beta-\omega^\beta}. \tag{2} $$

Definition 2. If $g(t)$ is a differentiable function on a finite open interval, then we define the fractal-fractional derivative of $g(t)$ in the Caputo-Fabrizio sense as

$$ {}_0^{\text{FF-CF}}\mathbb{D}_t^{\rho,\tau} g(t) = \frac{L(\rho)}{1-\rho}\int_0^t \frac{dg(\omega)}{dt^\tau}\exp\left[-\frac{\rho}{1-\rho}(t-\omega)\right]d\omega, \quad n-1<\rho,\tau\le n, \tag{3} $$

provided $K(0) = K(1) = 1$.

Definition 3. If $g(t)$ is a differentiable function on a finite open interval, then we define the fractal-fractional derivative of $g(t)$ in the Atangana-Baleanu sense as

$$ {}_0^{\text{FF-AB}}\mathbb{D}_t^{\rho,\tau} g(t) = \frac{\mathbf{AB}(\rho)}{1-\rho}\int_0^t \frac{dg(\omega)}{dt^\tau}E_\rho\left[-\frac{\rho}{1-\rho}(t-\omega)\right]d\omega, \quad n-1<\rho,\tau\le n, \tag{4} $$

where $E_\rho(.)$ is the Mittag-Leffler function, defined by

$$ E_\rho(\omega) = \sum_{k=0}^{\infty}\frac{\omega^k}{\Gamma(\rho k+1)}, \quad \rho>0. \tag{5} $$

This function is an essential function in the modeling of physical processes using fractional calculus concepts. We know that Equation (5) reduces to the exponential function e^x if one takes $\rho = 1$. Also, $\mathbf{AB}(\cdot)$ is a function used for normalization, and it satisfies the property $\mathbf{AB}(0) = \mathbf{AB}(1) = 1$. One of the most popular definitions of $\mathbf{AB}(\cdot)$ is

$$ \mathbf{AB}(\rho) = 1-\rho+\frac{\rho}{\Gamma(\rho)}. $$

Definition 4. If $g(t)$ is a differentiable function on a finite open interval, then we define the fractal-fractional integral of $g(t)$ in the Caputo sense [31] as

$$ {}_0^{\text{FF-C}}\mathbb{I}_t^{\rho,\tau} g(t) = \frac{\tau}{\Gamma(\rho)}\int_0^t \frac{\omega^{\tau-1}g(\omega)\,d\omega}{(t-\omega)^{1-\rho}}. \tag{6} $$

Definition 5. If $g(t)$ is a differentiable function on a finite open interval, then we define the fractal-fractional integral of $g(t)$ in the Caputo-Fabrizio sense [31] as

$$ {}_0^{\text{FF-CF}}\mathbb{I}_t^{\rho,\tau} g(t) = \frac{\tau\rho}{M(\rho)}\int_0^t \frac{g(\omega)\,d\omega}{\omega^{1-\rho}} + \frac{\tau(1-\rho)t^{\tau-1}g(t)}{M(\rho)}. \tag{7} $$

Definition 6. If $g(t)$ is a differentiable function on a finite open interval, then we define the fractal-fractional integral of $g(t)$ in the Atangana-Baleanu sense [31] as

$$ {}_0^{\text{FF-AB}}\mathbb{I}_t^{\rho,\tau} g(t) = \frac{\tau\rho}{\mathbf{AB}(\rho)}\int_0^t \frac{\omega^{\tau-1}g(\omega)\,d\omega}{(t-\omega)^{1-\rho}} + \frac{\tau(1-\rho)t^{\tau-1}g(t)}{\mathbf{AB}(\rho)}. \tag{8} $$

3. THE PROPOSED NUMERICAL METHODS

In what follows, the main aim is to construct two equations involving fractal-fractional derivatives,

$$ {}_0^{\text{FF}}\mathbb{D}_t^{\rho,\tau}\xi(t) = \mathbb{N}(t,\xi(t)), \quad t\in[t_0,T], \tag{9} $$

with the initial condition $\xi(0) = \xi_0$.

3.1. The Caputo Fractal-Fractional Derivative

From the results in Atangana and Qureshi [31], Equation (9) reduces to the following Caputo fractional representation:

$$ {}_{0}^{C}\mathbb{D}_t^{\rho}\xi(t) = \tau t^{\tau-1}\mathbb{N}(t,\xi(t)). \tag{10}$$

Now, taking into account the fundamental theorem of calculus, one gets

$$\xi(t)-\xi(t_0) = \frac{\tau}{\Gamma(\rho)}\int_0^t \frac{\omega^{\tau-1}\mathbb{N}(\omega,\xi(\omega))\,d\omega}{(t-\omega)^{1-\rho}}. \tag{11}$$

Then, inserting $t = t_n = t_0 + n\Delta t$ into (17) leads to the following expression:

$$\xi(t_n) = \xi(t_0) + \frac{\tau}{\Gamma(\rho)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \frac{\omega^{\tau-1}\mathbb{N}(\omega,\xi(\omega))\,d\omega}{(t_n-\omega)^{1-\rho}}, \quad 1 \le n \le N, \tag{12}$$

where $\Delta t = \frac{T-t_0}{N}$ is the time step for the discretization points.

Next, taking the linear Lagrange interpolation into account for the function of $f(\omega) = \omega^{\tau-1}\mathbb{N}(\omega,\xi(\omega))$, we obtain

$$\begin{aligned}&\omega^{\tau-1}\mathbb{N}(\omega,\xi(\omega)) \approx t_{i+1}^{\tau-1}\mathbb{N}(t_{i+1},\xi_{i+1})\\ &+\frac{\omega-t_{i+1}}{\Delta t}\big(t_{i+1}^{\tau-1}\mathbb{N}(t_{i+1},\xi_{i+1}) - t_i^{\tau-1}\mathbb{N}(t_i,\xi_i)\big),\\ &\omega\in[t_i,t_{i+1}],\end{aligned} \tag{13}$$

where $\xi_i = \xi(t_i)$. At this point, the result obtained in formula (13) can be used in relation (12). This substitution results in the following relationship:

$$\xi_n = \xi_0 + \tau\Delta t^{\rho}\left(\xi_n t_0^{\tau-1}\mathbb{N}(t_0,\xi_0) + \sum_{i=0}^{n}\eth_{n-i}t_i^{\tau-1}\mathbb{N}(t_i,\xi_i)\right), \tag{14}$$

where

$$\begin{aligned}\xi_n &= \frac{(n-1)^{\rho+1} - n^{\rho}(n-\rho-1)}{\Gamma(\rho+2)},\\ \eth_m &= \begin{cases}\dfrac{1}{\Gamma(\rho+2)}, & j=0,\\ \dfrac{(m-1)^{\rho+1}-2m^{\rho+1}+(m+1)^{\rho+1}}{\Gamma(\rho+2)}, & m=1,2,\ldots,n-1.\end{cases}\end{aligned} \tag{15}$$

3.2. The Atangana-Baleanu Fractal-Fractional Derivative

In this case we can convert Equation (9) to the following Atangana-Baleanu fractional form [31]:

$$ {}_{0}^{AB}\mathbb{D}_t^{\rho}\xi(t) = \tau t^{\tau-1}\mathbb{N}(t,\xi(t)). \tag{16}$$

Upon applying the integral operator to both sides of Equation (16), the following Volterra integral equation is constructed:

$$\begin{aligned}\xi(t)-\xi(t_0) &= \frac{\tau-\tau\rho}{\mathbf{AB}(\rho)}t^{\tau-1}\mathbb{N}(t,\xi(t))\\ &+\frac{\tau\rho}{\mathbf{AB}(\rho)\Gamma(\rho)}\int_0^t (t-\omega)^{\rho-1}\omega^{\tau-1}\mathbb{N}(\omega,\xi(\omega))\,d\omega.\end{aligned} \tag{17}$$

Taking $t = t_n = t_0 + n\Delta t$ in (17), we have

$$\begin{aligned}\xi(t_n) &= \xi(t_0) + \frac{\tau-\tau\rho}{\mathbf{AB}(\rho)}t_n^{\tau-1}\mathbb{N}(t_n,\xi(t_n))\\ &+\frac{\tau\rho}{\mathbf{AB}(\rho)\Gamma(\rho)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(t_n-\omega)^{\rho-1}\omega^{\tau-1}\mathbb{N}(\omega,\xi(\omega))\,d\omega.\end{aligned} \tag{18}$$

Now, substituting (13) into (18), we get the following implicit Atangana-Baleanu-Caputo scheme:

$$\begin{aligned}\xi_n &= \xi_0 + \frac{\tau-\tau\rho}{\mathbf{AB}(\rho)}t_n^{\tau-1}\mathbb{N}(t_n,\xi_n)\\ &+\frac{\tau\rho\Delta t^{\rho}}{\mathbf{AB}(\rho)}\left(\xi_n t_0^{\tau-1}\mathbb{N}(t_0,\xi_0) + \sum_{i=0}^{n}\eth_{n-i}t_i^{\tau-1}\mathbb{N}(t_i,\xi_i)\right),\end{aligned} \tag{19}$$

where ξ_n and $\eth_j$ are the coefficients defined in (15). A closer look at relations (14) and (19) shows that these expressions are implicit equations for determining y_n. The Newton iteration method is one of the most popular and efficient techniques for solving such problems. In this article, as in Garrappa [32], Ghanbari and Kumar [33], and Ghanbari et al. [34], we will use the Newton method to solve these equations. By solving these equations, approximate solutions to the original problem will be determined.

