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Canonical Quantum Gravity, Constructive QFT, and Renormalisation

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The canonical approach to quantum gravity has been put on a firm mathematical foundation in the recent decades. Even the quantum dynamics can be rigorously defined, however, due to the tremendously non-polynomial character of the gravitational interaction, the corresponding Wheeler–DeWitt operator-valued distribution suffers from quantisation ambiguities that need to be fixed. In a very recent series of works, we have employed methods from the constructive quantum field theory in order to address those ambiguities. Constructive QFT trades quantum fields for random variables and measures, thereby phrasing the theory in the language of quantum statistical physics. The connection to the canonical formulation is made via Osterwalder–Schrader reconstruction. It is well known in quantum statistics that the corresponding ambiguities in measures can be fixed using renormalisation. The associated renormalisation flow can thus be used to define a canonical renormalisation programme. The purpose of this article was to review and further develop these ideas and to put them into context with closely related earlier and parallel programmes.

Keywords: Canonical quantum gravity, lattice gauge field theory, constructive quantum field theory, renormalisation, Euclidian formulation

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1. INTRODUCTION

The canonical approach to quantum gravity has been initialised long time ago [1–14]. However, the mathematical foundations of the theory remained veiled due to the tremendous non-linearity of the gravitational interaction. This has much changed with the reformulation of general relativity as a Yang–Mills type gauge theory in terms of connection, rather than metric variables [15,16], and has culminated in a research programme now known as loop quantum gravity (LQG) (see e.g., Refs. 17–21 for monographs and recent reviews on the subject). The qualifier 'loop' stems from the fact that for gauge theories of Yang–Mills type, it has proved useful to formulate the theory in terms of holonomies of the connection along closed paths (loops) in order to maintain manifest gauge invariance. Such so-called (Wilson) loop variables are widely used, for instance, in (lattice) QCD [22].

LQG has succeeded in providing a rigorous mathematical framework: The representation theory of the canonical commutation relations and the * relations has been studied and a unique representation has been singled out [23–27] that allows for a unitary representation of the spatial diffeomorphism group. Moreover, the generators of temporal diffeomorphisms, sometimes referred to as Wheeler–DeWitt operators, could be rigorously quantised on the corresponding Hilbert space [28–32], and in contrast to the perturbative approach to quantum gravity [33, 34], no ultraviolet divergences were found. It should be emphasised that this was achieved 1) in the continuum, rather than on a lattice, that is, there is no artificial cut-off left over; 2) for the physical Lorentzian signature, rather than unphysical Euclidian one; and 3) non-perturbatively

and background independently, that is, one does not perturb around a classical background metric and then quantises the fluctuations which thus manifestly preserves the diffeomorphism covariance of all constructions.

However, the theory is not yet completed: Due to the tremendously non-polynomial nature of the gravitational interaction, the usual factor ordering ambiguity in the quantisation of operator-valued distributions which are non-linear in the fields is much more severe. Thus, the operators defined in Refs. 28–32 suffer from those ambiguities. Moreover, the following problem arises: In the classical theory, the canonical generators of space-time diffeomorphisms (i.e., their Hamiltonian vector fields) form a Lie algebroid (i.e., a Lie algebra except that the structure constants are replaced by structure functions on the phase space) known as the hypersurface algebroid [35]. The structure functions are themselves promoted to operator-valued distributions upon quantization; thus, it becomes even harder to find quantization of those generators such that the algebroid is represented without anomalies than it would be for an honest Lie algebra. Specifically, the commutator between two temporal diffeomorphism generators is supposed to 1) be proportional to a linear combination of spatial diffeomorphism generators with operator-valued distributions as coefficients and 2) in an ordering, such that the following holds: The image of any such commutator of a dense domain of vectors in the Hilbert space must be in the kernel of the space of spatially diffeomorphism-invariant distributions on that domain. In Ref. 37, it is shown that both conditions 1) and 2) hold; however, the coefficients in that linear combination do not qualify as quantisations of their classical counterpart. Thus, while the quantisation of the hypersurface algebroid closes, it does so with the wrong operator-valued distributions as coefficients.

Thus, the status of LQG can be summarised as follows:

As compared to Refs. 1–14, it is now possible to ask and answer precise questions about the mathematical consistency of the whole framework. As compared to the perturbative approach, the framework does not suffer from ultraviolet divergences and one does not have to worry about the convergence of a perturbation series due to the manifestly non-perturbative definition of LQG. However, just as in the perturbative approach, one needs further input in order to draw predictions from the theory, although of a different kind: In the perturbative approach, there are an infinite number of counter terms necessary due to non-perturbative non-renormalisability all of which come with coefficients that have to be measured, but one can argue that only a finite number of them is of interest for processes involving energies not exceeding a certain threshold (effective field theory point of view). In LQG, there are in principle infinitely many quantisation ordering prescriptions possible, each of which comes with definite coefficients in order to yield the correct naive continuum limit, but it is not clear which ordering to choose so that presently one resorts to the principle of least technical complexity.

Various proposals have been made in order to improve the situation. In Ref. 38, one exploits the fact that classically one can always trade a set of first-class constraints by a single weighted sum of their squares (called the master constraint). Since a single constraint always closes with itself and the weights can be chosen such that the master constraint commutes with spatial diffeomorphisms, one can now focus on the quantisation ambiguities involved in the master constraint without having to worry about anomalies. In Ref. 39, the case of general relativity coupled to perfect fluid matter was considered, which allows solving the constraints before quantisation so that the remaining quantisation ambiguity now only rests in the corresponding physical Hamiltonian that drives the time evolution of the physical (i.e., space-time diffeomorphism-invariant) observables. In Refs. 40–42, the constraints are quantised on a suitable space of distributions with respect to a dense domain of the Hilbert space, rather than the Hilbert space itself in order to find a representation of the hypersurface algebroid directly on that space of distributions which would at least partially fix the aforementioned ordering ambiguity.

It transpires that additional input is necessary in order to fix the quantisation ambiguity in the dynamics of LQG and thus to complete the definition of the theory. This would also put additional faith in applications of LQG, for instance to quantum cosmology [43–46] (where the amount of ambiguity is drastically reduced) which are believed to be approximations of LQG by enabling to make the connection between LQG and those approximations precise including an error control (see Refs. 47–53 for recent progress in that respect). In the recent proposal [54–57] which we intend to review in this article, the authors were inspired by Wilson's observation [54–57] that renormalisation methods help identify among the principally infinitely many interaction terms in Hamiltonians relevant for condensed matter physics the finitely many relevant ones that need to be measured. This insight implies that a theory may be perturbatively non-renomalisable but non-perturbatively renormalisable, also known as asymptotically safe [58]. The asymptotic safety approach to quantum gravity for Euclidian [59–68] and Lorentzian signature [69, 70] precisely rests on that idea and has received much attention recently. In fact, there is much in common between our proposal and asymptotically safe quantum gravity (especially for Lorentzian signature), and we will have the opportunity to spell out more precisely points of contact in the longer version of this article [196].

Also, there is a large body of work on renormalisation [71–75] in the so-called spin foam approach [85–92] and the related group field theory [76–81] and tensor model[1] [82–84] approach to quantum gravity. The spin foam approach is loosely connected to LQG in the following sense: The states of the Hilbert space underlying LQG are labelled by collections of loops, that is, 3D

[1]In principle, any field theory with a polynomial Lagrangian can be written as a (coloured) tensor model as follows: Pick any orthonormal basis with respect to the measure appearing in the action, expand the field in that basis, call the expansion coefficients a coloured (by the space-time or internal indices) tensor in an infinite-dimensional ℓ_2 space, and call the integral over polynomials in those basis functions that appear in the action upon expanding the fields interaction terms of those tensors. If the basis carries labels in $\mathbb{N}^{nd}$, we obtain a coloured tensor model with tensors of rank n.

graphs. A spin foam is an operator that maps such states excited on a graph to states excited on another graph. The operator depends on a specific class of 4D cell complex (foam) such that its boundary 3D complex is dual to the union of the two graphs corresponding to the incoming and outgoing Hilbert spaces. The operator is supposed to form the rigging map [93] of LQG, that is, a generalised projector onto the joint kernel of the Wheeler–Dewitt constraints. We say that the connection is loose because the rigging nature of current spin foams in 4D is not confirmed yet. In any case, a spin foam operator can be formulated as a state sum model, and thus, renormalisation ideas apply. (For applications of renormalisation group ideas in the cosmological sector of LQG, see Refs. 94–96.)

Most of the work on renormalisation is either within classical statistical physics (e.g., Ref. 97) or the Euclidian (also called constructive) approach to the quantum field theory [98–100]. In the Euclidian approach, the quantum field, which is an operator-valued distribution on Minkowski space, is replaced by a distribution-valued random variable on Euclidian space. While the dynamics in the Minkowski theory is given by Heisenberg equations, in the Euclidian theory, it is encoded in a measure on the space of random variables. We are then back in the realm of statistical physics because loosely speaking, the measure can be considered as a Gibbs factor for a Hamiltonian (sometimes called Euclidian action) in four spatial dimensions. How then should one use renormalisation ideas for quantum gravity? Quantum gravity is not a quantum field theory on Minkowski space (unless one works in the perturbative regime, but then it is non-renormalisable). Also, while the Minkowski and Euclidian signature of metrics are related by simple analytic rotation in time from the real to the imaginary axis, this does not even work for classical metrics with curvature, not to mention the quantum nature of the metric (in ordinary QFT, the metric is just a non-dynamic background structure). One can, of course, start with Euclidian signature GR and try to build a measure theoretic framework, but then the relation to the Lorentzian signature theory is unclear. Moreover, while as an ansatz for the Euclidian signature measure, we can take the exponential of the Euclidian Einstein–Hilbert action, that action is not bounded from below, and thus, the measure cannot be a probability measure which is one of the assumptions of constructive QFT. Finally, in contrast to constructive QFT, in quantum gravity expectation, values (operator language) or means (measure language) of basic operators (or random variables) such as the metric tensor have no direct physical meaning because coordinate transformations are considered as gauge transformations; hence, none of the basic fields correspond to observables.

In our approach [54–57], we will use the framework [39], that is, we do not consider vacuum GR but GR coupled to matter which acts as a dynamical reference field. This enables us 1) to solve the spatial diffeomorphism and Hamiltonian constraints classically, 2) to work directly on the physical Hilbert space (i.e., the generalised kernel of all constraints equipped with the inner product induced by the rigged Hilbert space structure, 3) to have at our disposal immediately the gauge-invariant degrees of freedom such that the physical Hilbert space is the representation space of a * representation of those observables, and 4) to be equipped with a physical Hamiltonian that drives the physical time evolution of those observables. Concretely and out of mathematical convenience, we use the perfect fluid matter suggested in Refs. 101 and 102, but for what follows, these details are not important. Important is only that it is possible to rephrase GR coupled to matter as a conservative Hamiltonian system and that all the machinery that was developed for LQG can be imported. Now, the quantisation ambiguity rests, of course, in the physical Hamiltonian and it is that object and its renormalisation on which we focus our attention.

As we just explained, we can bring GR coupled to matter somewhat closer to the usual setting of ordinary QFT or statistical physics, but still we cannot apply the usual path integral renormalisation scheme because we work in the canonical (or Hamiltonian) framework. The idea is then to make use of Feynman–Kac–Trotter–Wiener–like ideas in order to generate a Wiener measure theoretic framework from the Hamiltonian setting and vice versa to use Osterwalder–Schrader reconstruction to map the measure theoretic (or path integral) framework to the Hamiltonian one. This way we can map between the two frameworks and thus import path integral renormalisation techniques into the Hamiltonian framework which are strictly equivalent to those employed in path integral renormalisation. In order that this works one needs to check, of course, that the Wiener measure constructed obeys at least a minimal subset [103] of Osterwalder–Schrader axioms [104] in order for the reconstruction to be applicable, most importantly reflection positivity.

This was one of the goals of [54–57], namely, to define a renormalisation group flow directly within the Hamiltonian setting with strict equivalence to the path integral flow. Specifically, the flow is a flow of Osterwalder–Schrader triples $(\mathcal{H}, H, \Omega)$ consisting of a Hilbert space $\mathcal{H}$, a self-adjoint Hamiltonian H thereon bounded from below, and a vacuum vector $\Omega \in \mathcal{H}$ annihilated by H. While physically well-motivated, of course, one does not need to do this. Indeed, renormalisation techniques for Hamiltonians and vacua directly within the Hamiltonian setting were invented before, and we devote the next section for putting our framework into context with schemes closely related to ours. The fact that we have a precise relation between Hamiltonian and path integral renormalisation makes it possible to bring Hamiltonian formulations of quantum gravity such as LQG and path integral formulations, such as asymptotically safe quantum gravity, into closer contact.

The architecture of this article is as follows:

In the second section, we give an incomplete overview over and sketch Hamiltonian renormalisation frameworks closely related to ours and point out differences and similarities.

In the third section, we review how classical general relativity coupled to suitable matter can be brought into the form of a conservative Hamiltonian system and the LQG quantisation thereof. The necessity to remove quantisation ambiguities will be highlighted.

In the fourth section, we recall some background material on constructive QFT, the Feynman–Kac–Trotter–Wiener construction, and Osterwalder–Schrader reconstruction.

In the fifth section, we derive the natural relation between families of cylindrically defined measures, coarse graining, renormalisation group flows, and their fixed points. We then

use Osterwalder–Schrader reconstruction to map the flow into the Hamiltonian framework. This section contains new material as compared to [54–57] in the sense that we 1) develop some systematics in the choice of coarse graining maps that are motivated by naturally available structures in the classical theory, 2) clarify the importance of the choice of random variable or stochastic process when performing OS reconstruction, and 3) improve the derivation of the Hamiltonian renormalisation flow by adding the uniqueness of the vacuum as an additional assumption (also made in the OS framework of Euclidian QFT [98–100]) as well as some machinery concerning degenerate contraction semi-groups and associated Kato–Trotter formulae.

In the sixth section, we summarise, spell out implications of the renormalisation programme for the anomaly-free implementation of the hypersurface algebroid, and outline the next steps when trying to apply the framework to interacting QFT and finally canonical quantum gravity such as LQG.

The paper is supplemented by the following appendices:

In **Supplementary Appendix A**, we prove some properties for a coarse graining scheme appropriate for non-Abelian gauge theories; in **Supplementary Appendix B**, we prove a lemma on the existence of certain Abelian C^*–algebras needed for the construction of stochastic processes during OS reconstruction; in **Supplementary Appendix C**, we collect some renormalisation terminology for readers more familiar with actions, rather than measures; in **Supplementary Appendix D**, we give a proof for the Kato–Trotter product formula for semi-groups and projections in the simple case that the semi-group has a bounded generator; and in **Supplementary Appendix E**, we prove a strong limit identity between projections needed in **Section 5.3**.

In **Supplementary Appendix F**, we mention concrete points of contact between the scheme developed here and others in the context of density matrix, entanglement, and projective renormalisation.

In **Supplementary Appendix G**, we sketch a relation between Hamiltonian renormalisation via Osterwalder–Schrader reconstruction and the functional renormalisation group which is the underlying technique of the asymptotic safety programme. This article is the journal version of Ref. 196 which is organised slightly differently in the sense that Appendices F, G of this article are part of the main text of Ref. 196.

2. OVERVIEW OVER RELATED HAMILTONIAN RENORMALISATION SCHEMES

The purpose of this section is not to give a complete scan of the vast literature on the subject of Hamiltonian renormalisation but just to give an overview over those programmes that we believe are closest to ours. Also, we leave out many finer details as we just want to sketch their relation to our framework in broad terms. In sections 6 and 7 of Ref. 196, we will give a few more details on the connection between our approach and the density matrix and functional renormalisation group.

The starting point is, of course, the seminal works by Kadanoff [105] and Wilson [106, 107]. Kadanoff introduced the concept of a block spin transformation in statistical physics, that is, a coarse graining transformation in *real space* (namely, on the location of the spin degrees of freedom on the lattice), rather than in some more abstract space (e.g., momentum space blocking/suppressing as used, e.g., in the asymptotically safe quantum gravity approach). This kind of real-space coarse graining map is widely used not only in statistical physics but also in the path integral approach to QFT as, for instance, in lattice QCD [108]. On the other hand, Wilson introduced the concept of *Hamiltonian diagonalisation* to solve the Kondo problem (the low-temperature behaviour of the electrical resistance in metals with impurities). This defines a renormalisation group flow directly on the space of Hamiltonians and its lowest lying energy eigenstates. More precisely, one considers a family of Hamiltonians labelled by an integer-valued cut-off on the momentum mode label of the electron annihilation and creation operators. The renormalisation group flow is defined by diagonalising the Hamiltonian given by a certain cut-off label, and to use the eigenstates so computed to construct the matrix elements of the Hamiltonian at the next cut-off label. To make this practical, Wilson considered a *truncation*, at each renormalisation step, of the full energy spectrum to the 10^3 *lowest lying energy levels* which was sufficient for the low-temperature Kondo problem. This is in fact nothing but the concrete application of the Rayleigh–Ritz method. The concept of truncation plays an important role also in most other renormalisation schemes, as otherwise the calculations become unmanageable.

The next step was done by Wegner [109, 110] as well as Glazek and Wilson [111] which can be considered as a generalisation of the Hamiltonian methods of Refs. 106 and 107. It could be called *perturbative Hamiltonian block diagonalisation* and was applied in QFT already (e.g., Refs. 112 and 113 and references therein). Roughly speaking, one introduces a momentum cut-off on the modes of the annihilation and creation operators involved in the free part of the Hamiltonian, then perturbatively (with respect to the coupling constant) constructs unitarities which at least *block diagonalise* that Hamiltonian with respect to a basis defined by modes that lie below half the cut-off and those that lie between half and the full cut-off, and then projects the Hamiltonian onto the Hilbert space defined by the modes below half of the cut-off to define a new Hamiltonian at half the cut-off. This can be done for each value of the cut-off and thus defines a flow of Hamiltonians (and vacua defined as their ground states). Another branch of work closely related to this is the projective programme due to Kijowski [114, 115]. Here, a flow of Hamiltonians on Hilbert spaces for different resolutions is given by the partial traces of the corresponding density matrices given by minus their exponential (Gibbs factors, assuming that these are trace class). (See also Refs. 116–123 for more recent work on renormalisation building on this programme.)

In these developments, the spectrum of the Hamiltonian was directly used to define the flow. Another proposal was made by White [124] who defined the *density matrix renormalisation group*. This is a real-space renormalisation group flow which considers the reduced density matrix corresponding to the tensor product split of a vector (e.g., the ground state of a Hamiltonian) of the total Hilbert space into two factors corresponding to a block and the rest (or at least a much larger 'superblock'). This

density matrix is diagonalised, and then, the Hilbert space is *truncated* by keeping only a certain fixed number of *highest lying eigenvalues* of the reduced density matrix. Finally, the Hamiltonian corresponding to the block is projected, and then, the resulting structure is considered as the new structure on the coarser lattice resulting from collapsing the blocks to new vertices (we are skipping here some finer details). This method thus makes use of entanglement ideas since the reduced density matrix defines the degree of entanglement via its von Neumann entropy.

