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Algebraic Structure and Poisson Integral Method of Snake-Like Robot Systems

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The algebraic structure and Poisson's integral of snake-like robot systems are studied. The generalized momentum, Hamiltonian function, generalized Hamilton canonical equations, and their contravariant algebraic forms are obtained for snake-like robot systems. The Lie-admissible algebra structures of the snake-like robot systems are proved and partial Poisson integral methods are applied to the snake-like robot systems. The first integral methods of the snake-like robot systems are given. An example is given to illustrate the results.

Keywords: snake-like robot, algebraic structure, poisson integral, lagrange equation, hamiltonian function

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INTRODUCTION

The snake-like robot, which is based on the biological characteristics of snakes, constitutes an important branch of bionic robots [1]. Hirose developed the first snake robot in 1972 [2]. The snake robot is significantly different from a tracked robot, wheeled robot, and legged robot, being a mobile robot with high redundancy. Because of the multi-joint flexible structure design, a snake robot has the advantage of multi-gait motion and the ability to adapt to a complex unknown environment, and can be widely used in disaster rescue, underwater surveys, industrial testing, and other special environments that traditional robots or humans cannot enter; as a result, increasing attention is being paid to snake robots [3–6].

In 1946, Gray divided movement gaits into serpentine movement, rectilinear movement, concertina movement, and sidewinding movement in the study of the biological nature of snakes [7]. According to this study, there are two starting points to study the motion of the snake-like robot: One is to observe the movement rule of biological snakes from the perspective of bionics, and then apply the rule to the snake-like robot to verify its effectiveness and controllability; on the other hand, the physical models are established according to the actual physical systems, and based on the physical model a control law is proposed to make snake-like robots move in a serpentine motion. For example, Tang et al. [8] studied the control methods of snake-like robots in different environments. Hirose established a serpentine gait kinematics model with linkage structures based on the observation of biological snake movement processes and bone anatomy [2], Lilijeback et al. analyzed the position relationship between a snake robot and obstacles, proposed an obstacle assistant movement gait in planar motion, and built the kinematics and dynamics model for the snake robot [9, 10]. At present, the serpentine motion of many snake-like robots are realized on the passive wheel, while the passive wheel provides a non-holonomic constraint for the system of snake-like robots in dynamics, so it is necessary to analyze and discuss the constraint systems of snake-like robots. Ostrowski and Burdick [11] and Guo et al. [12] developed the kinematic model considering the constraint systems of snake-like robots.

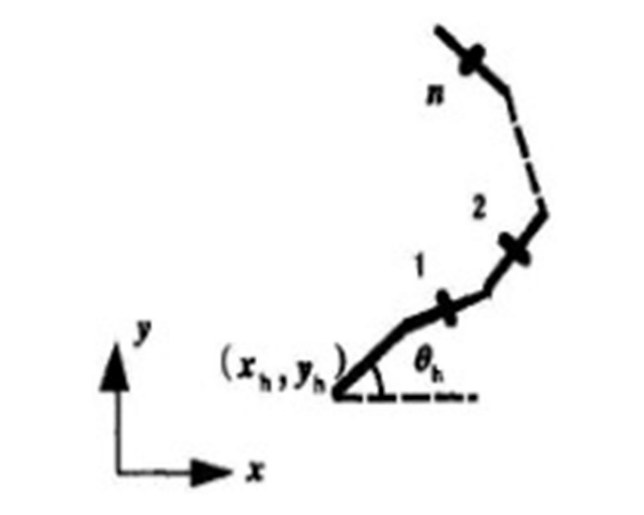

FIGURE 1 | Mechanical structure of snake-like robot.

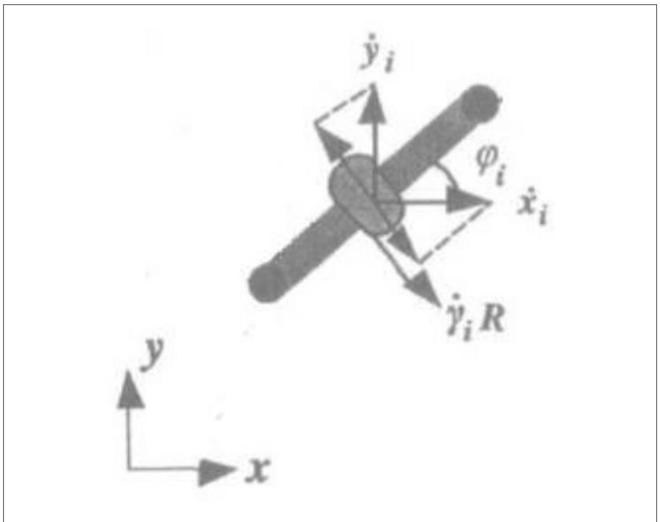

FIGURE 2 | A simplified model of one unit of the snake-like robot.

It is well-known that algebraic structure and Poisson's theory for constrained mechanical systems have been developed to seek invariants of mechanical and physical systems [13, 14]. Mei and Shi [15] have extended this method to non-holonomic constrained mechanical systems. Fu et al. have studied the algebraic structure and Poisson's theory of the relativistic Birkhoffian system, rotational relativistic dynamical system, mechanico-electrical coupling system, and f (R) cosmology system [16–18]. In this paper, we make an effort in this direction and demonstrate the applications of the algebraic structure and Poisson's theory of dynamical systems to snake-like robot system. This paper gives the algebraic structure and the first integral methods of snake-like robots. Firstly, the kinematics of the snake-like robot is analyzed, and then the generalized Lagrange equations and generalized Hamilton canonical equations of the snake-like robot systems are given. Secondly, contravariant algebraic forms of snake-like robot systems are obtained. Thirdly the Lie-admissible algebraic structure of the snake-like robot is researched. Fourthly, the Poisson integral methods are applied to the snake-like robot systems. Finally, an example is studied for a snake-like robot.