4. NUMERICAL SIMULATIONS FOR SOME CHAOTIC SYSTEMS

In this section, to show the validity of the proposed numerical technique, three chaotic systems involving Atangana-Baleanu-Caputo fractional derivatives are considered.

Example 1. Consider the following modified cyclically symmetric Thomas attractor given by [35]

$$\begin{aligned}{}_{0}^{\mathrm{FF}}\mathbb{D}_t^{\rho,\tau}x(t) &= \sin\big(\exp\big(y(t)\big)\big) - \mathbb{B}x(t),\\ {}_{0}^{\mathrm{FF}}\mathbb{D}_t^{\rho,\tau}y(t) &= \cos\big(\sin\big(z(t)\big)\big) - \mathbb{B}y(t),\\ {}_{0}^{\mathrm{FF}}\mathbb{D}_t^{\rho,\tau}z(t) &= \exp\big(\cos\big(x(t)\big)\big) - \mathbb{B}z(t).\end{aligned} \tag{20}$$

Now we take into account the iterative methods given by (14) and (19) to solve (20). Taking (14) into account, we get the iterative structure

$$
\begin{aligned}
x_n &= x_0 + \tau\Delta t^\rho \Bigg(\xi_n t_0^{\tau-1}\big[\sin\big(\exp(y_0)\big) - \mathbb{B}x_0\big] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1}\big[\sin\big(\exp(y_i)\big) - \mathbb{B}x_i\big]\Bigg), \\
y_n &= y_0 + \tau\Delta t^\rho \Bigg(\xi_n t_0^{\tau-1}\big[\cos\big(\sin(z_0)\big) - \mathbb{B}y_0\big] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1}\big[\cos\big(\sin(z_i)\big) - \mathbb{B}y_i\big]\Bigg), \\
z_n &= z_0 + \tau\Delta t^\rho \Bigg(\xi_n t_0^{\tau-1}\big[\exp\big(\cos(x_0)\big) - \mathbb{B}z_0\big] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1}\big[\exp\big(\cos(x_i)\big) - \mathbb{B}z_i\big]\Bigg).
\end{aligned} \tag{21}
$$

Moreover, from (19) we get

$$
\begin{aligned}
x_n &= x_0 + \frac{\tau-\tau\rho}{\mathbf{AB}(\rho)} t_n^{\tau-1}\big[\sin\big(\exp(y_n)\big) - \mathbb{B}x_n\big] \\
&\quad + \frac{\tau\rho\Delta t^\rho}{\mathbf{AB}(\rho)} \Bigg(\xi_n t_0^{\tau-1}\big[\sin\big(\exp(y_0)\big) - \mathbb{B}x_0\big] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1}\big[\sin\big(\exp(y_i)\big) - \mathbb{B}x_i\big]\Bigg), \\
y_n &= y_0 + \frac{\tau-\tau\rho}{\mathbf{AB}(\rho)} t_n^{\tau-1}\big[\cos\big(\sin(z_n)\big) - \mathbb{B}y_n\big] \\
&\quad + \frac{\tau\rho\Delta t^\rho}{\mathbf{AB}(\rho)} \Bigg(\xi_n t_0^{\tau-1}\big[\cos\big(\sin(z_0)\big) - \mathbb{B}y_0\big] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1}\big[\cos\big(\sin(z_i)\big) - \mathbb{B}y_i\big]\Bigg), \\
z_n &= z_0 + \frac{\tau-\tau\rho}{\mathbf{AB}(\rho)} t_n^{\tau-1}\big[\exp\big(\cos(x_n)\big) - \mathbb{B}z_n\big] \\
&\quad + \frac{\tau\rho\Delta t^\rho}{\mathbf{AB}(\rho)} \Bigg(\xi_n t_0^{\tau-1}\big[\exp\big(\cos(x_0)\big) - \mathbb{B}z_0\big] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1}\big[\exp\big(\cos(x_i)\big) - \mathbb{B}z_i\big]\Bigg).
\end{aligned} \tag{22}
$$

In **Figures 1–4** we plot the results of numerical simulations using the two iterative schemes (21) and (22) with $\mathbb{B} = 0.2$. For the numerical implementations we took $(x_0, y_0, z_0) = (0, 3, 11)$ as the initial condition. The results of the iterative scheme (21) are plotted in **Figures 5–8**. In each figure we have taken a fixed value of τ and different values of ρ. In these experiments we set $\Delta t = 10^{-3}$ and $T = 800$.

Example 2. In this example we consider the chaotic system

$$
\begin{aligned}
{}^{FF}_{0}\mathbb{D}_t^{\rho,\tau} x(t) &= x(t)z(t) - \mathbb{B}x(t) - \mathbb{D}y(t), \\
{}^{FF}_{0}\mathbb{D}_t^{\rho,\tau} y(t) &= \mathbb{D}x(t) + y(t)z(t) - \mathbb{B}y(t), \\
{}^{FF}_{0}\mathbb{D}_t^{\rho,\tau} z(t) &= \mathbb{C} + \mathbb{A}z(t) - \tfrac{z(t)^3}{3} - x(t)^2 + \mathbb{E}z(t)x(t)^3,
\end{aligned} \tag{23}
$$

To solve this chaotic system, we use the following iterative scheme obtained from (14):

$$
\begin{aligned}
x_n &= x_0 + \tau\Delta t^\rho \Bigg(\xi_n t_0^{\tau-1}\big[x_0z_0 - \mathbb{B}x_0 - \mathbb{D}y_0\big] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1}\big[x_iz_i - \mathbb{B}x_i - \mathbb{D}y_i\big]\Bigg), \\
y_n &= y_0 + \tau\Delta t^\rho \Bigg(\xi_n t_0^{\tau-1}\big[\mathbb{D}x_0 + y_0z_0 - \mathbb{B}y_0\big] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1}\big[\mathbb{D}x_i + y_iz_i - \mathbb{B}y_i\big]\Bigg), \\
z_n &= z_0 + \tau\Delta t^\rho \Bigg(\xi_n t_0^{\tau-1}\Big[\mathbb{C} + \mathbb{A}z_0 - \frac{z_0^3}{3} - x_0^2 + \mathbb{E}z_0x_0^3\Big] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1}\Big[\mathbb{C} + \mathbb{A}z_i - \frac{z_i^3}{3} - x_i^2 + \mathbb{E}z_ix_i^3\Big]\Bigg).
\end{aligned} \tag{24}
$$

Using (19) one gets

$$
\begin{aligned}
x_n &= x_0 + \frac{\tau-\tau\rho}{\mathbf{AB}(\rho)} t_n^{\tau-1}\big[x_nz_n - \mathbb{B}x_n - \mathbb{D}y_n\big] \\
&\quad + \frac{\tau\rho\Delta t^\rho}{\mathbf{AB}(\rho)} \Bigg(\xi_n t_0^{\tau-1}\big[x_0z_0 - \mathbb{B}x_0 - \mathbb{D}y_0\big] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1}\big[x_iz_i - \mathbb{B}x_i - \mathbb{D}y_i\big]\Bigg), \\
y_n &= y_0 + \frac{\tau-\tau\rho}{\mathbf{AB}(\rho)} t_n^{\tau-1}\big[\mathbb{D}x_n + y_nz_n - \mathbb{B}y_n\big] \\
&\quad + \frac{\tau\rho\Delta t^\rho}{\mathbf{AB}(\rho)} \Bigg(\xi_n t_0^{\tau-1}\big[\mathbb{D}x_0 + y_0z_0 - \mathbb{B}y_0\big] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1}\big[\mathbb{D}x_i + y_iz_i - \mathbb{B}y_i\big]\Bigg), \\
z_n &= z_0 + \frac{\tau-\tau\rho}{\mathbf{AB}(\rho)} t_n^{\tau-1}\Big[\mathbb{C} + \mathbb{A}z_n - \frac{z_n^3}{3} - x_n^2 + \mathbb{E}z_nx_n^3\Big] \\
&\quad + \frac{\tau\rho\Delta t^\rho}{\mathbf{AB}(\rho)} \Bigg(\xi_n t_0^{\tau-1}\Big[\mathbb{C} + \mathbb{A}z_0 - \frac{z_0^3}{3} - x_0^2 + \mathbb{E}z_0x_0^3\Big] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1}\Big[\mathbb{C} + \mathbb{A}z_i - \frac{z_i^3}{3} - x_i^2 + \mathbb{E}z_ix_i^3\Big]\Bigg).
\end{aligned} \tag{25}
$$

The approximate solutions obtained from (24) are plotted in **Figures 5**, **6**, and those obtained from (25) are plotted in **Figures 7**, **8**. The parameter values used in the model are $\mathbb{A} = 0.95$, $\mathbb{B} = 0.7$, $\mathbb{C} = 0.6$, $\mathbb{D} = 3.5$, and $\mathbb{E} = 0.1$. We used the initial guess $(x_0, y_0, z_0) = (0.1, 0, 0)$ for different fractional orders of ρ and τ.

