



Certain Concepts of Vague Graphs With Applications to Medical Diagnosis

Zehui Shao¹, Saeed Kosari^{1*}, Muhammad Shoaib² and Hossein Rashmanlou³

¹ Institute of Computing Science and Technology, Guangzhou University, Guangzhou, China, ² Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan, ³ Mazandaran Adib Institute of Higher Education, Sari, Iran

The purpose of this research study is to present and explore the key properties of some new operations on vague graphs, including rejection, maximal product, symmetric difference, and residue product. This article introduces the notions of degree of a vertex and total degree of a vertex in a vague graph. As well, this study outlines the specific conditions required for obtaining the degrees of vertices in vague graphs under the operations of maximal product, symmetric difference, and rejection. The article also discusses applications of vague sets in medical diagnosis.

Keywords: vague set, maximal product, rejection, symmetric difference, residue product, application

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*Correspondence:

Saeed Kosari
saeedkosari38@yahoo.com

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1. INTRODUCTION

Graph theory is an extremely useful tool for solving combinatorial problems in a wide range of fields, including geometry, algebra, number theory, topology, operations research, biology, and social systems. Graph theory also has many applications of great scope, such as in networking, image capture, clustering, handling uncertainty, image segmentation, finding communities in networks, bioscience, information technology, operations research, and social science networks consisting of points connected by lines. In fact, graph theory studies connections between objects, such as vertices and edges and the various relations between them. Fuzzy graph theory is finding an increasing number of applications in modeling real-time systems, where the amount of information inherent in the system varies with different levels of precision. In 1965, Zadeh [1] first proposed the theory of fuzzy sets. The fuzzy graph, with the approximate reasoning, enables many combinatorial problems in fields, such as topology and algebra to be solved more easily. The concept of fuzzy graphs is discussed by Rosenfeld [2] as well as by Bhattacharya [3, 4]. Fuzzy graphs date back to the nineteenth century, and their use has grown tremendously in recent years [5, 6]. Gau and Buehrer [7] proposed the concept of vague set in 1993, which replaces the value of an element in a set with a subinterval of [0, 1]. Specifically, a true-membership function $t_v(x)$ and a false-membership function $f_v(x)$ are used to describe the boundaries of the membership degree. Descriptions of real-world problems can be improved by using the theory of vague sets. Researchers have applied this theory to several real-world situations, such as decision-making and fuzzy control. The theory of vague sets is also helpful for fault diagnosis and knowledge discovery. Interval-valued fuzzy sets have a case vague set, which has been applied in different fields of mathematics. Ramakrishna [8] introduced the concept of vague graph and also studied related properties. Vague graphs have numerous applications in geometry and operations research and are also useful in many areas of computer science. Rashmanlou and Borzooei [9] studied new concepts relating to vague graphs, product vague graphs [10], regularity of vague graphs [11], and vague competition graphs [12]. Krishna and Lavanya [13] developed new concepts of coloring in vague graphs. Besides the

membership degree, the non-membership degree has been introduced as well, which is presented by Atanassove [14] in an intuitionistic fuzzy set, a type of extension of a fuzzy set. Parvathi and Karunambigai [15] discussed intuitionistic fuzzy graphs. Devi et al. [16] presented new concepts regarding intuitionistic fuzzy labeling graphs.

In this study we outline and explore the key properties of some new operations on vague graphs, including rejection, maximal product, symmetric difference, and residue product. We introduce new notions, such as degree of a vertex and total degree of a vertex in a vague graph. We also outline specific conditions for obtaining the degrees of vertices in vague graphs under the operations of maximal product, symmetric difference, and rejection. Furthermore, we explore applications of vague sets in medical diagnosis.

2. PRELIMINARIES

In this section we introduce the key preliminary notions and definitions that are used in this study.

Definition 2.1 ([17]). A graph is an ordered pair $G = (V, E)$, where V is the set of vertices of G and E is the set of all edges, arcs, or lines, which are two-element subsets of V (that is, an edge is related to two vertices and the relation is represented as an unordered pair $\{m, n\}$ of those vertices).

Note that for an edge $\{m, n\}$, graph theorists usually use the somewhat shorter notation mn . Two vertices m and n in an undirected graph G are said to be adjacent in G if mn is an edge of G . An edge whose endpoints are the same is called a loop. A graph without loops is called a simple graph.

Definition 2.2 ([7]). A vague set M is a pair $(T_M; F_M)$ of functions on a set V , where T_M and F_M are real-valued $V \rightarrow [0, 1]$ functions such that $T_M(m) + F_M(m) < 1$ for all $m \in V$. The interval $[T_M(m), 1 - F_M(m)]$ is known as the vague value of m in M .

In this definition, for m in M , $T_M(m)$ is the lower bound for the degree of membership and $F_M(m)$ is the lower bound for the negative of the degree of membership. Therefore, the degree of membership of $m \in M$ is given by the interval $[T_M(m), 1 - F_M(m)]$.

Definition 2.3 ([8]). Let $G = (V, E)$ be a crisp graph. A pair $\mathbf{G} = (M, N)$ is called a vague graph defined on the crisp graph $G = (V, E)$ if $M = (T_M, F_M)$ is a vague set on V and $N = (T_N, F_N)$ is vague set on $E \subseteq V \times V$ such that $T_N(mn) \leq \min(T_M(m), T_M(n))$ and $F_N(mn) \geq \max(F_M(m), F_M(n))$ for each edge mn in E .

Definition 2.4 ([9]). A vague graph \mathbf{G} is said to be strong if $T_N(mn) = \min(T_M(m), T_M(n))$ and $F_N(mn) = \max(F_M(m), F_M(n))$ for all $m, n \in V$.

Definition 2.5 ([9]). A vague graph \mathbf{G} is said to be complete if $T_N(mn) = \min(T_M(m), T_M(n))$ and $F_N(mn) = \max(F_M(m), F_M(n))$ for all $mn \in E$.

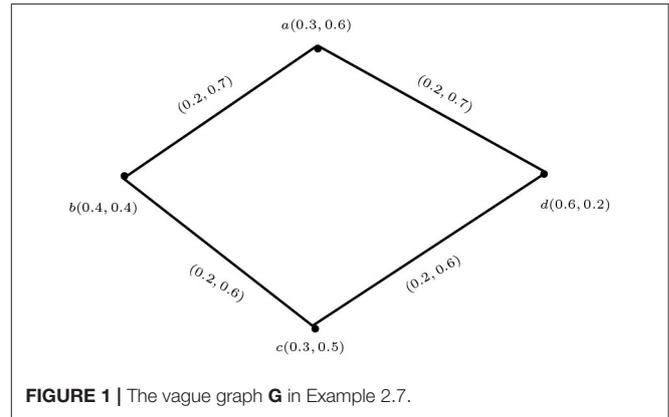


FIGURE 1 | The vague graph \mathbf{G} in Example 2.7.

Definition 2.6 ([11]). A vague graph \mathbf{G} is said to be connected if $T_N^\infty(m_i m_j) > 0$ and $F_N^\infty(m_i m_j) < 1$ for all $m_i, m_j \in V$. Also, we have

$$T_N^\infty(mn) = \sup\{T_N(mn_1) \wedge T_N(n_1 n_2) \wedge T_N(n_2 n_3) \wedge \dots \wedge T_N(n_{k-1} n) \mid m, n_1, n_2, \dots, n_{k-1}, n \in V\}$$

and

$$F_N^\infty(mn) = \inf\{F_N(mn_1) \vee F_N(n_1 n_2) \vee F_N(n_2 n_3) \vee \dots \vee F_N(n_{k-1} n) \mid m, n_1, n_2, \dots, n_{k-1}, n \in V\}.$$

Example 2.7. Consider a vague graph \mathbf{G} such that $V = \{a, b, c\}$, $E = \{ab, bc, cd, ad\}$, $M = \langle (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.3}, \frac{d}{0.6}), (\frac{a}{0.6}, \frac{b}{0.4}, \frac{c}{0.5}, \frac{a}{0.2}) \rangle$, and $N = \langle (\frac{ab}{0.2}, \frac{bc}{0.2}, \frac{cd}{0.2}, \frac{ad}{0.2}), (\frac{ab}{0.7}, \frac{bc}{0.6}, \frac{cd}{0.6}, \frac{ad}{0.7}) \rangle$.

By routine computations, it is easy to show that \mathbf{G} is a vague graph (Figure 1).

3. OPERATIONS ON VAGUE GRAPHS

In this section we define four new kinds of operations on vague graphs: the maximal product, residue product, rejection, and symmetric difference. We show that the maximal product, residue product, or rejection of two vague graphs is again a vague graph.

Definition 3.1. The maximal product $\mathbf{G}_1 * \mathbf{G}_2 = (M_1 * M_2, N_1 * N_2)$ of two vague graphs $\mathbf{G}_1 = (M_1, N_1)$ and $\mathbf{G}_2 = (M_2, N_2)$ is defined by

$$(i) \quad (T_{M_1} * T_{M_2})((m_1, m_2)) = \max\{T_{M_1}(m_1), T_{M_2}(m_2)\}, \\ (F_{M_1} * F_{M_2})((m_1, m_2)) = \min\{F_{M_1}(m_1), F_{M_2}(m_2)\} \\ \forall (m_1, m_2) \in (V_1 \times V_2);$$

$$(ii) \quad (T_{M_1} * T_{M_2})((m, m_2)(m, n_2)) = \max\{T_{M_1}(m), T_{N_2}(m_2 n_2)\}, \\ (F_{M_1} * F_{M_2})((m, m_2)(m, n_2)) = \min\{F_{M_1}(m), F_{N_2}(m_2 n_2)\} \\ \forall m \in V_1 \text{ and } m_2 n_2 \in E_2;$$

$$(iii) \quad (T_{M_1} * T_{M_2})((m_1, z)(n_1, z)) = \max\{T_{N_1}(m_1 n_1), T_{M_2}(z)\}, \\ (F_{M_1} * F_{M_2})((m_1, z)(n_1, z)) = \min\{F_{N_1}(m_1 n_1), F_{M_2}(z)\} \\ \forall z \in V_2 \text{ and } m_1 n_1 \in E_1.$$

Example 3.2. Consider the two vague graphs G_1 and G_2 shown in **Figures 2, 3**. Their maximal product $G_1 * G_2$ is shown in **Figure 4**.

For the vertex (a, d) , we find the membership and non-membership values as follows:

$$\begin{aligned} (T_{M_1} * T_{M_2})((a, d)) &= \max\{T_{M_1}(a), T_{M_2}(d)\} \\ &= \max\{0.4, 0.1\} = 0.4, \\ (F_{M_1} * F_{M_2})((a, d)) &= \min\{F_{M_1}(a), F_{M_2}(d)\} \\ &= \min\{0.5, 0.3\} = 0.3, \\ &\text{for } a \in V_1 \text{ and } d \in V_2. \end{aligned}$$

For the edge $(a, d)(a, e)$, we find the following membership and non-membership values:

$$\begin{aligned} (T_{M_1} * T_{M_2})((a, d)(a, e)) &= \max\{T_{M_1}(a), T_{N_2}(de)\} \\ &= \max\{0.4, 0.1\} = 0.4, \\ (F_{M_1} * F_{M_2})((a, d)(a, e)) &= \min\{F_{M_1}(a), F_{N_2}(de)\} \\ &= \min\{0.5, 0.6\} = 0.5, \\ &\text{for } a \in V_1 \text{ and } de \in E_2. \end{aligned}$$

Now, for edge $(a, g)(b, g)$ we have

$$\begin{aligned} (T_{M_1} * T_{M_2})((a, g)(b, g)) &= \max\{T_{N_1}(ab), T_{M_2}(g)\} \\ &= \max\{0.2, 0.3\} = 0.3, \\ (F_{M_1} * F_{M_2})((a, g)(b, g)) &= \min\{F_{N_1}(ab), F_{M_2}(g)\} \\ &= \min\{0.7, 0.4\} = 0.4, \\ &\text{for } g \in V_2 \text{ and } ab \in E_1. \end{aligned}$$

Similarly, we can find the membership and non-membership values for all the remaining vertices and edges.

Proposition 3.3. The maximal product of two vague graphs G_1 and G_2 is a vague graph.

Proof: Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs on crisp graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, and let $((m_1, m_2)(n_1, n_2)) \in E_1 \times E_2$. Then by Definition 3.1 we have two cases:

(i) If $m_1 = n_1 = m$, then

$$\begin{aligned} (T_{N_1} * T_{N_2})((m, m_2)(m, n_2)) &= \max\{T_{M_1}(m), T_{N_2}(m_2 n_2)\} \\ &\leq \max\{T_{M_1}(m), \min\{T_{M_2}(m_2), T_{M_2}(n_2)\}\} \\ &= \min\{\max\{T_{M_1}(m), T_{M_2}(m_2)\}, \max\{T_{M_1}(m), T_{M_2}(n_2)\}\} \\ &= \min\{(T_{M_1} * T_{M_2})(m, m_2), (T_{M_1} * T_{M_2})(m, n_2)\}, \\ (F_{N_1} * F_{N_2})((m, m_2)(m, n_2)) &= \min\{F_{M_1}(m), F_{N_2}(m_2 n_2)\} \\ &\geq \min\{F_{M_1}(m), \max\{F_{M_2}(m_2), F_{M_2}(n_2)\}\} \\ &= \max\{\min\{F_{M_1}(m), F_{M_2}(m_2)\}, \min\{F_{M_1}(m), F_{M_2}(n_2)\}\} \\ &= \max\{(F_{M_1} * F_{M_2})(m, m_2), (F_{M_1} * F_{M_2})(m, n_2)\}. \end{aligned}$$

(ii) If $m_2 = n_2 = z$, then

$$\begin{aligned} (T_{N_1} * T_{N_2})((m_1, z)(n_1, z)) &= \max\{T_{N_1}(m_1 n_1), T_{M_2}(z)\} \\ &\leq \max\{\min\{T_{N_1}(m_1 n_1), T_{M_2}(z)\}\} \\ &= \min\{\max\{T_{N_1}(m_1), T_{M_2}(z)\}, \max\{T_{M_1}(n_1), T_{M_2}(z)\}\} \\ &= \min\{(T_{M_1} * T_{M_2})(m_1, z), (T_{M_1} * T_{M_2})(n_1, z)\}, \\ (F_{N_1} * F_{N_2})((m_1, z)(n_1, z)) &= \min\{F_{N_1}(m_1 n_1), F_{M_2}(z)\} \\ &\geq \min\{\max\{F_{N_1}(m_1 n_1), F_{M_2}(z)\}\} \\ &= \max\{\min\{F_{M_1}(m_1), F_{M_2}(z)\}, \min\{F_{M_1}(n_1), F_{M_2}(z)\}\} \\ &= \max\{(F_{M_1} * F_{M_2})(m_1, z), (F_{M_1} * F_{M_2})(n_1, z)\}. \end{aligned}$$

Therefore, $G_1 * G_2$ is a vague graph. □

Theorem 3.4. The maximal product of two strong vague graphs G_1 and G_2 is a strong vague graph.