A variant of this is the *tensor renormalisation group approach* due to Levin and Nave [125]. It is based on the fact that each vector in a finite tensor product of *finite-dimensional Hilbert spaces* can be written as a *matrix product state*, that is, the coefficients of the vector with respect to the tensor product base can be written as a trace of a product of matrices of which there are, in general, as many as the dimensionality of the Hilbert space. One now performs a real-space renormalisation scheme directly in terms of those matrices which are considered to be located on a lattice with as many vertices as tensor product factors. Importantly, this work connects renormalisation to the powerful numerical machinery of tensor networks [126].

Finally, as observed by Vidal [127] and Evenbly and Vidal [128, 129], one can improve [124, 125] by building in an additional unitary disentanglement step into the tensor network renormalisation scheme. This is quite natural because a tensor network can also be considered as a quantum circuit with the truncation steps involved considered as isometries, but a quantum circuit in quantum computing [130] consists of a network of unitary gates, some of which have a disentangling nature depending on the state that they act upon. The resulting scheme is called multi-scale entanglement renormalisation ansatz (MERA).

As this brief and incomplete discussion reveals, there are numerous proposals in the literature for how to renormalise quantum systems. They crucially differ from each other in the choice of the coarse graining map. There are various aspects that discriminate between these maps, such as the following:

(1) Real space vs. other labels

The degrees of freedom to be coarse grained are labelled by points in space-time or else (momentum, energy, etc.).

(2) Kinematic vs. dynamical

Real-space block spin transformations are an example of a kinematic coarse graining, that is, the form of the action, a Hamiltonian, its vacuum vector, its associated reduced density matrix, and the corresponding degree of entanglement do not play any role. By contrast, Hamiltonian block diagonalisation, density matrix, and entanglement renormalisation take such dynamical information into account.

(3) Truncated vs. exact

In principle, any renormalisation scheme can be performed exactly, for example, in real-space path integral renormalisation, one can just integrate the excess degrees of freedom that live on the finer lattice but not on the coarser, thus obtaining the measure (or effective action) on the coarser lattice from that of the finer one. The same is true, for example, for the procedure followed in asymptotically safe quantum gravity. However, in practice, this may quickly become unmanageable, and thus, one resorts to approximation methods, for example, by truncation in the space of coupling constants, energy eigenstates, or reduced density matrix eigenstates.

For the newcomer to the subject, this plethora of suggestions may appear confusing. Which choice of coarse graining is preferred? Do different choices lead to equivalent physics? What can be said about the convergence of various schemes and what is the meaning of the fixed point(s) if it (they) exist(s)? The physical intuition is that different schemes should give equivalent results if 1) the corresponding fixed point conditions capture necessary and sufficient properties that the theory should have in order to qualify as a continuum theory and 2) *when performed exactly*. The first condition is obvious: we start from what we believe to be an initial guess for how the theory looks at different resolutions and then formulate a coarse graining flow whose fixed points are such that they qualify to define a continuum theory. The second condition entails that the coarse graining maps just differ in the separation of the total set of degrees of freedom into subsets corresponding to coarse and fine resolution, hence corresponds to choices of coordinate systems which, of course, can be translated into each other. However, when truncations come into play, this equivalence is lost because different schemes truncate different sets of degrees of freedom which are generically no longer in bijection. It is conceivable therefore that dynamically driven truncation schemes perform better at identifying the correct fixed point structure of the theory in the sense that they may converge faster and are less vulnerable to truncation errors or automatically pick the truncation of *irrelevant* couplings. This seems to be confirmed in spin system examples, but we are not aware of a general proof. Recently, the importance of the kinematic vs. dynamic issue has also been emphasised for the LQG and spin foam approach [131–133].

In our work, we currently are not concerned with issues of computationability, that is, we consider an exact scheme. Next, as far as the coarse graining map is concerned, we currently favour a kinematic scheme. The reason for doing this is that kinematic schemes are *naturally suggested by measure theoretic questions*, namely, measures on spaces of infinitely many degrees of freedom are never of the type of the exponential of some action times a normalisation constant times Lebesgue measure. Neither of these three ingredients is well defined. What is well defined are integrals of certain probe functions of the field with respect to that measure. These probe functions, in turn, are naturally chosen to depend on test functions that one integrates the field against. Thus, these test functions provide a natural notion of resolution, discretisation, and coarse graining. By integrating the measure against probe functions, one obtains a family of measures labelled by the test functions involved. The relation between test functions at different resolution induces a corresponding relation between members of the family of measures which *must hold exactly* for a

true measure of the continuum QFT. In turn, such consistency relations called *cylindrical consistency* can be used to *define* a measure on a space of infinitely many degrees of freedom [134], called a *projective limit*. The idea is then to formulate measure renormalisation in such a way that its fixed points solve the consistency relations. This approach has been advocated in Refs. 135 and 136 for Euclidian Yang–Mills theory and in Refs. 137 and 138 for spin foams. Note that spin foams, strictly speaking, do not construct measures but rather are supposed to construct a rigging map so that Hamiltonian methods come also into play. Indeed, in Refs. 131–133, it was shown that the cylindrically consistent coarse graining of the rigging map and its underlying space-time lattice, thought of as an anti-linear functional on the kinematical Hilbert space, induce a coarse graining of the spatial lattice on its boundary and thus the Hilbert space thereon, equipping it with a system of consistent embeddings, a structure similar to inductive limits of Hilbert spaces (an inductive structure requires in addition the injections to be isometric). That latter structure underlies the kinematical Hilbert space of LQG, and a renormalisation procedure based on inductive limits was already proposed in Refs. 139–141 due to the similarity of LQG to the lattice gauge theory.

Another reason for why picking real-space coarse graining schemes as compared to, say, momentum space–based ones is their background independence, which is especially important for quantum gravity. In our work, as we consider the version of LQG in which the constraints already have been solved, we will work with probability measures. As we will see, the connection between inductive limits of Hilbert spaces and projective limits of path integral measures can be made crystal clear in this case. The price we pay by using an exact, kinematical scheme is that the fixed point (or renormalised) Hamiltonian becomes spatially non-local at finite resolution. However, in the free QFT examples studied [54–57], which are spatially local in the continuum, by blocking the known fixed point theory from the continuum, one can see that this is natural and *must happen* for such schemes; hence, it is not a reason for concern but, in fact, physical reality. The degree of spatial non-locality, in fact, decreases as we increase the resolution scale.

When applying the framework to interacting QFT, one will have to resort to some kind of approximation scheme, and possibly, tools from entanglement renormalisation combined with tensor network techniques may prove useful. However, note that QFT of bosonic fields (gravity is an example) deals with infinite-dimensional Hilbert spaces even when the theory depends only on a finite number of degrees of freedom, say, by discretising it on a lattice and confining it to finite volume. Thus, to apply tensor network techniques which, to the best of our knowledge, require the factors in the tensor product to be finite-dimensional Hilbert spaces, one would have to cut off the dimensions of those Hilbert spaces right from the beginning, that is, one would have to work with *three* cut-offs, rather than two (see, e.g., Refs. 142 and 143 where quantum group representations are used in gauge theories, rather than classical group representations, and perform real-space renormalisation or [144] where one combines both the UV and the dimension cut-off into one by turning the dimension of tensor spaces in tensor models into a finite coarse graining parameter and otherwise performs the asymptotic safety programme which is often formulated in the presence of a cut-off anyway).

Some sort of truncation or approximation has to be made in practice when treating complex systems numerically. The physical insight behind the tensor network and density matrix/entanglement renormalisation developments, namely, the dynamically interesting vectors in a Hilbert space appear to lie in a 'tiny' subspace thereof is presumably a profound one, and the truncation of the Hilbert space to the corresponding subspaces appears to be well-motivated by the model (spin) systems studied so far. Still, what one would like to have is some sort of error control or convergence criteria on those truncations. We appreciate that this is a hard task for the future. For the time being, we phrase our framework without incorporating a cut-off on the dimension of Hilbert spaces as we are not yet concerned with numerical investigations; however, we may have to use some of these ideas in the future.

3. CANONICAL QUANTUM GRAVITY COUPLED TO REFERENCE MATTER

The physical idea is quite simple and goes back to Ref. 145: General relativity is a gauge theory, the gauge group being the space-time diffeomorphism group. Thus, the basic tensor and spinor fields in terms of which one writes the Einstein–Hilbert action and the action of the standard model coupled to the metric (or its tetrad) are not observable. However, the value of, say, a scalar field Φ at that space-time point X_y, at which four reference scalar fields $\phi^0, \ldots, \phi^3$ take values $y^0, .., y^3$, that is, $\Phi(X_y)$; $\phi^\mu(X_y) = y^\mu$ is space-time diffeomorphism-invariant. For this to work, the relation $\phi^\mu(X) = y^\mu$ must, of course, be invertible, in particular the reference scalar fields must not vanish anywhere or anytime. This seems to be a property of dark matter [146].

These kinds of relational observables have been further developed by various authors, in particular [147–153]. When one couples general relativity and such reference matter preserving general covariance, it becomes possible to formulate the theory in a manifestly gauge-invariant way. The form of that gauge-invariant formulation, of course, strongly depends on the type of reference matter used and its Lagrangian. In what follows, we use the concrete model [39] out of mathematical convenience, but we emphasise that the same technique works in a fairly general context. In the next subsection, that model will be introduced and the classical gauge-invariant formulation will be derived. After that, we quantise it using LQG methods which will be introduced in tandem.

3.1. Gaussian Dust Model

The Lagrangian of the theory takes the form

$$L = L_{EH} + L_{SM} + L_D, \tag{3.1}$$

where L_{EH} is the Einstein–Hilbert Lagrangian, L_{SM} is the standard model Lagrangian coupled to GR via the metric, its tetrad or its spin connection, and L_D is the Gaussian dust Lagrangian [101, 102]

$$L_D = -\frac{1}{2}\sqrt{|\det(g)|}\left\{g^{\mu\nu}\left[\rho\left(\nabla_\mu T\right)\left(\nabla_\nu T\right) + 2\left(\nabla_\mu T\right)\left(W_j\nabla_\nu S^j\right)\right] + \rho\right\}, \tag{3.2}$$

where g is the Lorentzian signature metric tensor, ∇_μ its Levi-Civita covariant differential, $\phi^0 := T$, $\phi^j := S^j$; $j = 1, 2, 3$ are the reference scalar fields introduced above, and ρ, W_j are additional four scalar fields. The latter four fields appear without derivatives and thus give rise to primary constraints in addition to those present even in vacuum GR. One can easily show that the contribution of L_D to the energy momentum tensor is of perfect fluid type. Further physical properties and motivations are discussed in Refs. 101 and 102. For what follows, it suffices to know that the equations of motion for T, S^j, say that $\nabla_\mu T$ is a time-like geodesic cotangent and that S^j is constant along the geodesic spray. Thus, those geodesics can be interpreted as world lines of dynamically coupled test observers.

The full constraint analysis of **Eq. 3.1** is carried out in Ref. 39. There are secondary constraints, and the full set of constraints contains those of the first and second classes (see Ref. 153 for a modern treatment of Dirac's algorithm [154]). One has to introduce a Dirac bracket and solve the second-class constraints in the course of which the variables ρ, W_j are eliminated. The remaining constraints are then of first class and read as

$$C^{\rm tot} = C + \frac{P - q^{ab}T_{,a}C_b}{\sqrt{1 + q^{ab}T_{,a}T_{,b}}},\ C_a^{\rm tot} = C_a + PT_{,a} + P_jS^j_{,a} \tag{3.3}$$

Here, C is the Wheeler–DeWitt constraint function (including standard matter) and C_a, $a = 1, 2, 3$ are the spatial diffeomorphism functions (including standard matter). The Dirac bracket reduces to the Poisson bracket on all the variables involved in **Eq. 3.3**, and P, P_j are the momenta conjugate to T, S^j, for example, $\{P(x), T(y)\} = \delta(x, y)$. Here, $a = 1, 2, 3$ are tensorial indices on the spatial hypersurface σ of the Arnowitt–Deser–Misner foliation underlying the Hamiltonian formulation of GR [155] with intrinsic metric tensor q_{ab}. For the moment, it is just necessary to know that C and C_a do not involve the variables T, P, S^j, and P_j.

The constraints (**Eq. 3.3**) encode the space-time diffeomorphism gauge symmetry in Hamiltonian form, in particular they represent the hypersurface deformation algebra [111]. It is possible to solve these remaining constraints to determine the complete set of gauge-invariant (the so-called Dirac) observables and to determine the physical Hamiltonian H that drives their physical time evolution [39]. Equivalently, we may gauge fix **Eq. 3.3**. The above interpretation of T and S^j suggest to use the gauge conditions $G = T - t$ and $G^a = \delta^a_j S^j - x^a$. The stabilisation of these gauge conditions fixes the Lagrange multipliers λ and λ^a in the gauge generator

$$K := C^{\rm tot}\left(\lambda, \vec{\lambda}\right) := \int_\sigma d^3x\left[\lambda C^{\rm tot} + \lambda^a C_a^{\rm tot}\right], \tag{3.4}$$

namely,

$$\dot{G}(t,x) = \{K, G(x)\} + \partial_t G(t,x) = \frac{\lambda(x)}{\sqrt{1 + q^{ab}T_{,a}T_{,b}}} + \lambda^a T_{,a} - 1 = 0,$$
$$\dot{G}^a(t,x) = \{K, G^a(x)\} + \partial_t G^a(t,x) = \lambda^b S^j_{,b}\delta^a_j = 0 \tag{3.5}$$

which when evaluated at $G = G^a = 0$ yields the unique solution $\lambda = 1$, $\lambda^a = 0$. Likewise, in this gauge, the constraints can be uniquely solved for $P = -C$ and $P_j = -\delta^a_j C_a$ while T and S^j are pure gauge. This shows that the physical degrees of freedom are those not involving T, P, S^j, and P_j.

For any function F independent of these variables, the reduced or physical Hamiltonian is that function on the phase space coordinatised by the physical degrees of freedom which generates the same time evolution as K when the constraints, gauge conditions, and stabilising Lagrange multipliers are installed

$$\{H, F\} := \{K, F\}_{C^{\rm tot} = \vec{C}^{\rm tot} = G = \vec{G} = \lambda - 1 = \vec{\lambda} = 0} = \left\{\int_\sigma d^3x C, F\right\}, \tag{3.6}$$

which shows that

$$H = \int_\sigma d^3x C \tag{3.7}$$

Thus, the final picture is remarkably simple: The physical phase space is simply coordinatised by all metric and standard matter degrees of freedom (and their conjugate momenta), while the physical Hamiltonian is just the integral of the usual Wheeler–DeWitt constraint. The influence of the reference matter now only reveals itself in the fact that H is not constrained to vanish as it only involves the geometry and standard matter contribution C of $C^{\rm tot}$ and that the number of physical degrees of freedom has increased by four as compared to the system without reference matter. This phenomenon is, of course, well-known from the electroweak interaction: One can solve the three isospin SU(2) Gauss constraints for three of the four degrees of freedom sitting in the complex-valued Higgs isodublett, leaving a single scalar Higgs field and three massive, rather than massless, vector bosons. (See Refs. 156 and 157 for further discussion.)

We close this subsection with three remarks:

First, a complete discussion requires to show that the gauge cut $G = G^a = 0$ on the constraint surface of the phase space be reachable from anywhere on the constraint surface. As **Eq. 3.5** shows, this requires that $S^j_{,a}$ be invertible. We thus impose this as an anholonomic constraint on the total phase space. One easily verifies from **Eq. 3.5** that this condition is gauge-invariant, that is, compatible with the dynamics.

Second, the simplicity of the final picture is due to the particular choice of reference matter. Other reference matter most likely will increase the complexity (see, e.g., Ref. 158), which produces a square root Hamiltonian! One may argue that the dust is a form of cold dark matter [146], but it is unclear whether this is physically viable. Nevertheless, the present model serves as a proof of principle, namely, that GR coupled to standard matter and reference can be cast into the form of a conservative Hamiltonian system.

Third, it should be appreciated that the reference matter helps us accomplish a huge step in the quantum gravity programme: It frees us from quantising and solving the constraints and constructing the physical inner product, the gauge-invariant observables, and their physical time evolution. All of these steps are of tremendous technical difficulty [17–21]. All we are left to do is to quantise the physical degrees of freedom and the physical Hamiltonian.

3.2. Loop Quantum Gravity Quantisation of the Reduced Physical System

In order to keep the technical complexity to a minimum, we consider just the contribution to H coming from the gravitational degrees of freedom (see [17–21, 28–32] for more detail on standard matter coupling). The Hamiltonian directly written in terms of $SU(2)$ gauge theory variables reads (we drop some numerical coefficients that are not important for our discussion)

$$\begin{aligned} H &= H_E + H_L \\ H_E &= \int_\sigma \mathrm{Tr}(F\wedge\{V,A\}) \\ V &= \int_\sigma \sqrt{|\det(E)|} \\ H_L &= \int_\sigma \mathrm{Tr}(\{\{H_E,V\},A\}\wedge\{\{H_E,V\},A\}\wedge\{V,A\}) \end{aligned} \tag{3.8}$$

Here, A is an SU(2) connection and E an SU(2) non-Abelian electric field that one would encounter also in an SU(2) Yang–Mills theory. However, the geometric interpretation of A and E is different, namely, $e^a_j := E^a_j/\sqrt{|\det(E)|}$ is a triad, that is, $q^{ab} = \delta^{jk}e^a_j e^b_k$ is the inverse spatial metric. Here, as before, $a, b, c, .. = 1, 2, 3$ denote spatial tensor indices, while now $j, k, l, .. = 1, 2, 3$ denote su(2) Lie algebra indices. Further, let Γ^j_a be the spin connection of e^a_j. Then, $K_{ab} := (A^j_a - \Gamma^j_a)e^k_b\delta_{jk}$ has the meaning of the extrinsic curvature of the ADM slices [155] on the kernel of the SU(2) Gauss constraint

$$C_j := \partial_a E^a_j + \epsilon_{jkl}A^k_a E^a_m \delta^m_l \tag{3.9}$$

The important quantity V is recognised as the total volume of the hypersurface σ, and H_E and H_L are known as the Euclidian and Lorentzian contributions to H. (See Refs. 28–32 for further details.) The Poisson brackets displayed are with respect to the standard symplectic structure

$$\begin{aligned} \{A^j_a(x), A^k_b(y)\} &= \{E^a_j(x), E^b_k(y)\} \\ &= \{E^a_j(x), A^k_b(y)\} - \kappa\delta^a_b\delta^k_j\delta(x,y) = 0, \end{aligned} \tag{3.10}$$

where $\hbar\kappa = \ell_P^2$ is the Planck area. The definition of the phase space is completed by the statement that the elementary fields A and E are real-valued

$$[A^j_a(x)]^* - A^j_a(x) = [E^a_j(x)]^* - E^a_j(x) = 0 \tag{3.11}$$

The traces involved in 3.2 are carried out by introducing the Lie algebra-valued 1-forms $A = A^j_a\,\tau_j\,dx^a$, where $2i\tau_j$ are the Pauli matrices and $F = 2(dA + A\wedge A)$ is the curvature of A. The non-polynomiality of GR is hidden in the Poisson brackets that appear in **Eq. 3.8**. The reason why we use these particular Poisson bracket structure will become clear only later.