Example 3. Consider the chaotic system given by [31]

$$
\begin{aligned}
{}^{FF}_{0}\mathbb{D}_t^{\rho,\tau} x(t) &= \mathbb{A}\left(y(t) - x(t)\right), \\
{}^{FF}_{0}\mathbb{D}_t^{\rho,\tau} y(t) &= (\mathbb{C} - \mathbb{A})\, x(t) - \big[\mathbb{S}_0 z(t) - \mathbb{S}_1 \sin(z(t))\big]\, x(t) + \mathbb{C}y(t), \\
{}^{FF}_{0}\mathbb{D}_t^{\rho,\tau} z(t) &= x(t)y(t) - \mathbb{B}z(t),
\end{aligned} \tag{26}
$$

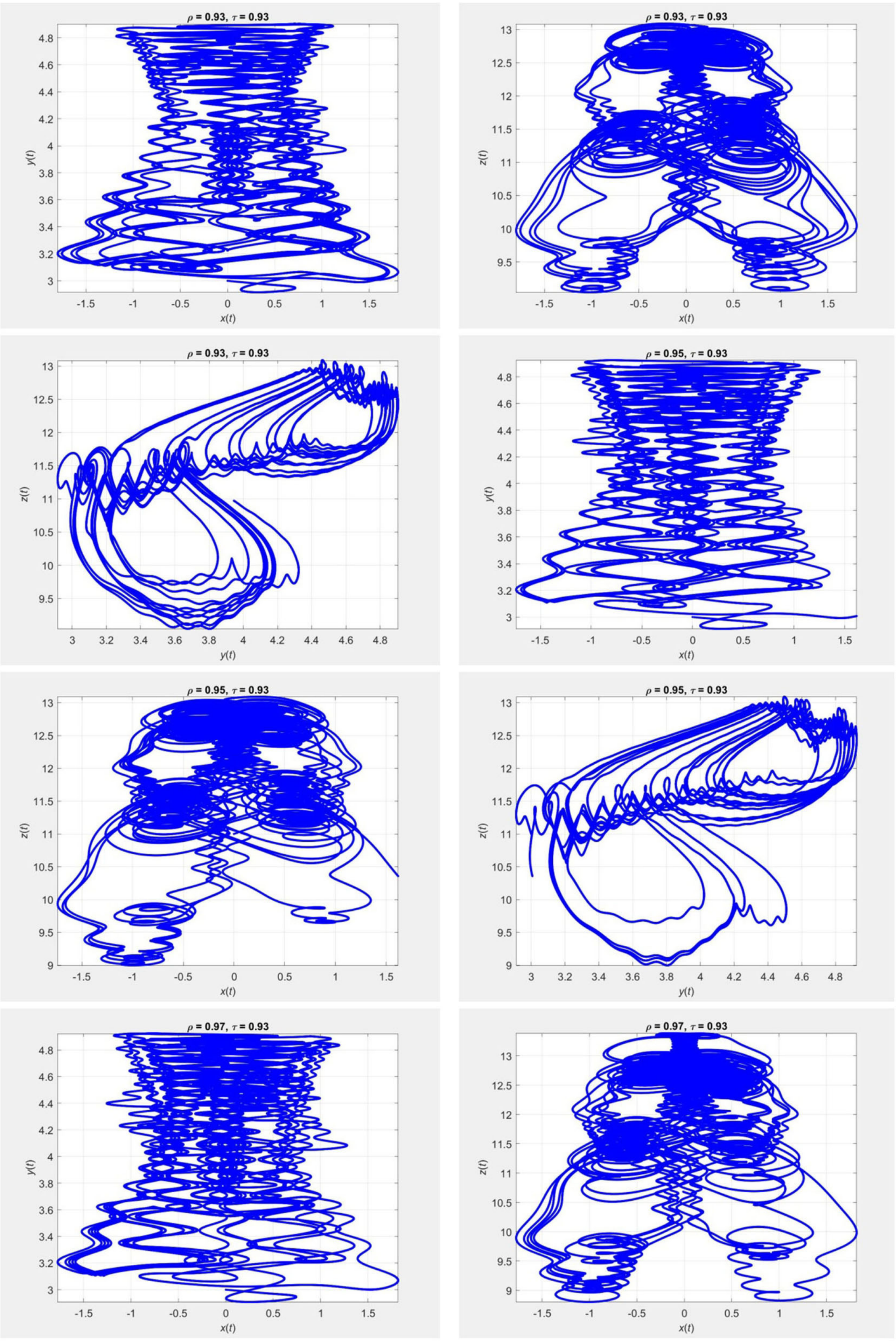

FIGURE 1 | Evolution of the chaotic system (20) for $\tau = 0.93$ and different values of ρ, using the iterative scheme (21).

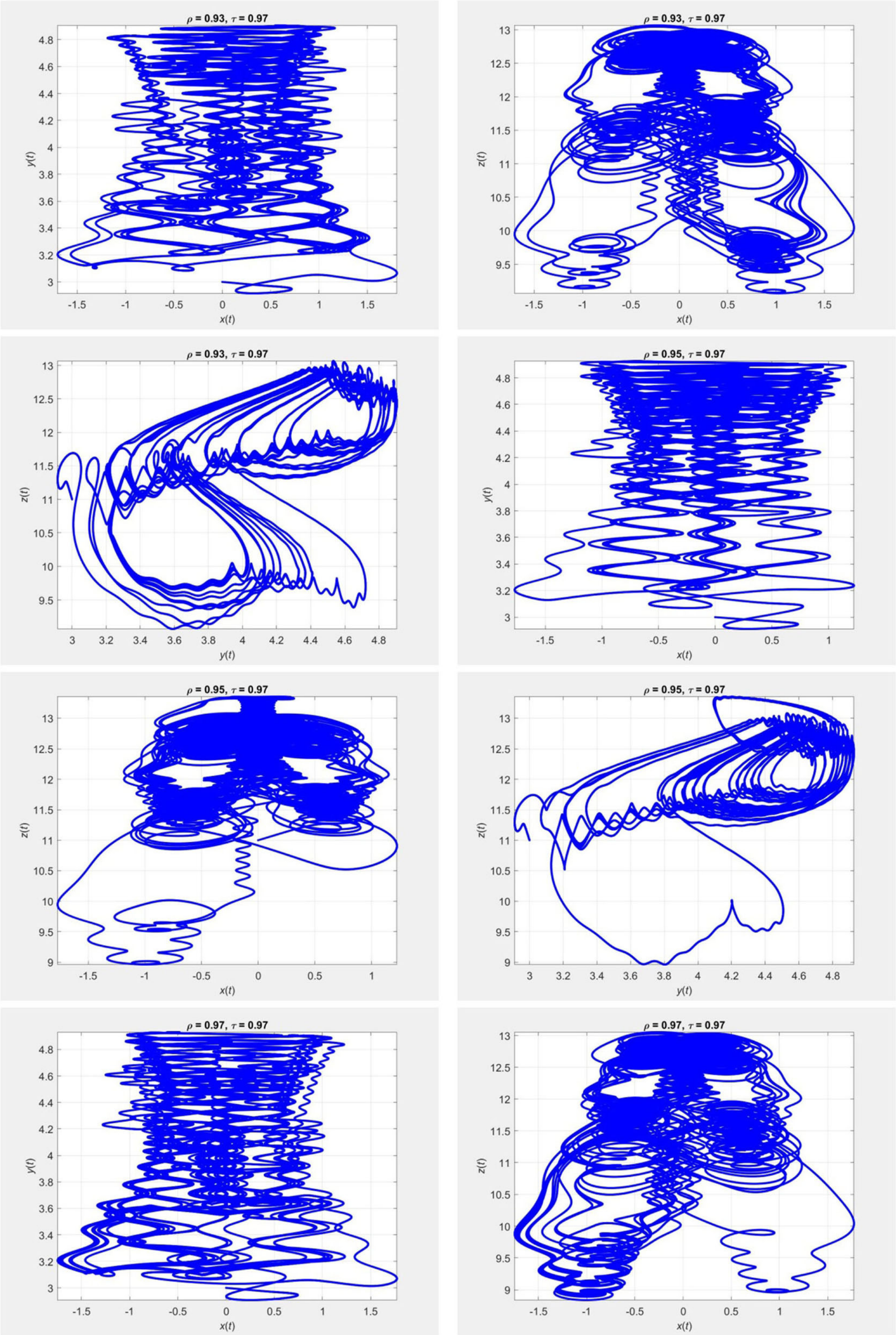

FIGURE 2 | Evolution of the chaotic system (20) for $\tau = 0.97$ and different values of ρ, using the iterative scheme (21).