LAGRANGIAN OF SNAKE-LIKE ROBOT SYSTEMS

In order to facilitate the analysis, the snake-like robot systems are simplified as a link mechanism, each link rod represents a joint of the snake-like robot (mechanical structure of snake-like robot systems are depicted in **Figure 1**). Then, according to **Figure 1**, we can get the analysis as follows: let (x_i , y_i) denote the center of gravity of each joint (each coordinate is set in the middle of each link), φ_i is the angle between the link and the horizontal direction, (x_h, y_h) and the θ_h denote the position of the snake-like robot and the absolute angle of snake-head, respectively, and m_h denotes the weight of the head of the robot. Length $2l$ and weight m are the same for each link.

We choose $q_i = \varphi_i, q_{n+1} = x_h, q_{n+2} = y_h, q_{n+3}=\theta_h$, $(i = 1, 2, \cdots, n)$ as the generalized coordinates.

From **Figure 1**, the relationship between the center of gravity of each joint (x_i, y_i) and the position of the snake head (x_h, y_h) can be given by

$$x_i = q_{n+1} + 2l\cos\left(q_{n+3}\right) + 2l\sum_{k=1}^{i-1}\cos\left(q_k\right) + l\cos\left(q_i\right), \qquad (i = 1, \cdots, n) \quad (1)$$

$$y_i = q_{n+2} + 2l\sin\left(q_{n+3}\right) + 2l\sum_{k=1}^{i-1}\sin\left(q_k\right) + l\sin\left(q_i\right),$$

so, the generalized velocities of snake-like systems can be given by

$$\dot{x}_i = 7\dot{q}_{n+1} - 2l\sin\left(q_{n+3}\right)\dot{q}_{n+3} - 2l\sum_{k=1}^{i-1}\sin\left(q_k\right)\dot{q}_k - l\sin\left(q_i\right)\dot{q}_i,$$

$$\dot{y}_i = \dot{q}_{n+2} + 2l\cos\left(q_{n+3}\right)\dot{q}_{n+3} + 2l\sum_{k=1}^{i-1}\cos\left(q_k\right)\dot{q}_k + l\cos\left(q_i\right)\dot{q}_i. \qquad (2)$$

We give a simplified model of one unit of the snake-like robot system as shown in **Figure 2**.

The constraint equations of the snake-like robot system are given by

$$\dot{x}_i\sin\left(q_i\right) - \dot{y}_i\cos\left(q_i\right) = \dot{q}_{n+3+i}R, \qquad (3)$$

where $q_{n+3+i} = \gamma_i$ denotes the rotating angle of each unit's sideslip, R denotes the turning radius of the unit;

submitting constraints (2) into (3), we can derive

$$f_\beta\left(q, \dot{q}\right) = \dot{q}_{n+1}\sin\left(q_i\right) - \dot{q}_{n+2}\cos\left(q_i\right) - 2l\cos\left(q_{n+3} - q_i\right)\dot{q}_{n+3}$$

$$-2l\sum_{k=1}^{i-1}\cos\left(q_i-q_k\right)\dot{q}_k-l\dot{q}_i-\dot{q}_{n+3+i}R=0. \tag{4}$$

The symbols cos and sin are written as C and S respectively, then the kinetic energy of the snake-like robot system is given by

$$\begin{aligned}T=&\frac{1}{2}\left[m_h\left(\dot{q}_{n+1}^2+\dot{q}_{n+2}^2\right)+J_h\dot{q}_{n+3}^2\right]+\frac{1}{2}m\sum_{i=1}^{n}\left[\dot{q}_{n+1}^2+\dot{q}_{n+2}^2\right.\\&+4l^2\dot{q}_{n+3}^2+4l^2\left[\left(\sum_{k=1}^{i-1}S\left(q_k\right)\dot{q}_k\right)^2+\left(\sum_{k=1}^{i-1}C\left(q_k\right)\dot{q}_k\right)^2\right]\\&+l^2\dot{q}_2^2+4l\dot{q}_{n+3}\left(\dot{q}_{n+3}C\left(q_{n+3}\right)-\dot{q}_{n+1}S\left(q_{n+3}\right)\right)\\&+4l\dot{q}_{n+2}\sum_{k=1}^{i-1}C\left(q_k\right)\dot{q}_k-4l\dot{q}_{n+1}\sum_{k=1}^{i-1}S\left(q_k\right)\dot{q}_k+2l\dot{q}_{n+2}C\left(q_i\right)\dot{q}_i\\&-2l\dot{q}_{n+1}S\left(q_i\right)\dot{q}_i+8l^2\dot{q}_{n+3}S\left(q_{n+3}\right)\sum_{k=1}^{i-1}S\left(q_k\right)\dot{q}_k+8l^2\dot{q}_{n+3}\\&C\left(q_{n+3}\right)C\dot{q}_k+4l^2\dot{q}_{n+3}\dot{q}_2S\left(q_{n+3}\right)S\left(q_k\right)\\&+4l^2\dot{q}_{n+3}\dot{q}_2C\left(q_{n+3}\right)C\left(q_k\right)+4l^2S\left(q_i\right)\dot{q}_i\sum_{k=1}^{i-1}S\left(q_k\right)\dot{q}_k\\&\left.+4l^2C\left(q_i\right)\dot{q}_i\sum_{k=1}^{i-1}C\left(q_k\right)\dot{q}_k\right]+\frac{1}{2}\sum_{i=1}^{n}J_y\dot{q}_i^2+J_\gamma\dot{q}_{n+3+i}^2,\\&\left(i=1,\cdots,n\right).\end{aligned} \tag{5}$$