Proof: Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two strong vague graphs on crisp graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, and let $((m_1, m_2)(n_1, n_2)) \in E_1 \times E_2$. Then, by Proposition 3.3, $G_1 * G_2$ is a vague graph. Now we have two cases:

(i) If $m_1 = n_1 = m$, then

$$\begin{aligned} (T_{N_1} * T_{N_2})((m, m_2)(m, n_2)) &= \max\{T_{M_1}(m), T_{N_2}(m_2 n_2)\} \\ &= \max\{T_{M_1}(m), \min\{T_{M_2}(m_2), T_{M_2}(n_2)\}\} \\ &= \min\{\max\{T_{M_1}(m), T_{M_2}(m_2)\}, \max\{T_{M_1}(m), T_{M_2}(n_2)\}\} \\ &= \min\{(T_{M_1} * T_{M_2})(m, m_2), (T_{M_1} * T_{M_2})(m, n_2)\}, \\ (F_{N_1} * F_{N_2})((m, m_2)(m, n_2)) &= \min\{F_{M_1}(m), F_{N_2}(m_2 n_2)\} \\ &= \min\{F_{M_1}(m), \max\{F_{M_2}(m_2), F_{M_2}(n_2)\}\} \\ &= \max\{\min\{F_{M_1}(m), F_{M_2}(m_2)\}, \min\{F_{M_1}(m), F_{M_2}(n_2)\}\} \\ &= \max\{(F_{M_1} * F_{M_2})(m, m_2), (F_{M_1} * F_{M_2})(m, n_2)\}. \end{aligned}$$

(ii) If $m_2 = n_2 = z$, then

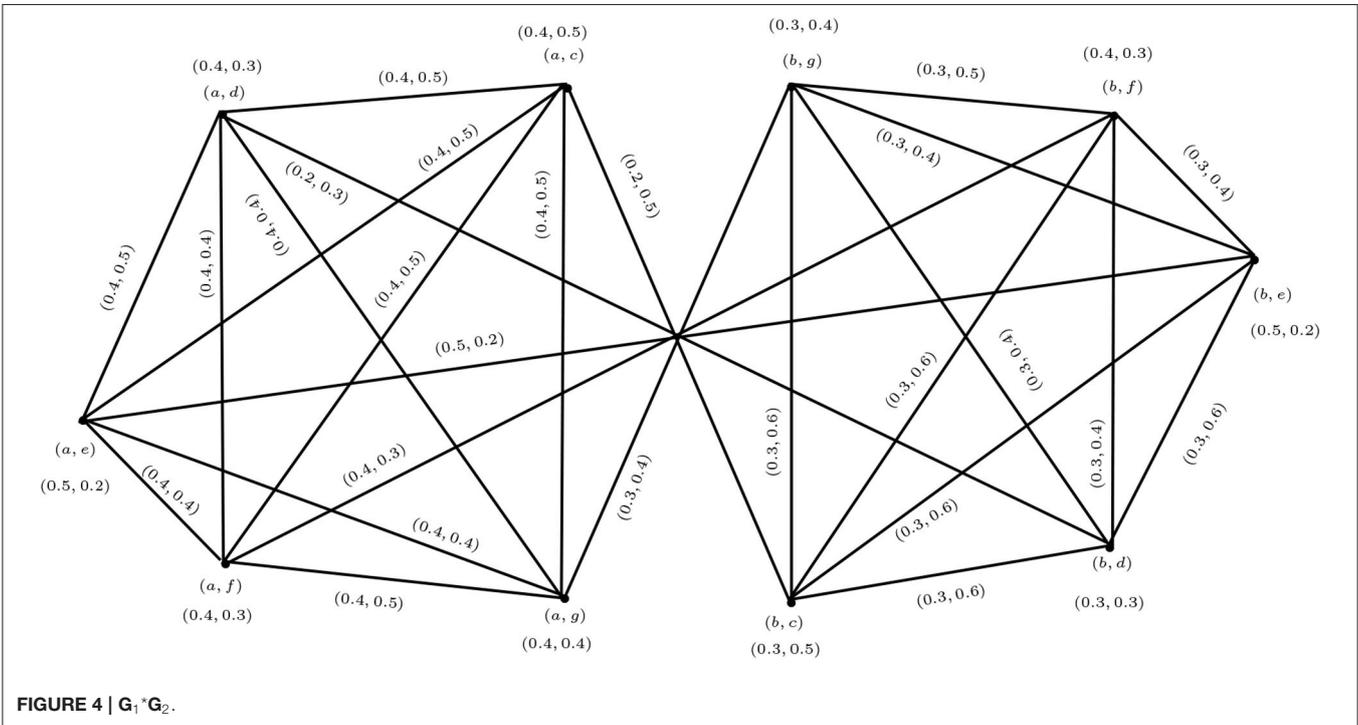
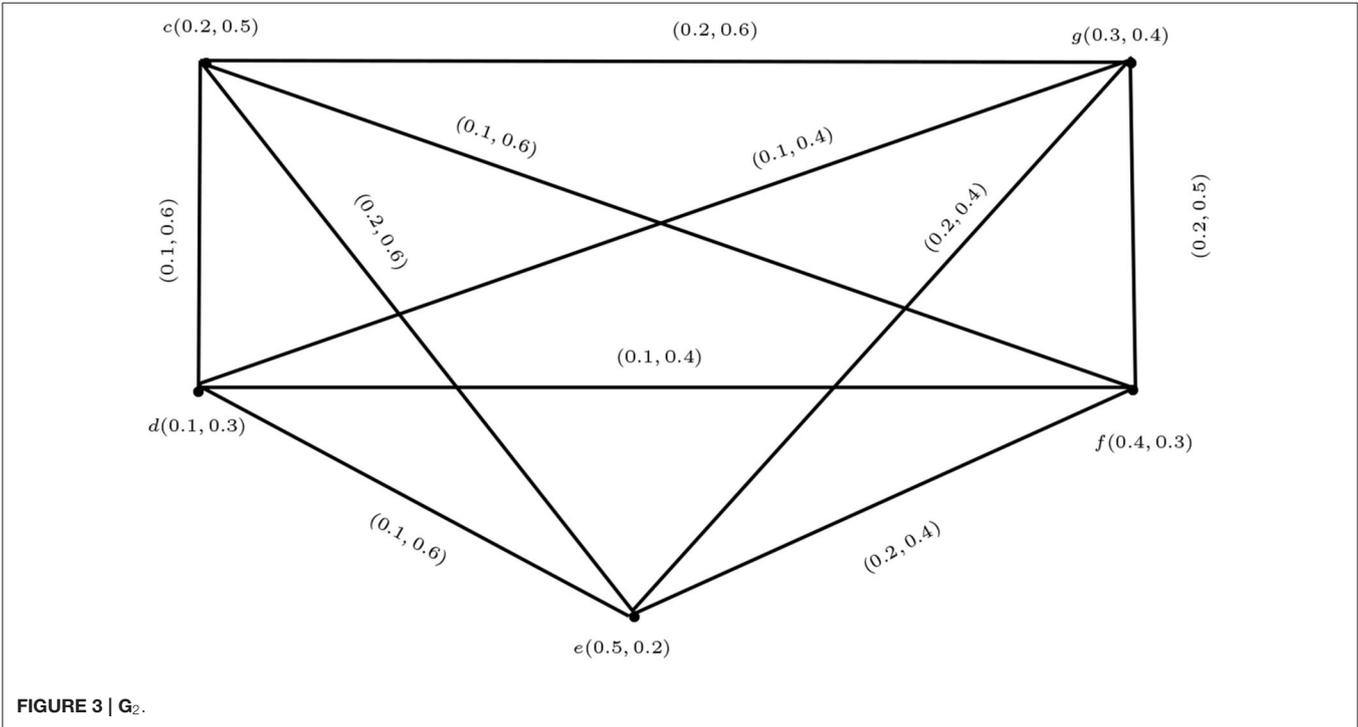
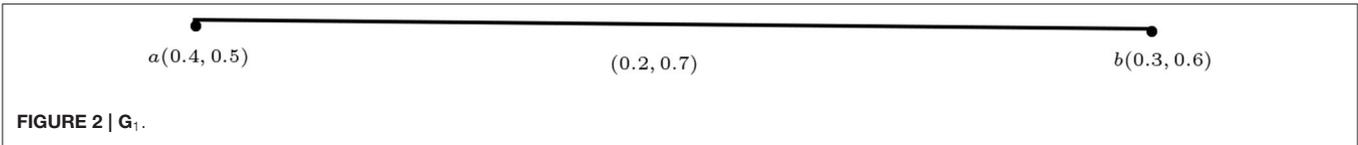
$$\begin{aligned} (T_{N_1} * T_{N_2})((m_1, z)(n_1, z)) &= \max\{T_{N_1}(m_1 n_1), T_{M_2}(z)\} \\ &= \max\{\min\{T_{N_1}(m_1 n_1), T_{M_2}(z)\}\} \\ &= \min\{\max\{T_{N_1}(m_1), T_{M_2}(z)\}, \max\{T_{M_1}(n_1), T_{M_2}(z)\}\} \\ &= \min\{(T_{M_1} * T_{M_2})(m_1, z), (T_{M_1} * T_{M_2})(n_1, z)\}, \\ (F_{N_1} * F_{N_2})((m_1, z)(n_1, z)) &= \min\{F_{N_1}(m_1 n_1), F_{M_2}(z)\} \\ &= \min\{\max\{F_{N_1}(m_1 n_1), F_{M_2}(z)\}\} \\ &= \max\{\min\{F_{M_1}(m_1), F_{M_2}(z)\}, \min\{F_{M_1}(n_1), F_{M_2}(z)\}\} \\ &= \max\{(F_{M_1} * F_{M_2})(m_1, z), (F_{M_1} * F_{M_2})(n_1, z)\}. \end{aligned}$$

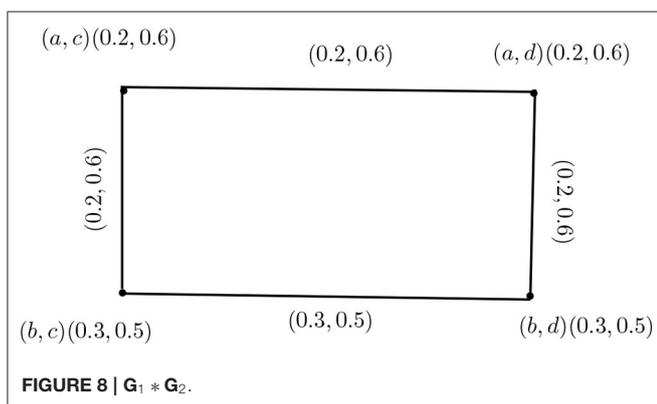
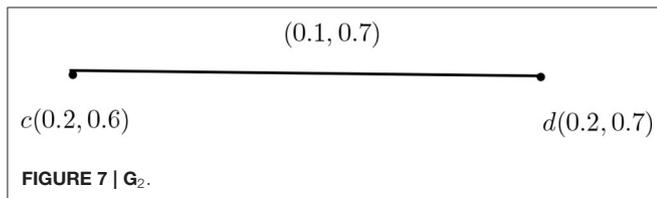
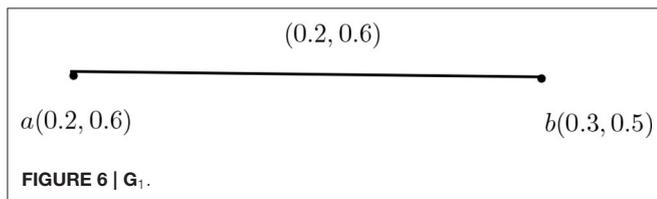
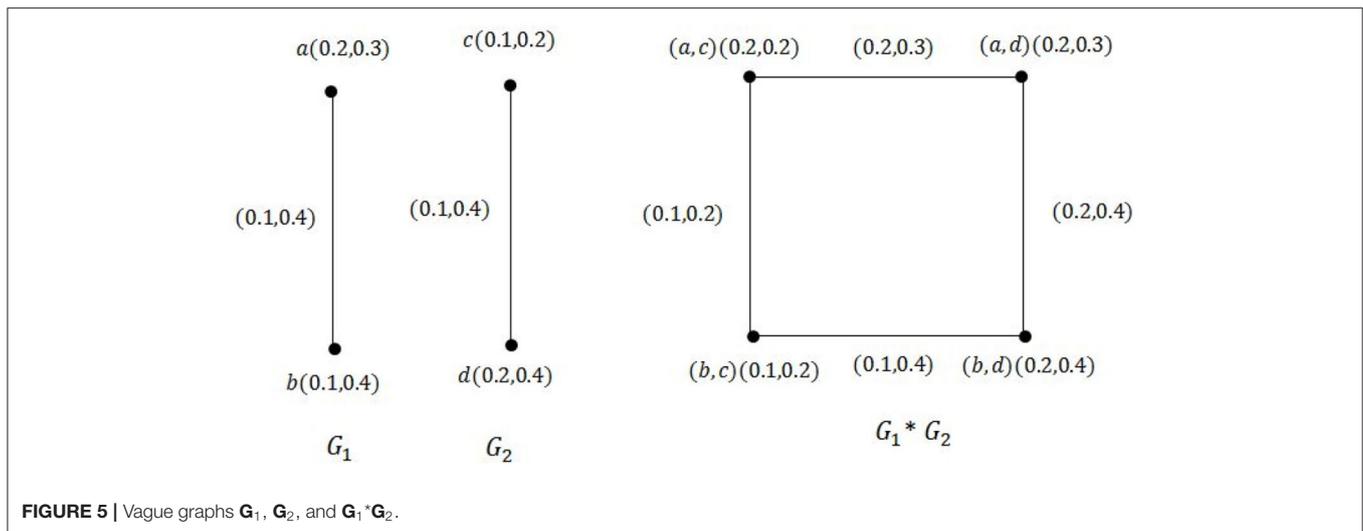
Therefore, $G_1 * G_2$ is a strong vague graph. □

Example 3.5. Consider the strong vague graphs G_1 and G_2 as in **Figure 5**.

It is easy to see that $G_1 * G_2$ is a strong vague graph too.

Remark 3.1. If the maximal product of two vague graphs $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ is a strong vague graph, G_1 and G_2 need not be strong in general.





Example 3.6. Consider the vague graphs G_1 and G_2 as in **Figures 6, 7**. The maximal product of G_1 and G_2 is $G_1 * G_2$ shown in **Figure 8**.

We can see that G_1 and $G_1 * G_2$ are strong vague graphs, but G_2 is not strong: since $T_{N_2}(m_2, n_2) = 0.1$ but

$\min\{T_{M_2}(m_2), T_{M_2}(n_2)\} = \min\{0.2, 0.2\} = 0.2$, we have $T_{N_2}(m_2, n_2) \neq \min\{T_{M_2}(m_2), T_{M_2}(n_2)\}$.

Theorem 3.7. The maximal product of two connected vague graphs is a connected vague graph.

Proof: Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two connected vague graphs on crisp graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, where $V_1 = \{m_1, m_2, \dots, m_k\}$ and $V_2 = \{n_1, n_2, \dots, n_s\}$. Then $T_{N_1}^\infty(m_i, m_j) > 0$ for all $m_i, m_j \in V_1$ and $T_{N_2}^\infty(n_i, n_j) > 0$ for all $n_i, n_j \in V_2$ (or $F_{N_1}^\infty(m_i, m_j) < 1$ for all $m_i, m_j \in V_1$ and $F_{N_2}^\infty(n_i, n_j) < 1$ for all $n_i, n_j \in V_2$). The maximal product of $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ can be taken as $G = (M, N)$. Now, consider the k subgraphs of G with the vertex set $\{(m_i, n_1), (m_i, n_2), \dots, (m_i, n_s)\}$ for $i = 1, 2, \dots, k$. Each of these subgraphs of G is connected, since the m_i 's are the same and G_2 is connected, so that each n_j is adjacent to at least one of the vertices in V_2 . Also, since G_1 is connected, each x_i is adjacent to at least one of the vertices in V_1 .

Hence, there exists at least one edge between any pair of the above k subgraphs. Thus, we have $T_N^\infty((m_i, n_j)(m_m, n_n)) > 0$ (or $F_N^\infty((m_i, n_j)(m_m, n_n)) < 1$) for all $((m_i, n_j)(m_m, n_n)) \in E$. Therefore, G is a connected vague graph. \square

Remark 3.2. The maximal product of two complete vague graphs is not a complete vague graph in general. This is because we do not include the case where $(m_1, m_2) \in E_1$ and $(n_1, n_2) \in E_2$ in the definition of the maximal product of two vague graphs.

Remark 3.3. The maximal product of two complete vague graphs is a strong vague graph.

Example 3.8. Consider the complete vague graphs G_1 and G_2 in **Figure 5**. A simple calculation yields that $G_1 * G_2$ is a strong vague graph.

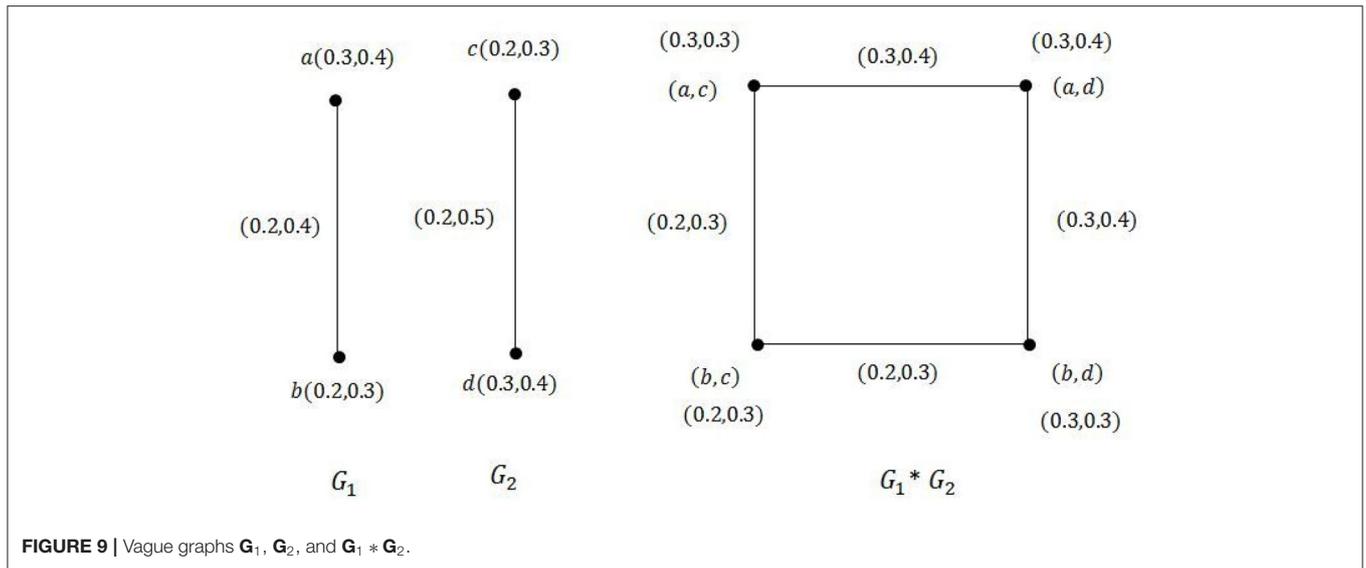


FIGURE 9 | Vague graphs G_1 , G_2 , and $G_1 * G_2$.

