



Rigidity of Complete Minimal Submanifolds in Spheres

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Let M be an n -dimensional complete minimal submanifold in an $(n + p)$ -dimensional sphere \mathbb{S}^{n+p} , and let h be the second fundamental form of M . In this paper, it is shown that M is totally geodesic if the L^2 norm of $|h|$ on any geodesic ball of M is of less than quadratic growth and the L^n norm of $|h|$ on M is less than a fixed constant. Further, under only the latter condition, we prove that M is totally geodesic. Moreover, we provide a sufficient condition for a complete stable minimal hypersurface to be totally geodesic.

OPEN ACCESS

Keywords: rigidity, minimal submanifold, totally geodesic submanifold, stable hypersurface, Sobolev inequality

Edited by:

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Specialty section:

This article was submitted to
Mathematical and Statistical Physics,
a section of the journal
Frontiers in Physics

Received: 10 June 2020

Accepted: 17 August 2020

Published: 26 October 2020

Citation:

Zhou J (2020) Rigidity of Complete
Minimal Submanifolds in Spheres.
Front. Phys. 8:571250.
doi: 10.3389/fphy.2020.571250

1. INTRODUCTION

Let $(x, u(x))$ be a minimal graph in $\mathbb{R}^2 \times \mathbb{R}$, which means that $u(x)$ solves the equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0.$$

The celebrated Bernstein theorem states that the complete minimal graphs in \mathbb{R}^3 are planes. The works of Fleming [9], Almgren [1], and Neto and Wang [16] tell us that the Bernstein theorem is valid for complete minimal graphs in \mathbb{R}^{n+1} provided that $n \leq 7$. Counterexamples to the theorem for $n \geq 8$ have been found by Bombieri et al. [2] and, later, by Lawson [13]. On the other hand, do Carmo and Peng [6] and Fischer-Colbrie and Schoen [10] proved independently that a completely stable minimal surface in \mathbb{R}^3 must be a plane, a result that generalizes the Bernstein theorem. For the high-dimensional case, it is an open question whether the completely oriented stable minimal hypersurfaces in \mathbb{R}^{n+1} (for $3 \leq n \leq 7$) are hyperplanes. However, it has been proved by do Carmo and Peng [6] that a complete stable minimal hypersurface M in \mathbb{R}^{n+1} is a hyperplane if

$$\lim_{R \rightarrow \infty} \frac{\int_{B_{x_0}(R)} |h|^2 dv}{R^{2q+2}} = 0, \quad q < \sqrt{\frac{2}{n}},$$

where $B_{x_0}(R)$ denotes the geodesic ball of radius R centered at $x_0 \in M$. Many interesting generalizations of the do Carmo-Peng theorem have been obtained (see, e.g., [7, 15, 16, 18]). By definition, the hyperbolic space \mathbb{H}^{n+1} is a Riemannian manifold with sectional curvature -1 which

is simply connected, complete, and $(n + p)$ -dimensional. In hyperbolic space, some results similar to the do Carmo-Peng theorem have been derived. Xia and Wang [20] studied complete minimal submanifolds in a hyperbolic space and obtained the following result.

THEOREM 1.1. [20] *For $n \geq 5$, let M be an n -dimensional complete immersed minimal submanifold in a hyperbolic space \mathbb{H}^{n+p} , and let h be the second fundamental form of M . Assume that*

$$\lim_{R \rightarrow \infty} \sup \frac{\int_{B_{x_0}(R)} |h|^2 dv}{R^2} = 0. \tag{1.1}$$

If there exists a positive constant C depending only on n and p such that

$$\int_M |h|^n dv < C,$$

then M is totally geodesic.

Recently, de Oliveira and Xia [8] improved Theorem 1.1 as follows.

