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On the *(k,s)*-Hilfer-Prabhakar Fractional Derivative With Applications to Mathematical Physics

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In this paper we introduce the (k, s)-Hilfer-Prabhakar fractional derivative and discuss its properties. We find the generalized Laplace transform of this newly proposed operator. As an application, we develop the generalized fractional model of the free-electron laser equation, the generalized time-fractional heat equation, and the generalized fractional kinetic equation using the (k, s)-Hilfer-Prabhakar derivative.

Keywords: modified (k, s) fractional integral operator, (k, s)-Prabhakar fractional derivative, (k, s)-Hilfer-Prabhakar fractional derivative, fractional heat equation, fractional kinetic equation

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1. INTRODUCTION

Fractional calculus is the area of mathematical analysis that deals with the study and application of integrals and derivatives of arbitrary order. In recent decades, fractional calculus has become of increasing significance due to its applications in many fields of science and engineering [1–5]. The first application of fractional calculus was given by Abel [6] and includes the solution to the tautocrone problem. Fractional calculus also has applications in biophysics, wave theory, polymers, quantum mechanics, continuum mechanics, field theory, Lie theory, group theory, spectroscopy, and other scientific areas [7–9]. Although this calculus has a long history, over the past few decades it has attracted greater attention because of the fascinating results obtained when it is used to model certain real-world problems [10–13]. What makes fractional calculus special is that there are numerous types of fractional operators, so any scientist modeling real-world phenomena can choose the operator that fits their purposes the best. Each classical fractional derivative is usually defined in terms of a specific integral. Among the most well-known concepts of fractional derivatives are the Riemann-Liouville, Caputo, Grünwald-Letnikov, and Hadamard derivatives [10, 14, 15], whose formulations involve single-kernel integrals and which are used to investigate, for example, memory effect problems [16].

The Riemann-Liouville fractional derivative is remarkable, but it has some drawbacks when used to model physical phenomena because of its improper physical conditions. Caputo's great contribution was to develop a concept of fractional derivative appropriate for physical conditions [17]. A number of other families of fractional operators have been established, such as the Liouville, Erdlyi-Kober, Hadamard, Grünwald-Letnikov, Hilfer, Hilfer-Prabhakar, and k-Hilfer-Prabhakar operators, to mention just a few [10, 18–20]. Because there are so many concepts of fractional operator, it has become necessary to define generic fractional operators, of which the classical ones are particular cases. One class of extensions of Riemann-Liouville fractional operators comprises the so-called k-Prabhakar integral operators, which can be found in [21]. Inspired by the definitions of k-Prabhakar integral operators and k-Hilfer-Prabhakar derivatives [20], the authors introduced the (k,s)-Hilfer fractional derivative, which unifies a large class of fractional operators [19, 20]; [Samraiz et al., accepted].

In recent years, the generalization of integral and differential operators has become an important subject of research in fractional calculus [9, 20, 22–28]. Different special functions, including the Gauss hypergeometric function, Mittag-Leffler-style functions, the Wright function, Meijer's G function, and Fox's H function, appear in the kernels of several generalizations of the integral operators. R. Hilfer introduced the Hilfer fractional derivative in [9], which is a generalization of the Riemann-Liouville and Caputo fractional derivatives.The Prabhakar integral and derivative operators are obtained from the Riemann-Liouville integral operator by extending its kernel to involve the three-parameter Mittag-Leffler function [19].

This paper is motivated by the rich applications of fractional differential equations (FDEs) in physics, economics, engineering, and many other branches of science [8, 10, 13, 17]. Since no general method exists that can be used to analytically solve every FDE, one of the most pressing and challenging tasks is to develop suitable methods for finding analytical solutions to certain classes of FDEs [29–31]. Researchers have become interested in fractional interpretations of the classical integral transforms, i.e., Laplace and Fourier transforms [32–34], in the past few years. It can be shown that integral transformations such as the Laplace, Fourier, generalized Laplace, and ρ-Laplace transforms are useful methods for obtaining analytical solutions to some classes of FDEs. In this framework, we use a generalized Laplace transform to obtain analytical solutions to certain classes of FDEs that contain (k,s)-Hilfer-Prabhakar fractional derivatives. Given the wide range of fractional operators available in the literature, it can be difficult to choose the most suitable approach for a given problem. It is therefore essential to consider generalizations of classical fractional operators to aid in choosing an appropriate operator.

Diaz et al. [35] defined k-gamma and k-beta functions as follows.

Definition 1.1. *The k-gamma function is a generalization of the classical Γ function given by*

$$\Gamma_k(\theta) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{\theta}{k}-1}}{(\theta)_{n,k}}, \quad k > 0,\ \mathrm{Re}(\theta) > 0,$$

where $(\theta)_{n,k} = \theta(\theta+k)(\theta+2k)\cdots(\theta+(n-1)k)$ for $n \geq 1$ is called the Pochhammer k symbol. The integral representation is

$$\Gamma_k(\theta) = \int_0^\infty x^{\theta-1} e^{\frac{-x^k}{k}}\, dx, \quad \mathrm{Re}(\theta) > 0.$$

Clearly, $\Gamma(\theta) = \lim_{k\to 1} \Gamma_k(\theta)$ and $\Gamma_k(\theta) = k^{\frac{\theta}{k}-1}\Gamma(\frac{\theta}{k})$.

Definition 1.2. *For $\mathrm{Re}(\theta) > 0$, $k > 0$, and $\mathrm{Re}(\zeta) > 0$, the k-beta function is given by*

$$B_k(\theta,\zeta) = \frac{1}{k}\int_0^1 \tau^{\frac{\theta}{k}-1}(1-\tau)^{\frac{\zeta}{k}-1}\, d\tau.$$

The functions Γ_k and B_k are related by an identity

$$B_k(\theta,\zeta) = \frac{\Gamma_k(\theta)\Gamma_k(\zeta)}{\Gamma_k(\theta+\zeta)}.$$

The k-Mittag-Leffler function given in [36] is defined as follows.

Definition 1.3. *Let $n \in N$, $k \in \mathbb{R}^+$, $\mu,\rho,\gamma \in \mathbb{C}$, $\mathrm{Re}(\rho) > 0$, and $\mathrm{Re}(\mu) > 0$. Then the k-Mittag-Leffler function is defined by*

$$E^{\gamma}_{k,\rho,\mu}(\theta) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\theta^n}{\Gamma_k(\rho n+\mu) n!}.$$

The modified (k,s)-fractional integral operator involving the k-Mittag-Leffler function given in [Samraiz et al., accepted] is defined as follows.