The potential energy of the snake-like robot system is assumed to be

$$U=0. \tag{6}$$

The dissipative functions of the snake-like robot system are assumed to be given by

$$\begin{aligned}D=&\frac{1}{2}D_{xy}\sum_{i=1}^{n}\left[\dot{q}_{n+1}^2+\dot{q}_{n+2}^2+4l^2\dot{q}_{n+3}^2\right.\\&+4l^2\left[\left(\sum_{k=1}^{i-1}S\left(q_k\right)\dot{q}_k\right)^2+\left(\sum_{k=1}^{i-1}C\left(q_k\right)\dot{q}_k\right)^2\right]\\&+l^2\dot{q}_2^2+4l\dot{q}_{n+3}\left(\dot{q}_{n+3}C\left(q_{n+3}\right)-\dot{q}_{n+1}S\left(q_{n+3}\right)\right)\\&+4l\dot{q}_{n+2}\sum_{k=1}^{i-1}C\left(q_k\right)\dot{q}_k-4l\dot{q}_{n+1}\sum_{k=1}^{i-1}S\left(q_k\right)\dot{q}_k+2l\dot{q}_{n+2}C\left(q_i\right)\dot{q}_i\\&-2l\dot{q}_{n+1}S\left(q_i\right)\dot{q}_i+8l^2\dot{q}_{n+3}S\left(q_{n+3}\right)\sum_{k=1}^{i-1}S\left(q_k\right)\dot{q}_k+8l^2\dot{q}_{n+3}\\&C\left(q_{n+3}\right)\sum_{k=1}^{i-1}C\left(q_k\right)\dot{q}_k+4l^2\dot{q}_{n+3}\dot{q}_2S\left(q_{n+3}\right)S\left(q_k\right)\\&+4l^2\dot{q}_{n+3}\dot{q}_2C\left(q_{n+3}\right)C\left(q_k\right)+4l^2S\left(q_i\right)\dot{q}_i\sum_{k=1}^{i-1}S\left(q_k\right)\dot{q}_k\\&\left.+4l^2C\left(q_i\right)\dot{q}_i\sum_{k=1}^{i-1}C\left(q_k\right)\dot{q}_k\right]+\frac{1}{2}\sum_{i=1}^{n}D_y\dot{q}_i^2.\end{aligned} \tag{7}$$

The Routh equation of the snake-like robot system can be given by

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{q}_s}-\frac{\partial T}{\partial q_s}=Q_s+\sum_{\beta=1}^{n}\lambda_\beta\frac{\partial f_\beta}{\partial\dot{q}_s}\quad\left(s=1,\cdots,2n+3;\ \beta=1,\cdots,n\right). \tag{8}$$

After derivation of the constraints of the snake-like robot system (4), we have

$$\dot{f}_\beta\left(q,\dot{q}\right)=0. \tag{9}$$

From Equation (8) and (9), the Lagrange multiply λ can be calculated as

$$\lambda=\left\{\lambda_1,\lambda_2,\ldots,\lambda_n\right\} \tag{10}$$

Submitting (10) into (8), we can derive

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{q}_s}-\frac{\partial T}{\partial q_s}=Q_s+\Lambda_s,\ \left(s=1,\cdots,2n+3\right) \tag{11}$$

where $\Lambda_s=\sum_{\beta=1}^{n}\lambda_\beta\frac{\partial f_\beta}{\partial\dot{q}_s}$.

The motion equation (12) of the complete system corresponding to the snake-like robot system (4, 11) can be given as

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s{}'+\Lambda_s, \tag{12}$$

where $Q_s{}'=-\frac{\partial D_s}{\partial\dot{q}_s}+\tau-\frac{\partial U}{\partial q_s}$ are the non-potential force, $\Lambda_s=\lambda_\beta\frac{\partial f_\beta}{\partial\dot{q}_s}$ the generalized nonholonomic constraint anti-forces, $L=T-U$ the Lagrangian of the snake-like robot system.

GENERALIZED HAMILTON CANONICAL EQUATIONS OF SNAKE-LIKE ROBOT SYSTEMS

Assume that the generalized momentums of snake-like robot systems are

$$\begin{aligned}p_1=&\frac{\partial L}{\partial\dot{q}_1}=\frac{1}{2}\left[\left(-4m\left(n-1\right)lS\left(q_{n+3}\right)-2mlS\left(q_{n+3}+q_1\right)\right)\dot{q}_{n+1}\right.\\&+\left(4m\left(n-1\right)lC\left(q_{n+3}\right)+2mlC\left(q_{n+3}+q_1\right)\right)\dot{q}_{n+2}\\&+\left(2m\left(2+4\left(n-1\right)\right)l^2C_{h1}\right)\dot{q}_{n+3}\\&\left.+\left(8m\left(n-1\right)l^2+2ml^2+2J_y\right)\dot{q}_1+\cdots+\left(4ml^2C_{1n}\right)\dot{q}_n\right],\\p_2=&\frac{\partial L}{\partial\dot{q}_2}=\frac{1}{2}\left[\left(-4m\left(n-2\right)lS\left(q_{n+3}\right)-2mlS\left(q_2\right)\right)\dot{q}_{n+1}\right.\\&+\left(4m\left(n-2\right)lC\left(q_{n+3}\right)+2mlC\left(q_{n+3}+q_1\right)\right)\dot{q}_{n+2}\\&+\left(2m\left(2+4\left(n-2\right)\right)l^2C_{h2}\right)\dot{q}_{n+3}\end{aligned} \tag{13}$$

$$+\left(2m\left(2+4\left(n-2\right)\right)l^2C_{12}\right)\dot{q}_1+\left(8m\left(n-2\right)l^2+2ml^2+2J_y\right)\dot{q}_2+\cdots+\left(4ml^2C_{2n}\right)\dot{q}_n\Big], \quad (14)$$

$$\vdots \quad \vdots \quad \vdots$$

$$p_n=\frac{\partial L}{\partial \dot{q}_n}=\frac{1}{2}\left[\left(-2mlS\left(q_{n+3}+\sum_{k=1}^{n}q_k\right)\right)\dot{q}_{n+1}+\left(-2mlC\left(q_{n+3}+\sum_{k=1}^{n}q_k\right)\right)\dot{q}_{n+2}+\left(4ml^2C_{h2}\right)\dot{q}_{n+3}+\left(4ml^2C_{1h}\right)\dot{q}_1+\cdots+\left(2ml^2+2J_y\right)\dot{q}_n\right], \quad (15)$$