To quantise the theory, we start from functions on the phase space that are usually employed in the lattice gauge theory (see, e.g., Ref. 159), namely, non-Abelian magnetic holonomy and electric flux variables

$$A(c) := \mathcal{P}\exp\left(\int_c A\right),\ E_f(S) = \int_S \mathrm{Tr}(f*E), \tag{3.12}$$

where $\mathcal{P}$ denotes path ordering, c is a piecewise analytic real curve, S is a piecewise real analytic surface, f is an su(2)-valued function, and $*\mathrm{E} = \epsilon_{abc}\mathrm{E}^a\mathrm{d}x^b\wedge\mathrm{d}x^c/2$ is the pseudo 2-form corresponding to the su(2)-valued vector density E. Note that $A(c)$ is SU(2)-valued, while $E_f(S)$ is su(2)-valued

$$A(c)^* = (A(c)^{-1})^T = A(c^{-1})^T,\ E_f(S)^* = -E_f(S)^T, \tag{3.13}$$

where c^{-1} is the same curve as c but with the opposite orientation. The simplest non-trivial Poisson brackets are

$$\{E_f(S), A(c)\} = \kappa A(c_1) f(c\cap S) A(c_2), \tag{3.14}$$

in case that $S\cap c$ is a single point in the interior of both c and S, see Refs. 23–27 for a complete discussion. The relations (3.13) and (3.14) are the defining relations of a non-commutative abstract $-^*$algebra $\mathfrak{A}$ generated by fluxes and complex-valued smooth functions F of a finite number of holonomy variables [23–27]. It is the free algebra generated by them and divided by the two-sided ideal generated by the canonical commutation relations $E_f(S)A(c) - A(c)E_f(S) = i\hbar\{E_f(S),\ F\}$ and the adjointness relations (**Eq. 3.13**). (See Refs. 23–27 for more details.)

Interestingly, the physical Hamiltonian H has a large symmetry group, namely, it is invariant under the group $\mathfrak{G} = SU(2)_{\mathrm{loc}} \rtimes Diff(\sigma)$, where $SU(2)_{\mathrm{loc}}$ denotes the group of local SU(2)-valued gauge transformations and $Diff(\sigma)$ denotes the group of (piecewise real analytic) diffeomorphisms of σ. An element of $\mathfrak{G}$ is given by a pair $\mathfrak{g} = (g, \varphi)$, which acts on the basic variables as

$$\alpha_{(g,\varphi)}(A) = -dgg^{-1} + g[\varphi^* A]g^{-1},\ \alpha_{(g,\varphi)}(*\mathrm{E}) = g[\varphi^*(*\mathrm{E})]g^{-1}, \tag{3.15}$$

where φ^* denotes the pull-back action of diffeomorphisms on differential forms. This action lifts to the algebra $\mathfrak{A}$, specifically

$$\begin{aligned} \alpha_{(g,\varphi)}(A(c)) &= g(b(c))\,A(\varphi(c))\,g(f(c))^{-1},\ \alpha_{(g,\varphi)}(E_f(S)) \\ &= E_{[g^{-1}fg]\circ\varphi^{-1}}(\varphi(S)), \end{aligned} \tag{3.16}$$

where $b(c)$ and $f(c)$ denote the beginning and final points of c, and this simple covariant transformation behaviour was part of the reason why the particular 'smearing' of A along curves involved in holonomies is used. Note also the different character of the two groups: While we still have to find the gauge-invariant observables with respect to the Gauss constraint, the diffeomorphism constraint is already solved. The

diffeomorphisms in $\mathfrak{G}$ are thus to be considered as active diffeomorphisms, rather than passive ones.

The mathematical problem in quantising the theory consists in constructing a * representation of $\mathfrak{A}$, that is, a representation $(\pi, \mathcal{H})$ of elements $a \in \mathfrak{A}$ as operators $\pi(a)$ densely defined on a common, invariant domain $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ such that the * relations are implemented as adjointness relations and such that the canonical commutation relations are implemented as commutators between them. Thus, we want, in particular, that

$$\begin{aligned}\pi(a^*) = [\pi(a)]^\dagger,\ \pi(a+b) = \pi(a) + \pi(b),\ \pi(ab)\\ = \pi(a)\pi(b),\ \pi(za) = z\pi(a),\ [\pi(a), \pi(b)] = \pi(c);\end{aligned} \tag{3.17}$$

for all $a, b, c \in \mathfrak{A}$, $z \in \mathbb{Z}$ if $ab - ba = c$. In QFT, this problem is known to typically have an uncountably infinite number of unitarily inequivalent solutions; there is no Stone–von Neumann uniqueness theorem when the number of degrees of freedom in infinite. Hence, to make progress, we must use additional physical input. That input can only come from the Hamiltonian. Thus, we require in addition that the representation supports H as a self-adjoint operator (H is real-valued) also densely defined on $\mathcal{D}$ and such that $\mathcal{H}$ carries a unitary representation U of $\mathfrak{G}$ (such that its generators are self-adjoint by Stone's theorem). Using the powerful machinery of the Gel'fand–Naimark–Segal construction [160], the representation property and the unitarity property can be granted if we find a positive linear and $\mathfrak{G}$ invariant functional ω: $\mathfrak{A} \to \mathbb{C}$ on $\mathfrak{A}$, that is,

$$\omega \circ \alpha_{\mathfrak{g}} = \omega,\ \omega(a^* a) \geq 0 \tag{3.18}$$

In Refs. 23–27, it was found that there is a *unique* ω satisfying (3.18). While the derivation is somewhat involved, the final result can be described in a compact form. The dense domain $\mathcal{D}$ consists of functions of the form

$$\psi(A) = \psi_\gamma\Big(\{A(c)\}_{c\in E(\gamma)}\Big);\ \psi_\gamma \in C^\infty\Big(SU(2)^{|E(\gamma)|}, \mathbb{C}\Big), \tag{3.19}$$

that is, ψ_γ is complex-valued, smooth functions of a finite number of holonomy variables. The union of the curves of these holonomies forms a finite graph γ, where $E(\gamma)$ denotes the set of its edges. Note that the elements of $\mathfrak{A}$ that just depend on the connection are themselves of the form **(Eq. 3.19)**, and thus, their action by multiplication

$$[\pi(f)\psi](A) := f(A)\,\psi(A) \tag{3.20}$$

is densely defined. The fluxes are densely defined when acting by derivation

$$\big[\pi\big(E_f(S)\big)\psi\big](A) := i\hbar\big\{E_f(S), \psi(A)\big\} \tag{3.21}$$

which also solves the canonical commutation relations.

To see that the adjointness conditions hold, we need the inner product. To define it, we note that graphs defined by finitely many piecewise analytic curves are partially ordered by set theoretic inclusion, and they are directed in the sense that for any two graphs γ_1, γ_2, there exists γ_3 with γ_1, $\gamma_2 \subset \gamma_3$, for instance $\gamma_3 = \gamma_1 \cup \gamma_2$. Then, we can decompose all edges of γ_1, γ_2 with respect to the edges of γ_3 and use the algebraic relations of the holonomy $A(c^{-1}) = A(c)^{-1}$ and $A(c \circ c\prime) = A(c)A(c\prime)$, where $c \circ c\prime$ is the composition of curves $f(c) = b(c\prime)$ in order to write ψ_1 and ψ_2 excited over γ_1 and γ_2, respectively, as functions excited over γ_3. Thus, it is sufficient to know the inner product of functions excited over the same graph γ which is given by

$$\langle \psi, \psi' \rangle_{\mathcal{H}} := \int_{SU(2)^{|E(\gamma)|}} \prod_{k=1}^{|E(\gamma)|} d\mu_H(h_k)\, \overline{\psi(\{h_k\})}\, \psi'(\{h_k\}), \tag{3.22}$$

where μ_H is the Haar measure on SU(2). One can check that the adjointness relations are indeed satisfied, in fact $\pi(E_f(S))$ is an unbounded but essentially self-adjoint operator (i.e., a symmetric operator with unique self-adjoint extension).

In fact, **Eq. 3.22** defines a cylindrical family of measures μ_γ, one for every graph γ. One has to check that **Eq. 3.22** is well-defined because a function excited on γ can be written also as a function excited over any finer graph $\gamma\prime$ by extending it trivially to the additional edges. This is, in fact, the case [23–27]. Then, the Kolmogorov-type extension theorems grant that the family extends to an honest continuum measure μ on the quantum configuration space $\overline{\mathcal{A}}$ of distributional connections. We will not go into the details here which can be found in Refs. 23–27 but just mentioning for the interested reader that this space coincides with the so-called Gel'fand spectrum of the Abelian C* algebra that one obtains by completing the space of functions (**Eq. 3.19**) in the sup norm. It follows that the Hilbert space is given by $\mathcal{H} = L_2(\overline{\mathcal{A}}, d\mu)$.

By construction, the Hilbert space $\mathcal{H}$ carries a unitary representation U of $\mathfrak{G}$ given by

$$(U(\mathfrak{g})\psi)(A) = \psi_\gamma\Big(\{\alpha_{\mathfrak{g}}(A(c))\}_{c\in E(\gamma)}\Big) \tag{3.23}$$

To check this, one uses the properties of the Haar measure (translation invariance) and the diffeomorphism invariance of **Eq. 3.22** which does not care about the location and shape of the curves involved.

The Hilbert space comes equipped with an explicitly known orthonormal basis called spin network functions (SNWFs). This makes use of harmonic analysis on compact groups G [161], in particular the Peter and Weyl theorem which states that the matrix element functions of the irreducible representations of G, which are all finite-dimensional and unitary without loss of generality, are mutually orthogonal, unless equivalent, with respect to the inner product defined by the Haar measure on G; moreover, they span the whole Hilbert space. As the irreducible representations of SU(2) are labelled by spin quantum numbers, the name SNWF comes at no surprise. More in detail, an SNWF $T_{\gamma, j, \iota}$ is labelled by a graph γ, a tuple $j = \{j_c\}_{c\in E(\gamma)}$ of spin quantum numbers decorating the edges, and a tuple $\iota = \{\iota_v\}_{v\in V(\gamma)}$ of intertwiners decorating the vertices v in the vertex set $V(\gamma)$ of γ. Here, an intertwiner ι_v projects the tensor product of irreducible representations corresponding to the edges incident at v onto one of the irreducible representations appearing in its

decomposition into irreducibles (Clebsch–Gordan theory). Besides providing an ONB convenient for concrete calculations, SNWFs make it easy to solve the Gauss constraint: A detailed analysis [17–21] shows that **Eq. 3.9** can be quantised in the given representation and just imposes that the space of intertwiners be restricted to those projecting on the trivial (spin zero) representation. We call such intertwiners gauge-invariant. Hence, the joint kernel of the Gauss constraints is a closed subspace of $\mathcal{H}$ which is explicitly known. We will abuse the notation and will not distinguish between that subspace and $\mathcal{H}$ and henceforth consider the Gauss constraint as solved. All operators considered in what follows are manifestly gauge invariant and preserve that subspace.

As a historical remark, solutions of the Gauss constraint are excited on closed graphs since there is no non-trivial intertwiner between the trivial representation and a single irreducible one; hence, open ends are forbidden. For closed graphs, one can alternatively label SNWFs by homotopically independent closed paths (loops) with a common starting point (vertex) on that graph. Originally, one used loops as labels, hence the name loop quantum gravity (LQG).

One of the many unfamiliar features of $\mathcal{H}$ is that it is not separable which easily follows from the uncountable cardinality of the set of graphs. This is a direct consequence of the diffeomorphism invariance of the inner product: Two graphs that are arbitrarily close but disjoint are simultaneously also arbitrarily far apart under the inner product. Thus, if the measure clusters for far apart support of the smearing functions (here the graphs), then the orthogonality of the corresponding spin network functions comes at no surprise. A direct consequence of this is that the diffeomorphism operators $U(\varphi)$ do not act (strongly) continuously; hence, a generator of infinitesimal diffeomorphisms generated by the integral curves of vector fields cannot exist. Yet another direct consequence is that the connection operator A itself does not exist; only its holonomies do.

The remaining task is to quantise the Hamiltonian, and it is at this point where the aforementioned quantisation ambiguities arise. The strategy followed in Refs. 28–32 is as follows: It turns out that the volume operator appearing in **Eq. 3.2** can be quantised on $\mathcal{H}$ as an essentially self-adjoint operator whose spectrum is pure point (discrete) [162–164]. It is densely defined on the span of the SNWF, and it acts vertex-wise, with no contribution from gauge-(in)variant vertices that are not at least three (four) valent or from vertices whose incident edges have tangents in a common two-dimensional or one-dimensional space. Next, the holonomy along an open curve c can be expanded as $A(c) = 1_2 + \int_c A + \ldots$ and along a closed curve α as $A(\alpha) = 1_2 + \int_{S,} \partial S = \alpha F$ so that the functions A and F that appear in **Eq. 3.8** can be approximated by suitable holonomies where the approximation is in terms of the 'length' of the curves involved which are matched with the coordinate volume assigned by the Lebesgue measure d^3x appearing in **Eq. 3.8**, approximating the integral by a Riemann sum (this is a regularisation step). Suppose then that somehow a well-defined operator H_E can be defined by replacing the classical functions by operators and the Poisson brackets by commutator times $\hbar$. Then, the same argument can be applied to the Lorentzian piece. As a final piece of information, one uses the observation that a spatially diffeomorphism-invariant operator, densely defined on the span of SNWF, cannot have non-trivial matrix elements between SNWF excited over different graphs [36]. This has the following consequence: Let $\mathcal{H}_\gamma$ be the closed linear span of SNWF excited precisely over γ. Then, if H is supposed to preserve its classical diffeomorphism invariance upon quantisation, we necessarily have

$$H = \oplus_\gamma H_\gamma,\ \mathcal{H} = \oplus_\gamma \mathcal{H}_\gamma, \tag{3.24}$$

where each H_γ is self-adjoint on $\mathcal{H}_\gamma$, in particular it preserves this space. Let now $P_\gamma\colon\ \mathcal{H} \to \mathcal{H}_\gamma$ be the orthogonal projection. Then, the following concrete expression for H can be given [39] (again we drop some numerical coefficients and set $\hbar = 1$)

$$\begin{aligned}
&H_{E,\gamma} = P_\gamma\, H'_{E,\gamma}\, P_\gamma\\
&H'_{E,\gamma} = i \sum_{v\in V(\gamma)} \sum_{c_1,c_2,c_3\in E(\gamma);\, c_1\cap c_2\cap c_3 = v}\\
&\epsilon^{IJK}\mathrm{Tr}\Big(\Big[A\big(\alpha_{\gamma,v,c_I,c_J}\big) - A\big(\alpha_{\gamma,v,c_I,c_J}\big)^{-1}\Big]\, A(c_K)\,[V, A(c_K)^{-1}]\Big) + h.c.\\
&H_{L,\gamma} = P_\gamma\, H'_{L,\gamma}\, P_\gamma\\
&H'_{L,\gamma} = i \sum_{v\in V(\gamma)} \sum_{c_1,c_2,c_3\in E(\gamma);\, c_1\cap c_2\cap c_3 = v}\\
&\epsilon^{IJK}\mathrm{Tr}\Big(A(c_I)\Big[[H'_{E,\gamma}, V], A(c_I)^{-1}\Big]\, A(c_J)\Big[[H'_{E,\gamma}, V\Big],\\
&A(c_J)^{-1}]\, A(c_K)\,[V, A(c_K)^{-1}]\Big) + h.c.
\end{aligned} \tag{3.25}$$

The sum is over vertices of γ and triples of edges incident at them (taken with outgoing orientation). For each vertex v and pairs of edges c, c' outgoing from v, one defines $\alpha_{\gamma,}$ v, c, c' as that loop within γ starting at v along c and ending at v along $(c')^{-1}$ with the minimal number of elements of $E(\gamma)$ used (if that loop is not unique, we average over them). It has been shown that the concrete expression (**Eq. 3.25**) has the correct semi-classical limit in terms of expectation values with respect to semi-classical coherent states [165–168] on sufficiently fine graphs of cubic topology [166–172].

Remarkably, **Eq. 3.25** defines an essentially self-adjoint, diffeomorphism-invariant, continuum Hamiltonian operator for Lorentzian quantum gravity in four space-time dimensions, densely defined on the physical continuum Hilbert space $\mathcal{H}$ *which is manifestly free of ultraviolet divergences*, that is, while for each given graph γ, the theory looks like a lattice gauge theory on γ; the theory is defined on all lattices simultaneously, which makes it a continuum theory. Moreover, note that the vector $\Omega = 1$ has norm unity and that $H\Omega = 0$.

Yet, one cannot be satisfied with **Eq. 3.25** for the following reasons:

1. While it is true that one can give a better motivated derivation than we could sketch here for reasons of space, there are some ad hoc steps involved.
2. There are several ordering ambiguities involved in **Eq. 3.25**: Not only could we have written the factors in different orders but instead of using the fundamental representation to approximate connections in terms of holonomies, we could have used higher spin representations [173] or an average over several of them, and in each case, we would have different coefficients appearing in front of these terms.
3. Of particular concern is definition of the minimal loop. While this gives good semi-classical results on sufficiently fine lattices, the theory lives on all lattices, also those which are very coarse, and on those, expression **Eq. 3.25** is doubtful because the Riemann approximation mentioned above would suggest to use a much finer loop. In fact, one is supposed to take the regulator (i.e., the coordinate volume ϵ of the Riemann approximants) away, and in that limit, the loop would shrink to zero. One can justify that this does not happen by using a sufficiently weak operator topology [28–32], namely, there exist diffeomorphism-invariant distributions (linear functionals) l on the dense span of SNWF ψ [36], and we define an operator O_ϵ to converge to an operator O in that topology if $l([O_\epsilon - O]\psi) \to 0$ for all l, ψ. Now, due to diffeomorphism invariance, we can deform for any ϵ the small loop to any diffeomorphic one as long as we do not cross other edges of the graph, in particular we can deform it as close as we want to the minimal one. Then, the result mentioned above about the matrix elements of diffeomorphism-invariant operators, in fact, forces us to choose that loop precisely, not only approximately. Of course, while the diffeomorphism symmetry of H makes the space of diffeomorphism-invariant distributions a natural space to consider, it is still not perfectly justified to use it in order to define a topology.
4. The naive dequantisation of **Eq. 3.25** will perform poorly on very coarse graphs and will be far from the continuum expression **Eq. 3.8**, but one could argue that that vectors supported on coarse graphs simply do not qualify as good semi-classical states.
5. Using the same argument as in (3), there is nothing sacred about the minimal loop, and one could take again other loops and/or average of over them with certain weights. However, then the locality of **Eq. 3.25** is lost.
6. The block diagonal or superselection structure (**Eq. 3.24**) which is forced on us by the non-separability of the Hilbert space and its spatial diffeomorphism covariance appears unphysical, and one would expect that the Hamiltonian creates also new excitations.