Definition 1.4. *Let $s \in \mathbb{R}\backslash\{-1\}$, $k \in \mathbb{R}^+$, $\mu,\rho,\omega,\gamma \in \mathbb{C}$, $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\mu) > 0$, and $\Phi \in L^1[0,\beta]$. Then the modified (k,s)-fractional integral operator involving the k-Mittag-Leffler function is given by*

$$({}^s_k\mathfrak{J}^{\omega,\gamma}_{0^+;\rho,\mu}\Phi)(\theta) = \frac{(s+1)^{1-\frac{\mu}{k}}}{k} \int_0^\theta (\theta^{s+1}-\zeta^{s+1})^{\frac{\mu}{k}-1}\zeta^s E^{\gamma}_{k,\rho,\mu}(\omega(\theta^{s+1}-\zeta^{s+1})^{\frac{\rho}{k}}) \Phi(\zeta)\, d\zeta. \tag{1.1}$$

Definition 1.5 ([Samraiz et al., accepted])**.** *Let $s \in \mathbb{R}\backslash\{-1\}$, $k \in \mathbb{R}^+$, $\mu,\rho,\omega,\gamma \in \mathbb{C}$, $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(\mu) > 0$, $n = \left[\frac{\mu}{k}\right]+1$, and $\Phi \in L^1[0,\beta]$. Then the (k,s)-Prabhakar fractional derivative operator with the k-Mittag-Leffler function as its kernel is given by*

$$({}^s_k\mathfrak{D}^{\omega,\gamma}_{0^+;\rho,\mu}\Phi)(\theta) = \left(\frac{1}{\theta^s}\frac{d}{d\theta}\right)^n k^n ({}^s_k\mathfrak{J}^{\omega,-\gamma}_{0^+;\rho,nk-\mu}\Phi)(\theta). \tag{1.2}$$

Definition 1.6 ([Samraiz et al., accepted])**.** *Let $s \in \mathbb{R}\backslash\{-1\}$, $k \in \mathbb{R}^+$, $\mu,\rho,\omega,\gamma \in \mathbb{C}$, $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(\mu) > 0$, $n = \left[\frac{\mu}{k}\right]+1$, and $\Phi \in C^n[0,\beta]$ with $0 < \theta < \beta < \infty$. Then the regularized version of the (k,s)-Prabhakar derivative is*

$${}^C_{s,k}\mathfrak{D}^{\omega,\gamma}_{0^+;\rho,\mu}\Phi(\theta) = k^{ns}{}_k\mathfrak{J}^{\omega,-\gamma}_{0^+;\rho,nk-\mu}\left(\frac{1}{\theta^s}\frac{d}{d\theta}\right)^n\Phi(\theta). \tag{1.3}$$

Definition 1.7. *Let $g \in C^n[\alpha,\beta]$ such that $g'(\zeta) > 0$ on $[\alpha,\beta]$. Then*

$$AC^n_g[\alpha,\beta] = \left\{\Phi:[\alpha,\beta]\to\mathbb{C} \text{ with } \Phi^{[n-1]} \in AC[\alpha,\beta]\right\},$$

where $\Phi^{[n-1]} = \left(\frac{1}{g'(\zeta)}\frac{d}{d\zeta}\right)^{n-1}\Phi$.

The generalized Laplace transform introduced by Jarad et al. [34] is presented in the following definition.

DEFINITION 1.8. *Let Φ and g be real-valued functions on $[\alpha,\infty)$ such that $g(\zeta)$ is continuous and $g'(\zeta) > 0$ on $[\alpha,\infty)$. The generalized Laplace transform of Φ is*

$$L_g\{\Phi(\theta)\}(u) = \int_\alpha^\infty e^{-u(g(\theta)-g(\alpha))}\Phi(\theta)g'(\theta)\,d\theta$$

for all values of u.

DEFINITION 1.9 ([34]). *Let Φ and Ψ be two piecewise-continuous functions on each interval $[0,T]$ that are of exponential order. The generalized convolution of Φ and Ψ is given by*

$$(\Phi *_g \Psi)(\theta) = \int_\alpha^\theta \Phi(\zeta)\Psi\big(g^{-1}(g(\theta)+g(\alpha)-g(\zeta))\big)g'(\zeta)\,d\zeta.$$

THEOREM 1.10 ([34]). *Let $\Phi \in C_g^{n-1}[\alpha,T]$ be such that $\Phi^{[1]}$ is of g-exponential order. Let $\Phi^{[1]}$ be a piecewise-continuous function on the interval $[\alpha,T]$. Then the generalized Laplace transform of $\Phi^{[1]}(\zeta)$ exists and*

$$L_g\{\Phi^{[1]}(\theta)\}(u) = sL_g\{\Phi(\theta)\}(u) - \Phi(\alpha).$$

PROPOSITION 1.11. *Let $s \in \mathbb{R}\backslash\{-1\}$, $k \in \mathbb{R}^+$, $\mu,\rho,\omega,\gamma \in \mathbb{C}$, $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(\mu) > 0$, and $\beta > 0$. Then the integral operator ${}^s_k\mathfrak{J}^{\omega,\gamma}_{0^+;\rho,\mu}$ is bounded on $C[0,\beta]$, i.e.,*

$$|({}^s_k\mathfrak{J}^{\omega,\gamma}_{0^+;\rho,\mu}\Phi)(\theta)| \le G\|\Phi\|_{C[0,\beta]},$$

where

$$\|\Phi\|_{C[0,\beta]} = \max\{|\Phi| : 0 < x < \beta\}$$

and

$$G = \frac{(s+1)^{-\frac{\mu}{k}}(\beta^{s+1})^{\mathrm{Re}(\frac{\mu}{k})}}{k} \sum_{n=0}^\infty \frac{|(\gamma)_{n,k}\omega^n|}{|\Gamma_k(\rho n+\mu)|n!}\frac{(\beta^{s+1})^{\mathrm{Re}(\frac{\rho}{k})n}}{\big[n\mathrm{Re}(\frac{\rho}{k})+\mathrm{Re}(\frac{\mu}{k})\big]}. \tag{1.4}$$