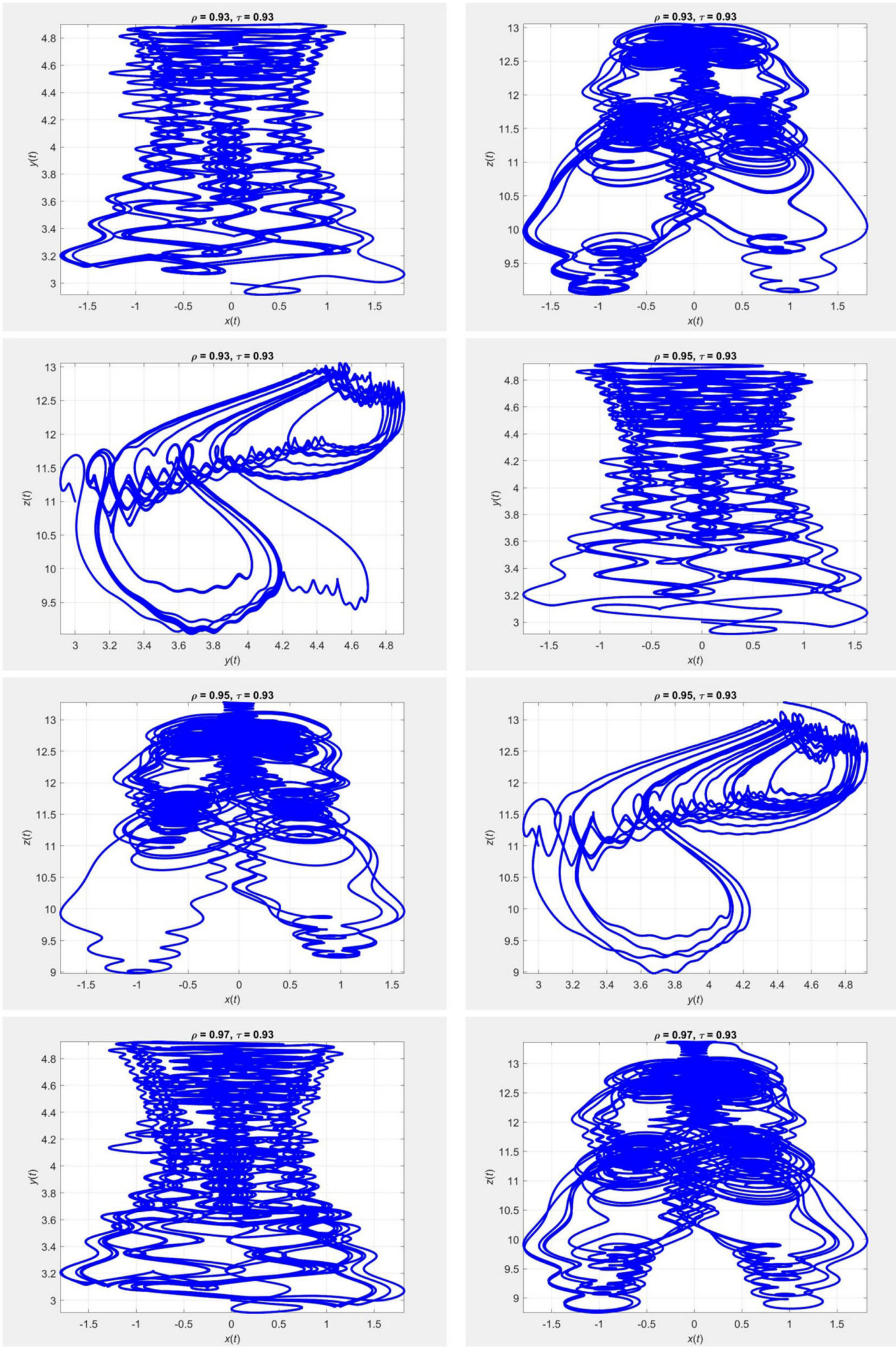

FIGURE 3 | Evolution of the chaotic system (20) for $\tau = 0.93$ and different values of ρ, using the iterative scheme (22).

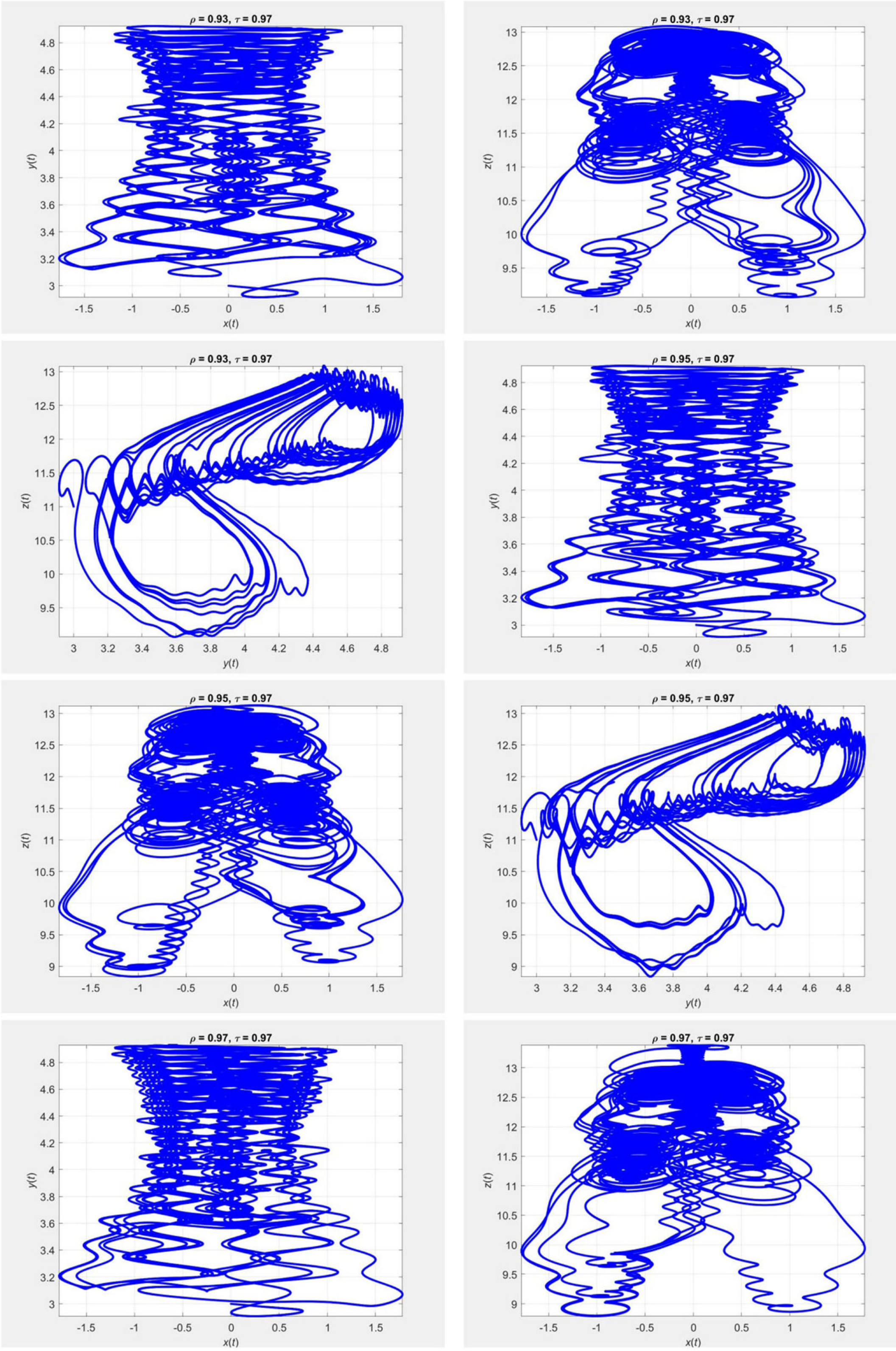

FIGURE 4 | Evolution of the chaotic system (20) for $\tau = 0.97$ and different values of ρ, using the iterative scheme (22).