$$p_{n+1}=\frac{\partial L}{\partial \dot{q}_{n+1}}=\frac{1}{2}\Big[(2m_h+2mn)\,\dot{q}_{n+1}+(0)\,\dot{q}_{n+2}+\left(-4mnlS\left(q_{n+3}\right)\right)\dot{q}_{n+3}+\left(-4m\left(n-1\right)lS\left(q_{n+3}\right)-2mlS\left(q_{n+3}+q_1\right)\right)\dot{q}_1+\cdots+\left(-2mlS\left(q_{n+3}+\sum_{k=1}^{n}q_k\right)\right)\dot{q}_n\Big], \quad (16)$$

$$p_{n+2}=\frac{\partial L}{\partial \dot{q}_{n+2}}=\frac{1}{2}\Big[(0)\,\dot{q}_{n+1}+(0)\,\dot{q}_{n+2}+\left(4mnlC\left(q_{n+3}\right)\right)\dot{q}_{n+3}+\left(4m\left(n-1\right)lC\left(q_{n+3}\right)+2mlC\left(q_{n+3}+q_1\right)\right)\dot{q}_1+\cdots+\left(-2mlC\left(q_{n+3}+\sum_{k=1}^{n}q_k\right)\right)\dot{q}_n\Big], \quad (17)$$

$$p_{n+3}=\frac{\partial L}{\partial \dot{q}_{n+3}}=\frac{1}{2}\Big[\left(-4mnlS\left(q_{n+3}\right)\right)\dot{q}_{n+1}+\left(4mnlC\left(q_{n+3}\right)\right)\dot{q}_{n+2}+(2m_h+2mn)\,\dot{q}_{n+3} \quad (18)$$
$$+\left(2m\left(2+4\left(n-1\right)\right)l^2C_{h1}\right)\dot{q}_1+\cdots+\left(4ml^2C_{hn}\right)\dot{q}_n\Big],$$

$$p_{n+4}=\frac{\partial L}{\partial \dot{q}_{n+4}}=J_r\dot{q}_{n+4}, \quad (19)$$

$$\vdots$$

$$p_{2n+3}=\frac{\partial L}{\partial \dot{q}_{2n+3}}=J_r\dot{q}_{2n+3}. \quad (20)$$

And introduce the Hamiltonian of the snake-like robot system as

$$H\left(t,q,\dot{q}\right)=p_s\dot{q}_s-L=p_s\dot{q}_s\left(t,q,p\right)-L\left(t,q_s,\dot{q}_s\left(t,q,p\right)\right)=H\left(t,q,p\right), \quad (21)$$

where $p=\left\{p_1,p_2,\cdots,p_{2n+3}\right\}$ denotes generalized momentums, equation (16) can be partially regularized as

$$\dot{q}_s=\frac{\partial H}{\partial p_s},\dot{p}_s=-\frac{\partial H}{\partial q_s}+Q_s{}'+\Lambda_s|\dot{q}_s=\dot{q}_s\left(t,q,p\right),\quad (s=1,\cdots,2n+3) \quad (22)$$

which is called the generalized Hamilton canonical equation of snake-like robot systems.

CONTRAVARIANT ALGEBRAIC FORMS OF SNAKE-LIKE ROBOT SYSTEMS

For snake-like robot systems, we can introduce contravariant vectors

$$a^{\mu}=\begin{cases}q^{\mu} & (\mu=1,\cdots,2n+3),\\ p_{\mu-n} & (\mu=2n+4,\cdots,4n+6),\end{cases} \quad (23)$$

then the Hamiltonian of snake-like robot systems will be written in the form

$$H\left(t,q_s,p_s\right)=H\left(t,a^{\mu}\right). \quad (24)$$

For generalized Hamilton canonical equation (15) of snake-like robot systems, we let

$$\left(Q_s{}'+\Lambda_s\right)|q_s=q_s\left(t,q,p\right)=\Lambda'_s=-\Omega_{sk}\frac{\partial H}{\partial p_k}\quad \left(s,k=1,\cdots,2n+3\right), \quad (25)$$

where

$$\Omega_{sk}=\begin{pmatrix}\Omega_{11} & 0 & \cdots & 0\\ 0 & \Omega_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \Omega_{(2n+3)(2n+3)}\end{pmatrix}, \quad (26)$$

then Equation (15) can be expressed in the contravariant algebraic form

$$\dot{a}^{\mu}-S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}=0,\quad (\mu,\nu=1,\cdots,4n+6) \quad (27)$$

where

$$S^{\mu\nu}=\omega^{\mu\nu}+T^{\mu\nu} \quad (28)$$

$$\omega^{\mu\nu}=\begin{pmatrix}0_{(2n+3)(2n+3)} & I_{(2n+3)(2n+3)}\\ -I_{(2n+3)(2n+3)} & 0_{(2n+3)(2n+3)}\end{pmatrix}, \quad (29)$$

$$T^{\mu\nu}=\begin{pmatrix}0_{(2n+3)(2n+3)} & 0_{(2n+3)(2n+3)}\\ 0_{(2n+3)(2n+3)} & -\Omega_{kk}\end{pmatrix}. \quad (30)$$

It is obvious that the tensor $S^{\mu\nu}$ is composed of anti-symmetrical tensor $\omega^{\mu\nu}$ and symmetrical tensor $T^{\mu\nu}$.

ALGEBRAIC STRUCTURE OF SNAKE-LIKE ROBOT SYSTEMS

Firstly, we study the algebraic structure of snake-like robot systems.

Performing the full derivative of function $A(a)$ along Equation (27), and this derivative is defined as a product:

$$\dot{A}\left(a\right)=\frac{\partial A}{\partial a_{\mu}}S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}\overset{def}{=}A\circ H,\quad (\mu,\nu=1,\cdots,2n+3) \quad (31)$$

this product satisfies the right-hand assignment law, left-hand assignment law, and scalar law, so we can derive that the snake-like robot system possesses a compatible algebraic structure.