THEOREM 1.12. *Let $s \in \mathbb{R}\backslash\{-1\}$, $k \in \mathbb{R}^+$, $\mu,\rho,\omega,\gamma \in \mathbb{C}$, $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(\gamma) > 0$, and $\mathrm{Re}(\mu) > 0$. Let $\Phi \in L^1[0,\beta]$ be a piecewise-continuous function on each interval $[0,\theta]$ that is of $g(\theta)$-exponential order. Then*

$$L_g\{({}^s_k\mathfrak{J}^{\omega,\gamma}_{0^+;\rho,\mu}\Phi)(\theta)\}(u) = \big((s+1)(ku)\big)^{-\frac{\mu}{k}}\big(1-k\omega(ku)^{-\frac{\rho}{k}}\big)^{-\frac{\gamma}{k}}L_g\{\Phi(\theta)\}(u).$$

THEOREM 1.13 ([Samraiz et al., accepted]). *Let $k \in \mathbb{R}^+$, $s \in [0,\infty)$, $\mu,\rho,\omega,\gamma \in \mathbb{C}$, $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\mu) > 0$, and $g(\theta) = \theta^{s+1}$. Let $\Phi \in AC_g^n[0,\beta]$ and ${}^s_k\mathfrak{J}^{\omega,\gamma}_{0^+;\rho,nk-m-\mu}\Phi$ for $m = 0,1,2,\ldots,n-1$ be of $g(\theta)$-exponential order. Then*

$$L_g\{({}^s_k\mathfrak{D}^{\omega,\gamma}_{0^+;\rho,\mu}\Phi)(\theta)\}(u) = (s+1)^{-\frac{nk-\mu}{k}}(ku)^{\frac{\mu}{k}}\big(1-k\omega(ku)^{-\frac{\rho}{k}}\big)^{\frac{\gamma}{k}}L_g\{\Phi(t)\}(u) - \sum_{m=0}^{n-1}k^{n-m}u^{n-m-1}\Big({}^s_k\mathfrak{D}^{\omega,-\gamma}_{0^+;\rho,\mu-(n-m)k}\Phi\Big)(0^+),$$

with $|k\omega(ku)^{-\frac{\rho}{k}}| < 1$.

THEOREM 1.14 ([Samraiz et al., accepted]). *The generalized Laplace transform of the regularized version of the (k,s)-Prabhakar fractional derivative is*

$$L_g\{{}^C_{s,k}\mathfrak{D}^{\omega,\gamma}_{0^+;\rho,\mu}\Phi(\theta)\}(u) = (s+1)^{-\frac{nk-\mu}{k}}\Big[(ku)^{\frac{\mu}{k}}\big(1-k\omega(ku)^{-\frac{\rho}{k}}\big)^{\frac{\gamma}{k}}L_g\{\Phi(\theta)\}(u) - \sum_{m=0}^{n-1}k^{m+1}(ku)^{\frac{\mu-(m+1)k}{k}}\big(1-k\omega(ku)^{-\frac{\rho}{k}}\big)^{\frac{\gamma}{k}}\big(\Phi^{[m]}\big)(0^+)\Big],$$

with $|k\omega(ku)^{-\frac{\rho}{k}}| < 1$.

2. THE (k,s)-HILFER-PRABHAKAR FRACTIONAL DERIVATIVE AND GENERALIZED LAPLACE TRANSFORMS

In this section we introduce a new family of operators called the (k,s)-Hilfer-Prabhakar fractional derivative. The generalized Laplace transforms of these operators are also studied in this section.

DEFINITION 2.1. *Let $\Phi \in C^1[0,\beta]$, $0 < \theta < \beta < \infty$, $s \in \mathbb{R}\backslash\{-1\}$, $k,\rho > 0$, $\omega,\gamma \in \mathbb{R}$, $\mu \in (0,1)$, $\nu \in [0,1]$, and $\big(\Phi * {}^s_k\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\big)(\theta) \in AC^1[0,\beta]$. The (k,s)-Hilfer-Prabhakar derivative is defined as*

$${}^s_k\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\Phi(\theta) = k\Big({}^s_k\mathfrak{J}^{\omega,-\gamma\nu}_{0^+;\rho,\nu(k-\mu)}\Big(\frac{1}{\theta^s}\frac{d}{d\theta}\Big){}^s_k\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi\Big)(\theta).$$

Note that if we choose $\nu = 0$ in the above definition, we get (1.2) corresponding to $m = 1$; and if we take $\nu = 1$, we obtain (1.3) corresponding to $m = 1$.

THEOREM 2.2. *For $s \in \mathbb{R}\backslash\{-1\}$, $k,\rho > 0$, $\omega,\gamma \in \mathbb{R}$, $\mu \in (0,1)$, $\nu \in [0,1]$, and $\Phi \in L^1[0,\beta]$, the operator ${}^s_k\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}$ is bounded on $C[0,\beta]$, i.e.,*

$$\|{}^s_k\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\Phi(\theta)\| \le C_1C_2\|\Phi\|_{[0,\beta]},$$

where

$$C_1 = \frac{(s+1)^{-\frac{\nu(k-\mu)}{k}}(\beta^{s+1})^{\mathrm{Re}(\frac{\nu(k-\mu)}{k})}}{k} \sum_{n=0}^\infty \frac{|(-\gamma\nu)_{n,k}\omega^n|}{|\Gamma_k(\rho n+\nu(k-\mu))|n!}\frac{(\beta^{s+1})^{\mathrm{Re}(\frac{\rho}{k})n}}{\big[n\mathrm{Re}(\frac{\rho}{k})+\mathrm{Re}(\frac{\nu(k-\mu)}{k})\big]} \tag{2.1}$$

and

$$C_2 = \frac{(s+1)^{-\frac{(1-\nu)(k-\mu)-k}{k}}(\beta^{s+1})^{\mathrm{Re}(\frac{(1-\nu)(k-\mu)-k}{k})}}{k} \sum_{m=0}^\infty \frac{|(\gamma(\nu-1))_{m,k}\omega^m|}{|\Gamma_k(\rho m+(1-\nu)(k-\mu))|m!} \times \frac{(\beta^{s+1})^{\mathrm{Re}(\frac{\rho}{k})m}}{\big[m\mathrm{Re}(\frac{\rho}{k})+\mathrm{Re}(\frac{(1-\nu)(k-\mu)}{k})\big]}. \tag{2.2}$$

Proof Using the estimates in Proposition 1.11, we get

$$\begin{aligned}
&\|{}^{s}_{k}\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\Phi(\theta)\| \\
&= \left\| k\left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma\nu}_{0^+;\rho,\nu(k-\mu)}\left(\frac{1}{\theta^s}\frac{d}{d\theta}\right)\left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi\right)(\theta)\right\| \\
&\leq C_1\left\|\left(\frac{1}{\theta^s}\frac{d}{d\theta}\right)\left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi\right)(\theta)\right\| \\
&= C_1\left\|\left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)-k}\Phi\right)(\theta)\right\| \\
&\leq C_1C_2\|\Phi\|_{[0,\beta]},
\end{aligned}$$

where C_1 and C_2 are the constants defined by (2.1) and (2.2).