Definition 3.9. Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs. For any vertex $(m_1, m_2) \in V_1 \times V_2$ we define

$$\begin{aligned} (d_T)_{G_1 * G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (T_{N_1} * T_{N_2})((m_1, m_2)(n_1, n_2)) \\ &= \sum_{m_1=n_1, m_2=n_2 \in E_2} \max\{T_{M_1}(m_1), T_{N_2}(m_2 n_2)\} \\ &+ \sum_{m_1 n_1 \in E_1, m_2=n_2} \max\{T_{N_1}(m_1 n_1), T_{M_2}(m_2)\}, \end{aligned}$$

$$\begin{aligned} (d_F)_{G_1 * G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (F_{N_1} * F_{N_2})((m_1, m_2)(n_1, n_2)) \\ &= \sum_{m_1=n_1, m_2 n_2 \in E_2} \min\{F_{M_1}(m_1), F_{N_2}(m_2 n_2)\} \\ &+ \sum_{m_1 n_1 \in E_1, m_2=n_2} \min\{F_{N_1}(m_1 n_1), F_{M_2}(m_2)\}. \end{aligned}$$

Theorem 3.10. Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs. If $T_{M_1} \geq T_{N_2}$, $F_{M_1} \leq F_{N_2}$, $T_{M_2} \geq T_{N_1}$, and $F_{M_2} \leq F_{N_1}$, then $(d_T)_{G_1 * G_2}(m_1, m_2) = (d_{G_2}(m_2)T_{M_1}(m_1) + (d_{G_1}(m_1)T_{M_2}(m_2))$ and $(d_F)_{G_1 * G_2}(m_1, m_2) = (d_{G_2}(m_2)F_{M_1}(m_1) + (d_{G_1}(m_1)F_{M_2}(m_2))$.

Proof: From the definition of a vertex in the cartesian product, we have

$$\begin{aligned} (d_T)_{G_1 * G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (T_{N_1} * T_{N_2})((m_1, m_2)(n_1, n_2)) \\ &= \sum_{m_1=n_1, m_2 n_2 \in E_2} \max\{T_{M_1}(m_1), T_{N_2}(m_2 n_2)\} \\ &+ \sum_{m_1 n_1 \in E_1, m_2=n_2} \max\{T_{N_1}(m_1 n_1), T_{M_2}(m_2)\} \\ &= \sum_{m_2 n_2 \in E_2, m_1=n_1} T_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1, m_2=n_2} T_{N_1}(m_1 n_1) \\ &= (d_{G_2}(m_2)T_{M_1}(m_1) + (d_{G_1}(m_1)T_{M_2}(m_2), \end{aligned}$$

$$\begin{aligned} (d_F)_{G_1 * G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (F_{N_1} * F_{N_2})((m_1, m_2)(n_1, n_2)) \\ &= \sum_{m_1=n_1, m_2 n_2 \in E_2} \min\{F_{M_1}(m_1), F_{N_2}(m_2 n_2)\} \\ &+ \sum_{m_1 n_1 \in E_1, m_2=n_2} \min\{F_{N_1}(m_1 n_1), F_{M_2}(m_2)\} \\ &= \sum_{m_2 n_2 \in E_2, m_1=n_1} F_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1, m_2=n_2} F_{N_1}(m_1 n_1) \\ &= (d_{G_2}(m_2)F_{M_1}(m_1) + (d_{G_1}(m_1)F_{M_2}(m_2), \end{aligned}$$

as claimed. \square

Example 3.11. Consider the vague graphs G_1 , G_2 , and $G_1 * G_2$ as in **Figure 9**. Since $T_{M_1} \geq T_{N_2}$, $F_{M_1} \leq F_{N_2}$, $T_{M_2} \geq T_{N_1}$, and $F_{M_2} \leq F_{N_1}$, by Theorem 3.10 we have

$$\begin{aligned} (d_T)_{G_1 * G_2}(a, c) &= (d_{G_2}(c)T_{M_1}(a) + (d_{G_1}(a)T_{M_2}(c) = 1 \cdot (0.3) \\ &+ 1 \cdot (0.2) = 0.5, \end{aligned}$$

$$\begin{aligned} (d_F)_{G_1 * G_2}(a, c) &= (d_{G_2}(c)F_{M_1}(a) + (d_{G_1}(a)F_{M_2}(c) = 1 \cdot (0.4) \\ &+ 1 \cdot (0.3) = 0.7. \end{aligned}$$

$$\begin{aligned} (d_T)_{G_1 * G_2}(a, d) &= (d_{G_2}(d)T_{M_1}(a) + (d_{G_1}(a)T_{M_2}(d) = 1 \cdot (0.3) \\ &+ 1 \cdot (0.3) = 0.6, \end{aligned}$$

$$\begin{aligned} (d_F)_{G_1 * G_2}(a, d) &= (d_{G_2}(d)F_{M_1}(a) + (d_{G_1}(a)F_{M_2}(d) = 1 \cdot (0.4) \\ &+ 1 \cdot (0.4) = 0.8. \end{aligned}$$

$$\begin{aligned} (d_T)_{G_1 * G_2}(b, c) &= (d_{G_2}(c)T_{M_1}(b) + (d_{G_1}(b)T_{M_2}(c) = 1 \cdot (0.2) \\ &+ 1 \cdot (0.2) = 0.4, \end{aligned}$$

$$\begin{aligned} (d_F)_{G_1 * G_2}(b, c) &= (d_{G_2}(c)F_{M_1}(b) + (d_{G_1}(b)F_{M_2}(c) = 1 \cdot (0.3) \\ &+ 1 \cdot (0.3) = 0.6. \end{aligned}$$

$$\begin{aligned}
 (d_T)_{G_1 * G_2}(b, d) &= (d)_{G_2}(d)T_{M_1}(b) + (d)_{G_1}(b)T_{M_2}(d) = 1 \cdot (0.2) \\
 &+ 1 \cdot (0.3) = 0.5, \\
 (d_F)_{G_1 * G_2}(b, d) &= (d)_{G_2}(d)F_{M_1}(b) + (d)_{G_1}(b)F_{M_2}(d) = 1 \cdot (0.3) \\
 &+ 1 \cdot (0.4) = 0.7.
 \end{aligned}$$

By direct calculations we obtain

$$\begin{aligned}
 (d_T)_{G_1 * G_2}(a, c) &= 0.3 + 0.2 = 0.5, \\
 (d_F)_{G_1 * G_2}(a, c) &= 0.4 + 0.3 = 0.7, \\
 (d_T)_{G_1 * G_2}(a, d) &= 0.3 + 0.3 = 0.6, \\
 (d_F)_{G_1 * G_2}(a, d) &= 0.4 + 0.4 = 0.8, \\
 (d_T)_{G_1 * G_2}(b, c) &= 0.2 + 0.2 = 0.4, \\
 (d_F)_{G_1 * G_2}(b, c) &= 0.3 + 0.3 = 0.6, \\
 (d_T)_{G_1 * G_2}(b, d) &= 0.3 + 0.2 = 0.5, \\
 (d_F)_{G_1 * G_2}(b, d) &= 0.3 + 0.4 = 0.7.
 \end{aligned}$$

It is clear that the degrees of vertices calculated using the formula in Theorem 3.10 and by the direct method are the same.

Definition 3.12. Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs. For any vertex $(m_1, m_2) \in V_1 \times V_2$ we define

$$\begin{aligned}
 (td_T)_{G_1 * G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (T_{N_1} * T_{N_2})(m_1, m_2)(n_1, n_2) \\
 &+ (T_{M_1} * T_{M_2})(m_1, m_2) \\
 &= \sum_{m_1=n_1, m_2=n_2 \in E_2} \max\{T_{M_1}(m_1), T_{N_2}(m_2 n_2)\} \\
 &+ \sum_{m_1 n_1 \in E_1, m_2=n_2} \max\{T_{N_1}(m_1 n_1), T_{M_2}(m_2)\} \\
 &+ \max\{T_{M_1}(m_1), T_{M_2}(m_2)\}, \\
 (td_F)_{G_1 * G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (F_{N_1} * F_{N_2})(m_1, m_2)(n_1, n_2) \\
 &+ (F_{M_1} * F_{M_2})(m_1, m_2) \\
 &= \sum_{m_1=n_1, m_2=n_2 \in E_2} \min\{F_{M_1}(m_1), F_{N_2}(m_2 n_2)\} \\
 &+ \sum_{m_1 n_1 \in E_1, m_2=n_2} \min\{F_{N_1}(m_1 n_1), F_{M_2}(m_2)\} \\
 &+ \min\{F_{M_1}(m_1), F_{M_2}(m_2)\}.
 \end{aligned}$$

Example 3.13. In this example we find the degree and the total degree of vertices (a, c) and (a, d) in Example 3.2:

$$\begin{aligned}
 (d_T)_{G_1 * G_2}(a, c) &= (d)_{G_2}(a)T_{M_1}(c) + (d)_{G_1}(c)T_{M_2}(a) \\
 &= 1(0.2) + 4(0.4) = 0.2 + 1.6 = 1.8, \\
 (d_F)_{G_1 * G_2}(a, c) &= (d)_{G_2}(a)F_{M_1}(c) + (d)_{G_1}(c)F_{M_2}(a) \\
 &= 1(0.5) + 4(0.5) = 0.3 + 1.2 = 1.5.
 \end{aligned}$$

Therefore, $d_{G_1 * G_2}(a, c) = (1.8, 1.5)$. In addition, by the definition of the total vertex degree in the maximal product,

$$\begin{aligned}
 (td_T)_{G_1 * G_2}(a, c) &= (d)_{G_2}(a)T_{M_1}(c) + (d)_{G_1}(c)T_{M_2}(a) \\
 &+ \max\{T_{M_1}(a), T_{M_2}(c)\} \\
 &= 1(0.2) + 4(0.4) + \max(0.2, 0.4) = 2.2, \\
 (td_F)_{G_1 * G_2}(a, c) &= (d)_{G_2}(a)F_{M_1}(c) + (d)_{G_1}(c)F_{M_2}(a) \\
 &+ \min\{F_{M_1}(a), F_{M_2}(c)\} \\
 &= 1(0.5) + 4(0.5) + \min(0.3, 0.4) = 1.8.
 \end{aligned}$$

Therefore, $td_{G_1 * G_2}(a, c) = (2.2, 1.8)$.

We also have

$$\begin{aligned}
 (d_T)_{G_1 * G_2}(a, d) &= (d)_{G_2}(a)T_{M_1}(d) + (d)_{G_1}(d)T_{M_2}(a) \\
 &= 1(0.1) + 4(0.4) = 0.1 + 1.6 = 1.7, \\
 (d_F)_{G_1 * G_2}(a, d) &= (d)_{G_2}(a)F_{M_1}(d) + (d)_{G_1}(d)F_{M_2}(a) \\
 &= 1(0.3) + 4(0.5) = 0.3 + 2 = 2.3,
 \end{aligned}$$

$$\begin{aligned}
 (td_T)_{G_1 * G_2}(a, d) &= (d)_{G_2}(a)T_{M_1}(d) + (d)_{G_1}(d)T_{M_2}(a) \\
 &+ \max\{T_{M_1}(a), T_{M_2}(d)\} \\
 &= 1(0.1) + 4(0.4) + \max(0.4, 0.1) = 2.1, \\
 (td_F)_{G_1 * G_2}(a, d) &= (d)_{G_2}(a)F_{M_1}(d) + (d)_{G_1}(d)F_{M_2}(a) \\
 &+ \min\{F_{M_1}(a), F_{M_2}(d)\} \\
 &= 1(0.3) + 4(0.5) + \min(0.5, 0.3) = 2.6.
 \end{aligned}$$

Hence, $d_{G_1 * G_2}(a, d) = (1.7, 2.3)$ and $td_{G_1 * G_2}(a, d) = (2.1, 2.6)$.

Similarly, we can find the degree and the total degree of all vertices in $G_1 * G_2$.

Theorem 3.14. Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs. If $T_{M_1} \geq T_{N_2}$, $F_{M_1} \leq F_{N_2}$, $T_{M_2} \geq T_{N_1}$, and $F_{M_2} \leq F_{N_1}$, then $(td_T)_{G_1 * G_2}(m_1, m_2) = (d)_{G_2}(m_2)T_{M_1}(m_1) + (d)_{G_1}(m_1)T_{M_2}(m_2) + \max\{T_{M_1}(m_1), T_{M_2}(m_2)\}$ and $(td_F)_{G_1 * G_2}(m_1, m_2) = (d)_{G_2}(m_2)F_{M_1}(m_1) + (d)_{G_1}(m_1)F_{M_2}(m_2) + \min\{F_{M_1}(m_1), F_{M_2}(m_2)\}$.

Proof: From Definition 3.12 we have

$$\begin{aligned}
 (td_T)_{G_1 * G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (T_{N_1} * T_{N_2})(m_1, m_2)(n_1, n_2) \\
 &+ (T_{M_1} * T_{M_2})(m_1, m_2) \\
 &= \sum_{m_1=n_1, m_2=n_2 \in E_2} \max\{T_{M_1}(m_1), T_{N_2}(m_2 n_2)\} \\
 &+ \sum_{m_1 n_1 \in E_1, m_2=n_2} \max\{T_{N_1}(m_1 n_1), T_{M_2}(m_2)\} \\
 &+ \max\{T_{M_1}(m_1), T_{M_2}(m_2)\} \\
 &= \sum_{m_2 n_2 \in E_2, m_1=n_1} T_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1, m_2=n_2} T_{N_1}(m_1 n_1) \\
 &+ \max\{T_{M_1}(m_1), T_{M_2}(m_2)\} \\
 &= (d)_{G_2}(m_2)T_{M_1}(m_1) + (d)_{G_1}(m_1)T_{M_2}(m_2) \\
 &+ \max\{T_{M_1}(m_1), T_{M_2}(m_2)\}
 \end{aligned}$$

and

$$\begin{aligned}
 (td_F)_{G_1 * G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (F_{N_1} * F_{N_2})((m_1, m_2)(n_1, n_2)) \\
 &+ (F_{M_1} * F_{M_2})(m_1, m_2) \\
 &= \sum_{m_1=n_1, m_2 n_2 \in E_2} \min\{F_{M_1}(m_1), F_{N_2}(m_2 n_2)\} \\
 &+ \sum_{m_1 n_1 \in E_1, m_2=n_2} \min\{F_{N_1}(m_1 n_1), F_{M_2}(m_2)\} \\
 &+ \min\{F_{M_1}(m_1), F_{M_2}(m_2)\} \\
 &= \sum_{m_2 n_2 \in E_2, m_1=n_1} F_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1, m_2=n_2} F_{N_1}(m_1 n_1) \\
 &+ \min\{F_{M_1}(m_1), F_{M_2}(m_2)\} \\
 &= (d)_{G_2}(m_2)F_{M_1}(m_1) + (d)_{G_1}(m_1)F_{M_2}(m_2) \\
 &+ \min\{F_{M_1}(m_1), F_{M_2}(m_2)\},
 \end{aligned}$$

as asserted. \square

Example 3.15. Consider the vague graphs G_1 , G_2 , and $G_1 * G_2$ in **Figure 9**. The total degree of the vertex in the maximal product is calculated by the following formula:

$$\begin{aligned}
 (td_T)_{G_1 * G_2}(m_1, m_2) &= (d)_{G_2}(m_2)T_{M_1}(m_1) + (d)_{G_1}(m_1)T_{M_2}(m_2) \\
 &+ \max\{T_{M_1}(m_1), T_{M_2}(m_2)\}, \\
 (td_F)_{G_1 * G_2}(m_1, m_2) &= (d)_{G_2}(m_2)F_{M_1}(m_1) + (d)_{G_1}(m_1)F_{M_2}(m_2) \\
 &+ \min\{F_{M_1}(m_1), F_{M_2}(m_2)\}.
 \end{aligned}$$