THEOREM 1.2. [8] *For $n \geq 4$, let M be an n -dimensional complete immersed minimal submanifold in a hyperbolic space \mathbb{H}^{n+p} such that n and p satisfy $(n^2 - 6n + 1) + \frac{8}{p} > 0$. Assume that*

$$\lim_{R \rightarrow \infty} \sup \frac{\int_{B_{x_0}(R)} |h|^d dv}{R^2} = 0,$$

where d is a constant with the following properties:

(1) *if $p = 1$ and $n \geq 4$, then*

$$d \in \left(\frac{n-1}{n}, \frac{(n-2)(n-1)}{n} \right);$$

(2) *if $p > 1$ and $n > 5$, then*

$$d \in \frac{(n-1)^2}{2n} \left(1 - \sqrt{1 - \frac{4}{(n-1)^2} \left(n - \frac{2}{p} \right)}, 1 + \sqrt{1 - \frac{4}{(n-1)^2} \left(n - \frac{2}{p} \right)} \right).$$

Then there exists a positive constant C depending only on n , p , and d such that M is totally geodesic if

$$\int_M |h|^n dv < C.$$

The unit sphere \mathbb{S}^{n+p} is a Riemannian manifold with sectional curvature 1 which is simply connected, complete, and $(n + p)$ -dimensional. Many results are available on the classification of compact minimal submanifolds in the unit sphere. Simons [17] calculated the Laplacian of $|h|^2$ of minimal submanifolds in a space form. As a consequence of Simons' formula, if M is

a compact minimal submanifold in \mathbb{S}^{n+p} and $|h|^2 \leq \frac{np}{2p-1}$, then either M is totally geodesic or $|h|^2 = \frac{np}{2p-1}$. In the latter case, Chern et al. [3] further proved that M is either a Clifford hypersurface or a Veronese surface in \mathbb{S}^4 . Li and Li [14] and Chen and Xu [4] proved independently that M is either a totally geodesic submanifold or a Veronese surface in \mathbb{S}^4 if $|h|^2 \leq \frac{2}{3}n$ everywhere on M . This result improves the pinching constant in Simons' formula. Deshmukh [5] studied n -dimensional compact minimal submanifolds in \mathbb{S}^{n+p} with scalar curvature S satisfying the pinching condition $S > n(n - 2)$ and proved that for $p \leq 2$ these submanifolds are totally geodesic.

The above results are rigidity theorems valid in the unit sphere, which characterize the behavior of minimal submanifolds. In this paper, we use the methods of minimal submanifolds in Euclidean space and hyperbolic space to investigate the rigidity of complete minimal submanifolds in spherical space. The main theorems are as follows.

THEOREM 1.3. *For $n \geq 3$, let M be an n -dimensional complete minimal submanifold in the unit sphere \mathbb{S}^{n+p} . We further assume that (1.1) holds. If*

$$\int_M |h|^n dv < C^n(n, p)$$

with $C(n, p) = (c(n))^{-1}(n^2 + 7)^{-\frac{1}{2}} \sqrt{(2b(p))^{-1} + (npb(p))^{-1}}$, where $c(n) = 2^n(1 + n)^{1+\frac{1}{n}}(n - 1)^{-1} \omega_n^{-\frac{1}{n}}$, ω_n is the volume of the unit ball in \mathbb{R}^n , $b(1) = 1$, and $b(p) = \frac{3}{2}$ if $p > 1$, then M is totally geodesic.

In [20], Xia and Wang believed that the condition (1.1) is not necessary. It is therefore interesting to see whether we can remove condition (1.1) from Theorem 1.3. In this case, we get a positive answer.

THEOREM 1.4. *For $n \geq 3$, let M be an n -dimensional complete minimal submanifold in the unit sphere \mathbb{S}^{n+p} . If*

$$\int_M |h|^n dv < \tilde{C}^n(n, p)$$

with $\tilde{C}(n, p) = (nc(n))^{-1} \sqrt{\frac{2(n-1)}{(n^2+7)b(p)} + \frac{4}{n(n^2+7)pb(p)}}$, where $c(n) = 2^n(1 + n)^{1+\frac{1}{n}}(n - 1)^{-1} \omega_n^{-\frac{1}{n}}$, ω_n is the volume of the unit ball in \mathbb{R}^n , $b(1) = 1$, and $b(p) = \frac{3}{2}$ if $p > 1$, then M is totally geodesic.

Remark 1.5. By using Simons' formula and the technique developed in do Carmo and Peng's paper, we obtain Theorem 1.4. The constant $\tilde{C}(n, p)$ in Theorem 1.4 is smaller than $C(n, p)$ in Theorem 1.3.