Proposition 2.3. *Let $s\in\mathbb{R}\backslash\{-1\}$, $k,\rho,\lambda>0$, $\omega,\gamma,\sigma\in\mathbb{R}$, $\mu\in(0,1)$, $\nu\in[0,1]$, $\lambda>\mu+\nu k-\mu\nu$, and $\Phi\in L^1[0,\beta]$. Then*

$$\left({}^{s}_{k}\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\left({}^{s}_{k}\mathfrak{J}^{\omega,\sigma}_{0^+;\rho,\lambda}\Phi\right)\right)(\theta)=\left({}^{s}_{k}\mathfrak{J}^{\omega,\sigma-\gamma}_{0^+;\rho,\lambda-\mu}\Phi\right)(\theta).$$

In particular,

$$\left({}^{s}_{k}\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\left({}^{s}_{k}\mathfrak{J}^{\omega,\gamma}_{0^+;\rho,\mu}\Phi\right)\right)(\theta)=\Phi(\theta).$$

Proof. By using Definition 2.1 and the semigroup property of the modified (k,s)-fractional integral operator with the k-Mittag-Leffler function, we obtain

$$\begin{aligned}
&\left({}^{s}_{k}\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\left({}^{s}_{k}\mathfrak{J}^{\omega,\sigma}_{0^+;\rho,\lambda}\Phi\right)\right)(\theta) \\
&= k\left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma\nu}_{0^+;\rho,\nu(k-\mu)}\left(\frac{1}{\theta^s}\frac{d}{d\theta}\right){}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\left({}^{s}_{k}\mathfrak{J}^{\omega,\sigma}_{0^+;\rho,\lambda}\Phi\right)\right)(\theta) \\
&= k\left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma\nu}_{0^+;\rho,\nu(k-\mu)}\left(\frac{1}{\theta^s}\frac{d}{d\theta}\right)\left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)+\sigma}_{0^+;\rho,(1-\nu)(k-\mu)+\lambda}\Phi\right)\right)(\theta) \\
&= \left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma\nu}_{0^+;\rho,\nu(k-\mu)}\left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)+\sigma}_{0^+;\rho,(1-\nu)(k-\mu)+\lambda-k}\Phi\right)\right)(\theta) \\
&= \left({}^{s}_{k}\mathfrak{J}^{\omega,\sigma-\gamma}_{0^+;\rho,\lambda-\mu}\Phi\right)(\theta).
\end{aligned}$$

This completes the proof.

Theorem 2.4. *Let $s\in\mathbb{R}\backslash\{-1\}$, $k,\rho,\lambda>0$, $\omega,\gamma\in\mathbb{R}$, $\mu\in(0,1)$, $\nu\in[0,1]$, $\lambda>\mu+\nu k-\mu\nu$, and $\Phi\in L^1[0,\beta]$. Then*

$$\left({}^{s}_{k}J^{\lambda}_{0+}\left({}^{s}_{k}\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\Phi\right)\right)(\theta)=\left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma}_{0^+;\rho,\lambda-\mu}\Phi\right)(\theta).$$

Proof. By using Definition 2.1 and Theorem 2 in [Samraiz et al., accepted], we get

$$\begin{aligned}
&\left({}^{s}_{k}J^{\lambda}_{0+}\left({}^{s}_{k}\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\Phi\right)\right)(\theta) \\
&= k\left({}^{s}_{k}J^{\lambda}_{0+}\,{}^{s}_{k}\mathfrak{J}^{\omega,-\gamma\nu}_{0^+;\rho,\nu(k-\mu)}\left(\frac{1}{\theta^s}\frac{d}{d\theta}\right){}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi\right)(\theta) \\
&= k\left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma\nu}_{0^+;\rho,\nu(k-\mu)+\lambda}\left(\frac{1}{\theta^s}\frac{d}{d\theta}\right){}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi\right)(\theta) \\
&= \left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma\nu}_{0^+;\rho,\nu(k-\mu)+\lambda}\left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)-k}\Phi\right)\right)(\theta) \\
&= \left({}^{s}_{k}\mathfrak{J}^{\omega,-\gamma}_{0^+;\rho,\lambda-\mu}\Phi\right)(\theta),
\end{aligned}$$

and thus the result is proved.

Theorem 2.5. *The Laplace transform of the (k,s)-Hilfer-Prabhakar fractional derivative is*

$$\begin{aligned}
&L_g\{{}^{s}_{k}\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\Phi(\theta)\}(u) \\
&= (s+1)^{\frac{\mu-k}{k}}(ku)^{\frac{\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma}{k}} \\
&\quad\times L_g\{\Phi(\theta)\}(u)-k(s+1)^{-\frac{\nu(k-\mu)}{k}}(ku)^{\frac{-\nu(k-\mu)}{k}} \\
&\quad\times\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}}{}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi(0^+).
\end{aligned}$$

Proof. By using Definition 2.1, Theorem 1.12, and Theorem 1.10, we obtain

$$\begin{aligned}
&L_g\{{}^{s}_{k}\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\Phi(\theta)\}(u) \\
&= k(s+1)^{-\frac{\nu(k-\mu)}{k}}(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}} \\
&\qquad L_g\left\{{}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}{}^{[1]}\Phi(\theta)\right\}(u) \\
&= k(s+1)^{-\frac{\nu(k-\mu)}{k}}(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}} \\
&\quad\times\left[uL_g\{{}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi(\theta)\}(u)-{}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi(0^+)\right] \\
&= (ku)(s+1)^{-\frac{\nu(k-\mu)}{k}}(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}} \\
&\qquad L_g\{{}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi(\theta)\}(u) \\
&\quad -k(s+1)^{-\frac{\nu(k-\mu)}{k}}(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}}{}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi(0^+) \\
&= (ku)(s+1)^{-\frac{\nu(k-\mu)}{k}}(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}} \\
&\quad\times\left[(s+1)^{-\frac{(1-\nu)(k-\mu)}{k}}(ku)^{\frac{-(1-\nu)(k-\mu)}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma(1-\nu)}{k}}\right. \\
&\qquad\left. L_g\{\Phi(\theta)\}(u)\right] \\
&\quad -k(s+1)^{-\frac{\nu(k-\mu)}{k}}(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}}{}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi(0^+) \\
&= (s+1)^{\frac{\mu-k}{k}}(ku)^{\frac{\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma}{k}}L_g\{\Phi(\theta)\}(u) \\
&\qquad -k(s+1)^{-\frac{\nu(k-\mu)}{k}} \\
&\quad\times(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}}{}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi(0^+),
\end{aligned}$$

which proves the result.