Using the formula we find that

$$\begin{aligned}
 (td_T)_{G_1 * G_2}(a, c) &= (d)_{G_2}(c)T_{M_1}(a) + (d)_{G_1}(a)T_{M_2}(c) \\
 &+ \max\{T_{M_1}(a), T_{M_2}(c)\} \\
 &= 1 \cdot (0.3) + 1 \cdot (0.2) + \max\{0.2, 0.3\} \\
 &= 0.3 + 0.2 + 0.3 = 0.8, \\
 (td_F)_{G_1 * G_2}(a, c) &= (d)_{G_2}(c)F_{M_1}(a) + (d)_{G_1}(a)F_{M_2}(c) \\
 &+ \min\{F_{M_1}(a), F_{M_2}(c)\} \\
 &= 1 \cdot (0.4) + 1 \cdot (0.3) + \min\{0.3, 0.4\} \\
 &= 0.4 + 0.3 + 0.3 = 1. \\
 (td_T)_{G_1 * G_2}(a, d) &= (d)_{G_2}(d)T_{M_1}(a) + (d)_{G_1}(a)T_{M_2}(d) \\
 &+ \max\{T_{M_1}(a), T_{M_2}(d)\} \\
 &= 1 \cdot (0.3) + 1 \cdot (0.3) + \max\{0.3, 0.3\} \\
 &= 0.3 + 0.3 + 0.3 = 0.9, \\
 (td_F)_{G_1 * G_2}(a, d) &= (d)_{G_2}(d)F_{M_1}(a) + (d)_{G_1}(a)F_{M_2}(d) \\
 &+ \min\{F_{M_1}(a), F_{M_2}(d)\} \\
 &= 1 \cdot (0.4) + 1 \cdot (0.4) + \min\{0.4, 0.4\} \\
 &= 0.4 + 0.4 + 0.4 = 1.2.
 \end{aligned}$$

$$\begin{aligned}
 (td_T)_{G_1 * G_2}(b, c) &= (d)_{G_2}(c)T_{M_1}(b) + (d)_{G_1}(b)T_{M_2}(c) \\
 &+ \max\{T_{M_1}(b), T_{M_2}(c)\} \\
 &= 1 \cdot (0.2) + 1 \cdot (0.2) + \max\{0.2, 0.2\} \\
 &= 0.2 + 0.2 + 0.2 = 0.6, \\
 (td_F)_{G_1 * G_2}(b, c) &= (d)_{G_2}(c)F_{M_1}(b) + (d)_{G_1}(b)F_{M_2}(c) \\
 &+ \min\{F_{M_1}(b), F_{M_2}(c)\} \\
 &= 1 \cdot (0.3) + 1 \cdot (0.3) + \min\{0.3, 0.3\} \\
 &= 0.3 + 0.3 + 0.3 = 0.9. \\
 (td_T)_{G_1 * G_2}(b, d) &= (d)_{G_2}(d)T_{M_1}(b) + (d)_{G_1}(b)T_{M_2}(d) \\
 &+ \max\{T_{M_1}(b), T_{M_2}(d)\} \\
 &= 1 \cdot (0.2) + 1 \cdot (0.3) + \max\{0.2, 0.3\} \\
 &= 0.2 + 0.3 + 0.3 = 0.8, \\
 (td_F)_{G_1 * G_2}(b, d) &= (d)_{G_2}(d)F_{M_1}(b) + (d)_{G_1}(b)F_{M_2}(d) \\
 &+ \min\{F_{M_1}(b), F_{M_2}(d)\} \\
 &= 1 \cdot (0.3) + 1 \cdot (0.4) + \min\{0.3, 0.4\} \\
 &= 0.3 + 0.4 + 0.3 = 1.
 \end{aligned}$$

On the other hand, by direct calculations we obtain

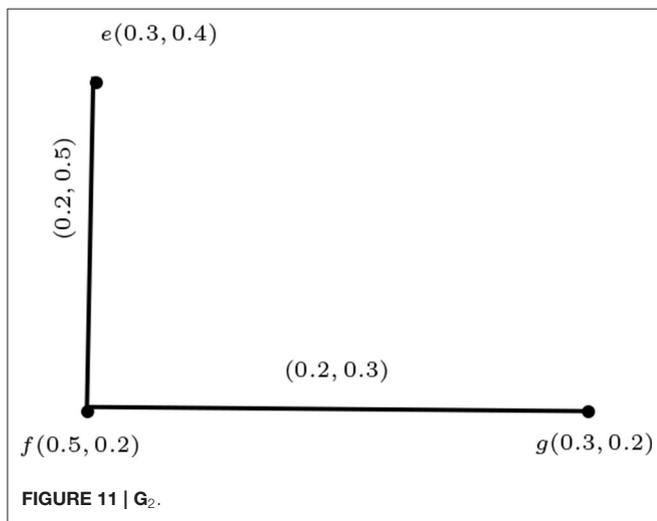
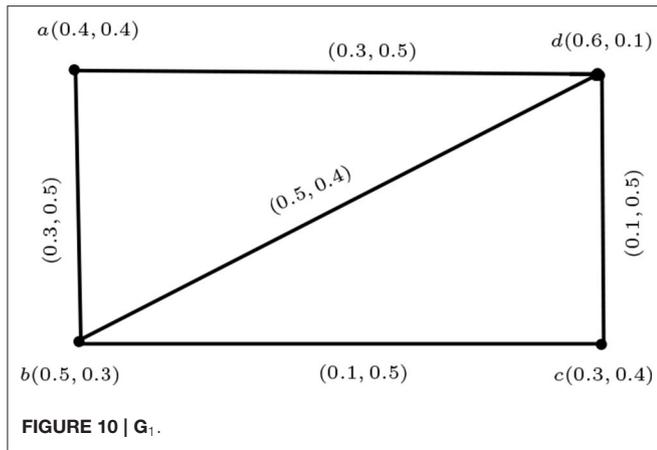
$$\begin{aligned}
 (td_T)_{G_1 * G_2}(a, c) &= 0.3 + 0.2 + 0.3 = 0.8, \\
 (td_F)_{G_1 * G_2}(a, c) &= 0.4 + 0.3 + 0.3 = 1, \\
 (td_T)_{G_1 * G_2}(a, d) &= 0.3 + 0.3 + 0.3 = 0.9, \\
 (td_F)_{G_1 * G_2}(a, d) &= 0.4 + 0.4 + 0.4 = 1.2, \\
 (td_T)_{G_1 * G_2}(b, c) &= 0.2 + 0.2 + 0.2 = 0.6, \\
 (td_F)_{G_1 * G_2}(b, c) &= 0.3 + 0.3 + 0.3 = 0.9, \\
 (td_T)_{G_1 * G_2}(b, d) &= 0.3 + 0.2 + 0.3 = 0.8, \\
 (td_F)_{G_1 * G_2}(b, d) &= 0.4 + 0.3 + 0.3 = 1.
 \end{aligned}$$

It is thus clear that the total degrees of vertices calculated using the formula and by the direct method are the same.

Definition 3.16. The rejection $G_1 | G_2 = (M_1 | M_2, N_1 | N_2)$ of two vague graphs $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ is defined as follows:

- (i) $(T_{M_1} | T_{M_2})((m_1, m_2)) = \min\{T_{M_1}(m_1), T_{M_2}(m_2)\},$
 $(F_{M_1} | F_{M_2})((m_1, m_2)) = \max\{F_{M_1}(m_1), F_{M_2}(m_2)\}$
 $\forall (m_1, m_2) \in (V_1 \times V_2);$
- (ii) $(T_{N_1} | T_{N_2})((m, m_2)(m, n_2)) = \min\{T_{M_1}(m), T_{M_2}(m_2), T_{M_2}(n_2)\},$
 $(F_{N_1} | F_{N_2})((m, m_2)(m, n_2)) = \max\{F_{M_1}(m), F_{M_2}(m_2), F_{M_2}(n_2)\}$
 $\forall m \in V_2 \text{ and } m_2 n_2 \notin E_2;$
- (iii) $(T_{N_1} | T_{N_2})((m, m_2)(m, n_2)) = \min\{T_{M_1}(m), T_{M_2}(m_2), T_{M_2}(n_2)\}$
 $(F_{N_1} | F_{N_2})((m, m_2)(m, n_2)) = \max\{F_{M_1}(m), F_{M_2}(m_2), F_{M_2}(n_2)\}$
 $\forall m \in V_2 \text{ and } m_1 n_1 \notin E_1;$
- (iv) $(T_{N_1} | T_{N_2})((m_1, m_2)(n_1, n_2)) = \min\{T_{M_1}(m_1), T_{M_1}(n_1), T_{M_2}(m_2), T_{M_2}(n_2)\},$
 $(F_{N_1} | F_{N_2})((m_1, m_2)(n_1, n_2)) = \max\{F_{M_1}(m_1), F_{M_1}(n_1), F_{M_2}(m_2), F_{N_2}(n_2)\}$
 $\forall m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \notin E_2.$

Example 3.17. Consider the vague graphs G_1 and G_2 in **Figures 10, 11**. The rejection of G_1 and G_2 , i.e., $G_1 | G_2$, is shown in **Figure 12**.



For the vertex (a, e) , we find the membership and non-membership values as follows:

$$\begin{aligned} (T_{M_1}|T_{M_2})((a, e)) &= \min\{T_{M_1}(a), T_{M_2}(e)\} \\ &= \min\{0.4, 0.3\} = 0.3, \\ (F_{M_1}|F_{M_2})((a, e)) &= \max\{F_{M_1}(a), F_{M_2}(e)\} \\ &= \max\{0.4, 0.4\} = 0.4 \end{aligned}$$

for $a \in V_1$ and $e \in V_2$.

For the edge $(e, c)(e, a)$, the membership and non-membership values are given by

$$\begin{aligned} (T_{N_1}|T_{N_2})((e, c)(e, a)) &= \min\{T_{M_1}(e), T_{M_2}(c), T_{M_2}(a)\} \\ &= \min\{0.3, 0.3, 0.4\} = 0.3, \\ (F_{N_1}|F_{N_2})((e, c)(e, a)) &= \max\{F_{M_1}(e), F_{M_2}(c), F_{M_2}(a)\} \\ &= \max\{0.4, 0.4, 0.4\} = 0.4 \end{aligned}$$

for $e \in V_2$ and $ac \notin E_1$.

For the edge $(e, c)(e, g)$ we have

$$\begin{aligned} (T_{N_1}|T_{N_2})((e, c)(e, g)) &= \min\{T_{M_1}(e), T_{M_2}(c), T_{M_2}(g)\} \\ &= \min\{0.3, 0.3, 0.3\} = 0.3, \\ (F_{N_1}|F_{N_2})((e, c)(e, g)) &= \max\{F_{M_1}(e), F_{M_2}(c), F_{M_2}(g)\} \\ &= \max\{0.4, 0.4, 0.2\} = 0.4 \end{aligned}$$

for $e \in V_2$ and $cg \notin E_2$.

Similarly, we can find the membership and non-membership values for all the remaining vertices and edges.

Proposition 3.18. The rejection of two vague graphs G_1 and G_2 is a vague graph.

Proof: Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs on crisp graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, and let $((m_1, m_2)(n_1, n_2)) \in E_1 \times E_2$. Then by Definition 3.16 we have the following:

(i) If $m_1 = n_1$ and $m_2 n_2 \notin E_2$, then

$$\begin{aligned} (T_{N_1}|T_{N_2})((m_1, m_2)(n_1, n_2)) &= \min\{T_{M_1}(m_1), T_{M_2}(m_2), T_{M_2}(n_2)\} \\ &= \min\{\min\{T_{M_1}(m_1), T_{M_2}(m_2)\}, \min\{T_{M_1}(n_1), T_{M_2}(n_2)\}\} \\ &= \min\{(T_{M_1}|T_{M_2})(m_1, m_2), (T_{M_1}|T_{M_2})(n_1, n_2)\}, \\ (F_{N_1}|F_{N_2})((m_1, m_2)(n_1, n_2)) &= \max\{F_{M_1}(m_1), F_{M_2}(m_2), F_{M_2}(n_2)\} \\ &= \max\{\max\{F_{M_1}(m_1), F_{M_2}(m_2)\}, \max\{F_{M_1}(n_1), F_{M_2}(n_2)\}\} \\ &= \max\{(F_{M_1}|F_{M_2})(m_1, m_2), (F_{M_1}|F_{M_2})(n_1, n_2)\}. \end{aligned}$$

(ii) If $m_2 = n_2$ and $m_1 n_1 \notin E_1$, then

$$\begin{aligned} (T_{N_1}|T_{N_2})((m_1, m_2)(n_1, n_2)) &= \min\{T_{M_1}(m_1), T_{M_1}(n_1), T_{M_2}(m_2)\} \\ &= \min\{\min\{T_{M_1}(m_1), T_{M_2}(m_2)\}, \min\{T_{M_1}(n_1), T_{M_2}(n_2)\}\} \\ &= \min\{(T_{M_1}|T_{M_2})(m_1, m_2), (T_{M_1}|T_{M_2})(n_1, n_2)\}, \\ (F_{N_1}|F_{N_2})((m_1, m_2)(n_1, n_2)) &= \max\{F_{M_1}(m_1), F_{M_1}(n_1), F_{M_2}(m_2)\} \\ &= \max\{\max\{F_{M_1}(m_1), F_{M_2}(m_2)\}, \max\{F_{M_1}(n_1), F_{M_2}(n_2)\}\} \\ &= \max\{(F_{M_1}|F_{M_2})(m_1, m_2), (F_{M_1}|F_{M_2})(n_1, n_2)\}. \end{aligned}$$

(iii) If $m_1 n_1 \notin E_1$ and $m_2 n_2 \notin E_2$, then

$$\begin{aligned} (T_{N_1}|T_{N_2})((m_1, m_2)(n_1, n_2)) &= \min\{T_{M_1}(m_1), T_{M_1}(n_1), T_{M_2}(m_2), T_{M_2}(n_2)\} \\ &= \min\{\min\{T_{M_1}(m_1), T_{M_2}(m_2)\}, \min\{T_{M_1}(n_1), T_{M_2}(n_2)\}\} \\ &= \min\{(T_{M_1}|T_{M_2})(m_1, m_2), (T_{M_1}|T_{M_2})(n_1, n_2)\}, \\ (F_{N_1}|F_{N_2})((m_1, m_2)(n_1, n_2)) &= \max\{F_{M_1}(m_1), F_{M_1}(n_1), F_{M_2}(m_2), F_{M_2}(n_2)\} \\ &= \max\{\max\{F_{M_1}(m_1), F_{M_2}(m_2)\}, \max\{F_{M_1}(n_1), F_{M_2}(n_2)\}\} \\ &= \max\{(F_{M_1}|F_{M_2})(m_1, m_2), (F_{M_1}|F_{M_2})(n_1, n_2)\}. \end{aligned}$$

Therefore, $G_1|G_2 = (M_1|M_2, N_1|N_2)$ is a vague graph. \square

Remark 3.4. The rejection of two complete vague graphs $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ is a complete vague graph.

