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A Relation Between Moore-Penrose Inverses of Hermitian Matrices and Its Application in Electrical Networks

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A novel relation between the Moore-Penrose inverses of two nullity-1 $n \times n$ Hermitian matrices which share a common null eigenvector is established, and its application in electrical networks is illustrated by applying the result to Laplacian matrices of graphs.

Keywords: resistance distance, electrical network, Hermitian matrix, Laplacian matrix, Moore-Penrose inverse

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1. INTRODUCTION

The Hermitian matrices are an important class of matrices arising in many contexts. A complex squared matrix is called a *Hermitian matrix* if it is equal to its conjugate transpose, in other words, for all i and j, its (i,j)-th element (i.e., the element in the i-th row and j-th column) is equal to the complex conjugate of its (j,i)-th element. It is widely known that all the eigenvalues of a Hermitian matrix are real. In addition, it is easily seen that Hermitian matrices contain real symmetric matrices as special cases.

Let M be an $n \times m$ matrix. An $m \times n$ matrix X is called the *Moore-Penrose (generalized) inverse* of M, if X satisfies the following equations:

$$MXM = M, XMX = X, (MX)^H = MX, (XM)^H = XM,$$

where X^H represents the conjugate transpose of the matrix M. It is well-known [1] that for any matrix M, the Moore-Penrose inverse of M does exist and is unique. For this reason, the unique Moore-Penrose inverse of M is denoted by M^+.

We proceed to introduce a special class of Hermitian matrices – the Laplacian matrices of graphs, which play a fundamental role in graph theory and electrical network theory. Let $G = (V, E)$ be a connected weighted graph of order n. For each edge e of G, we assign a positive real number w_e to e, and we call w_e the *weight* of e. Then the adjacency matrix of G, denoted by A, is a $n \times n$ matrix such that the (i,j)-th element of A is equal to the weight of the edge ij if i and j are connected by an edge and 0 otherwise. Suppose that D is the $n \times n$ diagonal matrix such that the i-th diagonal element is equal to the sum of the weights of the edges incident to i. Then the *Laplacian matrix* L of G is defined as $L = D - A$. It is easily seen that the Laplacian matrix is real and symmetric. Thus, the Laplacian matrix is a Hermitian matrix. According to the definition of the Laplacian matrix, we readily seen that the Laplacian matrix is singular and not invertible.

It is natural to consider a weighted graph G as a (resistive) electrical network $\mathcal{N}$ by viewing each edge e as a resistor such that the conductance of the resistor is w_e, where w_e is the weight on e. In this guise, the *resistance distance* [2] between any two vertices i and j of G, denoted by $\Omega(i,j)$, is defined as the net effective resistance between corresponding nodes i and j in $\mathcal{N}$. It should be mentioned that resistance distance, as an important component of circuit theory, has been studied for a long time, dating back to the classical work of Kirchhoff in 1847. It is amazing that the resistance distance turns out to have many purely mathematical interpretations, although it comes from physics and engineering, among which a fundamental one is the classical result which is given via the Moore-Penrose inverse of the Laplacian matrix [2]:

$$\Omega(i,j) = L^{+}_{ii} - 2L^{+}_{ij} + L^{+}_{jj}, \tag{1.1}$$

where L^{+}_{ij} denote the (i,j)-th element of L^{+}. Since the identification of resistance distance as a novel distance function on graphs, the resistance distance has been extensively studied in the literature of mathematics, physics, and chemistry. For more information on resistance distances, we refer the readers to recent papers [3–13] and references therein.

In this paper, a relation between the Moore-Penrose inverses of two nullity-1 $n \times n$ Hermitian matrices which share a common null eigenvector is established. Then its application in electrical networks is illustrated by applying the result to Laplacian matrices of graphs.

2. A RELATION BETWEEN MOORE-PENROSE INVERSES OF TWO HERMITIAN MATRICES

All the matrices considered in this section are square matrices of order n. For an invertible matrix M, we use M^{-1} to denote the inverse of M. Let I and $\mathbf{O}$ denote the identity matrix and zero matrix, respectively. This section is devoted to establish a relation between Moore-Penrose inverses of two Hermitian matrices of nullity-1 which share a common null eigenvector. To this end, we first give some properties on nullity-1 Hermitian matrices, which will be used in the later.