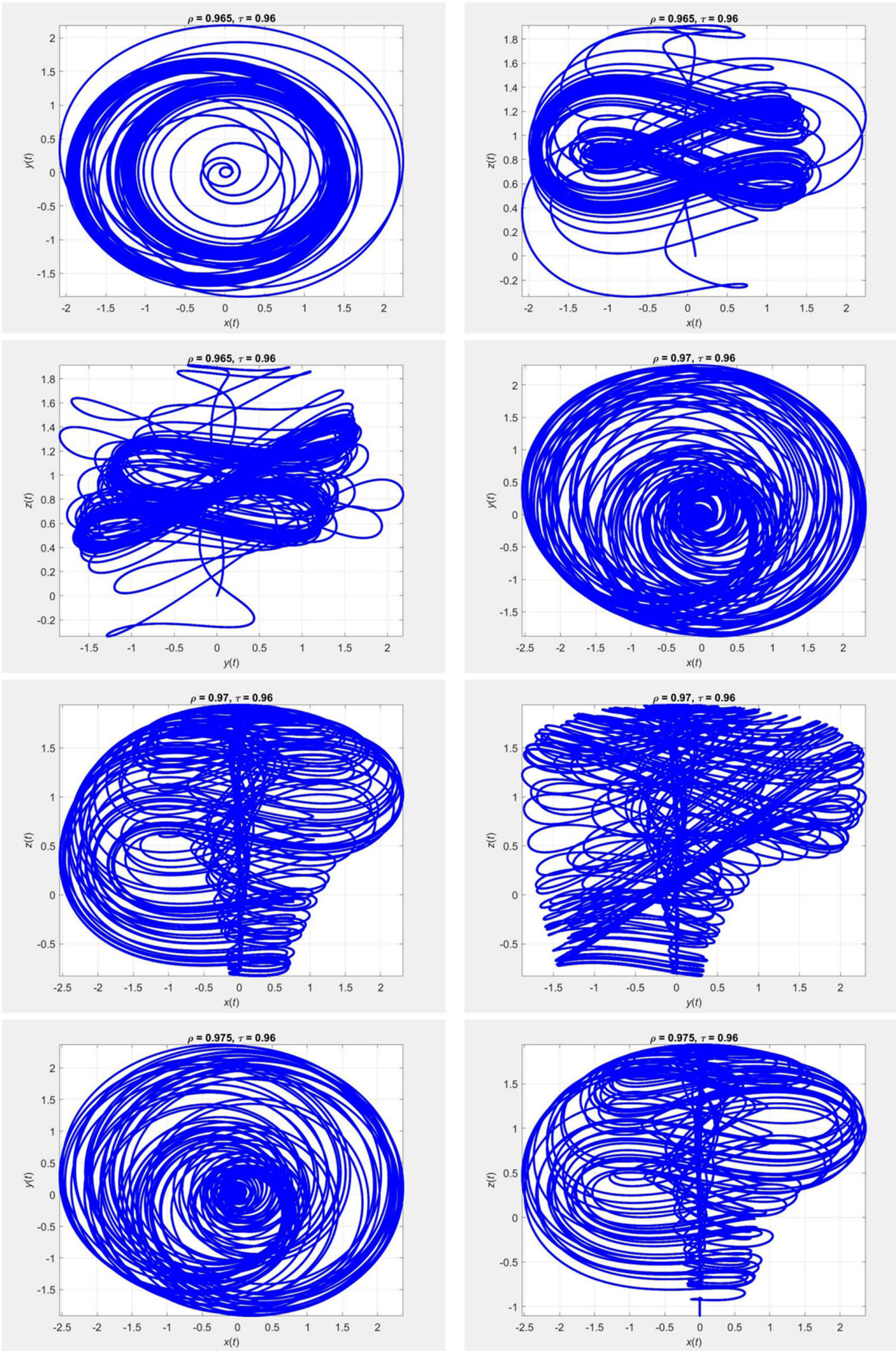

FIGURE 5 | Evolution of the chaotic system (23) for $\tau = 0.96$ and different values of ρ, using the iterative scheme (24).

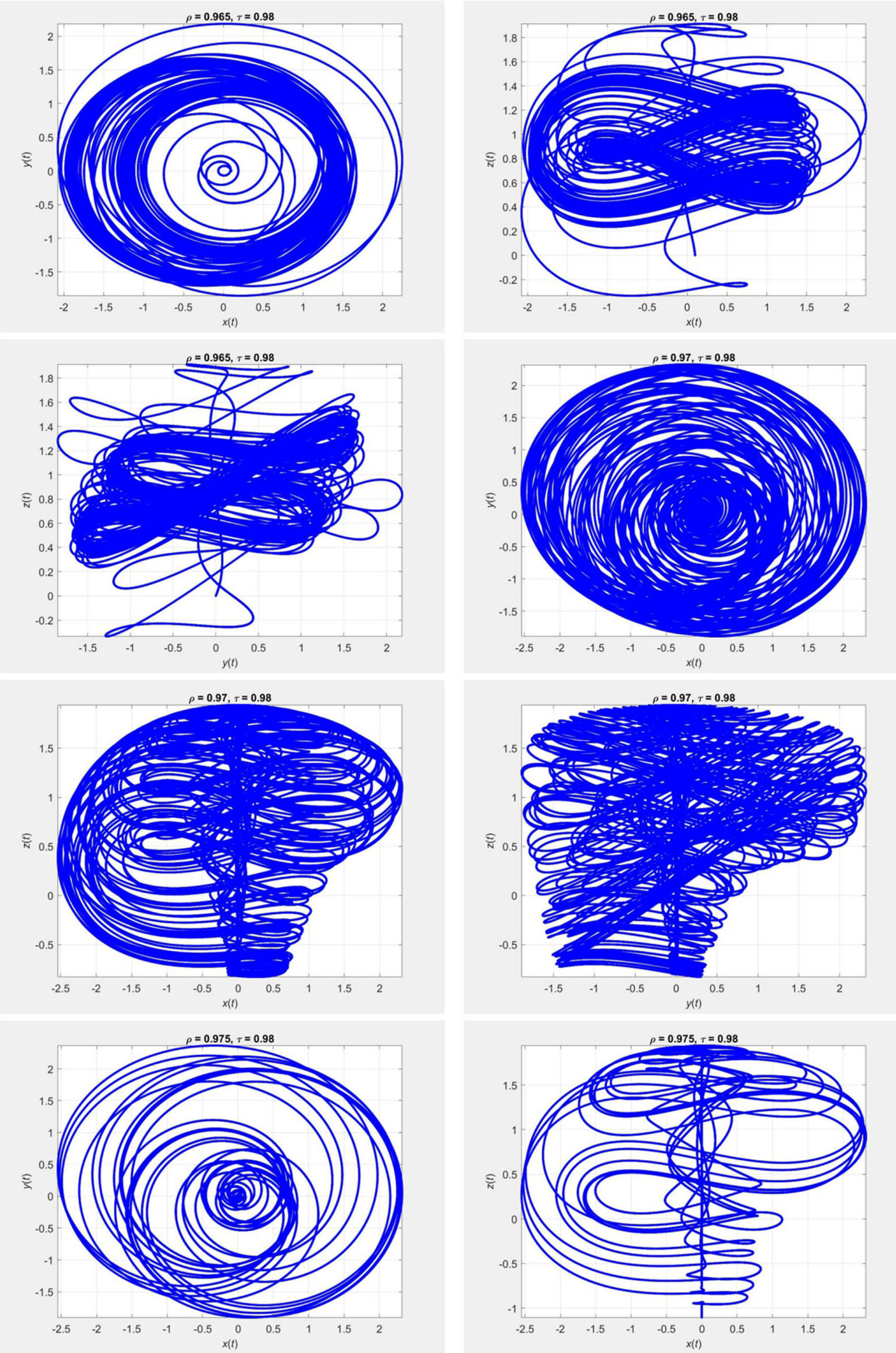

FIGURE 6 | Evolution of the chaotic system (23) for $\tau = 0.98$ and different values of ρ, using the iterative scheme (24).