We also investigate stable minimal hypersurfaces in the unit sphere and obtain a result similar to do Carmo and Peng's theorem. A minimal hypersurface M in a Riemannian manifold N is said to be stable if for each $f \in C_0^\infty(M)$,

$$\int_M \left(|\nabla f|^2 - (|h|^2 + \overline{\text{Ric}}(v, v))f^2 \right) dv \geq 0, \tag{1.2}$$

where $\overline{\text{Ric}}$ is the Ricci curvature of N and ν is the unit normal vector of M .

THEOREM 1.6. *For $n \geq 2$, let M be an n -dimensional complete stable minimal hypersurface in the unit sphere \mathbb{S}^{n+1} . If*

$$\limsup_{R \rightarrow \infty} \frac{\int_{B_{x_0}(R)} |h|^{2\delta} dv}{R^2} = 0, \quad 1 - \sqrt{\frac{2}{n}} < \delta < 1 + \sqrt{\frac{2}{n}}, \quad (1.3)$$

where $B_{x_0}(R)$ denotes the geodesic ball of radius R centered at $x_0 \in M$, then M is totally geodesic.

2. PRELIMINARIES

Let M be an n -dimensional complete submanifold in the $(n + p)$ -dimensional unit sphere \mathbb{S}^{n+p} . We will use the following convention on the range of indices unless specified otherwise:

$$1 \leq A, B, C, \dots \leq n + p, \quad 1 \leq i, j, k, \dots \leq n, \\ n + 1 \leq \alpha, \beta, \gamma, \dots \leq n + p.$$

We choose a local field of orthonormal frame $\{e_1, e_2, \dots, e_{n+p}\}$ in \mathbb{S}^{n+p} such that, restricted to M , $\{e_1, e_2, \dots, e_n\}$ is tangent to M and $\{e_{n+1}, \dots, e_{n+p}\}$ normal to M . Let $\{\omega_A\}$ be the field of dual frame and $\{\omega_{AB}\}$ the connection 1-form of \mathbb{S}^{n+p} . Restricting these forms to M , we have

$$\omega_i^\alpha = \sum_j h_{ij}^\alpha \omega_j, \quad h = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \quad \xi = \frac{1}{n} \sum_{i,\alpha} h_{ii}^\alpha e_\alpha,$$

where h and ξ are the second fundamental form and the mean curvature vector of M , respectively. We define

$$H = |\xi|, \quad |h|^2 = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2, \quad |\nabla h|^2 = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2,$$

where h_{ijk}^α is the component of the covariant derivative of h_{ij}^α . When M is minimal, we obtain the Simons' formula [3, 17]

$$\frac{1}{2} \Delta |h|^2 = |\nabla h|^2 + n|h|^2 - \sum_{i,j,k,l,\alpha,\beta} h_{ij}^\alpha h_{ij}^\beta h_{kl}^\alpha h_{kl}^\beta \\ - \sum_{i,j,\alpha,\beta} \left(\sum_k (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta) \right)^2. \quad (2.1)$$

The last terms in (2.1) can be estimated as [14]

$$- \sum_{i,j,k,l,\alpha,\beta} h_{ij}^\alpha h_{ij}^\beta h_{kl}^\alpha h_{kl}^\beta - \sum_{i,j,\alpha,\beta} \left(\sum_k (h_{ik}^\alpha h_{kj}^\beta - h_{jk}^\alpha h_{ki}^\beta) \right)^2 \geq -b(p)|h|^4, \quad (2.2)$$

with $b(1) = 1$ and $b(p) = \frac{3}{2}$ if $p > 1$. We need the following estimate:

LEMMA 2.1. [19] *Let M be an n -dimensional immersed submanifold with parallel mean curvature in the space form $M^{n+p}(k)$. Then*

$$|\nabla h|^2 - |\nabla |h||^2 \geq \frac{2}{np} |\nabla |h||^2.$$

We also need the following Hoffman-Spruck Sobolev inequality.