If the snake-like robot system in Equation (27) possesses the Lie algebraic structure, then Equation (31) satisfies the anti-symmetrical property

$$A\circ B+B\circ A=0, \quad (32)$$

and Jacobi identical equation

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{33}$$

When considering Equations (28)-(30), Equations (32) and (33) lead to a condition with respect to $T^{\mu\nu}$

$$T^{\mu\nu} + T^{\nu\mu} = 0 \tag{34}$$

$$T^{\tau\rho}\frac{\partial T^{\mu\nu}}{\partial a^{\rho}} + T^{\mu\rho}\frac{\partial T^{\nu\tau}}{\partial a^{\rho}} + T^{\nu\rho}\frac{\partial T^{\tau\mu}}{\partial a^{\rho}} = 0, \tag{35}$$
$$(\mu,\ \nu,\ \tau,\ \rho = 1, \cdots, 4n + 6).$$

Further, we have $\Omega_{kk} = 0,\ Q_s = 0, (s, k = 1, \cdots, 2n + 3)$. Then Equation (27) has no Lie algebraic structure, and we have

Conclusion 1: The snake like-robot systems do not have Lie algebraic structure.

For Equation (31), we define a new product

$$[A, B] \overset{def}{=} A \circ B - B \circ A, \tag{36}$$

we can prove that Equation (36) has anti-symmetrical property, and satisfies the Jacobi identical equation, namely, a new product with Lie algebraic structure, then Equation (27) has

Conclusion 2: The snake like-robot systems have Lie-admissible algebraic structure.

THE POISSON'S INTEGRAL METHOD OF SNAKE-LIKE ROBOT SYSTEMS

The classical Poisson integral method includes setting up the Poisson condition of the first integral for the Hamilton system, the Poisson theorem of third integrals is generated from two known integrals by Poisson brackets. The theoretical foundation of this method includes equations of motion of systems with Lie algebraic structure. Because the snake-like robot system has no Lie algebraic structure, it possesses Lie-admissible algebraic structure. Therefore, part of the Poisson integral methods can be applied to snake-like robot systems. Then we have the following conclusions:

Proposition 1: The necessary and sufficient condition on which $I(a^{\mu}, t) = c$ is the first integral of snake-like robot systems (27) in that the $I(a^{\mu}, t)$ satisfies

$$\frac{\partial I}{\partial t} + [I, H] = 0, \tag{37}$$

Proposition 2: $H = c$ is a first integral of snake-like robot systems (27), if the Hamiltonian of the system does not depend explicitly on time t, and forΛ'_s= 0.

Proposition 3: For snake-like robot systems (27), which possess a first integral $I(a^{\mu}, t)$ that contains t, but H and Λ'_sdo not depend explicitly on t, then

$$\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial t}\right) + \left[\frac{\partial I}{\partial t}H\right] + \frac{\partial \Lambda'_s}{\partial t}\frac{\partial I}{\partial a^{\mu}} + \frac{\partial I}{\partial a_{\mu}}S^{\mu\nu}\frac{\partial}{\partial a^{\nu}}\left(\frac{\partial H}{\partial t}\right) = 0 \tag{38}$$

i.e.,

$$\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial t}\right) + \left[\frac{\partial I}{\partial t}H\right] = 0. \tag{39}$$

Namely, $\frac{\partial I}{\partial t}$ is the first integral of the snake-like robot system (27), and $\frac{\partial I^2}{\partial t^2}, \ldots$, are also first integrals of the snake-like robot system.

Proposition 4: For snake-like robot systems (27), which possess a first integral $I(a^{\mu}, t)$ containing a^{ρ}, but H and Λ'_sdo not depend explicitly on a^{ρ}, $\frac{\partial I}{\partial a^{\rho}}, \frac{\partial I^2}{\partial a^{\rho 2}}, \cdots$, are also first integrals of the snake-like robot system.

EXAMPLE

We set up a simple snake-like robot which only has one joint, and choose $q_1 = \varphi_1, q_2 = x_h, q_3 = y_h, q_4{=}\theta_h, q_5 = \gamma_1$ as generalized coordinates.

The relationship between the center of gravity of the first joint (x_1, y_1) and the position of the snake head (x_h, y_h) can be given by

$$\begin{aligned} x_1 &= q_2 + 2lc(q_4) + lc(q_1), \\ y_1 &= q_3 + 2ls(q_4) + ls(q_1), \end{aligned} \tag{40}$$

so, the velocities of snake-like systems can be given by

$$\begin{aligned} \dot{x}_1 &= \dot{q}_2 - 2ls(q_4)\dot{q}_4 - ls(q_1)\dot{q}_1, \\ \dot{y}_1 &= \dot{q}_3 + 2lc(q_4)\dot{q}_4 + lc(q_1)\dot{q}_1. \end{aligned} \tag{41}$$

The constraints of the snake-like robot system are given by

$$\begin{aligned} f_{\beta}(q, \dot{q}) &= \dot{q}_2 s(q_1) - \dot{q}_3 c(q_1) - 2lc(q_4 - q_1)\dot{q}_4 \\ &\quad - l\dot{q}_1 - \dot{q}_5 R = 0. \end{aligned} \tag{42}$$

