



Dense Networks With Mixture Degree Distribution

Xiaomin Wang^{1,2*}, Fei Ma² and Bing Yao³

¹Key Laboratory of High-Confidence Software Technology, Peking University, Beijing, China, ²School of Electronics Engineering and Computer Science, Peking University, Beijing, China, ³College of Mathematics and Statistics, Northwest Normal University, Lanzhou, China

Complex networks have become a powerful tool to describe the structure and evolution in a large quantity of real networks in the past few years, such as friendship networks, metabolic networks, protein–protein interaction networks, and software networks. While a variety of complex networks have been published, dense networks sharing remarkable structural properties, such as the scale-free feature, are seldom reported. Here, our goal is to construct a class of dense networks. Then, we discover that our networks follow the mixture degree distribution; that is, there is a critical point above which the cumulative degree distribution has a power-law form and below which the exponential distribution is observed. Next, we also prove the networks proposed to show the small-world property. Finally, we study random walks on our networks with a trap fixed at a vertex with the highest degree and find that the closed form for the mean first-passage time increases logarithmically with the number of vertices of our networks.

OPEN ACCESS

Edited by:

Nuno A. M. Araújo,
University of Lisbon, Portugal

Reviewed by:

Haroldo V. Ribeiro,
State University of Maringá, Brazil
Rainer Klages,
Queen Mary University of London,
United Kingdom

*Correspondence:

Xiaomin Wang
wmxwm0616@163.com

Specialty section:

This article was submitted to
Mathematical and Statistical Physics,
a section of the journal
Frontiers in Physics

Received: 29 December 2020

Accepted: 15 February 2021

Published: 26 April 2021

Citation:

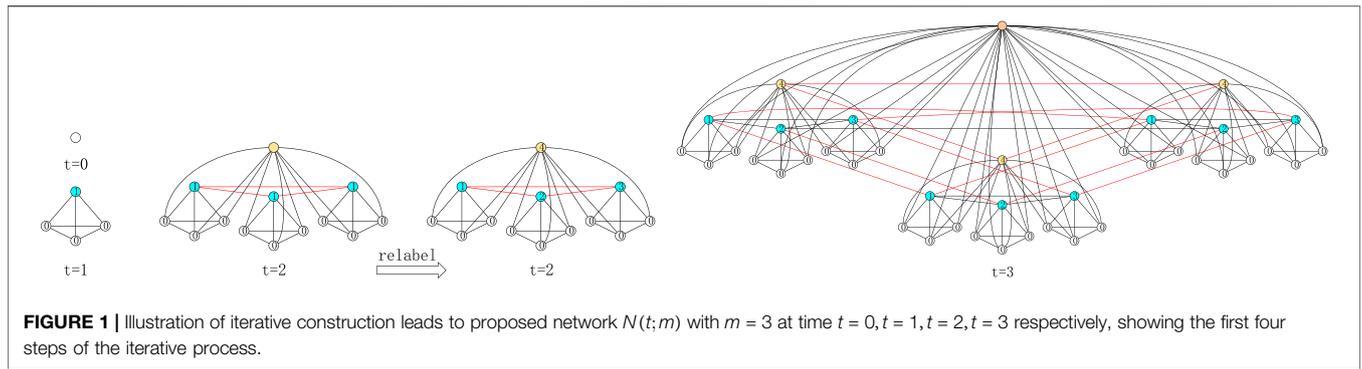
Wang X, Ma F and Yao B (2021) Dense
Networks With Mixture
Degree Distribution.
Front. Phys. 9:647346.
doi: 10.3389/fphy.2021.647346

Keywords: dense feature, scale-free property, small-world property, mean first-passage time, mixture degree distribution

1 INTRODUCTION

The exploding interest in complex networks during the several decades of the 21st century is rooted in the discovery that despite the diversity of complex networks, the structure and the evolution of each network are driven by a common set of basic laws and principles. Examples include the Internet and the World Wide Web [1], biological networks [2], social networks [3], and communication networks [4, 5], to mention but a few. There are two common considerable properties, the small-world effect and the scale-free feature. The well-known WS-network (Watts and Strogatz, WS) was proposed by Watts and Strogatz in *Nature* to explain small-world phenomena in diverse networks by two indices, diameter and clustering coefficient [6]. The most pioneering of generally studied networks is the BA-network (Barabási and Albert, BA) built by Barabási and Albert in *Science* using two rules, namely, growth and preferential attachment [7].