Lemma 2.1. *Let M be a nullity-1 Hermitian $n\times n$ matrix. Suppose that $0 = \lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of M with corresponding orthonormal eigenvectors $u_1, u_2, \ldots, u_n$. Then*

$$M^{+} = (M + u_1 u_1^{H})^{-1} - u_1 u_1^{H}. \tag{2.1}$$

$$MM^{+} = M^{+}M = I - u_1 u_1^{H}. \tag{2.2}$$

$$u_1 u_1^{H} M^{+} = \mathbf{O}. \tag{2.3}$$

Proof: Let $U = (u_1, u_2, \ldots, u_n)$ and $\Lambda = \mathrm{diag}\{0, \lambda_2, \ldots, \lambda_n\}$. Then

$$M = U\Lambda U^{H}.$$

As $u_1 u_1^{H} = U\mathrm{diag}\{1, 0, \ldots, 0\}U^{H}$, it follows that

$$\begin{aligned} M + u_1 u_1^{H} &= U\Lambda U^{H} + U\mathrm{diag}\{1, 0, \ldots, 0\}U^{H} \\ &= U\mathrm{diag}\{1, \lambda_2, \ldots, \lambda_n\}U^{H}. \end{aligned}$$

Thus $M + u_1 u_1^{H}$ is invertible with

$$(M + u_1 u_1^{H})^{-1} = U\mathrm{diag}\left\{1, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}\right\} U^{H}.$$

Consequently,

$$(M + u_1 u_1^{H})^{-1} - u_1 u_1^{H} = U\mathrm{diag}\left\{0, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}\right\} U^{H}.$$

Thus it is easily verified by the definition of the Moore-Penrose inverse that

$$M^{+} = (M + u_1 u_1^{H})^{-1} - u_1 u_1^{H}.$$

To prove Equation (2.2), note first that

$$\begin{aligned} MM^{+} &= U\Lambda U^{H} U\Lambda_0 H^{H} = U\Lambda\Lambda_0 H^{H} \quad \text{and} \\ M^{+}M &= U\Lambda_0 U^{H} U\Lambda U^{H} = U\Lambda_0\Lambda U^{H}, \end{aligned}$$

where $\Lambda_0 = U\mathrm{diag}\left\{1, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}\right\} U^{H}$. Then, note that

$$\Lambda\Lambda_0 = \Lambda_0\Lambda = \mathrm{diag}\{0, 1, \ldots, 1\}.$$

Thus we have

$$\begin{aligned} MM^{+} = M^{+}M &= U(\mathrm{diag}\{0, 1, \ldots, 1\})U^{H} \\ &= U(I - \mathrm{diag}\{1, 0, \ldots, 0\})U^{H} \\ &= UU^{H} - U\mathrm{diag}\{1, 0, \ldots, 0\}U^{H} = I - u_1 u_1^{H}. \end{aligned}$$

For Equation (2.3), by the above arguments we have

$$\begin{aligned} u_1 u_1^{H} M^{+} &= (U\mathrm{diag}\{1, 0, \ldots, 0\}U^{H}) \left(U\mathrm{diag}\{0, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}\}U^{H} \right) \\ &= U\mathrm{diag}\{1, 0, \ldots, 0\}\mathrm{diag}\{0, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}\}U^{H} = \mathbf{O}, \end{aligned}$$

as required. ■

According to the properties given in Lemma 2.1, a relation between Moore-Penrose inverses of two Hermitian matrices of nullity-1 which share a common null eigenvector could be established, as given in the following result.

Theorem 2.2. *Let M and M' be two nullity-1 Hermitian $n \times n$ matrices which share a common null eigenvector. Then*

$$(M')^{+} = M^{+}[I + (M' - M)M^{+}]^{-1}. \tag{2.4}$$

Proof. For the sake of simplicity, set $\Delta := M' - M$ and $\nabla := (M')^{+} - M^{+}$. Then

$$M'(M')^{+} = (M + \Delta)(M^{+} + \nabla) = MM^{+} + M\nabla + \Delta M^{+} + \Delta\nabla. \tag{2.5}$$