We obtain the kinetic energy of the snake-like robot system

$$\begin{aligned} T =& \frac{1}{2}(m + m_h)(\dot{q}_2^2 + \dot{q}_3^2) + \frac{1}{2}(J_h + 2ml^2)\dot{q}_4^2 + \frac{1}{2}\left(J_y + \frac{1}{2}ml^2\right) \\ & \dot{q}_1^2 + 2ml\dot{q}_4(\dot{q}_2 s(q_4) - \dot{q}_3 c(q_4)) \\ & + ml\dot{q}_1[\dot{q}_2 s(q_1) - \dot{q}_3 c(q_1)] + 2ml^2\dot{q}_4\dot{q}_1(\dot{q}_4 + \dot{q}_1) \\ & s(q_4 + q_1) + \frac{1}{2}J_r\dot{q}_5^2, \end{aligned} \tag{43}$$

potential energy

$$U = 0, \tag{44}$$

and dissipative function

$$\begin{aligned} D =& \frac{1}{2}D_y\dot{q}_1^2 + \frac{1}{2}D_{xy}[\dot{q}_2^2 + \dot{q}_3^2 + 4l\dot{q}_4^2 + l^2\dot{q}_1^2 + 4l\dot{q}_1(\dot{q}_2 s(q_4) \\ & - \dot{q}_3 c(q_4)) \\ & + 2l\dot{q}_1(\dot{q}_2 s(q_1) - \dot{q}_3 c(q_1)) + 4l^2\dot{q}_4\dot{q}_1\ s(q_4 + q_1)]. \end{aligned} \tag{45}$$

The Lagrange function of the system is written in the form

$$L = \frac{1}{2}(m + m_h)(\dot{q}_2^2 + \dot{q}_3^2) + \frac{1}{2}(J_h + 2ml^2)\dot{q}_4^2 + \frac{1}{2}\left(J_y + \frac{1}{2}ml^2\right)$$

$$\dot q_1^2+2ml\dot q_4\left(\dot q_2 s\left(q_4\right)-\dot q_3 c\left(q_4\right)\right)$$
$$+ml\dot q_1\left[\dot q_2 s\left(q_1\right)-\dot q_3 c\left(q_1\right)\right]+2ml^2\dot q_4\dot q_1\left(\dot q_4+\dot q_1\right)$$
$$s\left(q_4+q_1\right)+\frac{1}{2}J_r\dot q_5^2. \tag{46}$$

Taking the generalized momenta of snake-like robot systems

$$\begin{aligned}
p_1 &= \left(J_y+ml^2\right)\dot q_1+ml\left(\dot q_2 s\left(q_1\right)-\dot q_3 s\left(q_1\right)\right)+2ml^2\dot q_4\\
&\quad s\left(q_4+q_1\right),\\
p_2 &= m_h\dot q_2+2ml\dot q_4 s\left(q_4\right)+ml\dot q_1 s\left(q_1\right),\\
p_3 &= m_h\dot q_3-2ml\dot q_4 s\left(q_4\right)-ml\dot q_1 c\left(q_1\right),\\
p_4 &= \left(J_h+4ml^2\right)\dot q_4+2ml\left(\dot q_2 s\left(q_4\right)-\dot q_3 c\left(q_4\right)\right)+2ml^2\dot q_1\\
&\quad s\left(q_4+q_1\right),\\
p_5 &= J_r\dot q_5,
\end{aligned} \tag{47}$$

using Equation (47), we can obtain the generalized velocities in the form

$$\begin{aligned}
\dot q_1 &= \frac{mls\left(q_1\right)-\left(J_y+ml^2\right)mlc\left(q_1\right)}{\left(J_y+ml^2\right)mlc\left(q_1\right)}p_1,\\
\dot q_2 &= \frac{2mlc\left(q_1\right)+ml\,\mathrm{s}\left(q_4+q_1\right)}{ml\left(J_h+4ml^2s\left(q_1\right)+ml\right)}p_2,\\
\dot q_3 &= \frac{\left(J_y+ml^2\right)\left(J_y+ml^2\right)ml+2ml^2\,\mathrm{s}\left(q_4+q_1\right)mlc\left(q_1\right)}{4ml\left(s\left(q_1\right)-s\left(q_1\right)\right)\left(J_y+ml^2\right)}p_3,\\
\dot q_4 &= \frac{m_h mls\left(q_4\right)\left(s\left(q_1\right)-s\left(q_1\right)\right)}{2ml}p_4,\\
\dot q_5 &= \frac{P_5}{J_r}.
\end{aligned} \tag{48}$$

The Hamiltonian of the snake-like robot system can be expressed as

$$\begin{aligned}
H=&\frac{1}{2}\frac{mls\left(q_1\right)-\left(J_y+ml^2\right)mlc\left(q_1\right)}{\left(J_y+ml^2\right)mlc\left(q_1\right)}p_1^2\\
&+\frac{mlc\left(q_1\right)+mls\left(q_4+q_1\right)}{ml\left(J_h+4ml^2s\left(q_1\right)+ml\right)}p_2^2\\
&+\frac{\left(J_y+ml^2\right)\left(J_y+ml^2\right)ml+2ml^2s\left(q_4+q_1\right)mlc\left(q_1\right)}{4ml\left(s\left(q_1\right)-s\left(q_1\right)\right)\left(J_y+ml^2\right)}p_3^2\\
&+\frac{ml\left(J_y+ml^2\right)+\left(s\left(q_1\right)-s\left(q_1\right)\right)}{J_h+4ml^2s\left(q_1\right)}p_4^2+\frac{1}{2}\frac{{P_5}^2}{J_r}.
\end{aligned} \tag{49}$$