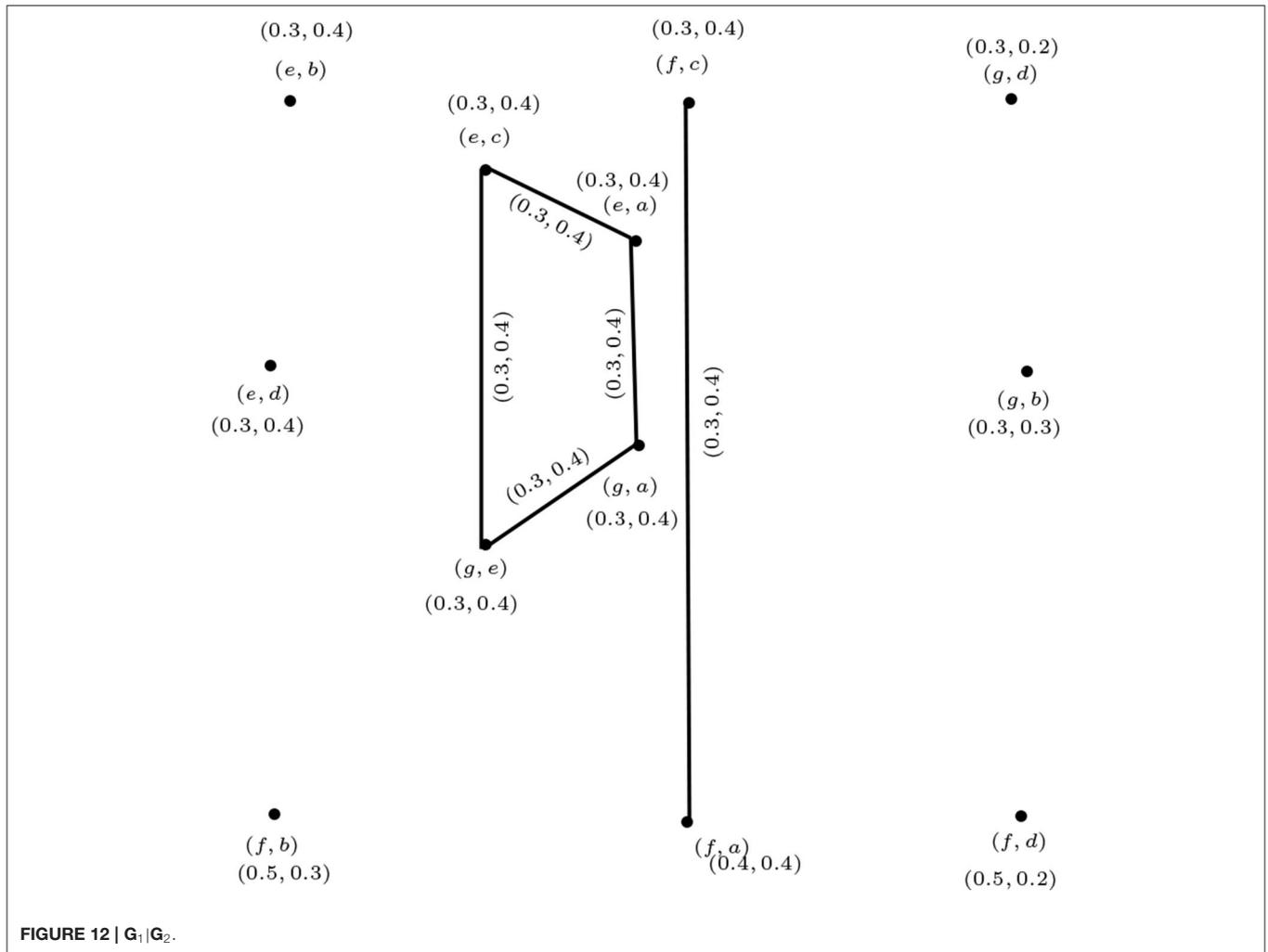


FIGURE 12 | $G_1|G_2$.

Definition 3.19. Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs. For any vertex $(m_1, m_2) \in V_1 \times V_2$ we define

$$\begin{aligned}
 (d_T)_{G_1|G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (T_{N_1}|T_{N_2})((m_1, m_2)(n_1, n_2)) \\
 &= \sum_{m_1=n_1, m_2 n_2 \notin E_2} \min\{T_{M_1}(m_1), T_{M_2}(m_2), T_{M_2}(n_2)\} \\
 &+ \sum_{m_2=n_2, m_1 n_1 \notin E_1} \min\{T_{M_1}(m_1), T_{M_1}(n_1), T_{M_2}(m_2)\} \\
 &+ \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \notin E_2} \min\{T_{M_1}(m_1), T_{M_1}(n_1), T_{M_2}(m_2), T_{M_2}(n_2)\},
 \end{aligned}$$

$$\begin{aligned}
 (d_F)_{G_1|G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (F_{N_1}|F_{N_2})((m_1, m_2)(n_1, n_2)) \\
 &= \sum_{m_1=n_1, m_2 n_2 \notin E_2} \max\{F_{M_1}(m_1), F_{M_2}(m_2), F_{M_2}(n_2)\} \\
 &+ \sum_{m_2=n_2, m_1 n_1 \notin E_1} \max\{F_{M_1}(m_1), F_{M_1}(n_1), F_{M_2}(m_2)\} \\
 &+ \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \notin E_2} \max\{F_{M_1}(m_1), F_{M_1}(n_1), F_{M_2}(m_2), F_{M_2}(n_2)\}.
 \end{aligned}$$

Definition 3.20. Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs. For any vertex $(m_1, m_2) \in V_1 \times V_2$ we define

$$\begin{aligned}
 (td_T)_{G_1|G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (T_{N_1}|T_{N_2})((m_1, m_2)(n_1, n_2)) \\
 &+ (T_{M_1}|T_{M_2})(m_1, m_2) \\
 &= \sum_{m_1=n_1, m_2 n_2 \notin E_2} \min\{T_{M_1}(m_1), T_{M_2}(m_2), T_{M_2}(n_2)\} \\
 &+ \sum_{m_2=n_2, m_1 n_1 \notin E_1} \min\{T_{M_1}(m_1), T_{M_1}(n_1), T_{M_2}(m_2)\} \\
 &+ \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \notin E_2} \min\{T_{M_1}(m_1), T_{M_1}(n_1), T_{M_2}(m_2), T_{M_2}(n_2)\},
 \end{aligned}$$

$$\begin{aligned}
 (td_F)_{G_1|G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (F_{N_1}|F_{N_2})((m_1, m_2)(n_1, n_2)) \\
 &+ (F_{M_1}|F_{M_2})(m_1, m_2) \\
 &= \sum_{m_1=n_1, m_2 n_2 \notin E_2} \max\{F_{M_1}(m_1), F_{M_2}(m_2), F_{M_2}(n_2)\} \\
 &+ \sum_{m_2=n_2, m_1 n_1 \notin E_1} \max\{F_{M_1}(m_1), F_{M_1}(n_1), F_{M_2}(m_2)\} \\
 &+ \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \notin E_2} \max\{F_{M_1}(m_1), F_{M_1}(n_1), F_{M_2}(m_2), F_{M_2}(n_2)\}.
 \end{aligned}$$

Example 3.21. In this example we find the degree and total degree of the vertex (e, a) in Example 3.17:

$$\begin{aligned} (d_T)_{G_1|G_2}(e, a) &= \min\{T_{M_2}(e), T_{M_1}(a), T_{M_1}(c)\} \\ &+ \min\{T_{M_2}(e), T_{M_1}(a), T_{M_2}(g)\} \\ &= \min\{0.3, 0.4, 0.3\} + \min\{0.3, 0.4, 0.3\} \\ &= 0.3 + 0.3 \\ &= 0.6, \end{aligned}$$

$$\begin{aligned} (d_F)_{G_1|G_2}(e, a) &= \max\{F_{M_2}(e), F_{M_1}(a), F_{M_1}(c)\} \\ &+ \max\{F_{M_2}(e), F_{M_1}(a), F_{M_2}(g)\} \\ &= \max\{0.4, 0.4, 0.4\} + \max\{0.4, 0.4, 0.2\} \\ &= 0.4 + 0.4 \\ &= 0.8. \end{aligned}$$

Therefore, $d_{G_1|G_2}(a, c) = (0.6, 0.8)$.

In addition, by the definition of the total vertex degree in the maximal product,

$$\begin{aligned} (td_T)_{G_1|G_2}(e, a) &= \min\{T_{M_2}(e), T_{M_1}(a), T_{M_1}(c)\} \\ &+ \min\{T_{M_2}(e), T_{M_1}(a), T_{M_2}(g)\} \\ &+ \min\{T_{M_2}(e), T_{M_1}(a)\} \\ &= \min\{0.3, 0.4, 0.3\} + \min\{0.3, 0.4, 0.3\} \\ &+ \min\{0.3, 0.4\} \\ &= 0.3 + 0.3 + 0.3 \\ &= 0.9, \end{aligned}$$

$$\begin{aligned} (td_F)_{G_1|G_2}(e, a) &= \max\{F_{M_2}(e), F_{M_1}(a), F_{M_1}(c)\} \\ &+ \max\{F_{M_2}(e), F_{M_1}(a), F_{M_2}(g)\} \\ &+ \max\{F_{M_2}(e), F_{M_1}(a)\} \\ &= \max\{0.4, 0.4, 0.4\} + \max\{0.4, 0.4, 0.2\} \\ &+ \max\{0.3, 0.4\} \\ &= 0.4 + 0.4 + 0.4 \\ &= 1.2. \end{aligned}$$

Therefore, $td_{G_1|G_2}(a, c) = (0.9, 1.2)$.

Similarly, we can find the degree and the total degree of all vertices in $G_1|G_2$.

Definition 3.22. The symmetric difference $G_1 \oplus G_2 = (M_1 \oplus M_2, N_1 \oplus N_2)$ of two vague graphs $G_1 = (M_1, N_1)$ and $G_2 =$

(M_2, N_2) is defined as follows:

$$\begin{aligned} \text{(i)} \quad (T_{M_1} \oplus T_{M_2})((m_1, m_2)) &= \min\{T_{M_1}(m_1), T_{M_2}(m_2)\}, \\ (F_{M_1} \oplus F_{M_2})((m_1, m_2)) &= \max\{F_{M_1}(m_1), F_{M_2}(m_2)\} \\ &\forall (m_1, m_2) \in (V_1 \times V_2); \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad (T_{N_1} \oplus T_{N_2})((m, m_2)(m, n_2)) &= \min\{T_{M_1}(m), T_{N_2}(m_2 n_2)\}, \\ (F_{N_1} \oplus F_{N_2})((m, m_2)(m, n_2)) &= \max\{F_{M_1}(m), F_{N_2}(m_2 n_2)\} \\ &\forall m \in V_1 \text{ and } m_2 n_2 \in E_2; \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad (T_{N_1} \oplus T_{N_2})((m_1, z)(n_1, z)) &= \min\{T_{N_1}(m_1 n_1), T_{M_2}(z)\}, \\ (F_{N_1} \oplus F_{N_2})((m_1, z)(n_1, z)) &= \max\{F_{N_1}(m_1 n_1), F_{M_2}(z)\} \\ &\forall z \in V_2 \text{ and } m_1 n_1 \in E_1; \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad (T_{N_1} \oplus T_{N_2})((m_1, m_2)(n_1, n_2)) &= \begin{cases} \min\{T_{M_1}(m_1), T_{M_1}(n_1), T_{N_2}(m_2 n_2)\} \\ \quad \forall m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2, \\ \min\{T_{M_2}(m_2), T_{M_2}(n_2), T_{N_1}(m_1 n_1)\} \\ \quad \forall m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2, \end{cases} \\ (F_{N_1} \oplus F_{N_2})((m_1, m_2)(n_1, n_2)) &= \begin{cases} \max\{F_{M_1}(m_1), F_{M_1}(n_1), F_{N_2}(m_2 n_2)\} \\ \quad \forall m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2, \\ \max\{F_{M_2}(m_2), F_{M_2}(n_2), F_{N_1}(m_1 n_1)\} \\ \quad \forall m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2. \end{cases} \end{aligned}$$

Example 3.23. Consider the vague graphs G_1 and G_2 as in Figures 13, 14. The symmetric difference of G_1 and G_2 , i.e., $G_1 \oplus G_2$, is shown in Figure 15.

For the vertex (a, f) , we find the membership and non-membership values as follows:

$$\begin{aligned} (T_{M_1} \oplus T_{M_2})((a, f)) &= \min\{T_{M_1}(a), T_{M_2}(f)\} \\ &= \min\{0.2, 0.3\} = 0.2, \\ (F_{M_1} \oplus F_{M_2})((a, f)) &= \max\{F_{M_1}(a), F_{M_2}(f)\} \\ &= \max\{0.4, 0.3\} = 0.4 \end{aligned}$$

for $a \in V_1$ and $f \in V_2$.

For the edge $(a, d)(a, e)$, the membership and non-membership values are given by

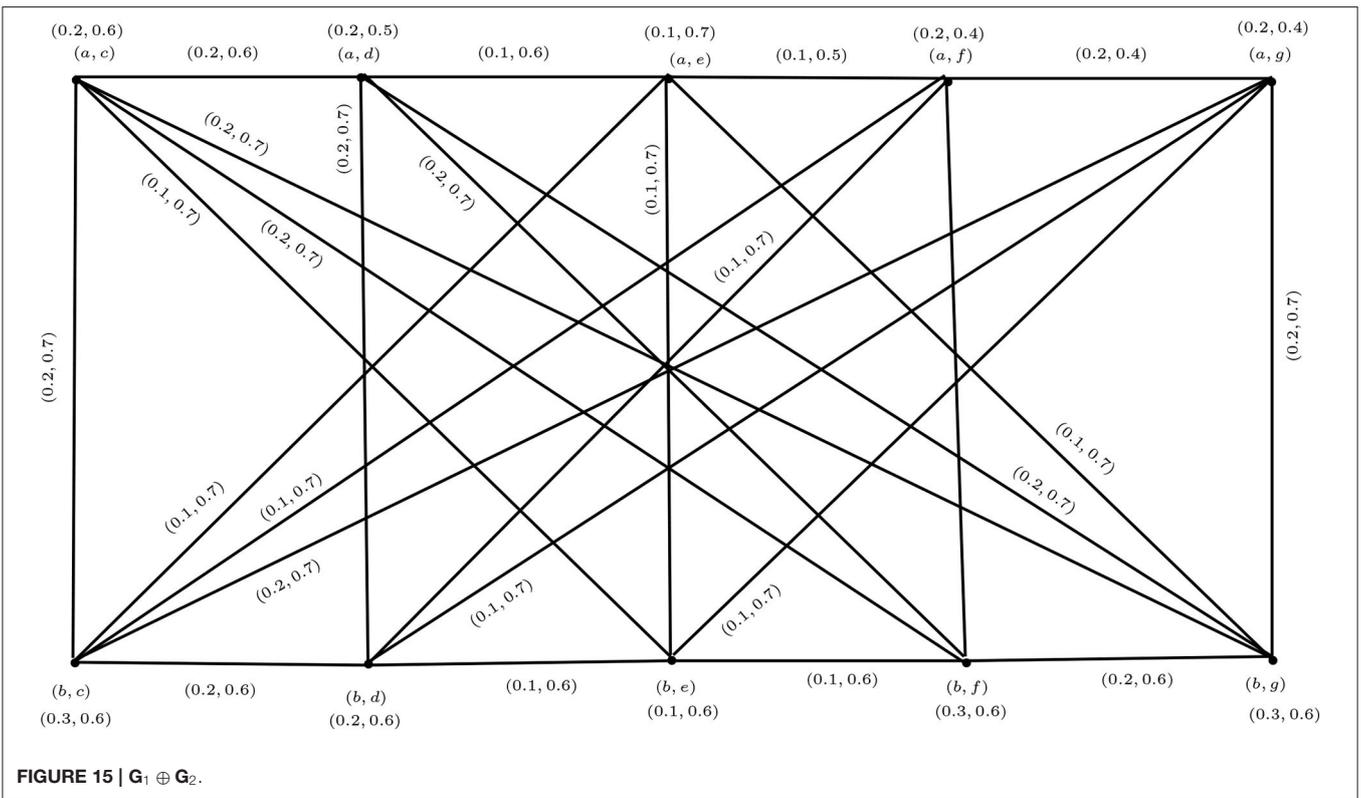
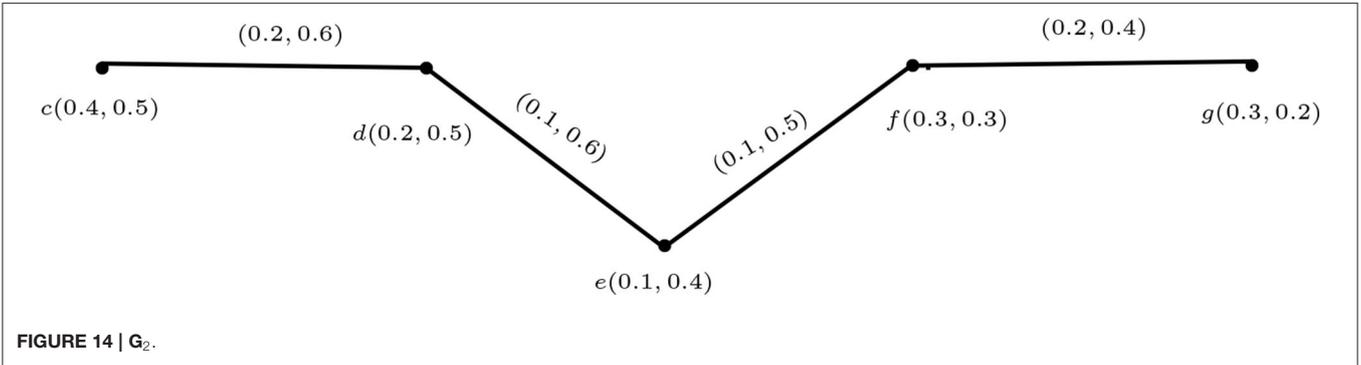
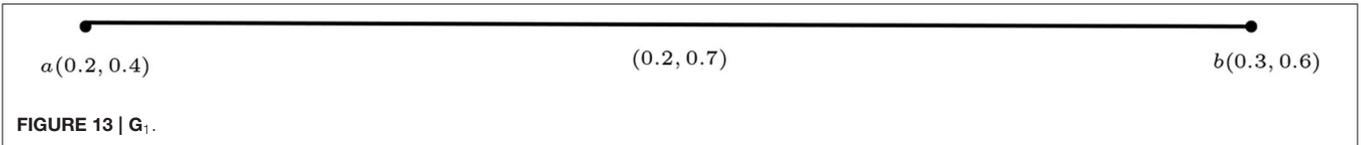
$$\begin{aligned} (T_{N_1} \oplus T_{N_2})((a, d)(a, e)) &= \min\{T_{M_1}(a), T_{N_2}(de)\} \\ &= \min\{0.2, 0.1\} = 0.1, \\ (F_{N_1} \oplus F_{N_2})((a, d)(a, e)) &= \max\{F_{M_1}(a), F_{N_2}(de)\} \\ &= \max\{0.4, 0.6\} = 0.6 \end{aligned}$$

for $a \in V_1$ and $de \in E_2$.