Let u_1 be the common null eigenvector shared by M and M'. Then by Lemma 2.1, we know that

$$M'(M')^{+} = MM^{+} = I - u_1 u_1^{H}.$$

Thus, Equation (2.5) gives

$$M\nabla + \Delta M^{+} + \Delta\nabla = \mathbf{O},$$

that is,

$$M'\nabla = -\Delta M^{+}.$$

Left-multiply both sides of the above equation by $(M')^{+}$, we get

$$(M')^{+}M'\nabla = -(M')^{+}\Delta M^{+}.$$

Bearing in mind that $(M')^{+}M' = I - u_1u_1^H$ and that $(M')^{+} = M^{+} + \nabla$, we arrive at

$$(I - u_1u_1^H)\nabla = -(M^{+} + \nabla)\Delta M^{+},$$

that is,

$$\nabla - u_1u_1^H\nabla = -(M^{+} + \nabla)\Delta M^{+}. \tag{2.6}$$

Since it is shown in Lemma 2.1 that

$$u_1u_1^H(M')^{+} = u_1u_1^H M^{+} = \mathbf{O},$$

we have

$$u_1u_1^H\nabla = u_1u_1^H[(M')^{+} - M^{+}] = \mathbf{O}.$$

Hence Equation (2.6) becomes

$$\nabla = -M^{+}\Delta M^{+} - \nabla\Delta M^{+},$$

or equivalently,

$$\nabla(I + \Delta M^{+}) = -M^{+}\Delta M^{+}.$$

So if $I + \Delta M^{+}$ is invertible, then by right-multiplying the above equation by $(I + \Delta M^{+})^{-1}$, we could obtain

$$\nabla = -M^{+}\Delta M^{+}(I + \Delta M^{+})^{-1},$$

which yields

$$\begin{aligned}(M')^{+} &= M^{+} + \nabla = M^{+} - M^{+}\Delta M^{+}(I + \Delta M^{+})^{-1}\\ &= M^{+}\left[I - \Delta M^{+}(I + \Delta M^{+})^{-1}\right]\\ &= M^{+}\left[I - (I + \Delta M^{+})(I + \Delta M^{+})^{-1} + (I + \Delta M^{+})^{-1}\right]\\ &= M^{+}\left[I - I + (I + \Delta M^{+})^{-1}\right]\\ &= M^{+}(I + \Delta M^{+})^{-1}\end{aligned}$$

It remains to verify that $I + \Delta M^{+}$ is invertible. As

$$M^{+} = (M + u_1u_1^H)^{-1} - u_1u_1^H,$$

it follows that

$$\begin{aligned}I + \Delta M^{+} &= I + \Delta[(M + u_1u_1^H)^{-1} - u_1u_1^H]\\ &= I + \Delta(M + u_1u_1^H)^{-1} - \Delta u_1u_1^H\\ &= I + \Delta(M + u_1u_1^H)^{-1} - (M' - M)u_1u_1^H\\ &= I + \Delta(M + u_1u_1^H)^{-1} - M'u_1u_1^H + Mu_1u_1^H.\end{aligned}$$

Noticing that u_1 is an 0-eigenvalue eigenvector of M and M', it gives that

$$\begin{aligned}I + \Delta M^{+} &= (M + u_1u_1^H)(M + u_1u_1^H)^{-1} + \Delta(M + u_1u_1^H)^{-1}\\ &= (M + u_1u_1^H + \Delta)(M + u_1u_1^H)^{-1}\\ &= (M' + u_1u_1^H)(M + u_1u_1^H)^{-1}.\end{aligned}$$

As $M + u_1u_1^H$ is non-singular, by the same reason we know that $M' + u_1u_1^H$ is non-singular, so that $I + \Delta M^{+}$ is invertible. The proof is complete. ■

Obviously, the Laplacian matrix is a Hermitian matrix. In addition, all the Laplacian matrices of connected graphs of the same order are nullity-1 and share the same eigenvector. Hence, Theorem 2.2 can be directly applied to Laplacian matrices. Let G and G' be weighted connected graphs of order n. As a straightforward consequence of Theorem 2.2, we have