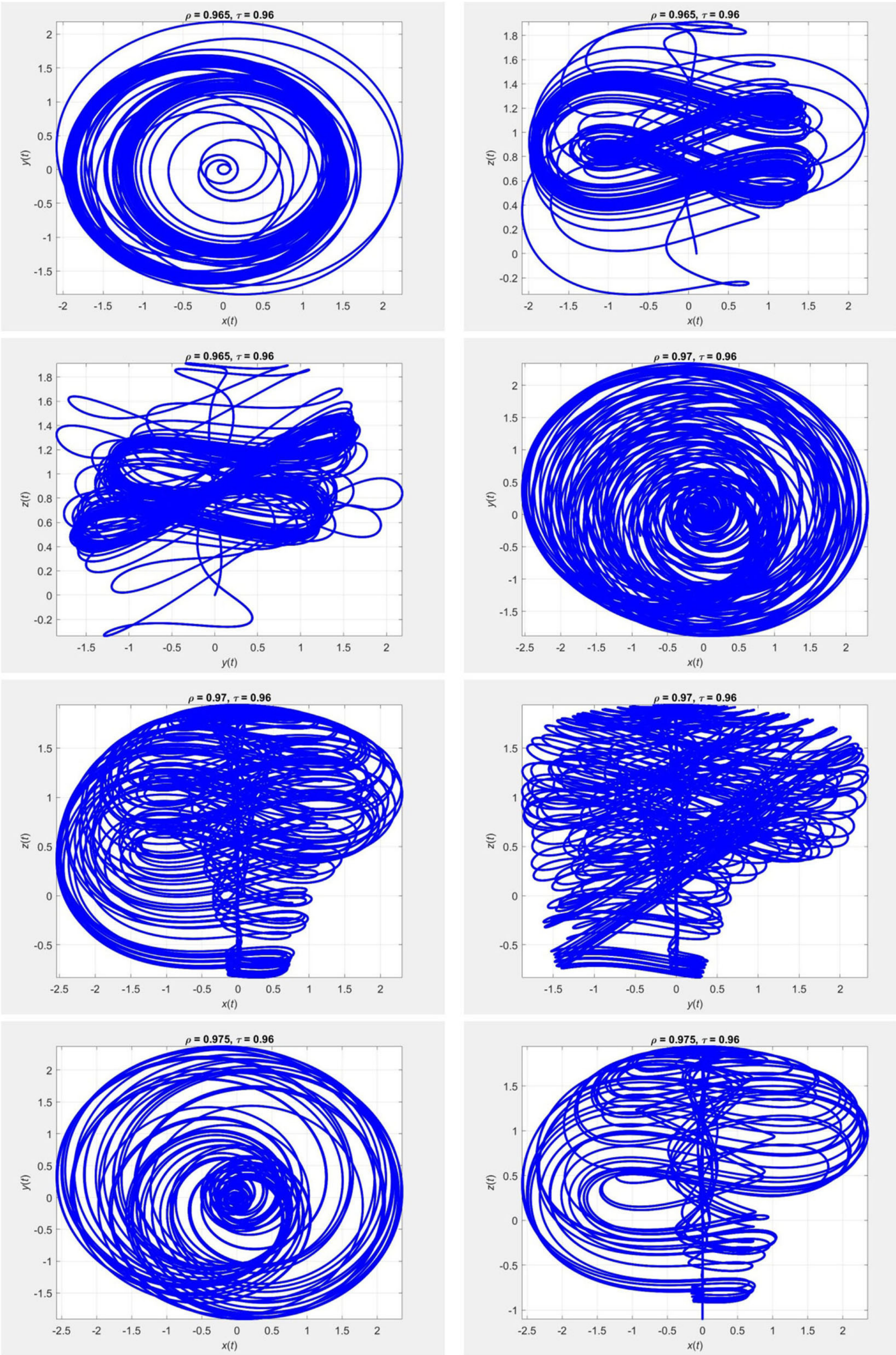

FIGURE 7 | Evolution of the chaotic system (23) for $\tau = 0.96$ and different values of ρ, using the iterative scheme (25).

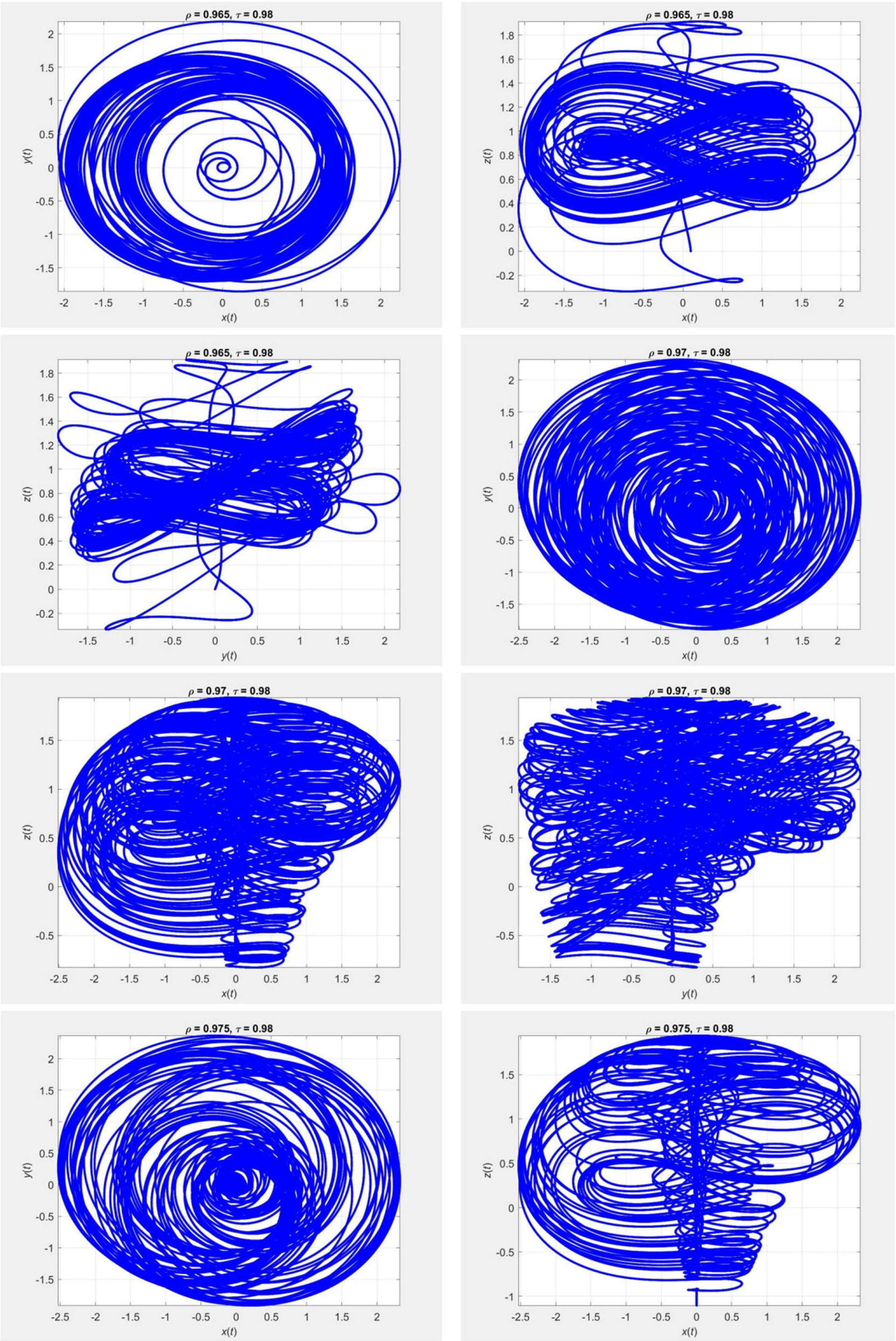

FIGURE 8 | Evolution of the chaotic system (23) for $\tau = 0.98$ and different values of ρ, using the iterative scheme (25).

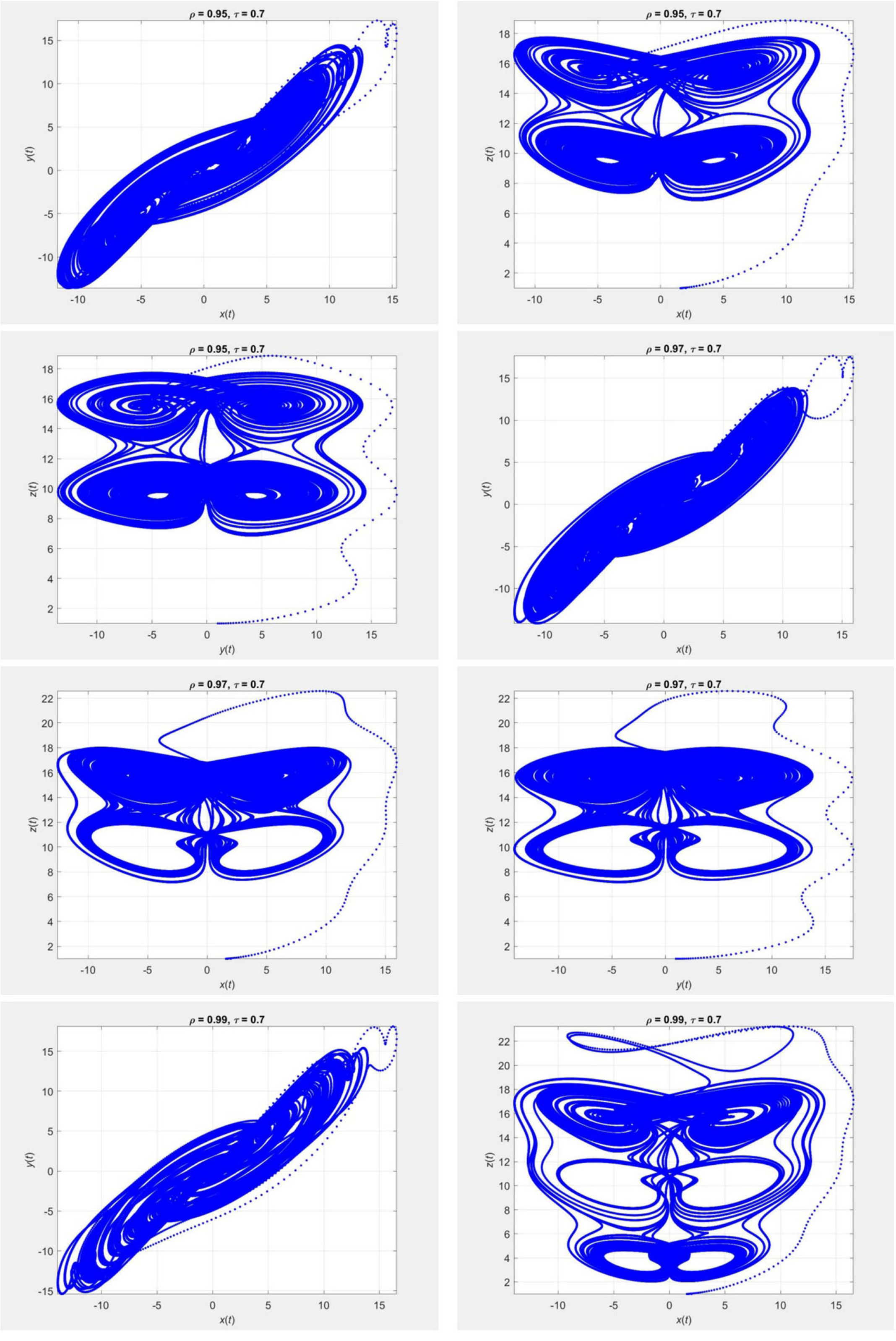

FIGURE 9 | Evolution of the chaotic system (26) for $\tau = 0.7$ and different values of ρ, using the iterative scheme (27).