3. GENERALIZATION OF THE FREE-ELECTRON LASER EQUATION

The integrodifferential free-electron laser equation describes the unsaturated behavior of the free-electron laser. Several attempts have been made to solve the generalized fractional integrodifferential free-electron laser equation in recent years. In this section, we develop a generalized fractional model of the free-electron laser equation that involves the novel (k,s)-Hilfer-Prabhakar derivative.

Theorem 3.1. *The solution of the Cauchy problem*

$${}^{s}_{k}\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\Phi(\theta)=\lambda\,{}^{s}_{k}\mathfrak{J}^{\sigma,\omega}_{0^+;\rho,\mu}\Phi(\theta)+f(\theta), \tag{3.1}$$

$${}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi(0^+)=C,\quad C\geq 0, \tag{3.2}$$

where $\theta \in (0,\infty)$, $f \in L^1[0,\infty)$, $\mu \in (0,1)$, $\nu \in [0,1]$, $\omega, \lambda \in \mathbb{R}$, $\rho > 0$, *and* $\gamma, \sigma \geq 0$, *is given by*

$$\Phi(\theta) = C\sum_{m=0}^{\infty} \lambda^m (s+1)^{-\frac{\nu(k-\mu)+\mu-k}{k}} (\theta^{s+1})^{\frac{\nu(k-\mu)+\mu(1+2m)}{k}-1} \times E^{(\gamma+\sigma)m-\gamma(\nu-1)}_{k,\rho,\nu(k-\mu)+\mu(1+2m)}\left(\omega(\theta^{s+1})^{\frac{\rho}{k}}\right) + \sum_{m=0}^{\infty} \lambda^m (s+1)^{2m} \left({}^s_k\mathfrak{J}^{\omega,(\gamma+\sigma)m+\gamma}_{k,\rho,\mu(1+2m)} f\right)(\theta).$$

Proof. By applying the generalized Laplace transform to both sides of (3.1) and using Theorems 2.5 and 1.12, we get

$$L_g\{{}^s_k\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\Phi(\theta)\}(u) = \lambda L_g\{{}^s_k\mathfrak{J}^{\sigma,\omega}_{0^+;\rho,\mu}\Phi(\theta)\}(u) + L_g\{f(\theta)\}(u),$$

which can also be written as

$$L_g\{\Phi(\theta)\}(u) = \frac{Ck(s+1)^{-\frac{\nu(k-\mu)}{k}}(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-k\omega(ku)^{\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}}}{(s+1)^{\frac{\mu-k}{k}}(ku)^{\frac{\mu}{k}}\left(1-k\omega(ku)^{\frac{-\rho}{k}}\right)^{\frac{\gamma}{k}}} + \frac{L_g\{f(\theta)\}(u)}{(s+1)^{\frac{\mu-k}{k}}(ku)^{\frac{\mu}{k}}\left(1-k\omega(ku)^{\frac{-\rho}{k}}\right)^{\frac{\gamma}{k}}} \left(1-\lambda(ku)^{-\frac{2\mu}{k}}\left(1-k\omega(ku)^{\frac{-\rho}{k}}\right)^{-\frac{\gamma+\sigma}{k}}\right)^{-1}.$$

Using the binomial expansion gives

$$\begin{aligned} L_g\{\Phi(\theta)\}(u) &= \frac{Ck(s+1)^{-\frac{\nu(k-\mu)}{k}}(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-k\omega(ku)^{\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}}}{(s+1)^{\frac{\mu-k}{k}}(ku)^{\frac{\mu}{k}}\left(1-k\omega(ku)^{\frac{-\rho}{k}}\right)^{\frac{\gamma}{k}}} + \frac{L_g\{f(\theta)\}(u)}{(s+1)^{\frac{\mu-k}{k}}(ku)^{\frac{\mu}{k}}\left(1-k\omega(ku)^{\frac{-\rho}{k}}\right)^{\frac{\gamma}{k}}} \\ &\quad \sum_{m=0}^{\infty}\lambda^m (ku)^{-2\frac{\mu m}{k}}\left(1-k\omega(ku)^{\frac{-\rho}{k}}\right)^{-\frac{(\gamma+\sigma)}{k}m} \\ &= Ck\sum_{m=0}^{\infty}\lambda^m (s+1)^{-\frac{\nu(k-\mu)+\mu-k}{k}}(ku)^{-\frac{\nu(k-\mu)+\mu(1+2m)}{k}} \left(1-k\omega(ku)^{\frac{-\rho}{k}}\right)^{-\frac{(\gamma+\sigma)m-\gamma(\nu-1)}{k}} \\ &\quad + \sum_{m=0}^{\infty}\lambda^m (s+1)^{-\frac{\mu-k}{k}}(ku)^{-\frac{\mu(1+2m)}{k}} \left(1-k\omega(ku)^{\frac{-\rho}{k}}\right)^{-\frac{\gamma+m(\gamma+\sigma)}{k}} L_g\{f(\theta)\}(u). \end{aligned}$$

Applying the inverse Laplace transform, we obtain

$$\Phi(\theta) = C\sum_{m=0}^{\infty} \lambda^m (s+1)^{-\frac{\nu(k-\mu)+\mu-k}{k}} (\theta^{s+1})^{\frac{\nu(k-\mu)+\mu(1+2m)}{k}-1} E^{(\gamma+\sigma)m-\gamma(\nu-1)}_{k,\rho,\nu(k-\mu)+\mu(1+2m)}\left(\omega(\theta^{s+1})^{\frac{\rho}{k}}\right) + \sum_{m=0}^{\infty} \lambda^m (s+1)^{2m} \left({}^s_k\mathfrak{J}^{\omega,(\gamma+\sigma)m+\gamma}_{k,\rho,\mu(1+2m)} f\right)(\theta),$$

hence the result.