Using Equations (47)–(49), we can obtain

$$\begin{aligned}
\dot p_1=&\frac{1}{2}\frac{mlc\left(q_1\right)+\left(J_y+ml^2\right)}{\left(J_y+ml^2\right)mlc^2\left(q_1\right)}p_1^2\\
&+\frac{mls\left(q_1\right)+mlc\left(q_4+q_1\right)}{ml\left(J_h+4ml^2s^2\left(q_1\right)c\left(q_1\right)+ml\right)}p_2^2+\\
&\frac{2ml^2c\left(q_4+q_1\right)mlc\left(q_1\right)}{4ml\left(c\left(q_1\right)-s^2\left(q_1\right)\right)\left(J_y+ml^2\right)}p_3^2\\
&+\frac{ml\left(J_y+ml^2\right)+\left(c\left(q_1\right)+s\left(q_1\right)\right)}{J_h+4ml^2c\left(q_1\right)}p_4^2\\
&+\frac{1}{2}D_yp_1mlc\left(q_1\right)+D_{xy}p_3p_4s\left(q_4+q_1\right)+ml^2p_1c\left(q_1\right),\\
\dot p_2=&+\frac{mlc\left(q_1\right)+mls\left(q_4+q_1\right)}{ml\left(J_h+4ml^2s\left(q_1\right)+ml\right)}p_2^2\\
&+\frac{ml\left(J_y+ml^2\right)+\left(s\left(q_1\right)-s\left(q_1\right)\right)}{4ml^2s\left(q_1\right)}p_4^2\\
&+D_ymlp_5p_1c\left(q_4\right)+D_{xy}p_3+mlp_2s\left(q_1\right)p_2,\\
\dot p_3=&\frac{mls\left(q_1\right)-\left(J_y+ml^2\right)mlc\left(q_1\right)}{\left(J_y+ml^2\right)mlc\left(q_1\right)}\\
&+\frac{\left(J_y+ml^2\right)\left(J_y+ml^2\right)ml+2ml^2s\left(q_4+q_1\right)mlc\left(q_1\right)}{4ml\left(s\left(q_1\right)-s\left(q_1\right)\right)\left(J_y+ml^2\right)}p_3^2\\
&+D_ymlp_5p_1c\left(q_4\right)+D_{xy}p_3+J_h+4p_2ml^2s\left(q_1\right)+ml,\\
\dot p_4=&\frac{mls\left(q_1\right)-mlc\left(q_4+q_1\right)}{ml\left(J_h+4ml^2s\left(q_1\right)+ml\right)}p_2^2\\
&+\frac{\left(J_y+ml^2\right)\left(J_y+ml^2\right)ml+2ml^2c\left(q_4+q_1\right)mlc\left(q_1\right)}{4ml\left(s\left(q_1\right)-s\left(q_1\right)\right)\left(J_y+ml^2\right)}p_3^2\\
&+D_yml^2p'_4p_1s\left(q_1+q_4\right)+2mlD_{xy}p_2+J_y+ml^2,\\
\dot p_5=&J_rp_5+D_ym^2l^2p_2p_1c\left(q_4\right)+D_{xy}p_2p_4\\
&+mlp_2s\left(q_1\right)p_2,
\end{aligned} \tag{50}$$

We call Equations (48) and (50) the generalized Hamilton canonical equations of the snake-like robot system.

In which

$$\begin{cases}
\Lambda'_1=\frac{1}{2}D_yp_1mlc\left(q_1\right)+D_{xy}p_3p_4\mathrm{s}\left(q_4+q_1\right)+ml^2p_1c\left(q_1\right),\\
\Lambda'_2=D_ymlp_5p_1c\left(q_4\right)+D_{xy}p_3+mlp_2s\left(q_1\right)p_2,\\
\Lambda'_3=D_ymlp_5p_1c\left(q_4\right)+D_{xy}p_3+J_h+4p_2ml^2s\left(q_1\right)+ml,\\
\Lambda'_4=D_yml^2p'_4p_1s\left(q_1+q_4\right)+2mlD_{xy}p_2+J_y+ml^2,\\
\Lambda'_5=\mathrm{D}_ym^2l^2p_2p_1c\left(q_4\right)+D_{xy}p_2p_4+mlp_2s\left(q_1\right)p_2.
\end{cases} \tag{51}$$

Let

$$\begin{aligned}
&a_1=q_1,a_2=q_2,a_3=q_3,a_4=q_4,a_5=q_5,\\
&a_6=p_1,a_7=p_2,a_8=p_3,a_9=p_4,a_{10}=p_5,
\end{aligned} \tag{52}$$

then Equation (50) can be expressed in the contravariant algebraic form

$$\dot a^\mu-S^{\mu\nu}\frac{\partial H}{\partial a^\nu}=0\,(\mu\nu=1\cdots5,) \tag{53}$$

where

$$\begin{aligned}
S^{\mu\nu} &= \omega^{u\nu}+T^{u\nu}\\
\omega^{\mu\nu} &= \begin{pmatrix}0_{5\times5} & I_{5\times5}\\ -I_{5\times5} & 0_{5\times5}\end{pmatrix},T^{\mu\nu}=\begin{pmatrix}0_{5\times5} & 0_{5\times5}\\ 0_{5\times5} & -\Omega_{k\times k}\end{pmatrix}
\end{aligned} \tag{54}$$

using Equations (25) and (51), we have

$$\begin{cases}
-\Omega_{11}=\frac{1}{2}D_ya_1mlc\left(a_1\right)+D_{xy}p_3p_4s\left(a_4+a_1\right)+2s\left(a_4\right)ml^2p_1c\left(a_1\right)\\
-\Omega_{22}=D_ymlp_5p_1c\left(a_4\right)+D_{xy}p_1+mla_4s\left(a_1\right)a_5\\
-\Omega_{33}=2D_ymlp_5a_3c\left(q_4\right)+D_{xy}p_1+J_h+4a_4ml^2s\left(a_1\right)+ml\\
-\Omega_{44}=D_yml^2p_3a_1s\left(a_1+a_4\right)+2mlD_{xy}p_1+J_y+ml^2\\
-\Omega_{55}=\mathrm{D}_ym^2l^2p_3p_1c\left(a_4\right)+D_{xy}p_1p_4+mla_7s\left(a_1\right)p_2.
\end{cases} \tag{55}$$