LEMMA 2.2. [12] *Let M be an n -dimensional complete submanifold in a Hadamard manifold and let $\psi \in C_0^1(M)$. Then*

$$\left(\int_M \psi^{\frac{n}{n-1}} dv \right)^{\frac{n-1}{n}} \leq c(n) \int_M (|\nabla \psi| + n|H|\psi) dv,$$

where $c(n) = 2^n(1+n)^{1+\frac{1}{n}}(n-1)^{-1}\omega_n^{-\frac{1}{n}}$ and ω_n is the volume of the unit ball in \mathbb{R}^n .

From Lemma 2.2, we have the following estimate.

LEMMA 2.3. [11] *For $n \geq 3$, let M be an n -dimensional complete minimal submanifold in \mathbb{S}^{n+p} and let $\psi \in C_0^1(M)$. Then*

$$\left(\int_M \psi^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \leq 2(n^2 + 7)c^2(n) \int_M (|\nabla \psi|^2 + |\psi|^2) dv.$$

3. PROOFS OF THE MAIN THEOREMS

PROOF OF THEOREM 1.3: Noting that

$$\frac{1}{2} \Delta |h|^2 = |\nabla |h||^2 + |h| \Delta |h|,$$

it follows from (2.1) and (2.2) that

$$|\nabla |h||^2 + |h| \Delta |h| \geq |\nabla h|^2 + n|h|^2 - b(p)|h|^4.$$

From Lemma 2.1, we have

$$|h| \Delta |h| \geq \frac{2}{np} |\nabla |h||^2 + n|h|^2 - b(p)|h|^4. \quad (3.1)$$

Given $\eta \in C_0^\infty(M)$, multiplying (3.1) by η^2 and integrating over M gives

$$\int_M \eta^2 |h| \Delta |h| dv + \int_M b(p) \eta^2 |h|^4 dv \geq \frac{2}{np} \int_M |\nabla |h||^2 \eta^2 dv \\ + n \int_M |h|^2 \eta^2 dv, \quad (3.2)$$

which implies

$$- \int_M 2\eta |h| \langle \nabla |h|, \nabla \eta \rangle dv + \int_M b(p) \eta^2 |h|^4 dv \\ \geq \left(1 + \frac{2}{np} \right) \int_M |\nabla |h||^2 \eta^2 dv + n \int_M |h|^2 \eta^2 dv. \quad (3.3)$$

Further, applying Hölder's inequality and taking $\psi = |h|\eta$ in Lemma 2.3, one verifies that

$$\begin{aligned} \int_M \eta^2 |h|^4 dv &\leq \left(\int_M |h|^n dv \right)^{\frac{2}{n}} \left(\int_M (\eta|h|)^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \quad (3.4) \\ &\leq 2(n^2 + 7)c^2(n) \left(\int_M |h|^n dv \right)^{\frac{2}{n}} \int_M (|\nabla(\eta|h|)|^2 \\ &\quad + |h|^2 \eta^2) dv. \end{aligned}$$

Setting

$$l = 2b(p)(n^2 + 7)c^2(n) \left(\int_M |h|^n dv \right)^{\frac{2}{n}},$$

from (3.3) and (3.5) we may estimate

$$\begin{aligned} l \int_M |\nabla \eta|^2 |h|^2 dv + (l-1) \int_M 2\eta|h|\langle \nabla|h|, \nabla \eta \rangle dv \\ \geq \left(1 + \frac{2}{np} - l \right) \int_M |\nabla|h||^2 \eta^2 dv + (n-l) \int_M |h|^2 \eta^2 dv. \end{aligned} \quad (3.5)$$

By assumption,

$$\left(\int_M |h|^n dv \right)^{\frac{1}{n}} < c^{-1}(n)(n^2 + 7)^{-\frac{1}{2}} \sqrt{(2b(p))^{-1} + (npb(p))^{-1}},$$

and it is easy to see that

$$1 + \frac{2}{np} - l > 0.$$

Therefore, we can find a $\theta > 0$ such that

$$1 + \frac{2}{np} - l \geq \theta.$$

On the other hand, for any $\varepsilon > 0$ we have

$$\begin{aligned} (l-1) \int_M 2\eta|h|\langle \nabla|h|, \nabla \eta \rangle dv &\leq |l-1|\varepsilon \int_M \eta^2 |\nabla|h||^2 dv \\ &\quad + |l-1|\varepsilon^{-1} \int_M |\nabla \eta|^2 |h|^2 dv. \end{aligned} \quad (3.6)$$