For the edge $(a, d)(b, d)$ we have

$$\begin{aligned} (T_{N_1} \oplus T_{N_2})((a, d)(b, d)) &= \min\{T_{N_1}(ab), T_{M_2}(d)\} \\ &= \min\{0.2, 0.2\} = 0.2, \\ (F_{N_1} \oplus F_{N_2})((a, d)(b, d)) &= \max\{F_{N_1}(ab), F_{M_2}(d)\} \\ &= \max\{0.7, 0.4\} = 0.7 \end{aligned}$$

for $ab \in E_1$ and $d \in V_2$.



For the edge $(a, c)(b, f)$, the membership and non-membership values are

$$(T_{N_1} \oplus T_{N_2})((a, c)(b, f)) = \min\{T_{M_2}(c), T_{M_2}(f), T_{N_1}(ab)\} = \min\{0.4, 0.3, 0.2\} = 0.2,$$

$$(F_{N_1} \oplus F_{N_2})((a, c)(b, f)) = \max\{F_{M_2}(c), F_{M_2}(f), F_{N_1}(ab)\} = \max\{0.5, 0.3, 0.7\} = 0.7$$

for $ab \in E_1$ and $cf \notin E_2$.

In the same way, we can find the membership and non-membership values for all remaining vertices and edges.

Proposition 3.24. The symmetric difference of two vague graphs G_1 and G_2 is a vague graph.

Proof: Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs on crisp graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, and let $((m_1, m_2)(n_1, n_2)) \in E_1 \times E_2$. Then by Definition 3.22 we have the following cases:

(i) If $m_1 = n_1 = m$, then

$$\begin{aligned} (T_{N_1} \oplus T_{N_2})((m, m_2)(m, n_2)) &= \min\{T_{M_1}(m), T_{N_2}(m_2 n_2)\} \\ &\leq \min\{T_{M_1}(m), \min\{T_{M_2}(m_2), T_{M_2}(n_2)\}\} \\ &= \min\{\min\{T_{M_1}(m), T_{M_2}(m_2)\}, \min\{T_{M_1}(m), T_{M_2}(n_2)\}\} \\ &= \min\{(T_{M_1} \oplus T_{M_2})(m, m_2), (T_{M_1} \oplus T_{M_2})(m, n_2)\}, \\ (F_{N_1} \oplus F_{N_2})((m, m_2)(m, n_2)) &= \max\{F_{M_1}(m), F_{N_2}(m_2 n_2)\} \\ &\geq \max\{F_{M_1}(m), \max\{F_{M_2}(m_2), F_{M_2}(n_2)\}\} \\ &= \max\{\max\{F_{M_1}(m), F_{M_2}(m_2)\}, \max\{F_{M_1}(m), F_{M_2}(n_2)\}\} \\ &= \max\{(F_{M_1} \oplus F_{M_2})(m, m_2), (F_{M_1} \oplus F_{M_2})(m, n_2)\}. \end{aligned}$$

(ii) If $m_2 = n_2 = z$, then

$$\begin{aligned} (T_{N_1} \oplus T_{N_2})((m_1, z)(n_1, z)) &= \min\{T_{N_1}(m_1 n_1), T_{M_2}(z)\} \\ &\leq \min\{\min\{T_{N_1}(m_1 n_1), T_{M_2}(z)\} \\ &= \min\{\min\{T_{M_1}(m_1), T_{M_2}(z)\}, \min\{T_{M_1}(n_1), T_{M_2}(z)\}\} \\ &= \min\{(T_{M_1} \oplus T_{M_2})(m_1, z), (T_{M_1} \oplus T_{M_2})(n_1, z)\}, \\ (F_{N_1} \oplus F_{N_2})((m_1, z)(n_1, z)) &= \max\{F_{N_1}(m_1 n_1), F_{M_2}(z)\} \\ &\geq \max\{\max\{F_{N_1}(m_1 n_1), F_{M_2}(z)\} \\ &= \max\{\max\{F_{M_1}(m_1), F_{M_2}(z)\}, \max\{F_{M_1}(n_1), F_{M_2}(z)\}\} \\ &= \max\{(F_{M_1} \oplus F_{M_2})(m_1, z), (F_{M_1} \oplus F_{M_2})(n_1, z)\}. \end{aligned}$$

(iii) If $m_1 n_1 \notin E_1$ and $m_2 n_2 \in E_2$, then

$$\begin{aligned} (T_{N_1} \oplus T_{N_2})((m_1, m_2)(n_1, n_2)) &= \min\{T_{M_1}(m_1), T_{M_1}(n_1), T_{N_2}(m_2 n_2)\} \\ &\leq \min\{T_{M_1}(m_1), T_{M_1}(n_1), \min\{T_{M_2}(m_2), T_{M_2}(n_2)\}\} \\ &= \min\{\min\{T_{M_1}(m_1), T_{M_2}(m_2)\}, \{T_{M_1}(m_1), T_{M_2}(n_2)\}\} \\ &= \min\{(T_{M_1} \oplus T_{M_2})(m_1, m_2), (T_{M_1} \oplus T_{M_2})(n_1, n_2)\}, \\ (F_{N_1} \oplus F_{N_2})((m_1, m_2)(n_1, n_2)) &= \max\{F_{M_1}(m_1), F_{M_1}(n_1), F_{N_2}(m_2 n_2)\} \\ &\geq \max\{F_{M_1}(m_1), F_{M_1}(n_1), \max\{F_{M_2}(m_2), F_{M_2}(n_2)\}\} \\ &= \max\{\max\{F_{M_1}(m_1), F_{M_2}(m_2)\}, \{F_{M_1}(m_1), F_{M_2}(n_2)\}\} \\ &= \max\{(F_{M_1} \oplus F_{M_2})(m_1, m_2), (F_{M_1} \oplus F_{M_2})(n_1, n_2)\}. \end{aligned}$$

(iv) If $m_1 n_1 \in E_1$ and $m_2 n_2 \notin E_2$, then

$$\begin{aligned} (T_{N_1} \oplus T_{N_2})((m_1, m_2)(n_1, n_2)) &= \min\{T_{M_2}(m_2), T_{M_2}(n_2), T_{N_1}(m_1 n_1)\} \\ &\leq \min\{T_{M_2}(m_2), T_{M_2}(n_2), \min\{T_{M_1}(m_1), T_{M_1}(n_1)\}\} \\ &= \min\{\min\{T_{M_1}(m_1), T_{M_2}(m_2)\}, \{T_{M_1}(n_1), T_{M_2}(n_2)\}\} \\ &= \min\{(T_{M_1} \oplus T_{M_2})(m_1, m_2), (T_{M_1} \oplus T_{M_2})(n_1, n_2)\}, \\ (F_{N_1} \oplus F_{N_2})((m_1, m_2)(n_1, n_2)) &= \max\{F_{M_2}(m_2), F_{M_2}(n_2), F_{N_1}(m_1 n_1)\} \\ &\geq \max\{F_{M_2}(m_2), F_{M_2}(n_2), \max\{F_{M_1}(m_1), F_{M_1}(n_1)\}\} \\ &= \max\{\max\{F_{M_2}(m_2), F_{M_1}(m_1)\}, \{F_{M_2}(m_2), F_{M_1}(n_1)\}\} \\ &= \max\{(F_{M_1} \oplus F_{M_2})(m_1, m_2), (F_{M_1} \oplus F_{M_2})(n_1, n_2)\}. \end{aligned}$$

Hence $G_1 \oplus G_2$ is a vague graph. \square

Remark 3.5. The symmetric difference of two connected vague graphs $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ is connected, because we include the case where $(m_1, m_2) \in E_1$ and $(n_1, n_2) \in E_2$ in the definition of the symmetric difference of two vague graphs.

Definition 3.25. Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs. For any vertex $(m_1, m_2) \in V_1 \times V_2$ we define

$$\begin{aligned} (d_T)_{G_1 \oplus G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (T_{N_1} \oplus T_{N_2})((m_1, m_2)(n_1, n_2)) \\ &= \sum_{m_1=n_1, m_2 n_2 \in E_2} \min\{T_{M_1}(m_1), T_{N_2}(m_2 n_2)\} \\ &\quad + \sum_{m_1 n_1 \in E_1, m_2=n_2} \min\{T_{N_1}(m_1 n_1), T_{M_2}(m_2)\} \\ &\quad + \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} \min\{T_{M_1}(m_1), T_{M_1}(n_1), T_{N_2}(m_2 n_2)\} \\ &\quad + \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} \min\{T_{N_1}(m_1 n_1), T_{M_2}(m_2), T_{M_2}(n_2)\} \end{aligned}$$

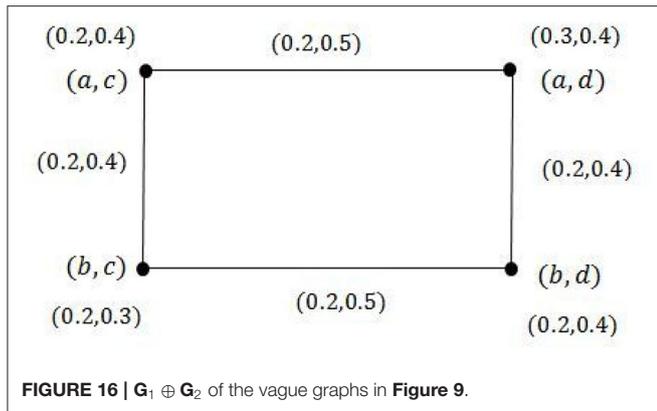
and

$$\begin{aligned} (d_F)_{G_1 \oplus G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (F_{N_1} \oplus F_{N_2})((m_1, m_2)(n_1, n_2)) \\ &= \sum_{m_1=n_1, m_2 n_2 \in E_2} \max\{F_{M_1}(m_1), F_{N_2}(m_2 n_2)\} \\ &\quad + \sum_{m_1 n_1 \in E_1, m_2=n_2} \max\{F_{N_1}(m_1 n_1), F_{M_2}(m_2)\} \\ &\quad + \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} \max\{F_{M_1}(m_1), F_{M_1}(n_1), F_{N_2}(m_2 n_2)\} \\ &\quad + \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} \max\{F_{N_1}(m_1 n_1), F_{M_2}(m_2), F_{M_2}(n_2)\}. \end{aligned}$$

Theorem 3.26. Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs. If $T_{M_1} \geq T_{N_2}$, $F_{M_1} \leq F_{N_2}$, $T_{M_2} \geq T_{N_1}$, and $F_{M_2} \leq F_{N_1}$, then for every $(m_1, m_2) \in V_1 \times V_2$ we have $(d)_{G_1 \oplus G_2}(m_1, m_2) = q(d)_{G_1}(m_1) + s(d)_{G_2}(m_2)$, where $s = |V_1| - (d)_{G_1}(m_1)$ and $q = |V_2| - (d)_{G_2}(m_2)$.

Proof: Using Definition 3.25,

$$\begin{aligned} (d_T)_{G_1 \oplus G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (T_{N_1} \oplus T_{N_2})((m_1, m_2)(n_1, n_2)) \\ &= \sum_{m_1=n_1, m_2 n_2 \in E_2} \min\{T_{M_1}(m_1), T_{N_2}(m_2 n_2)\} \\ &\quad + \sum_{m_1 n_1 \in E_1, m_2=n_2} \min\{T_{N_1}(m_1 n_1), T_{M_2}(m_2)\} \\ &\quad + \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} \min\{T_{M_1}(m_1), T_{M_1}(n_1), T_{N_2}(m_2 n_2)\} \\ &\quad + \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} \min\{T_{N_1}(m_1 n_1), T_{M_2}(m_2), T_{M_2}(n_2)\} \\ &= \sum_{m_2 n_2 \in E_2} T_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1} T_{N_1}(m_1 n_1) \\ &\quad + \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} T_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} T_{N_1}(m_1 n_1) \\ &= q(d_T)_{G_1}(m_1) + s(d_T)_{G_2}(m_2), \end{aligned}$$



$$\begin{aligned}
 (d_F)_{G_1 \oplus G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (F_{N_1} \oplus F_{N_2})((m_1, m_2)(n_1, n_2)) \\
 &= \sum_{m_1=n_1, m_2=n_2 \in E_2} \max\{F_{M_1}(m_1), F_{N_2}(m_2 n_2)\} \\
 &\quad + \sum_{m_1 n_1 \in E_1, m_2=n_2} \max\{F_{N_1}(m_1 n_1), F_{M_2}(m_2)\} \\
 &\quad + \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} \max\{F_{M_1}(m_1), F_{M_1}(n_1), F_{N_2}(m_2 n_2)\} \\
 &\quad + \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} \max\{F_{N_1}(m_1 n_1), F_{M_2}(m_2), F_{M_2}(n_2)\} \\
 &= \sum_{m_2 n_2 \in E_2} F_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1} F_{N_1}(m_1 n_1) \\
 &\quad + \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} F_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} F_{N_1}(m_1 n_1) \\
 &= q(d_T)_{G_1}(m_1) + s(d_F)_{G_2}(m_2),
 \end{aligned}$$

and hence the result is proved. \square

Example 3.27. Consider the two vague graphs G_1 and G_2 in **Figure 9** and their symmetric difference in **Figure 16**. In **Figure 9**, $T_{M_1} \geq T_{N_2}$, $F_{M_1} \leq F_{N_2}$, $T_{M_2} \geq T_{N_1}$, and $F_{M_2} \leq F_{N_1}$. Then, the total degree of a vertex in the symmetric difference is calculated by the following formula:

$$\begin{aligned}
 (d_T)_{G_1 \oplus G_2}(m_1, m_2) &= q(d_T)_{G_1}(m_1) + s(d_T)_{G_2}(m_2), \\
 (d_F)_{G_1 \oplus G_2}(m_1, m_2) &= q(d_F)_{G_1}(m_1) + s(d_F)_{G_2}(m_2).
 \end{aligned}$$

Using the formula we find that

$$\begin{aligned}
 (d_T)_{G_1 \oplus G_2}(a, c) &= 1 \cdot (0.2) + 1 \cdot (0.2) = 0.4, \\
 (d_F)_{G_1 \oplus G_2}(a, c) &= 1 \cdot (0.4) + 1 \cdot (0.5) = 0.9, \\
 (d_T)_{G_1 \oplus G_2}(a, d) &= 1 \cdot (0.2) + 1 \cdot (0.2) = 0.4, \\
 (d_F)_{G_1 \oplus G_2}(a, d) &= 1 \cdot (0.4) + 1 \cdot (0.5) = 0.9.
 \end{aligned}$$

Hence, $(d)_{G_1 \oplus G_2}(a, c) = (0.4, 0.9)$ and $(d)_{G_1 \oplus G_2}(a, d) = (0.4, 0.9)$.

In the same way, we can show that $(d)_{G_1 \oplus G_2}(b, c) = (d)_{G_1 \oplus G_2}(b, d) = (0.4, 0.9)$. Direct calculations give

$$\begin{aligned}
 (d_T)_{G_1 \oplus G_2}(a, c) &= 0.2 + 0.2 = 0.4, \\
 (d_F)_{G_1 \oplus G_2}(a, c) &= 0.4 + 0.5 = 0.9, \\
 (d_T)_{G_1 \oplus G_2}(a, d) &= 0.2 + 0.2 = 0.4, \\
 (d_F)_{G_1 \oplus G_2}(a, d) &= 0.4 + 0.5 = 0.9, \\
 (d_T)_{G_1 \oplus G_2}(b, c) &= 0.2 + 0.2 = 0.4, \\
 (d_F)_{G_1 \oplus G_2}(b, c) &= 0.4 + 0.5 = 0.9, \\
 (d_T)_{G_1 \oplus G_2}(b, d) &= 0.2 + 0.2 = 0.4, \\
 (d_F)_{G_1 \oplus G_2}(b, d) &= 0.4 + 0.5 = 0.9.
 \end{aligned}$$

It is obvious from the above that the degrees of vertices calculated using the formula and by the direct method are the same.