It transpires that we must improve **Eq. 3.25**, and the discussion has indicated a possible solution: Blocking free QFT from the continuum (i.e., restricting the Hilbert space to vectors of finite spatial resolution) with respect to a kinematic real-space coarse graining scheme exactly produces such a high degree of non-locality at finite resolution even if the continuum measure or the continuum Hamiltonian is local [54–57, 71–75, 108]. This bears the chance that what we see in **Eq. 3.25** is nothing but a naive guess of a continuum Hamiltonian which is blocked from the continuum but whose off-block diagonal form we cannot determine with the technology used so far. Accordingly, this calls for shifting our strategy which was already started in Refs. 169–172, 174 (in the sense that the block diagonal structure was dropped, but only one infinite graph was kept):

We take the above speculation serious and consider the operators H_γ as projections onto the subspaces $\mathcal{H}_\gamma$ of $\mathcal{H}$ of a continuum Hamiltonian H, but we will drop the unphysical block diagonal structure 3.24 which arises from the non-separability of $\mathcal{H}$. Rather the relation between H_γ is to be imposed by a renormalisation scheme induced by the path integral renormalisation scheme adopted in quantum statistical physics. To do this, we must first derive a path integral measure μ_γ from the OS data, $\mathcal{H}_\gamma$, H_γ, and Ω_γ where Ω_γ is the vacuum of H_γ by the usual Feynman–Kac–Trotter–Wiener formalism. Then, we can compute the flow of $\mu_\gamma$in the usual way and then translate into a flow of OS data by OS reconstructing them from the measures. The fixed points of the flow will then define the possible continuum theories, and these may be 'phases' quite different from **Eq. 3.25**. The details of this programme will be the subject of the following sections.

4. CONSTRUCTIVE QFT, FEYNMAN–KAC–TROTTER–WIENER CONSTRUCTION AND OSTERWALDER–SCHRADER RECONSTRUCTION

The purpose of this section is to provide some background information on constructive QFT and related topics such as the Feynman–Kac–Trotter–Wiener construction of measures (path integrals) from a Hamiltonian formulation (operator formulation) and vice versa the Osterwalder–Schrader reconstruction of a Hamiltonian framework from a measure. Our description will be minimal. The prime textbook references are [98–100, 175].

4.1. Measure Theoretic Glossary

Let S be a set. A collection B of the so-called measurable subsets of S is called a σ−algebra if i. it is closed under taking complements with respect to S, ii. closed under taking countable unions, and iii. B contains the empty set $\varnothing$. The pair (S, B) is called a measurable space. A measure space is a triple (S, B, μ), where (S, B) is a measure space and μ is a positive set function μ: $B \to \mathbb{R}_0^+ \cup \{+\infty\}\, s \mapsto \mu(s)$ which is σ−additive, that is, for any pairwise disjoint $s_I \cap s_J = \varnothing$, $I \neq J$; $I, J = \in \mathbb{N}$, we have

$$\mu(\cup_I s_I) = \sum_I \mu(s_I) \tag{4.1}$$

The measure μ is called a probability measure if $\mu(S) = 1$. One uses the notation

$$\mu(s) = \int_s d\mu(p) = \int_S d\mu(p)\,\chi_s(p), \tag{4.2}$$

where $\chi_s(p) = 1$ if $p \in s$ and $\chi_s(p) = 0$, else is called the characteristic function of $s \in B$.

Consider now a second measurable space $(\widetilde{S}, \widetilde{B})$. A function $X\colon S \to \widetilde{S}$ is called measurable or a random variable if the pre-images $X^{-1}(\widetilde{s}) = \{p \in S; f(p) \in \widetilde{s}\}$ of measurable sets $\widetilde{s} \subset \widetilde{S}$ are measurable in S. Let $\mathcal{F}$ be the set of random variables $X\colon S \to \widetilde{s}$; then for $X \in \mathcal{F}$, the set function

$$\widetilde{\mu}(\widetilde{s}) := \mu\left(X^{-1}(\widetilde{s})\right),\ \widetilde{s} \in \widetilde{B} \tag{4.3}$$

defines also a probability measure called the distribution of X. We consider real-valued functions $f\colon \widetilde{S} \to \mathbb{R}$ of the simple form

$$f(\widetilde{p}) = \sum_n z_n \chi_{\widetilde{s}_n}(\widetilde{p});\ z_n \in \mathbb{R},\ \widetilde{s}_n \in \widetilde{B}, \tag{4.4}$$

where the sum is over at most finitely many terms and define their integral as

$$\begin{aligned}
\widetilde{\mu}(f) &= \sum_n z_n \mu\left(s'_n\right) = \int_{\widetilde{S}} d\widetilde{\mu}(\widetilde{p})\left[\sum_n z_n \chi_{\widetilde{s}_n}(\widetilde{p})\right] = \int_{\widetilde{S}} d\widetilde{\mu}(\widetilde{p}) f(\widetilde{p}) \\
&= \sum_n z_n \mu\left(X^{-1}(\widetilde{s}_n)\right) = \int_S d\mu(p)\left[\sum_n z_n \chi_{X^{-1}(\widetilde{s}_n)}(p)\right] \\
&= \int_S d\mu(p)\left[\sum_n z_n \chi_{\widetilde{s}_n}(X(p))\right] = \int_S d\mu(p)\,(f \circ X)(p) = \mu(f \circ X)
\end{aligned} \tag{4.5}$$

One can show that this identity extends from simple functions to Borel functions that is, measurable functions $f\colon \widetilde{S} \to \mathbb{R}$, where $\mathbb{R}$ is equipped with the Borel σ–algebra (the smallest σ algebra containing all open intervals). We can then also extend it to those complex functions whose real and imaginary parts are Borel by linearity.

A stochastic process indexed by an index set $\mathcal{I}$ is a family $\{X_i\}_{i \in \mathcal{I}}$of random variables $X_i\colon S \to \widetilde{S}$. For any finite subset $I = \{i_1, .., i_N\} \subset \mathcal{I}$, we have the joint distribution

$$\widetilde{\mu}_I(\widetilde{s}_1 \times .. \times \widetilde{s}_N) := \mu\left(\cap_{k=1}^N X_{i_k}^{-1}(\widetilde{s}_k)\right) \tag{4.6}$$

The probability measures μ_I are called cylinder measures. For any complex-valued Borel function $f\colon \widetilde{S}^N \to \mathbb{C}$, we have similarly as in **Eq. 4.4**

$$\int_S d\mu(p) f\left(\{X_{i_k}(p)\}_{k=1}^N\right) = \int_{\widetilde{S}^N} d\widetilde{\mu}_I(\widetilde{p}_1, .., \widetilde{p}_n) f(\widetilde{p}_1, .., \widetilde{p}_n) \tag{4.7}$$

Functions on S of the form $f_I(p) = f(\{X_{i_k}(p)\}_{k=1}^N)$ are called cylinder functions.

In what follows, we assume that for each $N \in \mathbb{N}_0$, there exists a distinguished system $\mathcal{W}_N$ of complex-valued, bounded elementary functions W on N copies of $\widetilde{S}$ such that the corresponding cylinder functions enjoy the following properties:

(1) They generate an Abelian * algebra, that is, for all $I, I' \in \mathcal{I}$, the product $W_I\, W'_{I'}$ is a finite, complex linear combination of suitable $W''_{I''}$, $I'' \in \mathcal{I}$, $\mathcal{W}'' \in \mathcal{W}_{|\mathcal{I}''|}$ and also $\overline{W_I}$ is of that form.
(2) $\mathcal{W}_N$ contains the constant function.
(3) For each $I \in \mathcal{I}$, the moments $\mu(W_I)$, $W \in \mathcal{W}_{|I|}$ determine $\widetilde{\mu}_I$ uniquely.
(4) These properties show that W_I are $L_2(d\widetilde{\mu}_I, \widetilde{S}^{|I|})$ functions. We require their span to be dense.
(5) We saw that a probability measure μ together with a stochastic process gives rise to a family of cylindrical probability measures $(\widetilde{\mu}_I)_{I \in \mathfrak{I}}$ on $\widetilde{S}^{|I|}$. The converse question is under which circumstances a cylindrical family of cylinder probability measures determines a measure μ. A necessary criterion is as follows: The set $\mathcal{I}$ is partially ordered and directed by inclusion, that is, for each $I, J \in \mathcal{I}$, we find $K \in \mathcal{I}$ such that $I, J \subset K$ (for instance, $K = I \cup J$). Suppose that $I \subset J$. Then,

$$\mu\left(X_I^{-1}(\widetilde{s}_I)\right) = \widetilde{\mu}_I(\widetilde{s}_I) = \mu\left(X_J^{-1}\left(\widetilde{s}_I \times \left(\widetilde{S}\right)^{|J|-|I|}\right)\right) = \widetilde{\mu}_J\left(\widetilde{s}_I \times \left(\widetilde{S}\right)^{|J|-|I|}\right), \tag{4.8}$$

where $X_I = \{X_i\}_{i \in I}$, $\widetilde{s}_I \subset \widetilde{B}^{|I|}$. Furthermore, for any permutation π on $N = |I|$ elements set, $\pi \cdot I = \{i_{\pi(1)}, .., i_{\pi(N)}\}$ and $\pi \cdot \widetilde{s}_I = \{(\widetilde{p}_{\pi(1)}, .., \widetilde{p}_{\pi(N)});\ (\widetilde{p}_1, .., \widetilde{p}_N) \in \widetilde{s}_I\}$. Then,

$$\mu(X_{\pi \cdot I}^{-1}(\pi \cdot \widetilde{s}_I) = \widetilde{\mu}_{\pi \cdot I}(\pi \cdot \widetilde{s}_I) = \mu(X_I^{-1}(\widetilde{s}_I) = \widetilde{\mu}_I(\widetilde{s}_I) \tag{4.9}$$

Even more generally, a partial order on the set $\mathfrak{I}$ of finite subsets I of $\mathcal{I}$ is a transitive, reflexive, and antisymmetric relation, that is, $I < J \wedge J < K \Rightarrow I < K$ and $I < I$ and $I < J \wedge J < I \Rightarrow I = J$ for all $I, J, K \in \mathfrak{I}$. The set $\mathfrak{I}$ is called directed with respect to $<$, provided that for all $I, J \in \mathfrak{I}$, we find $K \in \mathfrak{I}$ such that $I, J < K$. For $I < J$, we may have surjective maps $P_{JI}\colon \widetilde{S}^{|J|} \to \widetilde{S}^{|I|}$ such that $X_I(p) = P_{JI}(X_J(p))$ and such that for $I < J < K$, we have $P_{JI} \circ P_{KJ} = P_{KI}$. Then, similar as in **Eq. 4.8**, we necessarily must have for $I < J$

$$\widetilde{\mu}_I(\widetilde{s}_I) = \widetilde{\mu}_J\left(P_{JI}^{-1}(\widetilde{s}_I)\right) \tag{4.10}$$

It turns out that these two conditions, **Eqs 4.8**, **4.9**, **or 4.10** is also sufficient in fortunate cases (for instance, if $\widetilde{S} = \mathbb{R}$, which is the classical Kolmogorov theorem, see Ref. 134), that is, we can then reconstruct the measure space (S, B, μ) and a stochastic process $\{X_i\}_{i \in \mathcal{I}}$ such that $\widetilde{\mu}_I$ are the cylinder measures of μ. It follows that the $W_I \in \mathcal{W}_{|I|}$, $I \in \mathcal{I}$ lie dense in $L_2(S, d\mu)$.

Physical meaning: We consider the elements $p \in S$ to be space-time fields Φ or spatial fields ϕ, respectively. The index set $\mathcal{I}$ will have the meaning of a set of test functions or more generally distributions whose elements i label the random variable X_i. These map the fields smeared with test functions to a finite-dimensional manifold (usually copies of $\mathbb{R}$ or more generally of a Lie group). For instance, for a scalar field Φ, we may consider the random variable $X_F(\Phi) = \exp(i \int_{\mathbb{R} \times \sigma} d^4x F(x)\Phi(x))$ which takes values in $\widetilde{S} = U(1)$. It is also customary to consider the field $p = \Phi$ itself as a random variable indexed by the same index set or to simply write $X_i(p) = p(i)$ as an abbreviation.

4.2. Constructive QFT

The application of interest of the previous subsection is a stochastic process indexed by either $\mathbb{R}\times L$ or just by L, where the label set L is a certain set of distributions on the spatial manifold. We distinguish between random variables Φ indexed by a pair $(t, f)\in\mathbb{R}\times L$ and random variables ϕ indexed by $f\in L$. Some examples are as follows:

(1) Real quantum scalar fields with smooth smearing:

Consider $L=\mathcal{S}(\mathbb{R}^3)$, the space of smooth test functions of rapid decrease and $\widetilde{S}=\mathbb{R}$ equipped with the Borel σ−algebra. Then, $\phi(f)=\langle f,\ \phi\rangle=\int d^3x f(x)\,\phi(x)$ and $\Phi(t,f)=\langle f,\ \Phi(t,.)\rangle$. Given $F:=(f_1,\ ..,\ f_N)\in L^N$, consider $\phi(F)=(\phi(f_1),\ ..,\ \phi(f_N))\in\mathbb{R}^N$. The space $\mathcal{W}_N$ of elementary functions on N copies of $\mathbb{R}$ can be chosen to be generated by the exponentials

$$w_{r_1,..,r_N}(\phi(F))=\exp\left(i\sum_{k=1}^{N}r_k\langle f_k,\phi\rangle\right) \tag{4.11}$$

with $r_1,\ ..,\ r_N\in\mathbb{R}$ labelling the (necessarily one-dimensional) unitary irreducible representations of $U(1)$.

In fact, since in this case, the space L is a vector space, it is sufficient to consider the functions $w(\phi(f))=\exp(i\phi(f)),\ f\in L$. Analogously, the space of elementary functions for the time-dependent fields can be chosen as $(w_k\in\mathcal{W}_{N_k})$

$$W(\Phi(t_1,F_1),..,\Phi(t_T,F_T))=w_T(\Phi(t_T,F_T))..w_1(\Phi(t_1,F_1)), \tag{4.12}$$

which, of course, reduces to

$$\exp\left(i\Phi\left(t_T,f'_T\right)\right).\exp\left(i\Phi\left(t_1,f'_1\right)\right),\ f'_k\in L \tag{4.13}$$

for certain $f'_k\in L$. Obviously, the Abelian −*algebra and boundedness conditions are satisfied. That these elementary functions suffice to determine the cylindrical measures requires a more involved argument (Bochner's theorem, [134]).

(2) Real quantum scalar fields with distributional smearing:

Consider a subset $L:=\subset\mathcal{S}'(\mathbb{R}^3)$ of the tempered distributions and $\widetilde{S}=U(1)$ equipped with the Borel σ−algebra. In applications to scalar fields coupled to general relativity elements, $f\in L$ are typically δ−distributions supported at a single point.

Then, $\phi(f):=\exp(i\langle f,\ \phi\rangle)$ and $\Phi(t,f)=\exp(i\langle f,\ \Phi(t,.)\rangle)$, where $\langle f,\rangle>$ is the evaluation of $f\in L$ on ϕ. Given $F:=(f_1,..,f_N)\in L^N$, consider $\phi(F)=(\phi(f_1),..,\phi(f_N))\in U(1)^N$. The space $\mathcal{W}_N$ of elementary functions on N copies of $U(1)$ can be chosen to be generated by the exponentials

$$w_{r_1,..,r_N}(\phi(F))=\exp\left(i\sum_{k=1}^{N}r_k\langle f_k,\phi\rangle\right) \tag{4.14}$$

with $r_1,\ ..,\ r_N\in\mathbb{R}$ labelling the (necessarily one-dimensional) unitary irreducible representations of $U(1)$. Analogously, the space of elementary functions for the time-dependent fields can be chosen as $(w_k\in\mathcal{W}_{N_k})$

$$W(\Phi(t_1,F_1),..,\Phi(t_T,F_T))=w_T(\Phi(t_T,F_T)).w_1(\Phi(t_1,F_1)) \tag{4.15}$$

In this case, we could still equip L with the structure of a real vector space if we extend L to the finite real linear combinations $\widetilde{L}$ of its generating set L. Since this is no longer possible for the non-Abelian gauge theory example below, we will refrain from doing this, in order to highlight the structural similarity between the examples.

(3) Non-Abelian gauge fields for compact gauge groups G:

A form factor is a distribution

$$f^a_c(x)=\int_c dy^a\delta^{(3)}(x,y), \tag{4.16}$$

where c is a one-dimensional path in σ. We take $\widetilde{S}=G$ equipped with the natural Borel $\sigma-$*algebra* and

$$\begin{aligned}\phi(c):=\phi(f_c):&=\mathcal{P}\exp\left(\int_c\phi\right)=\mathcal{P}\exp(\phi(f_c));\ \phi(f_c)\\&=\int_\sigma d^3x f^a_c(x)\phi_a(x),\end{aligned} \tag{4.17}$$

where we have identified ϕ as a G connection and $\mathcal{P}$ denotes path ordering. Thus, **Eq. 4.17** is the direct analogue of the scalar field construction (note that the Lie generators are anti−self-adjoint since G is compact so that **Eq. 4.17** is unitary) and $\phi(f_c)$ is simply the holonomy of ϕ along c. Likewise,

$$\Phi(t,c):=\mathcal{P}(\exp\left(\int_c\Phi(t,.)\right) \tag{4.18}$$

Note that the form factors do not form a vector space; in general, they cannot be added (unless two curves share a boundary point), and they can never be multiplied by a non-integer real scalar (there is a certain groupoid structure behind this [17–21]). Accordingly, our space of generating set of elementary functions $\mathcal{W}_N$ on N copies of G need to be more sophisticated. We consider the space L of form factors, and for each $F=(f_{c_1},..,f_{c_N})\in L^N$, the 'pairing' $\phi(F)=(\phi(c_1),\ ..,\ \phi(c_N))\in G^N$. Then, a possible choice of generating set $\mathcal{W}_N$ of elementary functions is

$$w_\delta(\phi(F))=\prod_{k=1}^{N}\sqrt{d_{j_k}}\left[\pi_{j_k}(\phi(c_k))\right]_{m_k,n_k} \tag{4.19}$$

with $\delta:=\{(j_1,m_1,n_1),..,(j_N,m_N,n_N)\}$. In fact, it is sufficient to consider mutually disjoint (up to end points), piecewise real analytic curves c_k. Here, j labels an irreducible representation π_j of G of dimension d_j and $[\pi_j(g)]_{m,\ n};\ m,\ n=1,\ ..,\ d_j$ its matrix element functions. By the Peter and Weyl theorem, these functions suffice to determine the cylindrical measures uniquely at least if they are absolutely continuous with respect to the product Haar measure. Likewise, we consider the elementary functions

$$W_{\delta_1,..,\delta_T}(\Phi(t_1,F_1),..,\Phi(t_T,F_T))=w_{\delta_N}(\Phi(t_T,F_T)).w_{\delta_1}(\Phi(t_1,F_1)) \tag{4.20}$$

The fact that these functions satisfy all requirements is the statement of Clebsch–Gordan decomposition theory together with the properties of the holonomy to factorise along segments of a curve (note the piecewise analyticity condition).

This ends our list of examples. We will denote the measure related to the stochastic process $\{\Phi(t, f)\}$ by μ and the measure related to the stochastic process $\{\phi(f)\}$ by ν. As the notation suggests, Φ is a field defined on space-time $M = \mathbb{R} \times \sigma$, while ϕ is a field defined on space σ. Note that $M = \mathbb{R} \times \sigma$ with σ any 3D manifold is a consequence of the requirement of global hyperbolicity [176, 177].