Corollary 2.3. *Let G and G′ be connected weighted graphs of order n with Laplacian matrices L and L′, respectively. Then*

$$(L')^{+} = L^{+}[I + (L' - L)L^{+}]^{-1}. \tag{2.7}$$

3. AN APPLICATION TO ELECTRICAL NETWORKS

The Laplacian matrix, also known as the Kirchhoff matrix, or admittance matrix, has wide applications in electrical networks. As introduced in the first section, the resistance distance could be computed in terms of the Moore-Penrose inverse of the Laplacian matrix. Actually, the computation of resistance distances is a classical problem in circuit theory and electrical network theory. Besides, this problem is relevant to a number of problems ranging from Lattice Green's functions, harmonic functions to random walks on graphs. For this reason, many researchers devote themselves to the computation of the resistance distance. With the development of more than 170 years, various formulae and techniques have been established, such as the traditional techniques like series and parallel circuits, Kirchhoff's laws and star-triangle transformation, as well as newly developed techniques like (algebraic, probabilistic, and combinatorial) formulae, local and global sum rules, recursion relations. In [14], a novel recursion formula for computing resistance distance is obtained. It turns out that resistance distances in some networks could be computed very easily by the recursion formula. In addition, the recursion formula extends the famous Rayleigh's monotonicity law by giving quantitative characterization to the law.

In this section, we use Corollary 2.3 to give a new proof to the recursion formula on resistance distances proposed in [14].

Theorem 3.1. *[14] Let G and G′ be two weighted graphs which are the same except for the weights on an edge $e = ij$ are w_e and w'_e. For any two vertices p and q, denote the resistance distance between them in G and G′ by $\Omega(p,q)$ and $\Omega'(p,q)$, respectively. Then*

$$\Omega'(p,q) = \Omega(p,q) - \frac{\delta \cdot [\Omega(p,i) + \Omega(q,j) - \Omega(p,j) - \Omega(q,i)]^2}{4[1 + \delta \cdot \Omega(i,j)]}, \tag{3.1}$$

where $\delta \equiv w_e' - w_e$.

Proof. Denote the Laplacian matrices of G and G' respectively by L and L', and let $\mathbf{e}$ be the (column) vector of order n whose components are 0 except the i-th and j-th components are respectively 1 and -1. Then

$$L' = L + \delta \cdot \mathbf{e}\mathbf{e}^H.$$

By Corollary 2.3, we have

$$(L')^+ = L^+[I + (L' - L)L^+]^{-1} = L^+(I + \delta \cdot \mathbf{e}\mathbf{e}^H L^+)^{-1}.$$

To compute $(L')^+$, we first compute $(I + \delta \cdot \mathbf{e}\mathbf{e}^H L^+)^{-1}$. Note that the elements of $I + \delta \cdot \mathbf{e}\mathbf{e}^H L^+$ are given by

$$[I + \delta \cdot \mathbf{e}\mathbf{e}^H L^+]_{kl} = \begin{cases} 1, & \text{if } k = l \neq i, j, \\ \delta \cdot (L_{il}^+ - L_{jl}^+), & \text{if } k = i \text{ and } l \neq i, \\ 1 + \delta \cdot (L_{ii}^+ - L_{ji}^+), & \text{if } k = l = i, \\ -\delta \cdot (L_{il}^+ - L_{jl}^+), & \text{if } k = j \text{ and } l \neq j, \\ 1 - \delta \cdot (L_{ij}^+ - L_{jj}^+), & \text{if } k = l = j, \\ 0 & \text{otherwise.} \end{cases}$$

Simple algebraic calculation leads to

$$\det(I + \delta \cdot \mathbf{e}\mathbf{e}^H L^+) = 1 + \delta \cdot (L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+).$$

Then by the adjoint method, we could obtain the inverse of $I + \delta \cdot \mathbf{e}\mathbf{e}^H L^+$, whose elements are given by

$$[(I + \delta \cdot \mathbf{e}\mathbf{e}^H L^+)^{-1}]_{kl} = \begin{cases} 1, & \text{if } k = l \neq i, j, \\ -\dfrac{\delta \cdot (L_{il}^+ - L_{jl}^+)}{1 + \delta \cdot (L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+)}, & \text{if } k = i \text{ and } l \neq i, \\ 1 - \dfrac{\delta \cdot (L_{ii}^+ - L_{ji}^+)}{1 + \delta \cdot (L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+)}, & \text{if } k = l = i, \\ \dfrac{\delta \cdot (L_{il}^+ - L_{jl}^+)}{1 + \delta \cdot (L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+)}, & \text{if } k = j \text{ and } l \neq j, \\ 1 + \dfrac{\delta \cdot (L_{ij}^+ - L_{jj}^+)}{1 + \delta \cdot (L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+)}, & \text{if } k = l = j \\ 0, & \text{otherwise.} \end{cases}$$