Substituting Equations (53) and (54) into Equation (27) leads to the contravariant algebraic form of snake-like robot systems

$$\begin{cases} \dot{a}_1 = \frac{mls(a_1)-(J_y+ml^2)mlc(a_1)}{(J_y+ml^2)mlc(a_1)} p_1 + 2ml^2 \mathrm{s}\,(a_4+a_1)\, mlc\,(a_1)\, p_2 \\ \dot{a}_2 = \frac{2mlc(a_1)+mls(a_4+a_1)}{ml(J_h+4ml^2 s(a_1)+ml)} p_1 \\ \dot{a}_3 = \frac{(J_y+ml^2)(J_y+ml^2)ml+2ml^2 s(a_4+a_1)mlc(a_1)}{4ml(s(a_1)-s(a_1))(J_y+ml^2)} p_3, \\ \dot{a}_4 = \frac{m_h mls(a_4)(s(a_1)-c(a_1))}{2ml} p_1, \\ \dot{a}_5 = \frac{p_1}{J_r} + 2ml^2 c\,(a_4+a_1)\, mlc\,(a_1)\,, \\ \dot{p}_1 = \frac{1}{2} \frac{mlc(a_1)+(J_y+ml^2)}{(J_y+ml^2)mlc^2(a_1)} {p_1}^2 + \frac{mls(a_1)+mlc(a_4+a_1)}{ml(J_h+4ml^2 s^2(a_1)c(a_1)+ml)} {p_2}^2 + \\ \frac{2ml^2 c(a_4+a_1)mlc(a_1)}{4ml(c(a_1)-s^2(a_1))(J_y+ml^2)} {p_4}^2 + \frac{1}{2} D_y p_1 mlc\,(a_1) + D_{xy} p_4 p_3 \mathrm{s}\,(a_4+a_1)\,, \\ \dot{p}_2 = \frac{mlc(a_1)+mls(a_4+a_1)}{ml(J_h+4ml^2 s(a_1)+ml)} {p_1}^2 + \frac{ml(J_y+ml^2)+(s(a_1)-c(a_1))}{4ml^2 s(a_1)} {p_4}^2 \\ + D_y mlp_5 p_1 c\,(q_4) + D_{xy} p_3 + mlp_2 s\,(a_1)\, p_2, \\ \dot{p}_3 = \frac{mls(a_1)-(J_y+ml^2)mlc(a_1)}{(J_y+ml^2)mlc(a_1)} + D_y mlp_5 p_1 c\,(a_4) + D_{xy} p_3 + J_h \\ + 4p_2 ml^2 s\,(a_1)\,, \\ \dot{p}_4 = \frac{mls(a_1)-mlc(a_4+a_1)}{ml(J_h+4ml^2 s(a_1)+ml)} {p_2}^2 + \frac{(J_y+ml^2)ml+2ml^2 c(a_4+a_1)mlca_1}{4ml(s(a_1)-s(a_1))(J_y+ml^2)} {p_4}^2, \\ \dot{p}_5 = D_y m^2 l^2 p_2 p_1 c\,(a_4) + D_{xy} p_3 p_4 + mlp_2 s\,(a_1)\, p_3 \end{cases} \tag{56}$$

From Proposition (2), the Hamiltonian of the system is written in the form

$$\begin{aligned} H = & \frac{1}{2} \frac{mls\,(a_1) - \left(J_y + ml^2\right) mlc\,(a_1)}{\left(J_y + ml^2\right) mlc\,(a_1)} {p_1}^2 \\ & + \frac{mlc\,(a_1) + mls\,(a_4 + a_1)}{ml\left(J_h + 4ml^2 s\,(a_1) + ml\right)} {p_2}^2 + \\ & \frac{\left(J_y + ml^2\right)\left(J_y + ml^2\right) ml + 2ml^2 s\,(a_4 + a_1)\, mlc\,(a_1)}{4ml\,(s\,(a_1) - c\,(a_1))\left(J_y + ml^2\right)} {p_3}^2 \\ & + \frac{ml\left(J_y + ml^2\right) + (s\,(a_1) - c\,(a_1))}{J_h + 4ml^2 s\,(a_1)} {p_4}^2 + \frac{1}{2} \frac{{p_5}^2}{J_r} = \mathrm{C}_1, \end{aligned} \tag{57}$$

and is the first integral.

Using Proposition (1), we can obtain the following integrals:

$$\begin{aligned} I_1 = & \int \frac{mls\,(a_1) - \left(J_y + ml^2\right) mlc\,(a_1)}{\left(J_y + ml^2\right) mlc\,(a_1)} dt \\ & + 4ml\,(s\,(a_1) - c\,(a_1))\, \mathrm{s}\,(a_4 + a_1)\, {p_2}^2 = \mathrm{C}_2, \end{aligned} \tag{58}$$

$$\begin{aligned} I_2 = & \int 2ml^2 \mathrm{s}\,(a_4 + a_1)\, mlc\,(a_1) p_1 + D_y 4ml^2 s\,(a_1)\, dt \\ & + \frac{ml^2 D_{xy} s\,(a_4)\, p_2 a_5}{4ml^2 s\,(a_1)\, c\,(a_4 + a_1)} + 2mls\,(a_1)\, a_5 p_3 = C_3, \end{aligned} \tag{59}$$

$$\begin{aligned} I_3 = & \int 2D_y ml^2 p_2 \mathrm{s}\,(a_4 + a_1)\, mlc\,(a_1) dt \\ & + mlD_{xy}\left(J_h + 4ml^2 s\,(a_1) + ml\right) = C_4. \end{aligned} \tag{60}$$

The first integral I_2includes a_5.Using Proposition (4), from I_2 we obtain a new integral.

$$I_4 = \frac{ml^2 D_{xy} s\,(a_4)\, a_7}{4ml^2 s\,(a_1)\, c\,(a_4 + a_1)} + 2mls\,(a_1)\, p_3 = C_4. \tag{61}$$

CONCLUSION

In this paper, we have studied the algebraic structure and Poisson integral theory of snake-like robot systems. This method reduces the expression variables of the snake-like robot and makes the expression more concise. We can also obtain the algebraic structure and Poisson integral theory of other soft robots.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

FJ-L is responsible for the topic selection and research work of the paper. ML is responsible for the research work of the paper. XC participates in the research of the topic selection and the establishment of motion equation.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.