The measures μ underlying a relativistic QFT are not only probability measures. In addition, they need to satisfy a set of axioms [98–100, 104] called Osterwalder–Schrader axioms which, however, are tailored to $M = \mathbb{R}^4$, stochastic processes with L being a vector space and with an Euclidean background metric at one's disposal. In quantum gravity and more generally in non-Abelian gauge theories, one typically must or may want to drop some of these structures. As a consequence, we will only keep those axioms that can also be applied in this more general context.

Some of them generalise to stochastic processes not indexed by a vector space, and some do not. Some generalise from the manifold $\mathbb{R}^4$ to the general space-time manifold $\mathbb{R} \times \sigma$ allowed by global hyperbolicity, and some do not. Fortunately, those that do generalise are sufficient for the reconstruction process [103]. We call them the minimal OS axioms, and we call a probability measure that satisfies them an OS measure.

An important remark is that the measures for gauge theories (such as general relativity) are to be formulated in terms of observable (gauge-invariant) fields which are typically composites of the elementary fields. That is why we work in a manifestly gauge (diffeomorphism)-invariant (equivalently, gauge-fixed) context as outlined in **Section 3**. In fact, in Ref. 39, we find an explicit formula that relates the observable composite fields to the elementary ones. The crucial condition is that the algebra of those observable fields is under sufficient mathematical control in order that Hilbert space representations can be found. This is the case for the construction sketched in **Section 3**.

The minimal set of OS axioms can be phrased as follows:

Let $\theta(t,\ x) := (-t,\ x)$ and $T_s(t,\ x) := (t + s,\ x)$ denote time reflection and time translation, respectively. Let $w_k \in \mathcal{W}_{N_k},\ k = 1, .., T,\ F_k \in L^{N_k},\ t_k \in \mathbb{R}$ and

$$\begin{aligned} &W_{(t_1,F_1),..,(t_T,F_T)} := w_T(\Phi(t_T,F_T)) \ldots w_1(\Phi(t_1,F_1)) \\ &R \cdot W_{(t_1,F_1),..,(t_T,F_T)} = W_{(-t_1,f_1),..,(-t_T,f_T)}, \\ &U(s) \cdot W_{(t_1,F_1),..,(t_T,F_T)} = W_{(t_1+s,F_1),..,(t_T+s,F_T)} \end{aligned} \tag{4.21}$$

Then, we have the following conditions on the generating functional

$$\mu(W(t_1,F_1),..,(t_T,F_T)) \tag{4.22}$$

I. Time reflection invariance:

$$\mu\left(W_{(-t_1,F_1),..,(-t_N,F_T)}\right) = \mu\left(W_{(t_1,F_1),..,(t_T,F_T)}\right) \tag{4.23}$$

II. Time translation invariance

$$\mu\left(W_{(t_1+s,F_1),..,(t_N+s,F_T)}\right) = \mu\left(W_{(t_1,F_1),..,(t_N,F_T)}\right). \tag{4.24}$$

III. Time translation continuity

$$\lim_{s\to 0} \mu\Big(\left[W_{(t_1,F_1),..,(t_T,F_T)}\right]^* W_{\left(t'_1+s,F'_1\right),..,\left(t'_{T'}+s,F'_{T'}\right)}\Big)$$

$$= \mu\Big(\left[W_{(t_1,F_1),..,(t_T,F_T)}\right]^* W_{\left(t'_1,F'_1\right),..,\left(t'_{T'},F'_{T'}\right)}\Big) \tag{4.25}$$

IV. Reflection positivity

Consider the vector space V of the complex span of functions of the form $W(t_1, F_1), .., (t_T, F_T)$ with $t_1, .., t_T > 0$. Then, for any Ψ, $\Psi' \in V$,

$$\langle \Psi,\ \Psi' \rangle := \mu\Big(\overline{\Psi}\, R \cdot \Psi'\Big),\ \langle \Psi,\ \Psi \rangle \geq 0 \tag{4.26}$$

Note that the stochastic process indexed by $\mathbb{R} \times L$ considers random variables $\Phi(t, f)$ at sharp points of time. It is often argued that this index set provides an insufficient 'smearing' in the time direction and fails to cover interacting QFT at least in 3 + 1 space-time dimensions (in 1 + 1 and 2 + 1 dimensions, there are examples for which this works [178–180]). However, this argument rests on perturbative results as on 3 + 1-dimensional Minkowski space; so far, no interacting QFT (obeying the Wightman axioms) has been rigorously constructed. It is still conceivable [181] that in a non-perturbative construction of the theory, for which constructive QFT is designed, one can deal with fields at sharp time. One could, of course, be more general and consider stochastic processes indexed by some L which now also includes smearing in the time direction, and the formulation of reflection positivity will then constrain to elements of L with positive time support; however, then the Wiener measure construction sketched below will not work. Our viewpoint is that this more general situation can be obtained from the sharp time construction because integrals of smearing functions with respect to time can be approximated by Riemann sums, which in turn are nothing but integrals with respect to sharp time smearing functions.

At the moment, it is rather unclear how and why μ and Φ define a relativistic QFT. This will become clear in the next subsection.

4.3. Osterwalder–Schrader (OS) Reconstruction

The following abstract argument is standard [98–100]. (See Refs. 54–57 for a proof adapted to the notation in this article.) Due to reflection positivity, **Eq. 4.26** defines a positive semi-definite

sesquilinear form on V. We compute its null space N and complete the quotient of equivalence classes V/N in the inner product **Eq. 4.24** to a Hilbert space. Given $\Psi \in V$, we denote its equivalence class $\Psi + N$ by $[\Psi]_\mu$, where we keep track of the measure dependence of the quotient construction. By construction the $D = [V]_\mu$ is dense in $\mathcal{H}$. Since the constant function $\Psi = 1 \in V$, we define a 'vacuum' vector by $\Omega := [1]_\mu$. Finally, we define for $s \geq 0$

$$K(s)[\Psi]_\mu := [U(s)\Psi]_\mu \tag{4.27}$$

The constraint $s \geq 0$ is due to the time support condition in the definition of V. One must show that this is well-defined (independent of the representative) [98–100]. By virtue of their definition (**Eq. 4.2**), the $U(s)$ forms a one-parameter Abelian group of operators $U(s)U(s') = U(s+s')$ on $L_2(S, d\mu)$. This implies that the $K(s)$ forms a one-parameter Abelian semi-group due to the constraint $s \geq 0$ (again one must show that the definition is well-defined). Time translation continuity (**Eq. 3.23**) translates into weak continuity of the semi-group. Furthermore, by time translation invariance 4.24, $U(s)$ defines unitary, in particular bounded operators, on $L_2(S, d\mu)$ which translates into the statement that $K(s)$ forms a contraction semi-group. Thus [98–100], there exists a positive self-adjoint operator H, called 'Hamiltonian' on $\mathcal{H}$ such that $K(s) = e^{-sH}$. Obviously, $K(s)\Omega = \Omega$; thus, Ω is a ground state for H which justifies the name 'vacuum'.

This elegant argument is deceivingly simple. To actually compute the *Osterwalder–Schrader triple* $(\mathcal{H}, \Omega, H)$ from μ and to relate it to the fields and Hamiltonian in terms of which one would construct the quantum theory using canonical quantisation is not clear yet. However, one can again use the following abstract argument [54–57]. Suppose that there is an Abelian C^*–algebra $\mathfrak{B}$ of bounded operators on $\mathcal{H}$ such that $\mathfrak{B}\Omega$ is dense (the C^*– norm is inherited from the uniform operator topology). It is not difficult to show that this is always the case when $\mathcal{H}$ is separable which is the only case that we will consider in our application to renormalisation, but it also holds in many non-separable situations)see appendix B of Ref. 196 for a proof). Let $\Delta(\mathfrak{B})$ be its Gel'fand spectrum [182] (which is a compact space), that is, the space of all * homomorphisms $\phi: \mathfrak{B} \to \mathbb{C}$. Then, by Gel'fand's theorem, $\mathfrak{B}$ can be thought of as the space $C(\Delta(\mathfrak{B}))$, that is, the continuous functions on the spectrum which is an Abelian C^*–algebra with respect to the sup norm. The correspondence (Gel'fand isomorphism) is given by $\widehat{b}(\phi) = \phi(b)$ for all $\phi \in \Delta(\mathfrak{B})$, and in fact, this is an isometric isomorphism of C^*–algebras. Consider now the linear functional

$$\nu\left(\widehat{b}\right) := \langle \Omega, b\Omega \rangle \tag{4.28}$$

which by construction is positive $\nu(|\widehat{b}|^2) = ||b\Omega||^2$. By the Riesz–Markov theorem [175], there exists a (regular Borel) probability measure on $S_I := \Delta(\mathfrak{B})$ which by abuse of notation we also denote by ν such that

$$\nu\left(\widehat{b}\right) = \int_{\Delta(\mathfrak{B})} d\nu(\phi)\, \widehat{b}(\phi), \tag{4.29}$$

that is, to say, the Hilbert space $\mathcal{H}$ obtained from OS reconstruction can be thought of as $L_2(\Delta(\mathfrak{B}), d\nu)$ under the isomorphism $b\Omega \mapsto \widehat{b}$, in particular Ω corresponds to the constant function equal to 1. We thus have managed to cast $\mathcal{H}$ into the language of measure theory on the set $S' = \Delta(\mathfrak{B})$. The fields ϕ that come out of this construction are random variables indexed by some index set L', that is, we have shown that we can always construct such a measure and a corresponding stochastic process. We think of the field ϕ as the spatial configuration fields underlying a canonical quantisation approach. A priori, however, it is not clear what L' is, although it must be related in some way to $\mathbb{R}^+ \times L$. In the case of free fields, one can show that, in fact, one can choose $\mathfrak{B}$ in such a way that $L' = L$ due to the quotient construction involved in $\mathcal{H}$ but even then it is a priori not clear how $\Phi(t, f)$ and $\phi(f)$, $f \in L$ are related. Again, in the case of free fields, one shows that $\phi(f)$ can be thought of as $\Phi(0, f)$, the space-time field at sharp time zero. However, in general, the relation between the stochastic processes underlying Φ and ϕ may be more complex. In any case, the operator H translates in this language into the operator

$$\widehat{H}\widehat{b} := \widehat{Hb\Omega} \tag{4.30}$$

4.4. Feynman–Kac–Trotter–Wiener (FKTW) Construction

Given an OS triple $(\mathcal{H}, \Omega, H)$, we saw at the end of the previous subsection that without loss of generality, we can assume that $\mathcal{H} = L_2(S_I, d\nu)$, where ν is a probability measure on S equipped with a Borel σ–algebra and that we are given a stochastic process $\phi(f)$, $f \in L$ indexed by some index set L, at least when $\mathcal{H}$ is separable (which will be the case in our applications). Moreover, $\Omega = 1$ in this presentation of $\mathcal{H}$ is cyclic for some C^*–algebra of functions on S_I. We pick some set $\mathcal{W}_N$, $N \in \mathbb{N}_0$ of elementary functions $w \in \mathcal{W}_N$ subject to the conditions 1.-4. spelled out just after (4.7) and for $F = (f_1, .., f_N) \in L^N$ have $\phi(F) = (\phi(f_1), .., \phi(f_N)) \in (S_I)^N$ as well as

$$w_F(\phi) = w(\phi(F)) \tag{4.31}$$

Let now $T \in \mathbb{N}_0$, $t_1 < t_2 < .. < t_T$ and $F_k \in L^{N_k}$, $w_k \in \mathcal{W}_{N_k}$. We consider the expectation value functional

$$\langle \Omega, w_{T,F_T}\, e^{-(t_T - t_{T-1})H}\, w_{T-1,F_{T-1}}\, e^{-(t_{T-1}-t_{T-2})H} \ \ldots\ e^{-(t_2-t_1)H}\, w_{1,F_1}\, \Omega \rangle \tag{4.32}$$

Consider now a stochastic process $\Phi(s, f)$ indexed by $(s, f) \in \mathbb{R} \times L$ and the elementary functions

$$W_{(t_k,F_k)_{k=1}^T}(\Phi) = w_{T,F_T}(\Phi(t_N, .)) .. w_{1,F_1}(\Phi(t_1, .)) \tag{4.33}$$

Then, the Wiener measure μ, if it exists, evaluated on **Eq. 4.33**

$$\mu\left(W_{(t_k,F_k)_{k=1}^T}\right) \tag{4.34}$$

is supposed to equal **Eq. 4.31**. The non-trivial question is why this should be the case, under which circumstances, and how to construct μ. For this, we consider the integral kernel K_β of the operator $e^{-\beta H}$, $\beta > 0$, that is,

$$\left[e^{-\beta H}\psi\right](\phi) =: \int_S d\nu(\phi')\, K_\beta(\phi,\phi')\,\psi(\phi') \tag{4.35}$$

Note the semi-group property

$$\int_{S'} d\nu(\phi)\, K_{\beta_1}(\phi_1,\phi) K_{\beta_2}(\phi,\phi_2) = K_{\beta_1+\beta_2}(\phi_1,\phi_2) \tag{4.36}$$

Define $S := \prod_{t\in\mathbb{R}} S'$. For each $T \in \mathbb{N}_0$, consider $t_1 < .. < t_T$ and measurable sets $s'_{t_k} \subset S'$ and define the set function

$$\mu\left(\left[\times_{k=1}^T s'_{t_k}\right] \times \left[\times_{t\notin\{t_1,..,t_T\}} S'\right]\right) := \int_{[S']^T} d\nu(\phi_1)\ldots d\nu(\phi_T) \prod_{k=1}^T \chi_{s'_k}(\phi_k) \prod_{k=1}^{T-1} K_{t_{k+1}-t_k}(\phi_{k+1},\phi_k) \tag{4.37}$$

It is not clear that this is a positive set function, but when it is, it is called the Wiener measure generated by the OS triple. For sufficient criteria for this property called Nelson-Symanzik positivity in the case of scalar fields (see Refs. 183 and 184). Basically, one needs to show that matrix elements of $e^{-\beta H}$ between positive functions are positive. Note that for $s_K{}' = S'$ for all k, we get

$$\mu(S) = \langle\Omega,\, e^{-(s_T-s_1)H}\Omega\rangle = 1 \tag{4.38}$$

This shows that μ is a probability measure on S. For quantum mechanical Schrödinger Hamiltonians, one can use the Trotter product formula and the Wiener measure of the heat kernel to prove positivity [185] (Feynman–Kac formula).

One can now show the following [54–57]:

Theorem.

i. Suppose that OS data ($\mathcal{H}$, H, Ω) are given and that the corresponding Wiener measure μ exists. Then, μ is an OS measure and its OS reconstruction reproduces the given OS data up to unitary equivalence.
ii. Suppose that an OS measure μ is given thus producing OS data ($\mathcal{H}$, Ω, H). Then, the corresponding Wiener measure exists and reproduces μ up to equivalence of measure spaces.

Here, measure spaces (S_j, B_j, μ_j); $j = 1, 2$ are called equivalent if there exists a bijection F: $S_1 \to S_2$ such that both F and F^{-1} are measurable and such that $\mu_1 = \mu_2 \circ F$. The reason why we generically only reproduce an equivalent and not an identical starting point lies in the large freedom in the choice of the stochastic process ϕ when performing the OS reconstruction step.

5. RENORMALISATION

5.1. Motivation

Our motivation for renormalisation comes from the current state of affairs with respect to the definition of the quantum dynamics in LQG as outlined in **Section 3**. In that case, the Hilbert space $\mathcal{H} = L_2(S_\prime, d\nu)$ is precisely of the form we envisage here. Moreover, we have a vacuum Ω for a candidate Hamiltonian H that, however, we are not sure whether all steps of the quantisation process that led to H are justified, namely, we have defined H as H_γ on certain mutually orthogonal subspaces $\mathcal{H}_\gamma$ preserving it using a choice of discretisation of the classical continuum expression which has naively the correct dequantisation if the graph γ fills the spatial manifold σ sufficiently densely. The definition of elementary functions in **Eq. 4.19** precisely reproduces the SNWF, and thus, the spatial connection defines a stochastic process indexed by graphs.

As already mentioned at the end of **Section 3**, we would like to take a fresh look at the problem. As usual in constructive QFT, if σ is not already compact, we replace it with a compact manifold σ_R, where R is an infrared (IR) cut-off which we remove in the end $R \to \infty$ (thermodynamic limit). In order not to clutter the notation, the dependence on R of all considerations that follow will be suppressed. Next, we do not consider all finite graphs γ (taking all finite graphs leads to a non-separable Hilbert space) but only a controllable countable family $\mathcal{M}$, therein which, however, is such that the discretised classical variables (configuration and momentum fields) in terms of which we perform the quantisation separate the points of the classical phase space when all the graphs in $\mathcal{M}$ are at our disposal. The set $\mathcal{M}$ is supposed to be partially ordered and directed. The motivation for doing so stems from the spatial diffeomorphism invariance of the classical LQG Hamiltonian: The algebraic form of the Hamiltonian discretised on diffeomorphic graphs is identical. This is precisely the starting point of the algebraic quantum gravity proposal [169–172], where it was emphasised that one can quantise gravity in terms of abstract graphs which gain their physical meaning only after choosing an embedding supplied, for instance, by a semi-classical state.

To have some intuitive picture in mind, consider $\sigma = \mathbb{R}^3$ with toroidal compactification $\sigma_R = T^3$ (where each direction has length R with respect to the Euclidian background metric on $\mathbb{R}^3$ and with periodic boundary conditions installed) and Γ the set of all finite graphs σ_R of cubic topology. This is still an uncountable set which we now restrict to a countable one as follows. Each element of Γ is uniquely labelled by $M \in \mathbb{N}$, where M^3 is the number of vertices of the graph (one could generalise this and have different numbers of vertices in each direction). We pick once and for all a coordinate system and locate the vertices of γ_M at the points

$$m\epsilon_M,\ m \in \mathbb{Z}^3_M,\ \mathbb{Z}_M = \{0,1,..,M-1\},\ \epsilon_M = \frac{R}{M}, \tag{5.1}$$

where the edges of the graph are straight lines in the coordinate directions between the vertices. We equip $\mathcal{M} := \mathbb{N}$ with the following partial order: $M < M'$ iff $\frac{M'}{M} \in \mathbb{N}$. Note that this implies $\gamma_M \subset \gamma_{M'}$ since

$$m\epsilon_M = m\frac{M'}{M}\epsilon_{M'} =: m'\epsilon_{M'} \tag{5.2}$$

with $m' \in \mathbb{Z}^3_{M'}$ and because the edges of the graphs are straight lines in the coordinate directions. This is certainly not a linear order because not all natural numbers are in relation but still

equips Γ with a direction: Given M and M' take, for instance, $M'' = MM'$, then $M, M' < M''$ (more efficiently take M'' as the least common multiple). It is clear that for M sufficiently large discretised phase space variables obtained by integrating continuum variables over 0- or 1-dimensional subsets of γ_M (vertices or edges) or by integrating momentum variables over 3- or 2-dimensional subsets of the cell complex corresponding to γ_M (faces and cubes) will separate the points of the continuum phase space. Instead of γ_M, one could also use the cubic cell complex γ_M^* dual to γ_M defined by saying that the barycentres of the cubes of γ_M^* coincide with the vertices of γ_M. However, in the spirit of economy, we will not use the additional structure γ_M^* in what follows.