Thus, when $|l-1|\varepsilon \leq \frac{\theta}{2}$, we obtain

$$\begin{aligned} (l + |l-1|\varepsilon^{-1}) \int_M |\nabla \eta|^2 |h|^2 dv &\geq \frac{\theta}{2} \int_M |\nabla|h||^2 \eta^2 dv \\ &\quad + (n-l) \int_M |h|^2 \eta^2 dv. \end{aligned} \quad (3.7)$$

Fix a point $x_0 \in M$ and choose $\eta \in C_0^\infty(M)$ as

$$\eta = \begin{cases} 1 & \text{on } B_{x_0}(R), \\ 0 & \text{on } M \setminus B_{x_0}(2R), \\ |\nabla \eta| \leq \frac{1}{R} & \text{on } B_{x_0}(2R) \setminus B_{x_0}(R), \end{cases} \quad (3.8)$$

with $0 \leq \eta \leq 1$, where $B_{x_0}(R)$ denotes the geodesic ball of radius R centered at $x_0 \in M$. Substituting the above η into (3.7) and letting $R \rightarrow \infty$, we deduce that

$$\int_M |\nabla|h||^2 dv \leq 0, \quad \int_M |h|^2 dv \leq 0.$$

Hence $|h|^2 = 0$, that is, M^n is totally geodesic. \square

Proof of Theorem 1.4: Direct computation yields

$$\Delta|h|^\delta = \delta(\delta-1)|h|^{\delta-2}|\nabla|h||^2 + \delta|h|^{\delta-1}\Delta|h|. \quad (3.9)$$

Multiplying (3.9) by $|h|^\delta$ and using (3.1), we infer that

$$\begin{aligned} |h|^\delta \Delta|h|^\delta &= \frac{\delta-1}{\delta} |\nabla|h|^\delta|^2 + \delta|h|^{2\delta-2}|h|\Delta|h| \\ &\geq \frac{\delta-1}{\delta} |\nabla|h|^\delta|^2 + \frac{2\delta}{np} |\nabla|h||^2 |h|^{2\delta-2} + n\delta|h|^{2\delta} \\ &\quad - \delta b(p)|h|^{2\delta+2} \\ &= \left(\frac{\delta-1}{\delta} + \frac{2}{np\delta} \right) |\nabla|h|^\delta|^2 + (n\delta - b(p)\delta|h|^2)|h|^{2\delta}. \end{aligned} \quad (3.10)$$

Let $\eta \in C_0^\infty(M)$. Multiplying (3.10) by η^2 and integrating over M yields

$$\begin{aligned} \int_M \eta^2 |h|^\delta \Delta|h|^\delta dv &\geq \left(\frac{\delta-1}{\delta} + \frac{2}{np\delta} \right) \int_M \eta^2 |\nabla|h|^\delta|^2 dv \\ &\quad + n\delta \int_M \eta^2 |h|^{2\delta} dv \\ &\quad - b(p)\delta \int_M \eta^2 |h|^{2\delta+2} dv. \end{aligned} \quad (3.11)$$

It follows from the divergence theorem and (3.11) that

$$\begin{aligned} b(p)\delta \int_M \eta^2 |h|^{2\delta+2} dv - \int_M 2\eta|h|^\delta \langle \nabla \eta, \nabla|h|^\delta \rangle dv \\ \geq \left(2 + \frac{2-np}{np\delta} \right) \int_M \eta^2 |\nabla|h|^\delta|^2 dv + n\delta \int_M \eta^2 |h|^{2\delta} dv. \end{aligned} \quad (3.12)$$

Applying Hölder's inequality and taking $\psi = |h|^\delta \eta$ in Lemma 2.3, we have

$$\begin{aligned} \int_M \eta^2 |h|^{2\delta+2} dv &\leq \left(\int_M |h|^n dv \right)^{\frac{2}{n}} \left(\int_M (\eta|h|^\delta)^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \\ &\leq 2n^2 c^2(n) \left(\int_M |h|^n dv \right)^{\frac{2}{n}} \int_M (|\nabla(\eta|h|^\delta)|^2 \\ &\quad + |h|^{2\delta} \eta^2) dv. \end{aligned} \quad (3.13)$$