Remark 3.2. *If* $s = 0$, $k = 1$, $\gamma = \nu = 0$, $\rho = \sigma = 1$, $\mu \to 1$, $f(\theta) = 0$, $\omega = ir$, *and* $\lambda = -i\Pi p$ *(with* $r, p \in \mathbb{R}$*), then above Cauchy problem reduces to the following free-electron laser equation:*

$$\frac{d}{d\theta}\Phi(\theta) = -ip\Pi\int_0^\theta (\theta - t)e^{ir(\theta-t)}\Phi(t)\,dt, \quad \Phi(0) = 1.$$

Corollary 3.3. *If we take* $s = 0$ *and* $k = 1$*, then we get the Cauchy problem given in [19]:*

$$\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\Phi(\theta) = \lambda\mathfrak{J}^{\sigma,\omega}_{0^+;\rho,\mu}\Phi(\theta) + f(\theta), \tag{3.3}$$

$$\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\Phi(0^+) = C, \quad C \geq 0, \tag{3.4}$$

where $\theta \in (0,\infty)$, $f \in L^1[0,\infty)$, $\mu \in (0,1)$, $\nu \in [0,1]$, $\omega, \lambda \in \mathbb{R}$, $\rho > 0$, *and* $\gamma, \sigma \geq 0$, *and its solution is given by*

$$\Phi(\theta) = C\sum_{m=0}^{\infty}\lambda^m(\theta)^{\nu(1-\mu)+\mu(1+2m)-1}E^{(\gamma+\sigma)m-\gamma(\nu-1)}_{\rho,\nu(1-\mu)+\mu(1+2m)}(\omega(\theta)^\rho) + \sum_{m=0}^{\infty}\lambda^m\left(\mathfrak{J}^{\omega,(\gamma+\sigma)m+\gamma}_{0^+,\rho,\mu(1+2m)}f\right)(\theta).$$

4. THE TIME-FRACTIONAL HEAT EQUATION

Lately, numerous papers have been devoted to mathematical analysis of variations of the time-fractional heat equation and its applications in mathematical physics and probability theory [see, for example, [37, 38] and the references therein]. This section focuses on the generalized time-fractional heat equation involving the (k, s)-Hilfer-Prabhakar derivative.

Theorem 4.1. *The solution of the Cauchy problem*

$${}^s_k\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}V(\theta,\zeta) = G\frac{\partial^2}{\partial\zeta^2}V(\theta,\zeta), \quad \zeta > 0,\ \theta \in \mathbb{R}, \tag{4.1}$$

$$\left[{}^s_k\mathfrak{J}^{-\gamma(1-\nu),\omega}_{0^+;\rho,(1-\nu)(k-\mu)}V(\theta,\zeta)\right]_{\zeta=0^+} = h(\theta), \tag{4.2}$$

$$\lim_{\theta\to\infty}V(\theta,\zeta) = 0, \tag{4.3}$$

where $s \in [0,\infty)$, $\mu \in (0,1)$, $\nu \in [0,1]$, $\omega \in \mathbb{R}$, $G, k, \rho > 0$, *and* $\gamma \geq 0$, *is given by*

$$V(\theta,\zeta) = \int_{-\infty}^{+\infty} dp\, e^{-ip\theta}\hat{h}(p) \frac{1}{2\Pi}\sum_{m=0}^{\infty}(-G)^m(s+1)^{-\frac{(1-\nu)(k-\mu)-(\mu-k)m}{k}}(\zeta^{s+1})^{\frac{\mu(m+1)-\nu(\mu-k)}{k}-1} \times E^{\gamma(m+1-\nu)}_{k,\rho,\mu(m+1)-\nu(k-\mu)}\left(\omega(\zeta^{s+1})^{\frac{\rho}{k}}\right)p^{2m}\hat{h}(p).$$

Proof. Let $\hat{V}(p,t) = F(V)(p,\zeta)$ denote the Fourier transform with respect to the space variable θ. Taking the Fourier transform of (4.1) and using (4.3), we obtain

$$ {}_k^s\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\hat{V}(p,\zeta) = -Gp^2\hat{V}(p,\zeta). $$

Now, applying the generalized Laplace transform to both sides of above equation, we get

$$ \begin{aligned} &L_g\{{}_k^s\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}\hat{V}(p,\zeta)\} \\ &= -Gp^2 L_g\{\hat{V}(p,\zeta)\}(u)(s+1)^{\frac{\mu-k}{k}}(ku)^{\frac{\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma}{k}} \\ &L_g\{\hat{V}(p,\zeta)\}(u) \\ &\quad - k(s+1)^{-\frac{\nu(k-\mu)}{k}}(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}} \\ &\left[{}_k^s\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}\hat{V}(p,\zeta)\right]_{\zeta=0^+} \\ &= -Gp^2 L_g\{\hat{V}(p,\zeta)\}(u), \end{aligned} $$

which can be written as

$$ \begin{aligned} &L_g\{\hat{V}(p,\zeta)\}(u) \\ &= \frac{k(s+1)^{-\frac{\nu(k-\mu)}{k}}(ku)^{\frac{-\nu(k-\mu)}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}}\hat{h}(p)}{(s+1)^{\frac{\mu-k}{k}}(ku)^{\frac{\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma}{k}}+Gp^2} L_g\{\hat{V}(p,\zeta)\}(u) \\ &= k(s+1)^{\frac{\nu(\mu-k)-(\mu-k)}{k}}(ku)^{\frac{\nu(\mu-k)-\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma(\nu-1)}{k}}\hat{h}(p) \\ &\quad\times\left(1+\frac{Gp^2}{(s+1)^{\frac{\mu-k}{k}}(ku)^{-\frac{\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma}{k}}}\right)^{-1} \\ &= k(s+1)^{\frac{\nu(\mu-k)-(\mu-k)}{k}}(ku)^{\frac{\nu(\mu-k)-\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma(\nu-1)}{k}}\hat{h}(p) \\ &\quad\times\sum_{m=0}^{\infty}(-G)^m(s+1)^{\frac{(\mu-k)m}{k}}(ku)^{\frac{\mu}{k}m}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma}{k}m} \\ &= k\sum_{m=0}^{\infty}(-G)^m(s+1)^{\frac{(1-\nu)(k-\mu)+(k-\mu)m}{k}}(ku)^{\frac{\nu(\mu-k)-\mu(m+1)}{k}} \\ &\qquad\times\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{-\frac{\gamma(m+1-\nu)}{k}}\hat{h}(p). \end{aligned} $$