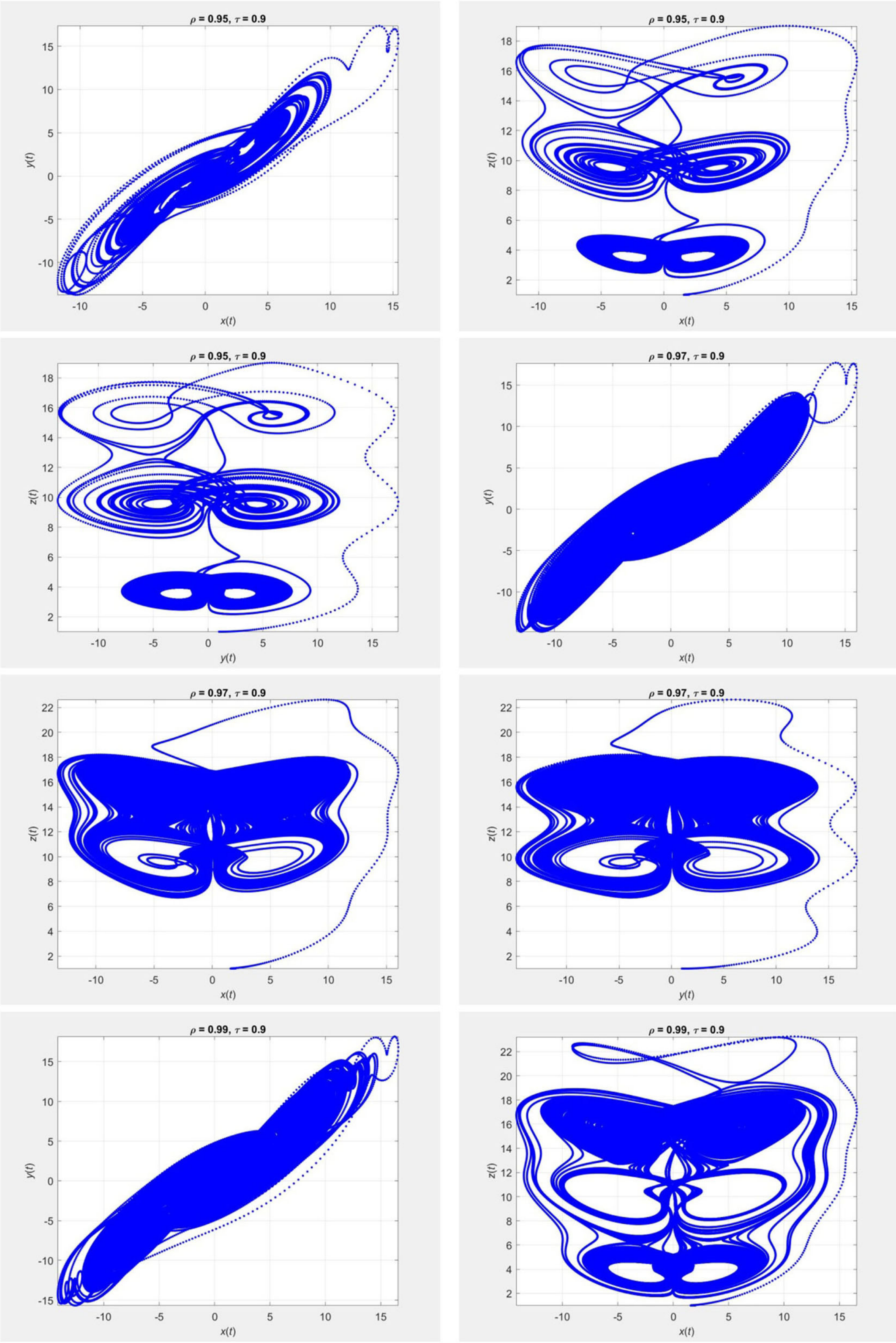

FIGURE 10 | Evolution of the chaotic system (26) for $\tau = 0.9$ and different values of ρ, using the iterative scheme (27).

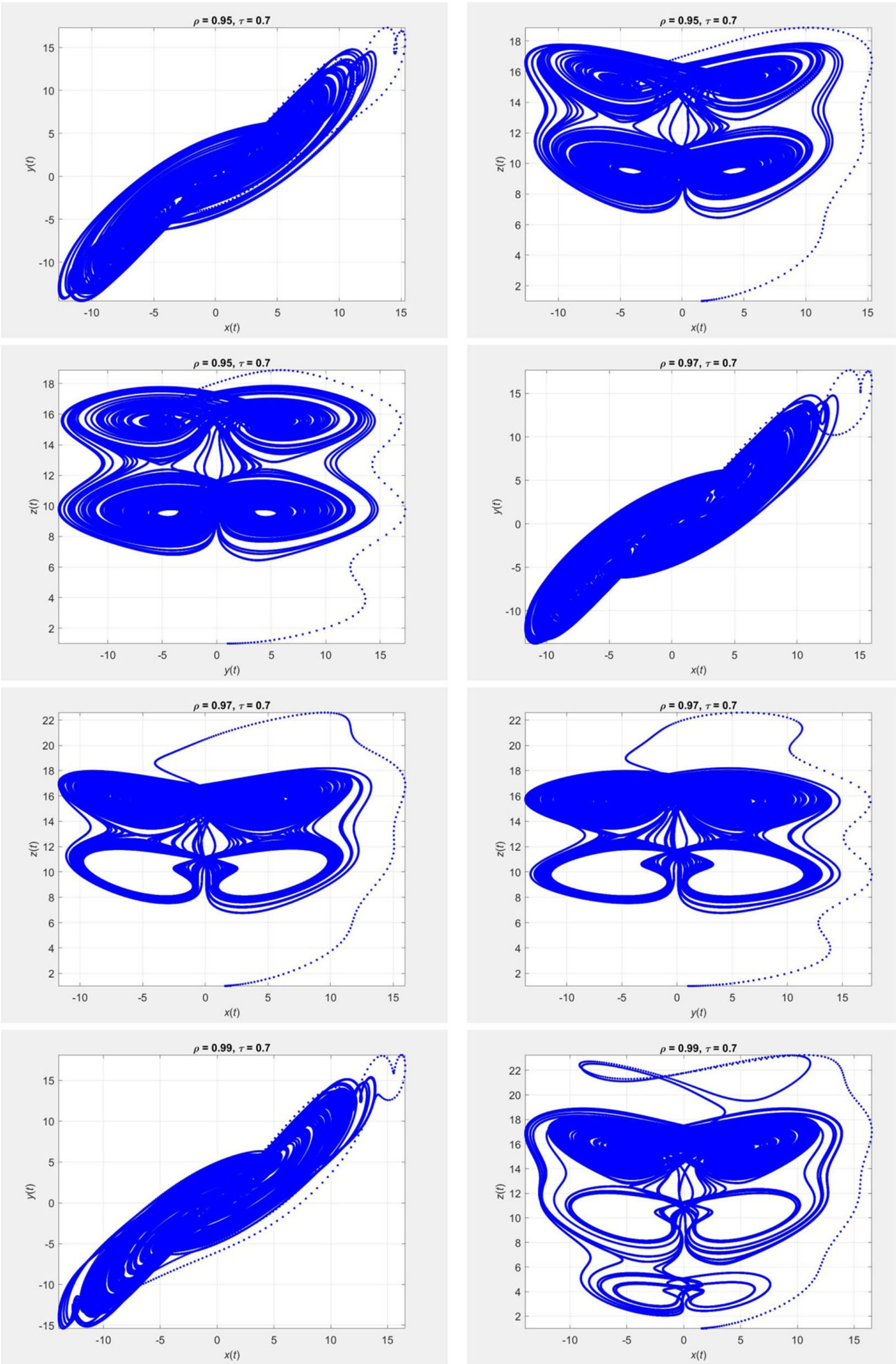

FIGURE 11 | Evolution of the chaotic system (26) for $\tau = 0.7$ and different values of ρ, using the iterative scheme (28).