Substituting (3.13) into (3.12) yields

$$\delta l \int_M |h|^{2\delta} |\nabla \eta|^2 dv + (\delta l - 1) \int_M 2\eta|h|^\delta \langle \nabla \eta, \nabla|h|^\delta \rangle dv$$

$$\begin{aligned} &\geq \left(2 + \frac{2-np}{np\delta} - l\delta\right) \int_M \eta^2 |\nabla|h|^\delta|^2 dv \\ &+ (n\delta - l\delta) \int_M \eta^2 |h|^{2\delta} dv, \end{aligned} \tag{3.14}$$

where $l = 2b(p)(n^2 + 7)c^2(n)(\int_M |h|^n)^{\frac{2}{n}}$. Further, using the Cauchy-Schwarz inequality, for each $\varepsilon > 0$ we obtain

$$\begin{aligned} (\delta l - 1) \int_M 2\eta|h|^\delta \langle \nabla\eta, \nabla|h|^\delta \rangle dv &\leq |\delta l - 1|\varepsilon \int_M \eta^2 |\nabla|h|^\delta|^2 dv \\ + |\delta l - 1|\varepsilon^{-1} \int_M |h|^{2\delta} |\nabla\eta|^2 dv. \end{aligned} \tag{3.15}$$

Therefore

$$\begin{aligned} &(\delta l + |\delta l - 1|\varepsilon^{-1}) \int_M |h|^{2\delta} |\nabla\eta|^2 dv \\ &\geq \left(2 + \frac{2-np}{np\delta} - l\delta - |\delta l - 1|\varepsilon\right) \int_M \eta^2 |\nabla|h|^\delta|^2 dv \\ &+ (n\delta - l\delta) \int_M \eta^2 |h|^{2\delta} dv. \end{aligned} \tag{3.16}$$

By the assumption in the theorem that

$$(nc(n))^{-1} \sqrt{\frac{2(n-1)}{(n^2+7)b(p)} + \frac{4}{n(n^2+7)pb(p)}} > \left(\int_M |h|^n dv\right)^{\frac{1}{n}},$$

we have

$$2 + \frac{2-np}{np\delta} - l\delta > 0.$$

Choosing ε sufficiently small, we can get

$$2 + \frac{2-np}{np\delta} - l\delta - |\delta l - 1|\varepsilon > 0.$$

Defining the cut-off function as in (3.8) and taking $\delta = \frac{n}{2}$ in (3.16), we obtain

$$\begin{aligned} &\left(\frac{n}{2}l + \left|\frac{n}{2}l - 1\right|\varepsilon^{-1}\right) \frac{\int_M |h|^n dv}{R^2} \\ &\geq \left(\frac{n}{2}l + \left|\frac{n}{2}l - 1\right|\varepsilon^{-1}\right) \frac{\int_{B_{x_0}(2R)} |h|^n dv}{R^2} \\ &\geq \left(2 + \frac{4-2np}{n^2p} - \frac{n}{2}l - \left|\frac{n}{2}l - 1\right|\varepsilon\right) \int_{B_{x_0}(R)} |\nabla|h|^{\frac{n}{2}}|^2 dv \\ &+ \left(\frac{1}{2}n^2 - \frac{n}{2}l\right) \int_{B_{x_0}(R)} |h|^n dv. \end{aligned} \tag{3.17}$$

Since

$$\left(\int_M |h|^n dv\right)^{\frac{1}{n}} < \tilde{C}(n, p),$$

upon taking $R \rightarrow \infty$ we have

$$\frac{\int_M |h|^n dv}{R^2} \rightarrow 0.$$

This and (3.17) imply $\nabla|h| = 0$ and $|h| = 0$, that is, M^n is totally geodesic. \square