Applying the inverse Laplace transform, we get

$$ \begin{aligned} \hat{V}(p,\zeta) = &\sum_{m=0}^{\infty}(-G)^m(s+1)^{\frac{(1-\nu)(k-\mu)+(k-\mu)m}{k}}(\zeta^{s+1})^{\frac{\mu(m+1)-\nu(\mu-k)}{k}-1} \\ &\times E^{\gamma(m+1-\nu)}_{k,\rho,\mu(m+1)-\nu(k-\mu)}(\omega(\zeta^{s+1})^{\frac{\rho}{k}})p^{2m}\hat{h}(p). \end{aligned} $$

Now, applying the inverse Fourier transform yields

$$ \begin{aligned} &V(\theta,\zeta) = \int_{-\infty}^{+\infty} dp\, e^{-ip\theta}\hat{h}(p) \\ &\frac{1}{2\Pi}\sum_{m=0}^{\infty}(-G)^m(s+1)^{-\frac{(1-\nu)(k-\mu)-(\mu-k)m}{k}}(\zeta^{s+1})^{\frac{\mu(m+1)-\nu(\mu-k)}{k}-1} \\ &\times E^{\gamma(m+1-\nu)}_{k,\rho,\mu(m+1)-\nu(k-\mu)}(\omega(\zeta^{s+1})^{\frac{\rho}{k}})p^{2m}\hat{h}(p). \end{aligned} $$

Remark 4.2. *If $s=0$, $k=1$, $\gamma=0$, and $\mu\to 1$, then the above Cauchy problem reduces to*

$$ \frac{\partial}{\partial\theta}V(\theta,\zeta) = G\frac{\partial^2}{\partial\theta^2}V(\theta,\zeta) $$

$$ \left[V(\theta,\zeta)\right]_{\zeta=0^+} = h(\theta), \quad \lim_{\theta\to\infty}V(\theta,\zeta)=0, $$

which is the heat equation.

Corollary 4.3. *If we take $s=0$ and $k=1$, we get the following Cauchy problem given in [19]:*

$$ \mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}V(\theta,\zeta) = G\frac{\partial^2}{\partial\theta^2}V(\theta,\zeta), \quad \zeta>0,\ \theta\in\mathbb{R}, $$

$$ \left[\mathfrak{J}^{-\gamma(1-\nu),\omega}_{0^+;\rho,(1-\nu)(1-\mu)}V(\theta,\zeta)\right]_{\zeta=0^+} = h(\theta), $$

$$ \lim_{\theta\to\infty}V(\theta,\zeta)=0, $$

where $\mu\in(0,1)$, $\nu\in[0,1]$, $\omega\in\mathbb{R}$, $R,\rho>0$, and $\gamma\geq 0$, with solution given by

$$ \begin{aligned} V(\theta,\zeta) = &\int_{-\infty}^{+\infty} dp\, e^{-ip\theta}\hat{h}(p)\frac{1}{2\Pi}\sum_{m=0}^{\infty}(-G)^m\zeta^{\mu(m+1)-\nu(\mu-1)-1} \\ &\times E^{\gamma(m+1-\nu)}_{\rho,\mu(m+1)-\nu(\mu-1)}(\omega\zeta^{\rho})p^{2m}\hat{h}(p). \end{aligned} $$

5. GENERALIZATION OF THE FRACTIONAL KINETIC DIFFERINTEGRAL EQUATION

Fractional differential equations are important tools for developing mathematical models of numerous phenomena in fields such as physics, dynamic systems, control systems, and engineering. In mathematical modeling, kinetic equations describe the continuity of the motion of a substance and are basic equations of mathematical physics and the natural sciences. In this section, we consider an equation that generalizes kinetic equations. For related literature, we refer the reader to [39–42].

Theorem 5.1. *Consider the Cauchy problem*

$$ a\,{}_k^s\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}N(t) - N_0 f(t) = b\,{}_k^s\mathfrak{J}^{\omega,\sigma}_{0^+;\rho,q}N(t), \quad f\in L^1[0,\infty), \tag{5.1} $$

$$ {}_k^s\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}N(0) = d, \quad d\geq 0, \tag{5.2} $$

where $s\in[0,\infty)$, $\nu\in[0,1]$, $\omega\in\mathbb{C}$, $a,b\in\mathbb{R}$ $(a\neq 0)$, $\mu,\rho,q,k>0$, and $\gamma,\sigma\geq 0$. The solution to the problem is

$$ \begin{aligned} N(t) = &\, d\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n(s+1)^{-\frac{\nu(k-\mu)+(\mu-k)(n+1)+qn}{k}}(t^{s+1})^{\frac{\nu(k-\mu)+\mu+(q+\mu)n}{k}-1} \\ &E^{(\gamma+\sigma)n+\gamma(1-\nu)}_{k,\rho,\nu(k-\mu)+\mu+(q+\mu)n}(\omega(t^{s+1})^{\frac{\rho}{k}}) \\ &+\frac{N_0}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n(s+1)^{n+1}\,{}_k^s\mathfrak{J}^{\omega,(\gamma+\sigma)n+\gamma}_{0^+;\rho,(q+\mu)n+\mu}f(t). \end{aligned} $$

Proof. Applying the generalized Laplace transform to both sides of (5.1), we get

$$ aL_g\{{}_k^s\mathfrak{D}^{\gamma,\mu,\nu}_{0^+;\rho,\omega}N(t)\}(u) - N_0 L_g\{f(t)\}(u) = bL_g\{{}_k^s\mathfrak{J}^{\omega,\sigma}_{0^+;\rho,q}N(t)\}(u). $$