5.2. Discretisation of Phase Space

In canonical quantisation, we start with a continuum phase space coordinatised by configuration fields ϕ^J and canonically conjugate momentum fields π_J in terms of which the classical continuum Hamiltonian H is formulated. Here, the index J corresponds to an internal symmetry and is typically Lie algebra valued. Now, we consider a discretisation of both the phase space and the Hamiltonian, one for each lattice M, while keeping track of how these fields ϕ_M^J and π_J^M are related to the continuum fields ϕ^J and π_J. The idea for how to do this stems from the observation that by construction of generally covariant field theories, the fields ϕ^J and π_J are *dual* in the sense that there is a natural bilinear form $\langle \pi, \phi\rangle_{I'\times K'} := \sum_J \langle \pi_J, \phi^J\rangle_{I\times K}$ on the phase space (usually a cotangent bundle T^*K') $I'\times K'$ of momentum and configuration fields, respectively, where $\langle .,.\rangle$ is spatially diffeomorphism-invariant. Note that $\langle .,.\rangle'$, $\langle .,.\rangle$ just differ by tracing over the internal directions in field space, that is, $I' = I'^d$, $K' = K'^d$, where d is the number of internal directions in field space.

For instance, the momentum of a scalar field is geometrically a scalar density of weight one, so that $\langle \pi, \phi\rangle' = \langle \pi, \phi\rangle := \int_\sigma d^3x\, \pi(x)\,\phi(x)$. The momentum of a G connection is geometrically a Lie algebra-valued vector field density so that $\langle \pi, \phi\rangle' = \sum_J \int_\sigma d^3x\, \pi_J^a(x)\phi_a^J(x)$ This also holds for higher p–forms as they occur in some supergravity theories as well as for (standard model or Rarita–Schwinger) fermions. Note that the bilinear form is in general not invariant under the internal symmetry group, but this will not be important for what follows. The fact that π and ϕ are conjugate is the statement, that their canonical brackets are

$$\{\langle \pi, k'\rangle', \langle i', \phi\rangle'\} = \langle i', k'\rangle' \tag{5.3}$$

for all $(i', k') \in I'\times K'$.

The fact that the bilinear form $\langle .,.\rangle$ is at our disposal motivates a natural choice for the index set L and L^* of the stochastic process ϕ, π. Namely, we choose L to be a certain distributional extension of I and likewise L^* as a certain distributional extension of K. These extensions should be such that $\langle i, k\rangle$ remains well-defined for $i \in L$, $k \in L^*$. For instance, for a scalar field we may choose L as the set of δ distributions $f_p(x) = \delta_p(x)$ with support at single points $p \in \sigma$ and L^* as the set of characteristic functions $g^R(x) = \chi_R(x)$ of connected D–dimensional submanifolds R of σ. For a compact G–connection, we can choose L as the set of form factors $f_c^a(x) := \int_C dy^a \delta(x,y)$ with support on (piecewise analytic) curves c. For L^*, we would consider the set of dual form factors of the form $g_a^S(x) := 1/(D-1)!\int_S \epsilon_{ab_1..b_{D-1}} dy^{b_1}\wedge..dy^{b_{D-1}}\delta(x,y)$ with support on (piecewise analytic) $D-1$ submanifolds S. We may also have opportunity to consider their Lie algebra-valued versions $f_c^{aJ}(x) = \tau^J f_c^a(x) \in L'$, $g_{aJ}^S(x) = \tau_J g_a^S(x) \in (L')^*$, where τ^J and τ_J are dual bases in the defining representation of the Lie algebra of G such that $\mathrm{Tr}(\tau^J\tau_K) = \delta_K^J$. Note that we deliberatively do not make use of the fact that these distributions are elements of vector spaces. This is because we aim at a uniform description of both linear and non-linear theories. In the case of linear theories, the description can be significantly simplified as we have done in Refs. 54–57.

The connection to **Section 4.2** is then as follows: For each $f \in L$, $\phi \in K$, we consider a map $(f, \phi)\mapsto \phi(f) \in \bar{S}$. For linear theories, one usually takes $\bar{S} = U(1)$, and for a G gauge theory, one takes $\bar{S} = G$. The object $\phi(f)$ exploits the existence of the natural bilinear form $\langle .,.\rangle$. For instance, for a scalar field, one considers $\phi(f_p) = \exp(i\langle f_p, \phi\rangle)$, while for a G connection, we consider the holonomy $\phi(f_c) = \mathcal{P}\exp(\langle f_c, \phi^J\rangle\tau_J)$. For each $N \in \mathbb{N}$, we consider $F = (f_1, .., f_N) \in L^N$ and define $\phi(F) = (\phi(f_1), .., \phi(f_N)) \in \bar{S}^N$. The space of elementary functions $\mathcal{W}_N$ consists of maps $\bar{S}\to\mathbb{C}$ subject to the conditions listed in the beginning of **Section 4.2**. We may generate $\mathcal{W}_N$ from monomials labelled by matrix element functions of finite-dimensional unitary representations of $\bar{S}$ (see **Eq. 4.19**).

For each $M \in \mathbb{N}$, let L_M be the space of discrete functions on the lattice consisting of M^D points with values in $\mathbb{R}^{\mathbf{t}}$, where $\mathbf{t}$ is tensorial number of configuration (or momentum) degrees of freedom per spatial point ($\mathbf{t} = 1$ for scalar fields, $\mathbf{t} = D$ for a G Yang–Mills theory in $D+1$ space-time dimensions, etc.). That is, an element $l_M \in L_M$ assigns to each point $m \in \mathbb{Z}_M^D$ a vector in $\mathbb{R}^{\mathbf{t}}$. The space L_M carries an auxiliary real Hilbert space structure (L_M is, of course, a finite-dimensional vector space), for example, for a G Yang–Mills theory,

$$\langle l_M, \tilde{l}_M\rangle_{L_M} = \sum_{m\in\mathbb{Z}_M^D}\sum_{a=1}^{\mathbf{t}} l_M(m,a)\,\tilde{l}_M(m,a) \tag{5.4}$$

for any $l_M, \tilde{l}_M \in L_M$, and we wrote $[l_M]^a(m) =: l_M(m, a)$.

Definition.

A discretisation of the continuum phase space $I\times K$ subordinate to $M \in \mathbb{N}$ is a pair of linear maps

$$I_M:\ L_M\to L;\ K_M:\ L_M\to L^* \tag{5.5}$$

with the following properties:

i. For any $l_M, l_M' \in L_M$

$$\langle I_M l_M, K_M l_M'\rangle_{I\times K} = \langle l_M, l_M'\rangle_{L_M} \tag{5.6}$$

That is, to say $I_M' K_M = K_M' I_M = \mathrm{id}_{L_M}$ where $I_M'\colon I\to L_M$, $K_M'\colon K\to L_M$ are the dual maps defined by

$$\langle I_M l_M, \phi\rangle_{I\times K} = \langle l_M, I_M'\phi\rangle_{L_M},\ \langle \pi, K_M l_M\rangle_{I\times K} = \langle K_M'\pi, l_M\rangle_{L_M}, \tag{5.7}$$

ii. For any $M < M'$ define the injection maps

$$I_{MM'} := K_{M'} I_M;\ K_{MM'} := I_{M'}' K_M:\ L_M \to L_{M'} \tag{5.8}$$

Then, we require

$$I_{M'} I_{MM'} = I_M,\ K_{M'} K_{MM'} = K_M \tag{5.9}$$

To see how this gives rise to discretised configuration and momentum variables let $\delta_M^{m,}\ a,\ \delta_{m,}\ a^M \in L_M$ with $m \in \mathbb{Z}_M^D,\ a = 1, ..\mathbf{t}$ be the Kronecker functions $[\delta_M^{m,}\ a]_b(\tilde{m}) :=\ \delta_b^a\ \delta_{m,}\ \tilde{m}$ and $[\delta_{m,}\ a^M]^b(\tilde{m}) :=\ \delta_a^b \delta_{m,}\ \tilde{m}$. Then, the following functions on the continuum phase space

$$(\pi_M)_J^a(m) := \langle \pi_J,\ K_M \delta_M^{m,a} \rangle_{I\times K},\ (\phi_M)_a^J(m) :=\ \langle I_M \delta_M^{m,a},\ \phi^J \rangle_{I\times K} \tag{5.10}$$

enjoy canonical brackets

$$\left\{ (\pi_M)_J^a(m),\ (\phi_M)_b^K(\tilde{m}) \right\} = \delta_b^a\ \delta_J^K\ \delta_{m,\tilde{m}} \tag{5.11}$$

where the first condition (**Eq. 5.6**) was used. Thus, **Eq. 5.6** makes sure that the discretisations **Eq. 5.10** enjoy canonical brackets, so we call **Eq. 5.6** the *symplectomorphism property*. The motivation for the second condition **Eq. 5.9** will become clear only later; however, we note that it implies that for all $M < M' < M''$

$$I_{M'M''} I_{MM'} = K_M' [I_{M'} I_{MM'}] = I_{MM''} \tag{5.12}$$

which we thus call *cylindrical consistency property*. Likewise, $K_{M'M''} K_{MM'} = K_{MM''}$. It says that injecting a function into the continuum is independent from which resolution scale M this is done.

Finally, we will impose a further restriction on the maps I_M, K_M, which amounts to a convenient choice of normalisation and thus is called *normalisation property*. Namely, we require that for all $M < M'$, the map $I_{MM'}:\ L_M \to L_{M'}$ restricts to $B_M \to B_{M'}$, where B_M is the set of functions on $\mathbb{Z}_M^D$ with values in the *bit space* $\{0, 1\}^{\mathbf{t}}$. This condition is only necessary in the non-Abelian case, and there avoids overcounting.

We note that **Eq. 5.9** defines elements $\pi_M = K_M \pi,\ \phi_M = I_M \phi$ of $L_M^{d\mathbf{t}}$ that we can now use to try to define a discretisation $H_M = H_M[\{(\pi_M)_j^a(m),\ (\phi_M)_a^j(m)\}_{a,j,m}]$ of the Hamiltonian $H = H[\pi,\ \phi]$. For instance, if the Hamiltonian depends only quadratically on the fields, then one may try (including discretisations of spatial derivatives and some spatial averages)

$$H_M := H\left[\pi = I_M \pi_M, \phi = K_M \phi_M\right] \tag{5.13}$$

For interacting Hamiltonians, more sophisticated approximations must be used. Certainly, the expression for H_M is in general plagued by a large amount of discretisation ambiguity beyond the choice of discretised variables. On the other hand, the fact that $\pi_M = K'_M \pi$ and $\phi_M = I_M' \phi$ are conjugate will be convenient when constructing H_M, and it is efficient to construct them motivated by the naturally available bilinear form $\langle ., . \rangle$' on the phase space.

To see that there are non-trivial examples for such maps, consider a scalar field in D spatial dimensions compactified on a torus with Euclidian coordinate length R in all directions. Then (recall $\epsilon_M = R/M$),

$$(I_M l_M)(x) :=\ \sum_{m \in \mathbb{Z}_M^D} l_M(m) \delta_{m\epsilon_M}(x),\ (K_M l_M)(x) :$$
$$=\ \sum_{m \in \mathbb{Z}_M^D} l_M(m) \chi_{m\epsilon_M}(x), \tag{5.14}$$

where

$$\chi_{m\epsilon_M}(x) = \prod_{a=1}^D \chi_{[m^a \epsilon_M, (m^a+1)\epsilon_M)}(x), \tag{5.15}$$

where the latter denotes the characteristic functions of left closed—right open—intervals. This clopen interval structure is very important in order that **Eqs 5.6** and **5.9** are satisfied [54–57]. Similar constructions work for gauge fields (see appendix A or [196]). Note that we changed here the notation as compared to [54–57]: The maps I_M, E_M used there are called here K_M, I'_M, respectively. The motivation for this change of notation is to make it manifest how much of the structure is in fact already canonically provided by the structure of the classical theory.

Given the lattice in D spatial dimensions labelled by $M \in \mathbb{N}$, we consider in general $N = M^D \mathbf{t}$ degrees of freedom $\phi(I_M l_M) =: \phi_M(l_M) :=\ \{\phi(I_M l_M^{m,a})\}_{m \in \mathbb{Z}_M^D,\ a = 1, .., \mathbf{t}} \in \widetilde{S}^N$, where $l_M^{m,a} = l_M\ \delta_M^{m,a}$ and l_M is restricted to the subset $B_M \subset L_M$ of functions $\mathbb{Z}_M^D \to F_2^{\mathbf{t}}$, where $F_2 = \{0, 1\}$ is the field in two elements (*bit space*). Thus, $l_M(m,\ a) \in \{0, 1\}$ is restricted to the information whether the degree of freedom $\phi(I_M l_M^{m,a})$ is excited or not. This is justified because 1. the missing information about the strength of the excitation is encoded in the representation label (see below) and 2. because the maps $I_{MM'}$ restrict to maps $B_M \to B_{M'}$ by assumption.

The space of elementary functions $\mathcal{W}_M$ on the lattice labelled by M is then generated by

$$w_{j,n,\tilde{n}}^M(\phi_M(l_M)) = w_{j,n,\tilde{n}}^M(\phi(I_M l_M)) = \prod_{m,a} \left[\pi_{j_{m,a}}(\phi(I_M l_M^{m,a}))\right]_{n_{m,a},\tilde{n}_{m,a}} \tag{5.16}$$

Here, $j_{m,a}$ labels an irreducible representation $\pi_{j_{m,a}}$ of G (one from each equivalence class), $d_{j_{m,a}}$ is its dimension, and $[\pi_{j_{m,a}}(.)]_{n_{m,a}}$, $\tilde{n}_{m,a}$ denote its matrix elements with $n_{m,a}, \tilde{n}_{m,a} \in \{1, .., d_{j_{m,a}}\}$.

To see how **Eq. 5.16** interacts with the map $I_{MM'}$ in the case of non-Abelian gauge theory, we note that the cylindrical consistency property of $I_{MM'}$ implies

$$w_{j,n,\tilde{n}}^M(\phi_M(l_M)) = w_{j,n,\tilde{n}}^M(\phi(I_M l_M)) = \sum_\alpha w_{j',n'_\alpha,\tilde{n}'_\alpha}^{M'}(\phi_{M'} I_{MM'} l_M)), \tag{5.17}$$

where the notation is as follows (see appendix A or [196]): $j_{m'a'} = [I_{MM'} l_M\ ](m', a) j_{[m'/M'M],\ a}$, where [.] denotes the Gauss bracket, $n'_{m',\ a} = n_{m,\ a}$ if $m' = M'/Mm,\ \tilde{n}'_{m,a} = \tilde{n}_{m,a}$ if $m'^a + 1 = M'/M(m^a + 1),\ m'^b = M'/Mm^b; b \neq a$, and otherwise the sum over α denotes the sum over all $\tilde{n}_{m'a'} = n_{m'+\delta_a,\ a'} \in \{1, .., d_{j'_{m',}\ a}\}$ with $[\delta_a]^b :=\ \delta_a^b$ that arise by writing the holonomy along the edge labelled by $m \in \mathbb{Z}_M^D,\ a = 1, .., D$ as products of holonomies along edges labelled by $m' \in \mathbb{Z}_{M'}^D,\ a$.

In general, therefore we see that for any generating function $w^M \in \mathcal{W}_M$, we have for all $M < M'$

$$w^M(\phi_M(l_M)) = \sum_\alpha z_\alpha\, w^{M'}_\alpha(\phi_{M'}(I_{MM'}l_{M'})), \tag{5.18}$$

where the sum over α involves a finite, unique set of generating functions $w^{M'} \in \mathcal{W}_{M'}$, and z_α are certain, definite complex numbers. Similar statements then, of course, hold for the stochastic process labelled by $\mathbb{R} \times L$ and for the functions

$$W^M\left(\Phi_M\left(t^{(1)}, l^{(1)}_M\right), .., \Phi_M\left(t^{(T)}, l^{(T)}_M\right)\right) = w^M_T\left(\Phi_M\left(t^{(T)}, l^{(T)}_M\right)\right) \ldots w^M_1\left(\Phi_M\left(t^{(1)}, l^{(1)}_M\right)\right) \tag{5.19}$$

5.3. Hamiltonian Renormalisation

Abstracting from the concrete lattice implementation and field content above, we are in the following situation: There is a partially ordered and directed label set $\mathcal{M}$, and for each $M \in \mathcal{M}$, we have a map I_M: $B_M \to L^{N(M)}$, where L is the index set of the stochastic process ϕ, $N(M) \in \mathbb{N}$ is the number of elements of L in the image of I_M, and $B_M = \{0, 1\}^{N(M)}$. Then, $\phi_M(l_M)$: $= \phi(I_M l_M) \in \widetilde{S}^{N(M)} =: \widetilde{S}_M$, and we have a generating set of elementary functions w^M: $\widetilde{S}^{N(M)} \to \mathbb{C}$.

Suppose that for each $M \in \mathcal{M}$, we have discretised the system somehow as sketched above and picked some OS triple $(\mathcal{H}^{(0)}_M, \Omega^{(0)}_M, H^{(0)}_M)$ with $\mathcal{H}^{(0)}_M = L_2(d\nu^{(0)}_M, \widetilde{S}_M)$. That is to say, we have a stochastic process $\{\phi_M(l_M)\}_{M\in\mathcal{M}}$ indexed by B_M and probability measures $\nu^{(0)}_M$ on $\widetilde{S}_M$. The Hamiltonian $H^{(0)}_M$ preserves $\mathcal{H}^{(0)}_M$ and annihilates the unit vector $\Omega^{(0)}_M \in \mathcal{H}^{(0)}_M$, which is cyclic. We consider a space of elementary functions $\mathcal{W}_M$ such that in particular $w^M(\phi_M(l_M))\Omega^{(0)}_M$; $w^M \in \mathcal{W}_M$, $l_M \in B_M$ lie dense in $\mathcal{H}^{(0)}_M$.

Using the Feynman–Kac–Trotter–Wiener (FKTW) construction, we obtain a family of OS measures $\mu^{(0)}_M$ on $S_M = \prod_{t\in\mathbb{R}} \widetilde{S}_M$, which can be probed using a stochastic process $\Phi_M(t, l_M)$ labelled by $\mathbb{R} \times B_M$. This measure family $\{\mu^{(0)}_M\}_{M\in\mathcal{M}}$ will generically not be cylindrically consistent and therefore does not define a continuum measure μ because of the discretisation ambiguities involved in the construction of $H^{(0)}_M$ which determines $\mu^{(0)}_M$. If it was, then we would have for $w_1, .., w_T \in \mathcal{W}_M$

$$\mu\left(w_T\left(\Phi\left(t_T, I_M l^T_M\right)\right) \ldots w_1\left(\Phi\left(t_1, I_M l^1_M\right)\right)\right) = \mu_M\left(w_T\left(\Phi_M\left(t_T, l^T_M\right).w_1\left(\Phi_M\left(t_1, l^1_M\right)\right)\right) \tag{5.20}$$

Using $I_M = I_{M'} I_{MM'}$ for $M < M'$, we would find the identity

$$\mu_M\left(w_T\left(\Phi_M\left(t_T, l^T_M\right).w_1\left(\Phi\left(t_1, l^1_M\right)\right)\right)\right) = \mu_{M'}\left(w_T\left(\Phi_{M'}\left(t_T, I_{MM'}l^T_M\right).w_1\left(\Phi_{M'}\left(t_1, I_{MM'}l^1_M\right)\right)\right)\right) \tag{5.21}$$

called the condition of cylindrical consistency.