Then, by algebraic calculation, we could obtain the product of L^+ and $(I + \delta \cdot \mathbf{e}\mathbf{e}^t L^+)^{-1}$. Thus, $(L')^+$ is obtained, whose elements are given below. For $1 \leq k, l \leq n$,

$$(L')_{kl}^+ = L_{kl}^+ - \frac{\delta \cdot (L_{ki}^+ - L_{kj}^+)(L_{il}^+ - L_{jl}^+)}{1 + \delta \cdot (L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+)}.$$

Now we are ready to prove Equation (3.1) according to the formula given in Equation (1.1). By Equation (1.1), we have

$$\begin{aligned} \Omega'(p, q) =& (L')_{pp}^+ + (L')_{qq}^+ - 2(L')_{pq}^+ = L_{pp}^+ + L_{qq}^+ - 2L_{pq}^+ \\ & - \frac{\delta \cdot (L_{pi}^+ - L_{pj}^+)^2 + \delta \cdot (L_{qi}^+ - L_{qj}^+)^2}{1 + \delta \cdot (L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+)} \\ & - \frac{2\delta \cdot [(L_{pi}^+ - L_{pj}^+)(L_{qi}^+ - L_{qj}^+)]}{1 + \delta \cdot (L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+)} \\ =& L_{pp}^+ + L_{qq}^+ - 2L_{pq}^+ - \frac{\delta \cdot [(L_{pi}^+ - L_{pj}^+)^2 + (L_{qi}^+ - L_{qj}^+)^2]}{1 + \delta \cdot (L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+)} \\ & - \frac{2\delta \cdot [(L_{pi}^+ - L_{pj}^+)(L_{qi}^+ - L_{qj}^+)]}{1 + \delta \cdot (L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+)} \\ =& L_{pp}^+ + L_{qq}^+ - 2L_{pq}^+ - \frac{\delta \cdot (L_{pi}^+ - L_{pj}^+ - L_{qi}^+ + L_{qj}^+)^2}{1 + \delta \cdot (L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+)} \\ =& L_{pp}^+ + L_{qq}^+ - 2L_{pq}^+ \\ & - \delta \cdot \frac{\begin{bmatrix} (L_{pi}^+ - L_{pj}^+ - L_{qi}^+ + L_{qj}^+) + \frac{1}{2}(L_{pp}^+ - L_{pp}^+ + L_{ii}^+ \\ -L_{ii}^+ + L_{qq}^+ - L_{qq}^+ + L_{jj}^+ - L_{jj}^+) \end{bmatrix}^2}{1 + \delta \cdot (L_{ii}^+ + L_{jj}^+ - 2L_{ij}^+)} \\ =& L_{pp}^+ + L_{qq}^+ - 2L_{pq}^+ \\ & - \delta \cdot \frac{\begin{bmatrix} (-\frac{1}{2}L_{pp}^+ + L_{pi}^+ - \frac{1}{2}L_{ii}^+) + (\frac{1}{2}L_{pp}^+ - L_{pj}^+ + \frac{1}{2}L_{jj}^+) \\ +(\frac{1}{2}L_{qq}^+ - L_{qi}^+ + \frac{1}{2}L_{ii}^+) + (-\frac{1}{2}L_{qq}^+ + L_{qj}^+ - \frac{1}{2}L_{jj}^+) \end{bmatrix}^2}{1 + \delta \cdot (L_{ii}^+ + L_{jj}^+ - L_{ij}^+ - L_{ji}^+)} \\ =& \Omega(p, q) - \frac{\delta \cdot [-\frac{1}{2}\Omega(p, i) + \frac{1}{2}\Omega(p, j) + \frac{1}{2}\Omega(q, i) - \frac{1}{2}\Omega(q, j)]^2}{[1 + \delta \cdot \Omega(i, j)]} \\ =& \Omega(p, q) - \frac{\delta \cdot [\Omega(p, i) + \Omega(q, j) - \Omega(p, j) - \Omega(q, i)]^2}{4[1 + \delta \cdot \Omega(i, j)]}. \end{aligned}$$

The proof is completed. ■

4. CONCLUSION

The Moore-Penrose inverse of the Hermitian matrix has various applications. In this paper, a relation between generalized inverses of two nullity-1 $n \times n$ Hermitian matrices which share a common null eigenvector is established, and a simple application in electrical networks is illustrated. Further applications of the relation needs to be revealed in the future.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.