Definition 3.28. Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs. For any vertex $(m_1, m_2) \in V_1 \times V_2$ we define

$$\begin{aligned}
 (td_T)_{G_1 \oplus G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (T_{N_1} \oplus T_{N_2})((m_1, m_2)(n_1, n_2)) \\
 &\quad + (T_{M_1} \oplus T_{M_2})(m_1, m_2) \\
 &= \sum_{m_1=n_1, m_2 n_2 \in E_2} \min\{T_{M_1}(m_1), T_{N_2}(m_2 n_2)\} \\
 &\quad + \sum_{m_1 n_1 \in E_1, m_2=n_2} \min\{T_{N_1}(m_1 n_1), T_{M_2}(m_2)\} \\
 &\quad + \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} \min\{T_{M_1}(m_1), T_{M_1}(n_1), T_{N_2}(m_2 n_2)\} \\
 &\quad + \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} \min\{T_{N_1}(m_1 n_1), T_{M_2}(m_2), T_{M_2}(n_2)\} \\
 &\quad + \min\{T_{M_1}(m_1), T_{M_2}(m_2)\}
 \end{aligned}$$

and

$$\begin{aligned}
 (td_F)_{G_1 \oplus G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (F_{N_1} \oplus F_{N_2})((m_1, m_2)(n_1, n_2)) \\
 &\quad + (F_{M_1} \oplus F_{M_2})(m_1, m_2) \\
 &= \sum_{m_1=n_1, m_2 n_2 \in E_2} \max\{F_{M_1}(m_1), F_{N_2}(m_2 n_2)\} \\
 &\quad + \sum_{m_1 n_1 \in E_1, m_2=n_2} \max\{F_{N_1}(m_1 n_1), F_{M_2}(m_2)\} \\
 &\quad + \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} \max\{F_{M_1}(m_1), F_{M_1}(n_1), F_{N_2}(m_2 n_2)\} \\
 &\quad + \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} \max\{F_{N_1}(m_1 n_1), F_{M_2}(m_2), F_{M_2}(n_2)\} \\
 &\quad + \max\{F_{M_1}(m_1), F_{M_2}(m_2)\}.
 \end{aligned}$$

Example 3.29. In this example we find the degree and total degree of the vertex (a, e) in Example 3.23. We have

$$\begin{aligned}
 s &= |V_1| - (d)_{G_1}(a) \\
 &= 2 - 1 = 1
 \end{aligned}$$

and, similarly,

$$q = |V_2| - (d)_{G_2}(e) = 5 - 2 = 3.$$

Therefore

$$(d_T)_{G_1 \oplus G_2}(a, e) = q(d_T)_{G_1}(a) + s(d_T)_{G_2}(e) = 3(0.2) + 1(0.1 + 0.1) = 0.6 + 0.2 = 0.8,$$

$$(d_F)_{G_1 \oplus G_2}(a, e) = q(d_F)_{G_1}(a) + s(d_F)_{G_2}(e) = 3(0.4) + 1(0.6 + 0.5) = 1.2 + 1.1 = 2.3.$$

So

$$(d)_{G_1 \oplus G_2}(a, e) = (0.8, 2.3).$$

In addition, by Definition 3.28 we have

$$(td_T)_{G_1 \oplus G_2}(a, e) = q(td_T)_{G_1}(a) + s(td_T)_{G_2}(e) - (s - 1)T_{G_2}(e) - (q - 1)T_{G_1}(a) - \max\{T_{G_1}(a), T_{G_2}(e)\} = 3(0.2 + 0.2) + 1(0.1 + 0.1 + 0.1) - (1 - 1)(0.1) - (3 - 1)(0.2) - \max\{0.2, 0.1\} = 3(0.4) + 0.3 - 0.4 - 0.2 = 0.9,$$

$$(td_F)_{G_1 \oplus G_2}(a, e) = q(td_F)_{G_1}(a) + s(td_F)_{G_2}(e) - (s - 1)F_{G_2}(e) - (q - 1)F_{G_1}(a) - \min\{F_{G_1}(a), F_{G_2}(e)\} = 3(0.2 + 0.2) + 1(0.4 + 0.5 + 0.6) - (1 - 1)(0.4) - (3 - 1)(0.4) - \min\{0.4, 0.4\} = 3(0.4) + 1.5 - 0.8 - 0.4 = 1.5.$$

Therefore,

$$(td)_{G_1 \oplus G_2}(a, e) = (0.9, 1.5).$$

Similarly, we can find the degree and total degree of all vertices in $G_1 \oplus G_2$.

Theorem 3.30. Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs.

(i) If $T_{M_1} \geq T_{N_2}$ and $T_{M_2} \geq T_{N_1}$, then for all $(m_1, m_2) \in V_1 \times V_2$,

$$(td_T)_{G_1 \oplus G_2}(m_1, m_2) = q(td_T)_{G_1}(m_1) + s(td_T)_{G_2}(m_2) - (q - 1)T_{G_1}(m_1) - \max\{T_{G_1}(m_1), T_{G_2}(m_2)\}.$$

(ii) If $F_{M_1} \leq F_{N_2}$ and $F_{M_2} \leq F_{N_1}$, then for all $(m_1, m_2) \in V_1 \times V_2$,

$$(td_F)_{G_1 \oplus G_2}(m_1, m_2) = q(td_F)_{G_1}(m_1) + s(td_F)_{G_2}(m_2) - (q - 1)F_{G_1}(m_1) - \min\{F_{G_1}(m_1), F_{G_2}(m_2)\}.$$

Here $s = |V_1| - (d)_{G_1}(m_1)$ and $q = |V_2| - (d)_{G_2}(m_2)$.

Proof: For all $(m_1, m_2) \in V_1 \times V_2$ we have

$$\begin{aligned} (td_T)_{G_1 \oplus G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (T_{N_1} \oplus T_{N_2})((m_1, m_2)(n_1, n_2)) + (T_{M_1} \oplus T_{M_2})(m_1, m_2) \\ &= \sum_{m_1=n_1, m_2 n_2 \in E_2} \min\{T_{M_1}(m_1), T_{N_2}(m_2 n_2)\} + \sum_{m_1 n_1 \in E_1, m_2=n_2} \min\{T_{N_1}(m_1 n_1), T_{M_2}(m_2)\} \\ &\quad + \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} \min\{T_{M_1}(m_1), T_{M_1}(n_1), T_{N_2}(m_2 n_2)\} \\ &\quad + \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} \min\{T_{N_1}(m_1 n_1), T_{M_2}(m_2), T_{M_2}(n_2)\} \\ &\quad + \max\{T_{M_1}(m_1), T_{M_2}(m_2)\} \\ &= \sum_{m_2 n_2 \in E_2} T_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1} T_{N_1}(m_1 n_1) + \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} T_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} T_{N_1}(m_1 n_1) \\ &\quad + \max\{T_{M_1}(m_1), T_{M_2}(m_2)\} \\ &= \sum_{m_2 n_2 \in E_2} T_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1} T_{N_1}(m_1 n_1) + \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} T_{N_2}(m_2 n_2) \\ &\quad + \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} T_{N_1}(m_1 n_1) + T_{M_1}(m_1) + T_{M_2}(m_2) - \max\{T_{M_1}(m_1), T_{M_2}(m_2)\} \\ &= q(td_T)_{G_1}(m_1) + s(td_T)_{G_2}(m_2) - (q - 1)T_{G_1}(m_1) - \max\{T_{G_1}(m_1), T_{G_2}(m_2)\}, \end{aligned}$$

$$\begin{aligned} (td_F)_{G_1 \oplus G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (F_{N_1} \oplus F_{N_2})((m_1, m_2)(n_1, n_2)) + (F_{M_1} \oplus F_{M_2})(m_1, m_2) \\ &= \sum_{m_1=n_1, m_2 n_2 \in E_2} \max\{F_{M_1}(m_1), F_{N_2}(m_2 n_2)\} + \sum_{m_1 n_1 \in E_1, m_2=n_2} \max\{F_{N_1}(m_1 n_1), F_{M_2}(m_2)\} \\ &\quad + \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} \max\{F_{M_1}(m_1), F_{M_1}(n_1), F_{N_2}(m_2 n_2)\} \\ &\quad + \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} \max\{F_{N_1}(m_1 n_1), F_{M_2}(m_2), F_{M_2}(n_2)\} \\ &\quad + \min\{F_{M_1}(m_1), F_{M_2}(m_2)\} \\ &= \sum_{m_2 n_2 \in E_2} F_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1} F_{N_1}(m_1 n_1) + \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} F_{N_2}(m_2 n_2) \\ &\quad + \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} F_{N_1}(m_1 n_1) \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} F_{N_1}(m_1 n_1) \\
 &+ \min\{F_{M_1}(m_1), F_{M_2}(m_2)\} \\
 = &\sum_{m_2 n_2 \in E_2} F_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1} F_{N_1}(m_1 n_1) \\
 &+ \sum_{m_1 n_1 \notin E_1 \text{ and } m_2 n_2 \in E_2} F_{N_2}(m_2 n_2) \\
 &+ \sum_{m_1 n_1 \in E_1 \text{ and } m_2 n_2 \notin E_2} F_{N_1}(m_1 n_1) + F_{M_1}(m_1) + F_{M_2}(m_2) \\
 &\quad - \min\{F_{M_1}(m_1), F_{M_2}(m_2)\} \\
 = &q(td_F)_{G_1}(m_1) + s(td_F)_{G_2}(m_2) \\
 &- (q-1)F_{G_1}(m_1) - \min\{F_{G_1}(m_1), F_{G_2}(m_2)\},
 \end{aligned}$$

where $s = |V_1| - (d)_{G_1}(m_1)$ and $q = |V_2| - (d)_{G_2}(m_2)$. □

Example 3.31. In this example, we calculate the total degree of the vertices in Example 3.27.

The total degree of a vertex in the symmetric difference is given by

$$\begin{aligned}
 (td_T)_{G_1 \oplus G_2}(m_1, m_2) &= q(td_T)_{G_1}(m_1) + s(td_T)_{G_2}(m_2) \\
 &- (q-1)T_{G_1}(m_1) - \max\{T_{G_1}(m_1), T_{G_2}(m_2)\}, \\
 (td_F)_{G_1 \oplus G_2}(m_1, m_2) &= q(td_F)_{G_1}(m_1) + s(td_F)_{G_2}(m_2) \\
 &- (q-1)F_{G_1}(m_1) - \min\{F_{G_1}(m_1), F_{G_2}(m_2)\}.
 \end{aligned}$$

Using the above formula, we calculate

$$\begin{aligned}
 (td_T)_{G_1 \oplus G_2}(a, c) &= 1 \cdot (0.5) + 1 \cdot (0.4) - (1-1) \cdot (0.3) \\
 &- \max\{0.2, 0.3\} = 0.6, \\
 (td_F)_{G_1 \oplus G_2}(a, c) &= 1 \cdot (0.8) + 1 \cdot (0.8) - (1-1) \cdot (0.4) \\
 &- \min\{0.3, 0.4\} = 1.3, \\
 (td_T)_{G_1 \oplus G_2}(a, d) &= 1 \cdot (0.5) + 1 \cdot (0.5) - (1-1) \cdot (0.3) \\
 &- \max\{0.3, 0.3\} = 0.7, \\
 (td_F)_{G_1 \oplus G_2}(a, d) &= 1 \cdot (0.8) + 1 \cdot (0.9) - (1-1) \cdot (0.4) \\
 &- \min\{0.4, 0.4\} = 1.3, \\
 (td_T)_{G_1 \oplus G_2}(b, c) &= 1 \cdot (0.4) + 1 \cdot (0.4) - (1-1) \cdot (0.2) \\
 &- \max\{0.2, 0.2\} = 0.6, \\
 (td_F)_{G_1 \oplus G_2}(b, c) &= 1 \cdot (0.7) + 1 \cdot (0.8) - (1-1) \cdot (0.2) \\
 &- \min\{0.3, 0.3\} = 1.2, \\
 (td_T)_{G_1 \oplus G_2}(b, d) &= 1 \cdot (0.4) + 1 \cdot (0.5) - (1-1) \cdot (0.2) \\
 &- \max\{0.2, 0.3\} = 0.6, \\
 (td_F)_{G_1 \oplus G_2}(b, d) &= 1 \cdot (0.7) + 1 \cdot (0.9) - (1-1) \cdot (0.2) \\
 &- \min\{0.3, 0.4\} = 1.3.
 \end{aligned}$$

By direct calculations, we find

$$\begin{aligned}
 (td_T)_{G_1 \oplus G_2}(a, c) &= 0.2 + 0.2 + 0.2 = 0.6, \\
 (td_F)_{G_1 \oplus G_2}(a, c) &= 0.4 + 0.5 + 0.4 = 1.3, \\
 (td_T)_{G_1 \oplus G_2}(a, d) &= 0.2 + 0.2 + 0.3 = 0.7, \\
 (td_F)_{G_1 \oplus G_2}(a, d) &= 0.4 + 0.5 + 0.4 = 1.3, \\
 (td_T)_{G_1 \oplus G_2}(b, c) &= 0.2 + 0.2 + 0.2 = 0.6, \\
 (td_F)_{G_1 \oplus G_2}(b, c) &= 0.4 + 0.5 + 0.3 = 1.2, \\
 (td_T)_{G_1 \oplus G_2}(b, d) &= 0.2 + 0.2 + 0.2 = 0.6, \\
 (td_F)_{G_1 \oplus G_2}(b, d) &= 0.4 + 0.5 + 0.4 = 1.3.
 \end{aligned}$$

It is clear that the total degrees of vertices calculated using the formula and by the direct method are the same.

Definition 3.32. The residue product $G_1 \bullet G_2 = (M_1 \bullet M_2, N_1 \bullet N_2)$ of two vague graphs $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ is defined as follows:

- (i) $(T_{M_1} \bullet T_{M_2})((m_1, m_2)) = \max\{T_{M_1}(m_1), T_{M_2}(m_2)\},$
 $(F_{M_1} \bullet F_{M_2})((m_1, m_2)) = \min\{F_{M_1}(m_1), F_{M_2}(m_2)\}$
 $\forall (m_1, m_2) \in (V_1 \times V_2);$
- (ii) $(T_{N_1} \bullet T_{N_2})((m_1, m_2)(n_1, n_2)) = T_{N_1}(m_1 n_1),$
 $(F_{N_1} \bullet F_{N_2})((m_1, m_2)(n_1, n_2)) = F_{N_1}(m_1 n_1)$
 $\forall m_1 n_1 \in E_1, m_2 \neq n_2.$

Example 3.33. Consider the vague graphs G_1 and G_2 in **Figures 17, 18**. The residue product of G_1 and G_2 , i.e., $G_1 \bullet G_2$, is shown in **Figure 19**.

For the vertex (b, e) , we find the membership and non-membership values as follows:

$$\begin{aligned}
 (T_{M_1} \bullet T_{M_2})((b, e)) &= \max\{T_{M_1}(b), T_{M_2}(e)\} \\
 &= \max\{0.2, 0.2\} = 0.2, \\
 (F_{M_1} \bullet F_{M_2})((b, e)) &= \min\{F_{M_1}(b), F_{M_2}(e)\} \\
 &= \min\{0.7, 0.6\} = 0.6
 \end{aligned}$$

for $b \in V_1$ and $e \in V_2$.

For the edge $(a, c)(b, d)$, we calculate the membership and non-membership values to be

$$\begin{aligned}
 (T_{N_1} \bullet T_{N_2})((a, c)(b, d)) &= T_{N_1}(ab) = 0.1, \\
 (F_{N_1} \bullet F_{N_2})((a, c)(b, d)) &= F_{N_1}(ab) = 0.8
 \end{aligned}$$

for $ab \in E_1$ and $c \neq d$.

Similarly, we can find the membership and non-membership values for all the remaining vertices and edges.

Proposition 3.34. The residue product of two vague graphs G_1 and G_2 is a vague graph.

Proof: Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs on crisp graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$,

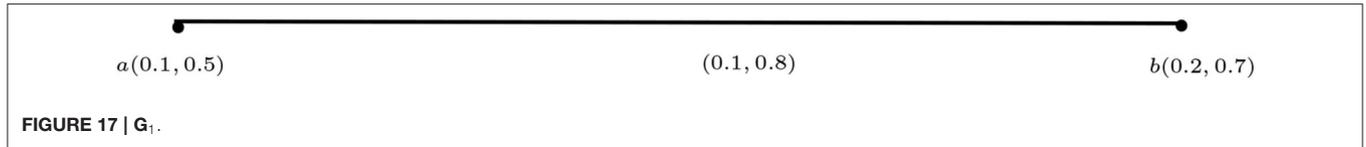


FIGURE 17 | G_1 .