Using Theorems 2.5 and 1.12, we get

$$a\Big[(s+1)^{\frac{\mu-k}{k}}(ku)^{\frac{\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma}{k}}L_g\{N(t)\}(u) - k(s+1)^{-\frac{\nu(k-\mu)}{k}}(ku)^{\frac{-\nu(k-\mu)}{k}} \times \left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}}\, {}^{s}_{k}\mathfrak{J}^{\omega,-\gamma(1-\nu)}_{0^+;\rho,(1-\nu)(k-\mu)}N(0^+)\Big] - N_0L_g\{f(t)\}(u) = b(s+1)^{-\frac{\mu}{k}}(ku)^{-\frac{\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{-\frac{\sigma}{k}}L_g\{N(t)\},$$

which can be written as

$$\left[\frac{a-b(s+1)^{-\frac{\mu-k+q}{k}}(ku)^{-\frac{\mu+q}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{-\frac{\gamma+\sigma}{k}}}{(s+1)^{-\frac{\mu-k}{k}}(ku)^{-\frac{\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{-\frac{\gamma}{k}}}\right] L_g\{N(t)\}(u) = akd(s+1)^{-\frac{\nu(k-\mu)}{k}}(ku)^{-\frac{\nu(k-\mu)}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma\nu}{k}} + N_0L_g\{f(t)\}(u),$$

$$L_g\{N(t)\}(u) = akd\left[\frac{(s+1)^{-\frac{\nu(k-\mu)+(\mu-k)}{k}}(ku)^{-\frac{\nu(k-\mu)+\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma(\nu-1)}{k}}}{a-b(s+1)^{-\frac{\mu-k+q}{k}}(ku)^{-\frac{\mu+q}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{-\frac{\gamma+\sigma}{k}}}\right] + \left[\frac{(s+1)^{-\frac{\mu-k}{k}}(ku)^{-\frac{\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{-\frac{\gamma}{k}}}{a-b(s+1)^{-\frac{\mu-k+q}{k}}(ku)^{-\frac{\mu+q}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{-\frac{\gamma+\sigma}{k}}}\right] N_0L_g\{f(t)\}(u).$$

Taking $\left|\frac{b}{a}(s+1)^{-\frac{\mu-k+q}{k}}(ku)^{-\frac{\mu+q}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{-\frac{\gamma+\sigma}{k}}\right| < 1$ gives

$$\begin{aligned} L_g\{N(t)\}(u) &= \Big[kd(s+1)^{-\frac{\nu(k-\mu)+(\mu-k)}{k}}(ku)^{-\frac{\nu(k-\mu)+\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{\frac{\gamma(\nu-1)}{k}} + (s+1)^{-\frac{\mu-k}{k}}(ku)^{-\frac{\mu}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{-\frac{\gamma}{k}}a^{-1}N_0L_g\{f(t)\}(u)\Big] \\ &\quad\times\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n(s+1)^{-\frac{(\mu-k+q)n}{k}}(ku)^{-\frac{(\mu+q)n}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{-\frac{(\gamma+\sigma)n}{k}} \\ &= dk\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n(s+1)^{-\frac{\nu(k-\mu)+(\mu-k)(n+1)+qn}{k}}(ku)^{-\frac{\nu(k-\mu)+\mu+(\mu+q)n}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{-\frac{(\gamma+\sigma)n+\gamma(1-\nu)}{k}} \\ &\quad+\frac{N_0}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n(s+1)^{-\frac{(\mu-k)(n+1)+qn}{k}}(ku)^{-\frac{\mu+(\mu+q)n}{k}}\left(1-k\omega(ku)^{-\frac{\rho}{k}}\right)^{-\frac{(\gamma+\sigma)n+\gamma}{k}}. \end{aligned}$$

Applying the inverse Laplace transform, we get

$$N(t) = d\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n(s+1)^{-\frac{\nu(k-\mu)+(\mu-k)(n+1)+qn}{k}} (t^{s+1})^{\frac{\nu(k-\mu)+\mu+(q+\mu)n}{k}-1}E^{(\gamma+\sigma)n+\gamma(1-\nu)}_{k,\rho,\nu(k-\mu)+\mu+(q+\mu)n}\left(\omega(t^{s+1})^{\frac{\rho}{k}}\right) + \frac{N_0}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n(s+1)^{n+1}\,{}^{s}_{k}\mathfrak{J}^{\omega,(\gamma+\sigma)n+\gamma}_{0^+;\rho,(q+\mu)n+\mu}f(t),$$

which is the required result.

Remark 5.2. *If we take $s=0$, $k=1$, $\nu=\gamma=\sigma=0$, $\mu\to 0$, $a=1$, and $b=-c^p$, then we get the following fractional kinetic equation given in [39]:*

$$N(t) - N_0f(t) = -c^pD^p_{0^+}N(t), \quad N(0)=d, \quad d\geq 0,$$

where $D^p_{0^+}$ is the Riemann-Liouville fractional integral operator, defined as

$$D^p_{0^+}N(t) = \frac{1}{\Gamma(p)}\int_0^t(t-\tau)^{p-1}N(\tau)\,d\tau.$$

Here $N(t)$ denotes the number density of a given species at time t, with $N_0=N(0)$ being the number density of that species at time $t=0$, c is a constant, and $f\in L^1[0;\infty)$.

Corollary 5.3. *If we take $s=0$ and $\nu=0$, then we get the following Cauchy problem given in [42]:*

$$a\,{}_k\mathfrak{D}^{\gamma,\mu}_{0^+;\rho,\omega}N(t) - N_0f(t) = b\,{}_k\mathfrak{J}^{\omega,\sigma}_{0^+;\rho,q}N(t), \quad f\in L^1[0,\infty),$$
$${}_k\mathfrak{J}^{\omega,-\gamma}_{0^+;\rho,k-\mu}N(0) = d, \quad d\geq 0,$$

where $\omega\in\mathbb{C}$, $a,b\in\mathbb{R}(a\neq 0)$, $\mu,\rho,q,k>0$, and $\gamma,\sigma\geq 0$. The solution to the problem is

$$N(t) = d\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n t^{\frac{\mu+(q+\mu)n}{k}-1}E^{(\gamma+\sigma)n+\gamma}_{k,\rho,\mu+(q+\mu)n}\left(\omega(t)^{\frac{\rho}{k}}\right) + \frac{N_0}{a}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n {}_k\mathfrak{J}^{\omega,(\gamma+\sigma)n+\gamma}_{0^+;\rho,(q+\mu)n+\mu}f(t).$$

6. CONCLUSION

A new generalized fractional derivative operator, referred to as the (k,s)-Hilfer-Prabhakar fractional derivative, is developed in this article. The generalized Laplace transform of the proposed operator is also studied. Potential applications of the proposed operator are discussed, which concern fractional models of the free-electron laser equation, heat equation, and kinetic equation that involve the new operator. The results in this article suggest that this novel operator can be used to solve various types of problems arising in mathematical physics and other fields.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

MS, KN, and DK: Conceptualization. MS, ZP, GR, and KN: Writing original draft. KN and DK: Methodology.

ZP, GR, and DK: Formal analysis. MS and GR: Validation. KN and DK: Revision and final check. All authors contributed to the article and approved the submitted version.

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