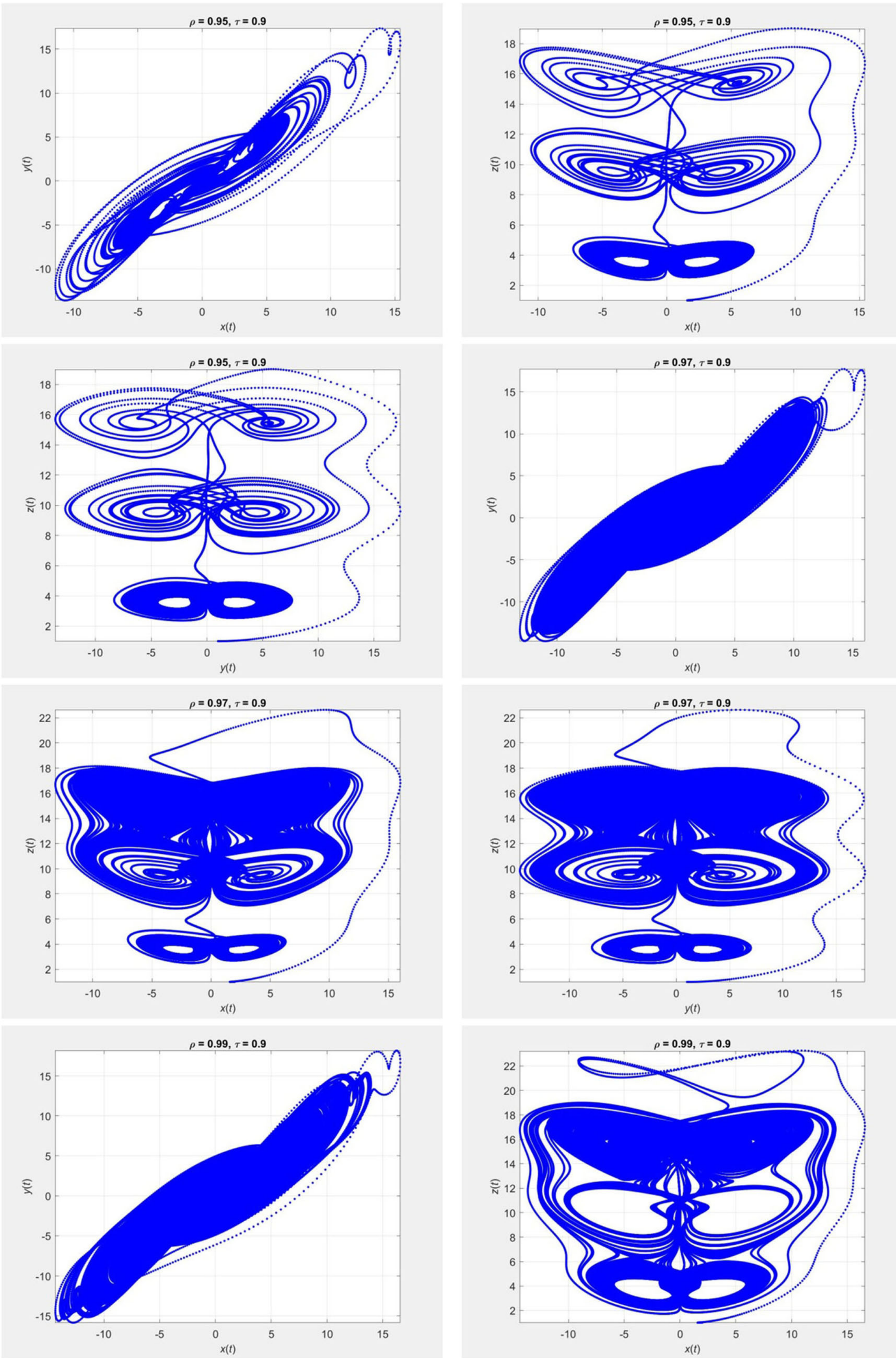

FIGURE 12 | Evolution of the chaotic system (26) for $\tau = 0.9$ and different values of ρ, using the iterative scheme (28).

For this system, from (14) we obtain the following iterative scheme:

$$
\begin{aligned}
x_n &= x_0 + \tau \Delta t^{\rho} \left(\xi_n t_0^{\tau-1} \left[\mathbb{A} \left(y_0 - x_0 \right) \right] + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1} \left[\mathbb{A} \left(y_i - x_i \right) \right] \right), \\
y_n &= y_0 + \tau \Delta t^{\rho} \Bigg(\xi_n t_0^{\tau-1} \left[\left(\mathbb{C} - \mathbb{A} \right) x_0 - \left[\mathbb{S}_0 z_0 - \mathbb{S}_1 \sin(z_0) \right] x_0 + \mathbb{C} y_0 \right] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1} \left[\left(\mathbb{C} - \mathbb{A} \right) x_0 - \left[\mathbb{S}_0 z_i - \mathbb{S}_1 \sin(z_i) \right] x_i + \mathbb{C} y_i \right] \Bigg), \\
z_n &= z_0 + \tau \Delta t^{\rho} \left(\xi_n t_0^{\tau-1} \left[x_0 y_0 - \mathbb{B} z_0 \right] + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1} \left[x_i y_i - \mathbb{B} z_i \right] \right).
\end{aligned} \tag{27}
$$

Moreover, from (19), the approximate solution is obtained as

$$
\begin{aligned}
x_n &= x_0 + \frac{\tau - \tau\rho}{\mathbf{AB}(\rho)} t_n^{\tau-1} \left[\mathbb{A} \left(y_n - x_n \right) \right] \\
&\quad + \frac{\tau\rho\Delta t^{\rho}}{\mathbf{AB}(\rho)} \left(\xi_n t_0^{\tau-1} \left[\mathbb{A} \left(y_0 - x_0 \right) \right] + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1} \left[\mathbb{A} \left(y_i - x_i \right) \right] \right), \\
y_n &= y_0 + \frac{\tau - \tau\rho}{\mathbf{AB}(\rho)} t_n^{\tau-1} \left[\left(\mathbb{C} - \mathbb{A} \right) x_0 - \left[\mathbb{S}_0 z_n - \mathbb{S}_1 \sin(z_n) \right] x_n + \mathbb{C} y_n \right] \\
&\quad + \frac{\tau\rho\Delta t^{\rho}}{\mathbf{AB}(\rho)} \Bigg(\xi_n t_0^{\tau-1} \left[\left(\mathbb{C} - \mathbb{A} \right) x_0 - \left[\mathbb{S}_0 z_0 - \mathbb{S}_1 \sin(z_0) \right] x_0 + \mathbb{C} y_0 \right] \\
&\quad + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1} \left[\left(\mathbb{C} - \mathbb{A} \right) x_0 - \left[\mathbb{S}_0 z_i - \mathbb{S}_1 \sin(z_i) \right] x_i + \mathbb{C} y_i \right] \Bigg), \\
z_n &= z_0 + \frac{\tau - \tau\rho}{\mathbf{AB}(\rho)} t_n^{\tau-1} \left[x_n y_n - \mathbb{B} z_n \right] \\
&\quad + \frac{\tau\rho\Delta t^{\rho}}{\mathbf{AB}(\rho)} \left(\xi_n t_0^{\tau-1} \left[x_0 y_0 - \mathbb{B} z_0 \right] + \sum_{i=0}^{n} \eth_{n-i} t_i^{\tau-1} \left[x_i y_i - \mathbb{B} z_i \right] \right).
\end{aligned} \tag{28}
$$

Figures 9–12 show the portraits corresponding to the chaotic system in (26) obtained using (27) and (28). The parameters for the model are $\mathbb{A} = 50, \mathbb{B} = 2, \mathbb{C} = 30, \mathbb{S}_1 = 1$, and $\mathbb{S}_2 = 20$. We used the initial guess $(x_0, y_0, z_0) = (2, 1, 1)$ for different fractional orders of ρ.

5. CONCLUSION

Although many numerical methods are available, the development of new efficient numerical schemes has always been one of the most important concerns in applied mathematics and engineering. A foremost reason for the widespread interest in new numerical methods is that they may reveal new facts about real-world phenomena. This paper has presented some efficient approximate methods for solving chaotic systems that use new definitions for the derivative, called fractal-fractional derivatives. The concept of memory is one of the most important features of these types of derivatives. With this valuable feature, the evolution of the phenomena modeled by such systems can be more accurately predicted. The proposed new techniques have been tested by using them to solve several important practical problems. Application of the methods to these problems revealed very interesting behaviors of the systems that have meaningful interpretations. The numerical methods presented in this article have the potential to be used for solving similar models. Since any new numerical method should be validated in terms of convergence, stability and consistency of solutions, these are important research directions left to future work.

DATA AVAILABILITY STATEMENT

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation, to any qualified researcher.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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