As reviewed in **Section 3**, condition **Eq. 5.21** grants the existence of μ under rather generic conditions. The strategy (see also Refs. 137 and 138) is therefore to construct an iterative sequence of measure families $\mathbb{N}_0 \ni n \mapsto \{\mu^{(n)}_M\}_{M\in\mathcal{M}}$ *called renormalisation (group) flow* with initial family as above such that the fixed point family satisfies **Eq. 5.21**. We refer to section C of [196] for the reader interested in more notions of the renormalisation group in the language of measure theory.

The scheme that we will employ in fact does not make use of **Eq. 5.21** for all $M < M'$ but only $M' = p^n M$, where p is a prime. The simplest choice is $p = 2$, but we have tested the formalism also for $p = 3, 5$ [54–57] and mixtures thereof in the case of free scalar QFT. This, in fact, does cover all possible M because any natural number can be written as kp^l; k, p relative prime, but the fixed point family could depend on k. Of course, one assumes that the fixed point family is independent of the choices of p, k as an expression of universality as confirmed again for simple systems [54–57]. Thus, we define as renormalisation flow

$$\mu^{(n+1)}_M\left(w_T\left(\Phi_M\left(t_T, l^N_M\right).w_1\left(\Phi\left(t_1, l^1_M\right)\right)\right)\right) = \mu^{(n)}_{M'}\left(w_T\left(\Phi_{M'}\left(t_T, I_{MM'}l^N_M\right)..w_1\left(\Phi_{M'}\left(t_1, I_{MM'}l^1_M\right)\right)\right)\right) \tag{5.22}$$

for M': $= 2M$.Having then obtained μ^* from cylindrically consistent projections μ^*_M, we want to construct the OS triple $(\mathcal{H}^*, \Omega^*, H^*)$ using OS reconstruction. However, while we are sure that $\mu^{(0)}_M$ is an OS measure for each M by theorem 4.4, we are a priori not granted that $\mu^{(n)}_M$ is an OS measure, that is, that the flow preserves the OS measure class. This is , in fact, shown in Refs. 54–57.

Theorem.

*The renormalisation flow (**Eq. 5.22**) preserves the OS measure class, and its fixed points define OS measures μ^*.*

Responsible for this result is the fact that the time operations that define an OS measure commute with the spatial coarse graining operation. Thus, in principle, we can perform renormalisation in the measure (or path integral) language and then carry out OS reconstruction in order to find the continuum Hamiltonian theory that we are interested in. On the other hand, the fact that FKTW construction and OS reconstruction are inverses of each other (theorem 4.4) allows for the possibility to map the renormalisation flow of measures directly into a renormalisation flow of OS triples. In detail,

Step 1: Identifying the stochastic processes

We need to work out the null space of the reflection positive sesquilinear form determined by the measure $\mu^{(n)}_M$ from the vector space V_M of finite linear combinations of vectors of the form

$$w^M_T\left(\Phi_M\left(t_T, l^T_M\right)\right)..\, w^M_T\left(\Phi_M\left(t_T, l^T_M\right)\right) \tag{5.23}$$

with $t_T > t_{T-1} > .. > t_1 > 0$ for $w^M_K \in \mathcal{W}_M$ (for coinciding points of time we can reduce the number of time steps by decomposing the products of elementary functions into linear combinations of those).

The Hilbert space $\mathcal{H}^{(n)}_M$ is then the completion of the span of equivalence classes $[\psi_M]_{(n)}_{\mu_M}$, $\psi_M \in V_M$, in particular the vacuum is $\Omega^{(n)}_M = [1]_{(n)}_{\mu_M}$. However, the abstract description in terms of equivalence classes is not very useful in practice, rather we wish to describe them concretely in terms of stochastic processes and measures $\nu^{(n)}_M$ as outlined in **Section 4.3**. As the Hilbert spaces we deal with are separable, this is always possible (see appendix B of [196]); however, that construction does not directly refer to the space-time stochastic process Φ we started from. The reason why

this happens is because of the appearing equivalence classes: To perform concrete calculations, one will work with representatives, which makes the construction non-canonical because the choice of such representatives is largely a matter of taste. In our setting, if $\mu_M^{(n)}$ is obtained by the FKTW construction from OS data, then, of course, $\Phi_M(0, .) =: \phi_M(.)$ is a possible choice. However, in the renormalisation step, we are to deduce the OS data at resolution M from the measure $\mu_M^{(n+1)}$ which was renormalised from $\mu_{M'}^{(n)}$, $M < M'$ via **Eq. 5.22**, and thus, it is not a priori clear how the stochastic process ϕ_M can be chosen, in particular it is not clear whether it can be chosen as $\phi_M(.) = \Phi_{M'}(0, I_{MM'}.)$ which appears to be a natural choice.

However, we are in a better situation than in the generic case because it is clear that $\mathcal{H}_M^{(n+1)}$ can be formulated in terms of the fields $\phi_M(t, l_M) = \Phi_{M'}(t, I_{MM'}l_M)$ for a minimal number of distinguished times $t \in \tau$, where the set τ is determined by the quotient process (see, e.g., Refs. 54–57). Alternatively, one can view the fields $\phi_M(t, l_M)$, $t \in \tau$ as fields at time zero $\widetilde{\phi}_M(\widetilde{l}_M)$, but in a larger space of fields, that is, a stochastic process $\widetilde{\phi}_M$ with a larger index set $\widetilde{B}_M = \tau \times B_M$ that still lives on the lattice labelled by M [54–57]. It follows that without further input, which will be provided below, $\phi_M(.) = \Phi_{M'}(0, I_{MM'}.)$ is in general a *compound field*, that is, a *collective degree of freedom* composed out of $\widetilde{\phi}_M$ which together with its momentum π_M is insufficient to define the Hamiltonian $H_M^{(n+1)}$ which will generically depend on the larger set of variables $\widetilde{\phi}_M$ and its conjugate momentum $\widetilde{\pi}_M$. Note that this compound field is composite out of other gauge-invariant fields as an effect of renormalisation and not because of reasons of gauge invariance.

Step 2: Working out the flow of OS triples

Using the correspondence between the Wiener measures $\mu^{(n)}$ and the corresponding operator expressions, we have for $t_T > .. > t_1$

$$\begin{aligned}
&\mu_M^{(n+1)}\left(w_T^M\left(\Phi_M\left(t_T, l_M^T\right).w_1^M\left(\Phi_M\left(t_1, l_1^M\right)\right)\right)\right.\\
&= \langle \Omega_M^{(n+1)},\ w_T^M\left(\phi_M\left(l_M^T\right)\right) e^{-(t_T-t_{T-1})H_M^{(n+1)}}\ w_{T-1}^M\left(\phi_M\left(l_M^{T-1}\right)\right) e^{-(t_{T-1}-t_{T-2})H_M^{(n+1)}}\ \ldots\\
&e^{-(t_2-t_1)H_M^{(n+1)}}\ w_1^M\left(\phi_M\left(l_M^1\right)\right)\Omega_M^{(n+1)}\rangle_{\mathcal{H}_M^{(n+1)}}\\
&= \mu_{M'}^{(n)}\left(w_T^M\left(\Phi_{M'}\left(t_T, I_{MM'}l_M^T\right)..w_1^M\left(\Phi_{M'}\left(t_1, I_{MM'}l_1^M\right)\right)\right)\right.\\
&= \langle \Omega_{M'}^{(n)}, w_T^M\left(\phi_{M'}\left(I_{MM'}l_M^T\right)\right) e^{-(t_T-t_{T-1})H_{M'}^{(n)}} w_{T-1}^M\left(\phi_{M'}\left(I_{MM'}l_M^{T-1}\right)\right) e^{-(t_{T-1}-t_{T-2})H_{M'}^{(n)}}\ldots\\
&e^{-(t_2-t_1)H_{M'}^{(n)}}\ w_1^M\left(\phi_{M'}\left(I_{MM'}l_M^1\right)\right)\Omega_{M'}^{(n)}\rangle_{\mathcal{H}_{M'}^{(n)}}
\end{aligned} \tag{5.24}$$

for all choices of $M \in \mathcal{M}$; $T \in \mathbb{N}_0$; $t_T > .. > t_1$; l_M^1, .., l_M^T (in practice, e.g., $M' = 2M$ is fixed).

We consider **Eq. 5.24** as the *master equation* from which everything must be deduced. To avoid the compound field phenomenon mentioned above, we use that **Eq. 5.24** i) is supposed to hold for an arbitrary number of time steps and ii) we add as further input one more OS axiom, namely, *uniqueness of the vacuum* which is, in fact, a standard axiom to impose in QFT on Minkowski space [98–100]. In terms of measures, it can be stated as *ergodicity of time translations*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} ds\, U(s)\Psi =_{\mu \text{ a.e. }} \mu(\Psi)\cdot 1,\ \Psi \in L_2(S, d\mu) \tag{5.25}$$

We separate this axiom from the minimal ones because it enters in a crucial way only at this last stage of the renormalisation process. The subsequent discussion considerably extends the arguments of Refs. 54–57.

First of all, going back to **Eq. 5.24** and picking $T = 1$, we find

$$\begin{aligned}
&\langle \Omega_M^{(n+1)}, w_T^M\left(\phi_M\left(l_M\right)\right)\Omega_M^{(n+1)}\rangle_{\mathcal{H}_M^{(n+1)}}\\
&= \langle \Omega_M^{(n+1)}, w^M\left(\phi_{M'}\left(I_{MM'}l_M\right)\right)\Omega_{M'}^{(n)}\rangle_{\mathcal{H}_{M'}^{(n)}}
\end{aligned} \tag{5.26}$$

Using the fact that $w_M \in \mathcal{M}$ form a $-^*$algebra, we can formulate 5.26 as follows: Assuming by induction that up to renormalisation step n, the vectors $w^M(\phi_M(l_M))\,\Omega_M^{(n)}$ and $w^M \in \mathcal{W}_M$ span a dense subspace of $\mathcal{H}_M^{(n)}$, consider the closed linear span $\widehat{\mathcal{H}_{M'}^{(n)}}$ of vectors of the form

$$w^M\left(\phi_{M'}\left(I_{MM'}l_M\right)\right)\Omega_{M'}^{(n)}, \tag{5.27}$$

which is a subspace of $\mathcal{H}_{M'}^{(n)}$. Then, **Eq. 5.26** is the statement that the map

$$\begin{aligned}
J_{MM'}^{(n)}:\ \mathcal{H}_M^{(n+1)} \to \widehat{\mathcal{H}_{M'}^{(n)}};\ w^M\left(\phi_M\left(l_M\right)\right)\Omega_M^{(n+1)} \mapsto w^M\\
\left(\phi_{M'}\left(I_{MM'}l_M\right)\right)\Omega_{M'}^{(n)}
\end{aligned} \tag{5.28}$$

is an *isometry*, that is,

$$\left[J_{MM'}^{(n)}\right]^\dagger J_{MM'}^{(n)} = 1_{\mathcal{H}_M^{(n+1)}} \tag{5.29}$$

which implies that

$$P_{MM'}^{(n)} := J_{MM'}^{(n)}\left[J_{MM'}^{(n)}\right]^\dagger:\ \mathcal{H}_{M'}^{(n)} \to \widehat{\mathcal{H}_{M'}^{(n)}} \tag{5.30}$$

is a projection.

Next for $T = 2$, $t_2 - t_1 = \beta$, we find from **Eq. 5.24**

$$\begin{aligned}
&\langle \Omega_M^{(n+1)}, w\left(\phi_M\left(l_M\right)\right) e^{-\beta H_M^{(n+1)}}\ w'\left(\phi_M\left(l'_M\right)\right)\Omega_M^{(n+1)}\rangle_{\mathcal{H}_M^{(n+1)}}\\
&= \langle \Omega_{M'}^{(n)}, w\left(\phi_{M'}\left(I_{MM'}l_M\right)\right) e^{-\beta H_{M'}^{(n)}}\ w'\left(\phi_M\left(I_{MM'}l'_M\right)\right)\Omega_{M'}^{(n)}\rangle_{\mathcal{H}_{M'}^{(n)}}
\end{aligned} \tag{5.31}$$

and using again the $-^*$ property of the algebra $\mathcal{W}_M$ and taking formally the first derivative of **Eq. 5.31** at $\beta = 0$, we conclude

$$H_M^{(n+1)} = \left[J_{MM'}^{(n)}\right]^\dagger H_{M'}^{(n)} J_{MM'}^{(n)} \tag{5.32}$$

Note that (choose $w = 1$ in **Eq. 5.28**)

$$H_M^{(n+1)}\Omega_M^{(n+1)} = \left[J_{MM'}^{(n)}\right]^\dagger H_{M'}^{(n)}\Omega_{M'}^{(n)} = 0; \tag{5.33}$$

hence, the new vacuum is automatically annihilated by the new Hamiltonian.

We notice that for finite β, **Eq. 5.31** is not implied by **Eq. 5.32,** unless $[H_{M'}^{(n)}, P_{MM'}^{(n)}] = 0$, and it is here where we use the condition

that the correspondence pt5.24 is to hold for an arbitrary number and choices of time as well as the uniqueness of the vacuum. Using the projection $P^{(n)}_{MM'}$ onto the closed linear span of the $w(\phi_{M'}(I_{MM'}l_M))\Omega^{(n)}_{M'}$, we see that the operators $w(\phi_{M'}(I_{MM'}l_M))$ on $\mathcal{H}^{(n)}_{M'}$ are block diagonal with respect to the decomposition

$$\mathcal{H}^{(n)}_{M'} = P^{(n)}_{MM'}\mathcal{H}^{(n)}_{M'} \oplus \left[P^{(n)}_{MM'}\right]^{\perp}\mathcal{H}^{(n)}_{M'} \tag{5.34}$$

since they together with their adjoints leave $P^{(n)}_{MM'}\mathcal{H}^{(n)}_{M'}$ invariant (the $w \in \mathcal{W}_M$ generate a*−algebra). Thus, $P^{(n)}_{MM'}w(\phi_{M'}(I_{MM'}l_M))$ $[P^{(n)}_{MM'}]^{\perp} = [P^{(n)}_{MM'}]^{\perp}w(\phi_{M'}(I_{MM'}l_M))P^{(n)}_{MM'} = 0$, but in general, $[P^{(n)}_{MM'}]^{\perp}w(\phi_{M'}(I_{MM'}l_M))[P^{(n)}_{MM'}]^{\perp} \neq 0$. Thus, it is not sufficient to insert w operators an arbitrary number of times and at arbitrary places into the correspondence **Eq. 5.24** in order to deduce (**Eq. 5.32**) from **Eq. 5.24**.

Let $b^{(n)}_{M'I}$ be an orthonormal basis of $P^{(n)}_{MM'}\mathcal{H}^{(n)}_{M'}$. Then, since $\Omega^{(n)}_{M'}$ is cyclic for the algebra $W^{(n)}_{MM'}$ generated by the $w(\phi_{M'}(I_{MM'}l_M))$ with respect to $P^{(n)}_{MM'}\mathcal{H}^{(n)}_{M'}$, we find $w'_I \in W^{(n)}_{MM'}$ such that $b^{(n)}_{M'I} = w'_I\Omega^{(n)}_{MM'}$ (or can be made at least arbitrarily close). Next, assume that $\Omega^{(n)}_{M'}$ is the unique ground state for $H^{(n)}_{M'}$, then

$$e^{-\beta H^{(n)}_{M'}} \to \Omega^{(n)}_{M'}\langle\Omega^{(n)}_{M'},.\rangle_{\mathcal{H}^{(n)}_{M'}} \tag{5.35}$$

becomes the projection on the ground state for $\beta \to \infty$. It follows in the limit $\beta \to \infty$

$$\begin{aligned} P^{(n)}_{MM'} &= \sum_I b^{(n)}_{M'I}\langle b^{(n)}_{M'I},.\rangle_{\mathcal{H}^{(n)}_{M'}} \\ &= \sum_I w_I'\Omega^{(n)}_{M'}\langle w_I'\Omega^{(n)}_{M'},.\rangle_{\mathcal{H}^{(n)}_{M'}} \to \sum_I w_I' e^{-\beta H^{(n)}_{M'}}\left(w_I'\right)^{\dagger} \end{aligned} \tag{5.36}$$

Let w_I be the element in the algebra generated by the $w(\phi_M(l_M))$ such that $J^{(n)}_{MM'}w_I\Omega^{(n+1)}_M = w_I'\Omega^{(n)}_{M'}$ (which exists because $P^{(n)}_{MM'}\mathcal{H}^{(n)}_{M'}$ is the closure of the image of $J^{(n)}_{MM'}$). Then, due to isometry $[J^{(n)}_{MM'}]^{\dagger}J^{(n)}_{MM'} = 1_{\mathcal{H}^{(n+1)}_M}$ (5.29), we have

$$\begin{aligned} &\sum_I w_I\Omega^{(n+1)}_M\langle w_I\Omega^{(n+1)}_M,.\rangle_{\mathcal{H}^{(n+1)}_M} = \left[J^{(n)}_{MM'}\right]^{\dagger}\sum_I w_I'\Omega^{(n)}_{M'}\langle w_I'\Omega^{(n)}_{M'},.\rangle_{\mathcal{H}^{(n)}_{M'}}J^{(n)}_{MM'} \\ &= \left[J^{(n)}_{MM'}\right]^{\dagger}P^{(n)}_{MM'}J^{(n)}_{MM'} = 1_{\mathcal{H}^{(n+1)}_M} \end{aligned} \tag{5.37}$$

On the other hand, if $\Omega^{(n+1)}_M$ is the unique ground state for $H^{(n+1)}_M$, we have by the same argument as in **Eqs 5.35** and **5.36** in the limit $\beta \to \infty$

$$1_{\mathcal{H}^{(n+1)}_M} \to \sum_I w_I e^{-\beta H^{(n+1)}_M} w_I^{\dagger} \tag{5.38}$$

Since the identity operator $1_{\mathcal{H}^{(n+1)}_M}$ can be inserted an arbitrary number of times and at arbitrary places on the left hand side of **Eq. 5.24** and since it can be written as (**Eq. 5.38**) which under the correspondence **Eq. 5.24** translates into **Eq. 5.36**, the correspondence **Eq. 5.24** is to hold also when we insert $P^{(n)}_{MM'}$ an arbitrary number of times and at arbitrary places on the right hand side of **Eq. 5.24**. In particular, this means that we must replace on the right hand side of **Eq. 5.24** the operator $e^{-\beta H^{(n)}_{M'}}$ by

$$\lim_{N\to\infty} P^{(n)}_{MM'}\left(e^{-\frac{\beta}{N}H^{(n)}_{M'}}P^{(n)}_{MM'}\right)^N \tag{5.39}$$

To see this, we write in **Eq. 5.24** for each $k = 2, .., T$ and for any $N \in \mathbb{N}$ on the lhs $e^{-(t_k-t_{k-1})H^{(n+1)}_M} = (e^{-(t_k-t_{k-1})H^{(n)}_M/\!/n+1)})^N$ and replace $1_{\mathcal{H}^{(n)}_M}$ by the approximants (5.38) or more precisely the $P(n, k, \beta)$ of appendix E of [196] for $P = 1_{\mathcal{H}^{(n+1)}_M}$. Using multi-linearity of **Eq. 5.24,** we can rewrite the resulting expression in terms of **Eq. 5.24** again, just that now we have not T insertions of w operators, but $T' = 2\,N\,T$ insertions at times $t'_1 < .. < t'_{T'}$ such that

$$t_{2kN+2l\prime} - t'_{kN+2l-1} = \frac{t_k - t_{k-1}}{N},\ t'_{2kN+2l-1} - t'_{kN+2l-2} = \beta, \tag{5.40}$$

for $k = 1, .., T-1; l = 1, .., N$. By the correspondence 5.24, this translates into the corresponding expressions on the right hand side with approximants (5.36) or more precisely the $P(n, k, \beta)$ of appendix E of [196] for $P = P^{(n)}_{MM'}$. Then, one takes strong limits in the appropriate order (see appendix E of [196]), in particular $\beta \to \infty$, keeping $t_k - t_{k-1}$ fixed. As this is to hold for all N, we take $N \to \infty$.