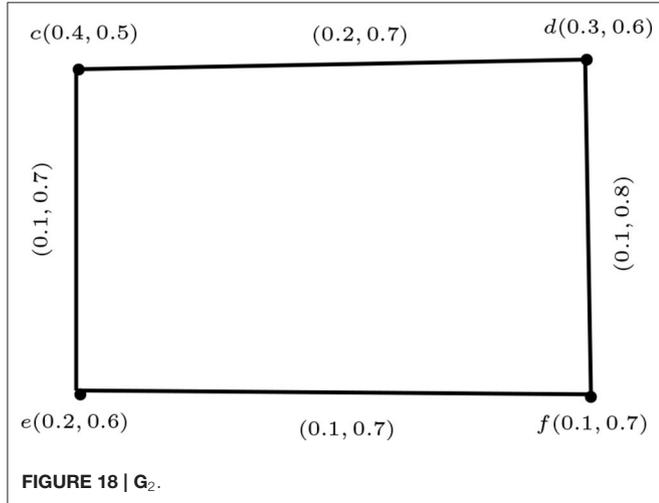


FIGURE 18 | G_2 .

respectively, and let $((m_1, m_2)(n_1, n_2)) \in E_1 \times E_2$. If $m_1 n_1 \in E_1$ and $m_2 \neq n_2$, then

$$\begin{aligned} (T_{N_1} \bullet T_{N_2})((m_1, m_2)(n_1, n_2)) &= T_{N_1}(m_1 n_1) \\ &\leq \min\{T_{M_1}(m_1), T_{M_1}(n_1)\} \\ &\leq \max\{\min\{T_{M_1}(m_1), T_{M_1}(n_1)\}, \min\{T_{M_2}(m_2), T_{M_2}(n_2)\}\} \\ &= \min\{\max\{T_{M_1}(m_1), T_{M_1}(n_1)\}, \max\{T_{M_2}(m_2), T_{M_2}(n_2)\}\} \\ &= \min\{(T_{M_1} \bullet T_{M_2})(m_1, m_2), (T_{M_1} \bullet T_{M_2})(n_1, n_2)\}, \\ (F_{N_1} \bullet F_{N_2})((m_1, m_2)(n_1, n_2)) &= F_{N_1}(m_1 n_1) \\ &\geq \max\{F_{M_1}(m_1), F_{M_1}(n_1)\} \\ &\geq \min\{\max\{F_{M_1}(m_1), F_{M_1}(n_1)\}, \max\{F_{M_2}(m_2), F_{M_2}(n_2)\}\} \\ &= \max\{\min\{F_{M_1}(m_1), F_{M_1}(n_1)\}, \min\{F_{M_2}(m_2), F_{M_2}(n_2)\}\} \\ &= \max\{(F_{M_1} \bullet F_{M_2})(m_1, m_2), (F_{M_1} \bullet F_{M_2})(n_1, n_2)\}, \end{aligned}$$

which completes the proof. \square

Definition 3.35. Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs. For any vertex $(m_1, m_2) \in V_1 \times V_2$ we define

$$\begin{aligned} (d_T)_{G_1 \bullet G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (T_{N_1} \bullet T_{N_2})((m_1, m_2)(n_1, n_2)) \\ &= \sum_{m_1 n_1 \in E_1, m_2 \neq n_2} T_{N_1}(m_1 n_1) \\ &= (d_T)_{G_1}(m_1), \end{aligned}$$

$$\begin{aligned} (d_F)_{G_1 \bullet G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (F_{N_1} \bullet F_{N_2})((m_1, m_2)(n_1, n_2)) \\ &= \sum_{m_1 n_1 \in E_1, m_2 \neq n_2} F_{N_1}(m_1 n_1) \\ &= (d_F)_{G_1}(m_1). \end{aligned}$$

Definition 3.36. Let $G_1 = (M_1, N_1)$ and $G_2 = (M_2, N_2)$ be two vague graphs. For any vertex $(m_1, m_2) \in V_1 \times V_2$ we define

$$\begin{aligned} (td_T)_{G_1 \bullet G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (T_{N_1} \bullet T_{N_2})((m_1, m_2)(n_1, n_2)) \\ &\quad + (T_{M_1} \bullet T_{M_2})(m_1, m_2) \\ &= \sum_{m_1 n_1 \in E_1, m_2 \neq n_2} T_{N_1}(m_1 n_1) + \min\{T_{M_1}(m_1), T_{M_2}(m_2)\} \\ &= \sum_{m_1 n_1 \in E_1, m_2 \neq n_2} T_{N_1}(m_1 n_1) + T_{M_1}(m_1) + T_{M_2}(m_2) \\ &\quad - \max\{T_{M_1}(m_1), T_{M_2}(m_2)\} \\ &= (td_T)_{G_1}(m_1) + T_{M_2}(m_2) - \max\{T_{M_1}(m_1), T_{M_2}(m_2)\}, \end{aligned}$$

$$\begin{aligned} (td_F)_{G_1 \bullet G_2}(m_1, m_2) &= \sum_{(m_1, m_2)(n_1, n_2) \in E_1 \times E_2} (F_{N_1} \bullet F_{N_2})((m_1, m_2)(n_1, n_2)) \\ &\quad + (F_{M_1} \bullet F_{M_2})(m_1, m_2) \\ &= \sum_{m_1 n_1 \in E_1, m_2 \neq n_2} F_{N_1}(m_1 n_1) + \max\{F_{M_1}(m_1), F_{M_2}(m_2)\} \\ &= \sum_{m_1 n_1 \in E_1, m_2 \neq n_2} F_{N_1}(m_1 n_1) + F_{M_1}(m_1) + F_{M_2}(m_2) \\ &\quad - \min\{F_{M_1}(m_1), F_{M_2}(m_2)\} \\ &= (td_F)_{G_1}(m_1) + F_{M_2}(m_2) - \min\{F_{M_1}(m_1), F_{M_2}(m_2)\}. \end{aligned}$$

Example 3.37. In this example we find the degree and total degree of the vertex (b, e) in Example 3.33:

$$\begin{aligned} (d_T)_{G_1 \bullet G_2}(b, e) &= (d_T)_{G_1}(b) \\ &= 0.1 + 0.1 = 0.2, \end{aligned}$$

$$\begin{aligned} (d_F)_{G_1 \bullet G_2}(b, e) &= (d_F)_{G_1}(b) \\ &= 0.8 + 0.8 = 1.6. \end{aligned}$$

Therefore,

$$(d)_{G_1 \bullet G_2}(b, e) = (0.2, 1.6).$$

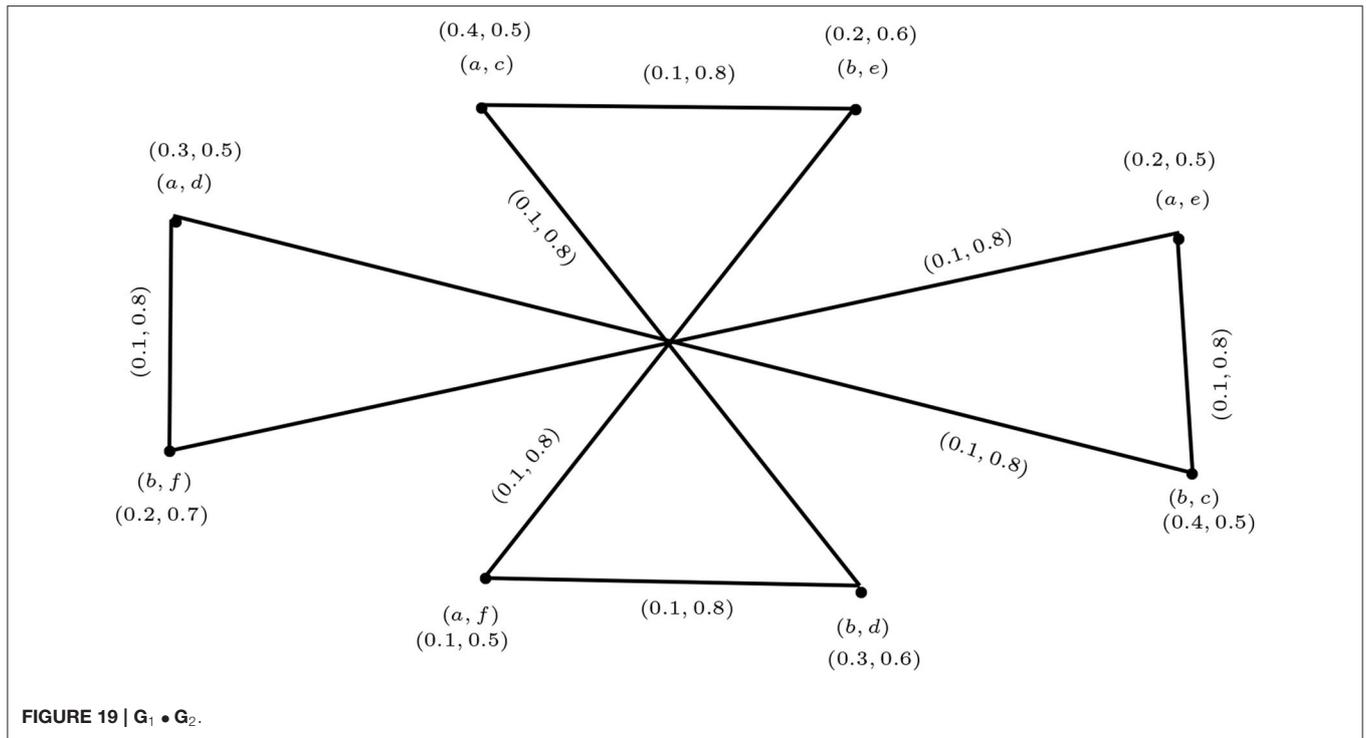


FIGURE 19 | $G_1 \bullet G_2$.

TABLE 1 | The vague relation $L(P \rightarrow S)$.

| L | Heartburn | Coughing | Pain during swallowing | Weight loss |
|---------|------------|------------|------------------------|-------------|
| Shahbaz | (0.8, 0) | (0.6, 0.2) | (0.2, 0.3) | (0.1, 0.6) |
| Faisal | (0.3, 0.1) | (0.3, 0.5) | (0.3, 0.6) | (0.2, 0.6) |
| Shoaib | (0.5, 0.2) | (0.1, 0.2) | (0.1, 0.8) | (0.3, 0.4) |
| Danish | (0.4, 0.4) | (0, 0.3) | (0.5, 0.4) | (0.5, 0.3) |

TABLE 2 | The vague relation $R(S \rightarrow D)$.

| R | Cancer of kidney | Cancer of colon | Cancer of breast | Cancer of bladder |
|------------------------|------------------|-----------------|------------------|-------------------|
| Heartburn | (0.1, 0.3) | (0.6, 0.1) | (0.2, 0.3) | (0.7, 0.1) |
| Coughing | (0.5, 0.4) | (0.3, 0.4) | (0.3, 0.5) | (0.2, 0.6) |
| Pain during swallowing | (0.3, 0.6) | (0.5, 0.2) | (0.6, 0.1) | (0.7, 0.1) |
| Weight loss | (0.2, 0.3) | (0.6, 0.3) | (0.5, 0.2) | (0.7, 0.1) |

The total degree of (b, e) is given by

$$\begin{aligned} (td_T)_{G_1 \bullet G_2}(b, e) &= (td_T)_{G_1}(b) + T_{M_2}(e) - \max\{T_{M_1}(b), T_{M_2}(e)\} \\ &= (0.2 + 0.2) + 0.2 - \max(0.2, 0.2) \\ &= 0.4, \end{aligned}$$

$$\begin{aligned} (td_F)_{G_1 \bullet G_2}(b, e) &= (td_F)_{G_1}(b) + F_{M_2}(e) - \min\{F_{M_1}(b), F_{M_2}(e)\} \\ &= (0.8 + 0.8) + 0.6 - \min(0.7, 0.6) \\ &= 1.6. \end{aligned}$$

Therefore,

$$(td)_{G_1 \bullet G_2}(b, e) = (0.4, 1.6).$$

Similarly, we can find the degree and total degree of all vertices in $G_1 \bullet G_2$.

4. APPLICATION OF VAGUE SETS TO MEDICAL DIAGNOSIS

Following the approach outlined by De et al. [18], we will apply vague sets to medical diagnosis by using a max-min-max composition in terms of vague relations. First, we use vague sets to define the disease symptoms. Then, we describe medical knowledge in terms of vague relations. Finally, we determine a diagnosis on the basis of vague relations. Consider four patients named Shahbaz, Shoaib, Faisal, and Danish, and define the set of patients $P = \{\text{Shahbaz, Shoaib, Faisal, Danish}\}$. Let the set of symptoms under consideration be $S = \{\text{heartburn, coughing, pain during swallowing, weight loss}\}$. A vague relation L is available from set P to set S , and this is summarized in Table 1.

Cancer is a group of dangerous and prevalent diseases, and represents one of humankind's greatest medical challenges. Many people are diagnosed with late-stage cancer that is difficult or

TABLE 3 | The composition $M(P \rightarrow D)$ of vague relations L and R .

| M | Cancer of kidney | Cancer of colon | Cancer of breast | Cancer of bladder |
|---------|------------------|-----------------|------------------|-------------------|
| Shahbaz | (0.5, 0.3) | (0.6, 0.1) | (0.2, 0.3) | (0.7, 0.1) |
| Faisal | (0.3, 0.3) | (0.3, 0.1) | (0.3, 0.3) | (0.3, 0.1) |
| Shoaib | (0.2, 0.3) | (0.5, 0.2) | (0.3, 0.3) | (0.5, 0.2) |
| Danish | (0.5, 0.3) | (0.5, 0.3) | (0.5, 0.3) | (0.5, 0.3) |

TABLE 4 | S_R , the best version of R determined by the formula $S_R = T_R - F_R \pi_R$.

| S_R | Cancer of kidney | Cancer of colon | Cancer of breast | Cancer of bladder |
|---------|------------------|-----------------|------------------|-------------------|
| Shahbaz | 0.44 | 0.57 | 0.05 | 0.68 |
| Faisal | 0.18 | 0.24 | 0.18 | 0.24 |
| Shoaib | 0.05 | 0.44 | 0.18 | 0.44 |
| Danish | 0.44 | 0.44 | 0.44 | 0.44 |

impossible to treat because of a lack of awareness of the disease symptoms. Mathematical models involving vague sets can be used to determine the most likely diagnosis given a set of symptoms that a patient presents with.

There are many different types of cancers; here we focus on a few of the more life-threatening kinds: (1) kidney cancer, (2) colon cancer, (3) breast cancer, and (4) bladder cancer. We define the set of diagnoses to be $D = \{\text{cancer of kidney, cancer of colon, cancer of breast, cancer of bladder}\}$. The vague relation $R(S \rightarrow D)$ from the set of symptoms to the set of diagnoses is given in **Table 2**. The composition $M(P \rightarrow D)$ of the vague relations L and R is shown in **Table 3**; it gives the diagnosis for each patient via the formulas

$$T_M(p_i, d_k) = \bigvee_{s \in S} [T_L(p_i, s) \wedge T_R(s, d_k)],$$

$$F_M(p_i, d_k) = \bigwedge_{s \in S} [F_L(p_i, s) \vee F_R(s, d_k)],$$

where p_i denotes the patients, d_k denotes the different diagnoses, $\wedge = \min$, and $\vee = \max$.

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Shown in **Table 4** is S_R , the best version of diagnosis for this set of patients, which is determined by the formula $S_R = T_R - F_R \pi_R$. It is very important because the max-min-max rule alone fails to provide exact information.

5. CONCLUSION

Compared with fuzzy models, vague models offer greater compatibility and flexibility. A vague graph is a type of extension of a fuzzy graph, and is used widely in the field of computer science. We have defined four new operations of a vague graph, called the maximal product, rejection, symmetric difference, and residue product. We have discussed their properties and provided examples on finding the degree of a vertex and the total degree of vertices of graphs that meet specific conditions. We have formulated and proved theorems for these graphs by using the concept of degree of a vertex and total degree of a vertex of a graph. Furthermore, we have presented an application of vague sets to the medical diagnosis of four types of cancer. In future work we will explore further properties relating to vague graphs and bipolar vague graphs.

DATA AVAILABILITY STATEMENT

All datasets generated for this study are included in the article/supplementary material.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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