Proof of Theorem 1.6: Since M is a stable minimal hypersurface in the unit sphere \mathbb{S}^{n+1} , (1.2) holds on M . Let $\eta \in C_0^\infty(M)$. Replacing f by $\eta|h|^\delta$ in (1.2) and taking $\text{Ric}(v, v) = n$ give

$$\int_M |\nabla(\eta|h|^\delta)|^2 dv \geq \int_M \eta^2 |h|^{2\delta+2} dv + n \int_M \eta^2 |h|^{2\delta} dv,$$

that is,

$$\begin{aligned} &\int_M |\nabla\eta|^2 |h|^{2\delta} dv + \int_M \eta^2 |\nabla|h|^\delta|^2 dv \\ &+ 2 \int_M \eta|h|^\delta \langle \nabla\eta, \nabla|h|^\delta \rangle dv - n \int_M \eta^2 |h|^{2\delta} dv \\ &\geq \int_M \eta^2 |h|^{2\delta+2} dv. \end{aligned} \tag{3.18}$$

Substituting (3.18) into (3.12) and noting that $b(1) = 1$, we obtain

$$\begin{aligned} &\delta \int_M |\nabla\eta|^2 |h|^{2\delta} dv + 2(\delta - 1) \int_M \eta|h|^\delta \langle \nabla\eta, \nabla|h|^\delta \rangle dv \\ &\geq \left(2 + \frac{2-n}{n\delta} - \delta\right) \int_M \eta^2 |\nabla|h|^\delta|^2 dv + 2n\delta \int_M \eta^2 |h|^{2\delta} dv. \end{aligned} \tag{3.19}$$

Using $1 - \sqrt{\frac{2}{n}} < \delta < 1 + \sqrt{\frac{2}{n}}$, we see that

$$2 + \frac{2-n}{n\delta} - \delta > 0.$$

Further, for any $\varepsilon > 0$, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} &\varepsilon|\delta - 1| \int_M \eta^2 |\nabla|h|^\delta|^2 dv + \varepsilon^{-1}|\delta - 1| \int_M |\nabla\eta|^2 |h|^{2\delta} dv \\ &\geq 2(\delta - 1) \int_M \eta|h|^\delta \langle \nabla\eta, \nabla|h|^\delta \rangle dv. \end{aligned} \tag{3.20}$$

Combining (3.20) and (3.19) gives

$$\begin{aligned} &(\delta + \varepsilon^{-1}|\delta - 1|) \int_M |\nabla\eta|^2 |h|^{2\delta} dv \\ &\geq \left(2 + \frac{2-n}{n\delta} - \delta - \varepsilon|\delta - 1|\right) \int_M \eta^2 |\nabla|h|^\delta|^2 dv \\ &+ 2n\delta \int_M \eta^2 |h|^{2\delta} dv. \end{aligned} \tag{3.21}$$

Choosing ε sufficiently small, we can obtain

$$2 + \frac{2-n}{n\delta} - \delta - \varepsilon|\delta - 1| > 0.$$

Furthermore, defining the cut-off function as in (3.8) and using the assumption (1.3) yield $\nabla|h| = 0$ and $|h| = 0$, that is, M^n is totally geodesic. \square

4. CONCLUSION

In this paper, by using Simons' formula, a Sobolev-type inequality as in Chen and Xu [4], and the technique of do Carmo and Peng, we obtain rigidity theorems for minimal submanifolds in S^{n+p} . Compared with Theorem 1.1, Theorem 1.4 removes the condition on the growth of the norm of the second fundamental form. Moreover, our results require only $n \geq 3$, whereas Theorems 1.1 and 1.2 require $n \geq 5$ and $n \geq 4$, respectively. Whether the pinching constant for the total curvature in Theorem 1.4 is optimal remains an open question and is a topic of future research.

DATA AVAILABILITY STATEMENT

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

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AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

FUNDING

This work was supported by the Natural Science Foundation of Anhui Province Education Department (grant nos. KJ2017A341 and KJ2018A0330), the Talent Project of Fuyang Normal University (grant no. RCXM201714), and the Key Laboratory of Applied Mathematics of Fujian Province University (Putian University) (grant no. SX201805).

ACKNOWLEDGMENTS

The author expresses sincere gratitude to the reviewers and the editors for their careful reading of the manuscript and constructive recommendations.

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Conflict of Interest: The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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