Equation 5.39 is known in the mathematics literature [189–192] as a degenerate case of a Kato–Trotter product [188], of which there are many versions. One of them states that for contraction semi-groups generated by self-adjoint operators A, B such that $A + B$ is essentially self-adjoint on the dense domain $D(A)\cap D(B)$, we have strong convergence

$$\lim_{N\to\infty}\left[e^{-A/N}\,e^{-B/N}\right]^N = e^{-(A+B)} \tag{5.41}$$

In our case, the second contraction semi-group, $s \mapsto e^{-sB}$ is replaced by the degenerate one $K(s) = K(0) = P^{(n)}_{MM'}$. In [189–192], sufficient criteria for the existence of a degenerate semi-group $K(\beta)$, $K(0)$ an invariant projection, rather than the identity, are studied, such that in, say, the strong operator topology $K(\beta) = \lim_{N\to\infty}(e^{-\beta/NA}P)^N$. Assuming that existence $K(\beta)$ of the limit (5.39) is secured, we deduce

$$\begin{aligned} H^{(n+1)}_M &:= -\left[J^{(n)}_{MM'}\right]^{\dagger}\left[\frac{d}{d\beta}K(\beta)\right]_{\beta=0}J^{(n)}_{MM'},\ K(\beta): \\ &= \lim_{N\to\infty}P^{(n)}_{MM'}\left[e^{-\frac{\beta}{N}H^{(n)}_{M'}}P^{(n)}_{MM'}\right]^N \end{aligned} \tag{5.42}$$

In particular, if the solution of **Eq. 5.42** is given by

$$K(\beta) = P^{(n)}_{MM'}e^{-\beta P^{(n)}_{MM'}H^{(n)}_{M'}P^{(n)}_{MM'}}, \tag{5.43}$$

we recover **Eq. 5.32**, since $P^{(n)}_{MM'}J^{(n)}_{MM'} = J^{(N)}_{MM'}$. In appendix D of Ref. 196, we prove **Eq. 5.43** for the case that $H^{(n)}_{MM'}$ is bounded, that is, **Eq. 5.43** is strictly true when replacing $H^{(n)}_{M'}$ by its bounded spectral projections $E^{(n)}_{M'}(B)$, B Borel.

In what follows, we will assume this to hold also when $e^{-\beta H^{(n)}_{M'}}$ is a general contraction semi-group. In Refs. 189–192, we find proofs

for existence of a resulting degenerate semi-group under special circumstances, but no concrete formulae in terms of the original projection and semi-group are given. Thus, for the time being, we will use **Eq. 5.32** as a plausible solution of the *exact relation* **Eq. 5.42** but keep in mind that **Eq. 5.42** may contain more information.

To conclude this step, under the assumption that uniqueness of the vacuum is preserved under the renormalisation flow and that the degenerate Kato–Trotter product formula applies to general contraction semi-groups, we can strictly derive **Eq. 5.29** and **Eq. 5.32** as equivalent to **Eq. 5.24**. Unfortunately, it is not possible to show that the uniqueness property is automatically preserved under the flow: Suppose that $H^{(n)}_{M'}$ has unique vacuum $\Omega^{(n)}_{M'}$ and that $H^{(n+1)}_{M}\nu_M = [J^{(n)}_{MM'}]^\dagger\, H^{(n)}_{M'} J^{(n)}_{MM'}\nu_M = 0$, then we can just conclude that $H^{(n)}_{M'} J^{(n)}_{MM'}\nu_M \in [P^{(n)}_{MM'}]^\perp \mathcal{H}^{(n)}_{M'}$. Hence, without further input, the uniqueness property must be checked self-consistently.

Step 3: Constructing the continuum theory from the fixed point data

Once we found a fixed point family $J_{MM'};(\mathcal{H}_M, \Omega_M, H_M)$ with $M < M'$, $M, M' \in \mathcal{M}$, we have an inductive limit structure $(J_{MM'}, \mathcal{H}_M)$ of Hilbert spaces since $J_{M'M''}J_{MM'} = J_{MM''}$ is inherited from $I_{M'M''}I_{MM'} = I_{MM''}$ for $M < M' < M''$ and therefore can define the continuum Hilbert space $\mathcal{H}$ as its inductive limit which always exists [160]. Thus, there exist isometries $J_M\colon\ \mathcal{H}_M \to \mathcal{H}$ such that $J_{M'}J_{MM'} = J_M$, $M < M'$. Moreover, there exists a consistently defined quadratic form H (not necessarily an operator) such that $H_M = J_M^\dagger H J_M$. Note that we can compute matrix elements of H between the subspaces $J_M\mathcal{H}_M$, $J_{M'}\mathcal{H}_{M'}$ of $\mathcal{H}$ for any M, M' without actually knowing H, just its known finite resolution projections are needed, by using any M, $M' < M''$

$$\begin{aligned}\langle J_M\psi_M, HJ_{M'}\psi_{M'}\rangle_{\mathcal{H}} &= \langle J_{M''}J_{MM''}\psi_M, HJ_{M''}J_{M'M''}\psi_{M'}\rangle_{\mathcal{H}}\\ &= \langle J_{MM''}\psi_M, H_{M''}J_{M'M''}\psi_{M'}\rangle_{\mathcal{H}_{M''}}\end{aligned} \tag{5.44}$$

We stress that H is *not* the inductive limit of H_M since that would require $H_{M'}J_{MM'} = J_{MM'}H_M$. This inductive limit condition is much stronger than the quadratic form condition $H_M = J^\dagger_{MM'}H_{M'}J_{MM'}$ which can be seen by multiplying the inductive limit condition from the left with $J^\dagger_{MM'}$ and using isometry. It is not possible to derive the inductive limit condition from the quadratic form condition because $J^\dagger_{MM'}$ has no left inverse.

We emphasise that this Hamiltonian renormalisation scheme can be seen as an *independent, real-space, kinematical* renormalisation flow different from the OS measure (or path integral) scheme even if the assumptions that were made during its derivation from the measure theoretic one are violated. Note that both schemes are *exact*, that is, make no truncation error. This is possible because we do not need to compute the spectra of the Hamiltonians (which is practically impossible to do analytically without error even at finite resolution), but only matrix elements which is computationally much easier and can often performed analytically, even if the Hilbert spaces involved are *infinite-dimensional* as is the case in bosonic QFT even at finite resolution.

As a final remark, recall that the reduction of **Eqs 5.24–5.29** and **Eq. 5.29** rests crucially on the assumption that the vacuum vectors $\Omega^{(n)}_M$ remain the unique ground states of the Hamiltonians $H^{(n)}_M$ in the course of the renormalisation, a condition which is difficult to keep track-off in practice and which, in fact, contains dynamical information. Is it possible that the OS measure flow and the Hamiltonian flow (**Eqs 5.29** and **5.32)** nevertheless deliver the same continuum theory, even if we drop the vacuum uniqueness assumption? In that respect, note that one arrives at **Eqs 5.29** and **(5.32** from **Eq. 5.24** by deleting by hand the off-block diagonal terms in $H^{(n)}_{M'}$ with respect to the decomposition (**Eq. 5.34**). When deleting those terms by hand, then **Eq. 5.24** indeed becomes equivalent to **Eqs 5.29** and **5.32**. This is reminiscent of the Raleigh–Ritz procedure of diagonalising a self-adjoint operator [188]: There the statement is that for any self-adjoint operator H bounded from below (which is precisely our situation) and any finite-dimensional projection P, $\dim(P)$ eigenvalues of PHP ordered by size are upper bounds of $\dim(P)$ eigenvalues, ordered by size, in the discrete part of the spectrum (i.e., isolated eigenvalues of finite multiplicity) of H. Here, we deal with an infinite projection, instead of a finite one, but the general setting is the same. The idea is that as we increase M, we approach the continuum Hamiltonian for which eventually there are no off-diagonal elements.

6. CONCLUSION

In this contribution, we have reviewed, extended, and clarified the proposal [54–57]. The extension consisted in i. an improved derivation of the renormalisation scheme (5.29) and (5.32) from OS reconstruction using an extended minimal set of OS axioms that also includes the uniqueness of the vacuum (which is, in fact, always assumed in QFT on Minkowski space) and ii. a much more systematic approach to the choice of coarse graining maps for a general QFT which are motivated by structures naturally provided already by the classical theory. The clarification consisted in separating off the null space quotient process imposed by OS reconstruction as an independent part of the renormalisation flow whose formulation naturally uses the language of stochastic processes.

We also had the opportunity to make several points of contact with other renormalisation programmes that are currently being further developed. For instance, the reduced density matrix approach on which entanglement renormalisation schemes rest occurs naturally in our scheme as well when looking at the flow of the vacuum and Hilbert space. Next, since we consider a real-space renormalisation scheme, when translated in terms of the flow of Wiener measures that we obtain from the flow of OS data, we are rather close to the asymptotic safety programme because our spatial lattices can, of course, be translated into momentum lattices by Fourier transformation that are used in the asymptotically safe quantum gravity programme. Finally, our proposal is obviously very close in language and methods to all other Hamiltonian renormalisation schemes, and while we currently focus

on a kinematical coarse graining scheme, our approach also contains dynamical components such as the flow of the vacuum.

In [54–57, 193], we have successfully applied our scheme to free QFT (scalar fields and Abelian gauge theories) exploiting their linear structure. Obviously, one should construct further solvable examples of interacting theories, for example, interacting 2D scalar QFT [178–180] or free Abelian gauge theories but artificially discretised in terms of non-linear holonomies in order to simulate the situation in loop quantum gravity (see [193] for further remarks).

Of course, the ultimate goal is to use Hamiltonian renormalisation to find a continuum theory for canonical quantum gravity. Here, we can use the LQG candidate as a starting point because it is rather far developed, but, of course, the flow scheme developed can be applied to any other canonical programme. However, using LQG and the concrete scheme that employs a fixed subset of graphs γ_M labelled by $M \in \mathbb{M}$ of cubical topology is, at each resolution M, precisely the algebraic quantum gravity (AQG) version of LQG [169–172]. Hence, we can already speculate on what can be expected from the renormalisation flow:

The Hamiltonian $H_M^{(0)}$ defined on the corresponding $\mathcal{H}_M^{(0)} = L_2(SU(2)^{3M^3}, d^{3M^3}\mu_H)$ (μ_H being the $SU(2)$ Haar measure) could be, but not needs to be, ordered in such a way as to annihilate the vacuum $\Omega_M^{(0)} = 1$ of a discretised volume operator $V_M^{(0)}$ as it is standard in current regularisations of the Hamiltonian constraint. In fact, it may be desirable to choose the vacuum of $H_M^{(0)}$ not to coincide with that of $V_M^{(0)}$ in order to imprint its algebraic structure. The operator $H_M^{(0)}$ preserves $\mathcal{H}_M^{(0)}$ but not each subspace defined by sub-lattices of γ_M and is thus not super-local in contrast to the definition [28–32]; for instance, it will use volume operators local to a vertex and holonomies along plaquettes incident at that vertex (next neighbour interaction). When running the renormalisation scheme, next-to-next neighbour interactions will be switched on (this is exactly what happens in the examples [54–57, 193]), and upon reaching the fixed point, the Hamiltonian H_M will involve all possible interactions *with precise coefficients* and thus be spatially non-local but hopefully quasi-local (i.e., the interactions die off exponentially with the distance between vertices defined by the 3D taxi driver metric on the graph (each edge counting one unit)). Note that this quasi-locality at finite resolution can be straightforwardly computed in the examples [54–57, 193] by using the spatially local continuum Hamiltonian and projecting it with J_M, $J_M^\dagger$ (blocking from the continuum) and is thus *physically correct*. In other words, spatial locality in the continuum is not in conflict with spatial non-locality at finite resolution. In fact, we even *expect* a high degree of spatial non-locality for very small M for which the naive dequantisation of $H_M^{(0)}$ at any phase space point p will be far off the classical value $H(p)$ which matches with the remarks made at the end of **Section 3**.

Several questions arise from this picture should the flow display any fixed points: First, for compact σ and if indeed we use a countable set of lattices γ_M as above, the resulting inductive limit Hilbert space could be separable (since there is a countable basis defined by vectors at finite resolution), thus would not be the standard LQG representation space $\mathcal{H}_{\rm LQG} = L_2(\overline{\mathcal{A}}, d\mu_{\rm AL})$ of square integrable functions with respect to the Ashtekar–Lewandowski measure μ_{AL} on a space $\overline{\mathcal{A}}$ of distributional connections[2]. In view of the uniqueness theorem [23–27], one of its assumptions will then be violated. The most likely possibility is that the corresponding vacuum expectation value functional is not spatially diffeomorphism-invariant since the diffeomorphism symmetry was explicitly broken in the renormalisation process. If the continuum Hamiltonian is still spatially diffeomorphism-invariant, we would be in the situation of spontaneous symmetry breakdown and could view this as a phase transition from the symmetric $\mathcal{H}_{\rm LQG}$ phase to this broken phase. Note that in our gauge-fixed situation, the diffeomorphism group is considered as a continuous symmetry group and not as a gauge group.

Next, precisely due to this separability, the resulting theory may not suffer from the discontinuity of holonomy operators which otherwise gives rise to what has been called the 'staircase problem' in the literature [194]: The cubical graphs γ_M contain paths only along the coordinate axes. Since all $M \in \mathcal{M}$ are allowed, these paths separate the points of the classical configuration space but not of the distributional space $\overline{\mathcal{A}}$. In particular, any path that is not a 'staircase' path cannot be accommodated at any finite resolution. Yet, the continuum Hamiltonian in the example [193] does not care about the fact that it was defined as a fixed point of a flow of its finite resolution projections of cubical lattices only; it also knows how to act on states which are excited on non-'staircase' paths. The reason for why this happens is as follows: Consider any path c and some staircase approximant $\widetilde{c}$ with the same end points as c which has zero winding number with respect to c so that $c \circ \widetilde{c}^{-1} = \partial S$ bounds a surface. Then, for an Abelian connection A, we have $\int_\nu A - \int_{\widetilde{c}} A = \int_S dA$, and in the classical theory, the surface integral converges to zero. In the quantum theory, a similar calculation can be made because the Hilbert space measure is supported on a different kind of distributional connections than $\overline{\mathcal{A}}$.

Finally, although the scheme, strictly speaking, was derived for theories with gauge-fixed space-time diffeomorphism constraints and a true physical Hamiltonian bounded from below, we may, of course, 'abuse' it and also consider constraint operators $C(f)$ as Hamiltonians, define their finite resolution expressions $C(f)_M^{(0)}$, and let them flow (here, f is a test function on the spatial manifold σ)[3]. This will involve as a new ingredient also a discretisation of the smearing function f which could be done using the maps I_M, K_M for scalar fields (see Ref. 193). Suppose then that for all f fixed point families, $\{C(f)_M\}_{M \in \mathbb{N}_0}$ can be obtained. Should we expect that the $C(f)_M$ represent a finite resolution version of the classical continuum constraint (hypersurface deformation) algebra $\{C(f), C(g)\} = C(h(f, g))$, where $h(f, g)$ is another (in general, phase space–dependent) smearing function? The answer is in the negative! Namely, what we want is that the continuum

[2]In the non-compact case, one may need to take the infinite tensor product extension [69] which is also non-separable but in a different sense, and there one regains separability by passing to irreducible representations of the observable algebra.

[3]In fact, the physical Hamiltonian of **Section 3** is not manifestly bounded from below, hence we to abuse the formalism in the sense that we assumed the semi-boundedness.

operators obey $[C(f),\ C(g)] = iC(h(f,\ g))$ (with appropriate orderings of C, $h(f,\ g)$ in place). But if $C(f)_M = J_M^\dagger C(f) J_M$, then

$$\left[C(f)_M, C(g)_M\right] = J_M^\dagger\left[C(f)P_M C(g) - C(g)P_M C(f)\right]J_M$$
$$= C\left(h(f,g)\right)_M + J_M^\dagger\left[C(f)\left[P_M, C(g)\right]\right]J_M \quad (6.1)$$

Thus, even if the continuum algebra closes, one *does not see this at any finite resolution,* unless $[C(f),\ P_M] = 0$ for all f, M. This will generically not hold because not even $C(f)_{M'}$ preserves $J_{MM'}\mathcal{H}_M$, $M < M'$, unless $J_{MM'}C(f)_M = C(f)_{M'}J_{MM'}$, that is, $C(f)$ forms an inductive family which is not expected. Of course, the correction term in **Eq. 6.1** is expected to become 'small' in the limit $M \to \infty$ in which $P_M \to 1_{\mathcal{H}}$, and thus, an appropriate criterion for closure of the continuum algebra using only finite resolution projections can be formulated (see Refs. 54–57 for the simpler case of rotational invariance). Note that the quantisation performed for spatially diffeomorphism-invariant Hamiltonian operators on the Hilbert space $\mathcal{H}_{\mathrm{LQG}}$ displayed in **Section 3** was *forced* to have the unphysical property $[H,\ P_M] = 0$ (see the statement just before 3.24). But the underlying theorem exploits in a crucial way the non-separability of $\mathcal{H}_{\mathrm{LQG}}$, and thus fortunately does not hold on separable Hilbert spaces.

Before closing, note that even if this approach of taking the UV limit can be completed and unless the manifold σ is compact, we still must take the thermodynamic or infrared limit and remove the IR cut-off R (compactification scale). As is well-known from statistical quantum field theory [160], interesting phenomena related to phase transitions can happen here. Moreover, constructible examples of low dimensional interacting QFT show that the thermodynamic limit requires techniques that go beyond what was displayed here [178–180]. However, we consider this momentarily as a 'higher order' problem and reserve it for future research.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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SUPPLEMENTARY MATERIAL

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fphy.2020.548232/full#supplementary-material.

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