In currently existing studies, the main concentration is to create networks which have scale-free and small-world properties as mentioned above. However, most networks are sparse, which means that the average degree of networks is asymptotically equal to a positive constant under the limitation of a large number of vertices. The principal reason is that numbers of real-world networks have been found to indicate sparsity. On the contrary, a few scholars have discovered the existence of dense networks in [8, 9]. For probing such networks, several available networks have been presented and analytically explored with primary methods, including mean-field theory [10, 11], generate function [12], and rate equation [13, 14]. In particular, the vast majority of these networks are proved to have no scale-free feature. To put this in another way, the degree distribution of these networks does not obey the power-law distribution. Therefore, it is of interest to develop new theoretical frameworks for



producing networks with both the density feature and the scale-free feature. Here, we propose a class of networks with the appropriate structural properties mentioned above. Specifically speaking, these networks are precisely proved to be not only scale-free and small-world but also dense. Throughout this article, because graphs are abstract representations of networks, there is no need to distinguish between graph and network, which indicates that these two terms are considered to be the same.

The remainder of this article can be organized into the following sections. In **Section 2**, our task is to introduce network construction and discuss some extensively reported structural properties, including the average degree, mixture degree distribution, small-world property, and mean first-passage time. Among them, our networks are analytically proved to show both the density and scale-free features since the power-law exponent of cumulative degree distribution is equal to constant 2. Afterward, all the networks have a smaller diameter and a higher clustering coefficient, suggesting that they share the small-world property. Then, we derive an explicit expression of the mean first-passage time on our networks associated with the trapping problem. Finally, we close this article with a Conclusion and Discussions in the last section.

2 NETWORK CONSTRUCTION AND TOPOLOGICAL PROPERTIES

In this section, our intention is to generate a class of networks: let $N(t; m)$ denote the network with parameter m and step t ($t \geq 0$). Then, we investigate some topological properties of our networks under consideration, including the density feature, mixture degree distribution, small-world property, and mean first-passage time.

A Network Construction

Here, we will present our network, $N(t; m)$ ($m \geq 2$). The network is built in an iterative fashion, with each iteration, with respect to the elements, generated in the previous steps. We divide this process into the following steps.

Step 0: We start from a single vertex, we designate it as the active vertex of the network, and the active vertex is placed in the layer zero, denoted by $L = 0$.

Step 1: We add a cycle C_m with m vertices and m edges, each vertex in the cycle is labeled as 0, and we connect each vertex labeled as 0 to the active vertex; then, we obtain $N(1; m)$, all the cycle's vertices of network $N(1; m)$ at the layer $L = 1$.

Step 2: We generate m replicas of $N(1; m)$, each replica identical to the network created in the previous iteration (step 1), add a new vertex regarded as an active vertex, connect each of the bottom vertices of these m replicas to the new active vertex, and attach each of the non-bottom vertices of the m replicas to form an m cycle, so we acquire $N(2; m)$, where the bottom vertices of network $N(1; m)$ at the layer $L = 2$. **Figure 1** explains the process of obtaining our network $N(2; m)$.

Step 3: We add m replicas of $N(2; m)$ and create a new active vertex, and then join each of the bottom vertices of these m replicas with the active vertex, and connect each of the non-bottom vertices of these m replicas to form $3^{3-1} - 1/2$ cycles; thus, we get $N(3; m)$.

These steps can be easily generalized. In fact, step t will involve the following operation:

Step t : We add m replicas of $N(t-1; m)$, with each being identical to the network created in the previous iteration (step $t-1$); create a new vertex treated as an active vertex; link each of the bottom vertices of these m replicas with the active vertex; and connect each of the non-bottom vertices of these m replicas to produce $3^{t-1} - 1/2$ cycles; thus, we arrive at the network $N(t; m)$.

Indefinitely repeating the steps of replication and connection, apparently, we obtain a class of networks. As an illustrative example, a network $N(3; 3)$ for the particular case of $m = 3$ is shown in the right of **Figure 1**.

Now, we compute some related basic parameters of our networks by construction. Let $V(t; m)$ and $E(t; m)$ denote the set of vertices and edges, with the number of vertices $|V(t; m)|$ and the number of edges $|E(t; m)|$ of $N(t; m)$, respectively. According to the growth steps of network $N(t; m)$, we can easily compute and get a pair of equations satisfied by the number of vertices $|V(t; m)|$ and the number of edges $|E(t; m)|$ at step t as follows:

$$\begin{cases} |V(t; m)| = m|V(t-1; m)| + 1 \\ |E(t; m)| = m|E(t-1; m)| + \frac{m^{t+1} - m}{m-1} \end{cases} \quad (1)$$

At step $t = 0$, clearly, there is $|V(0; m)| = 1$ and $|E(0; m)| = 0$. At step $t = 1$, we have $|V(1; m)| = m + 1$ and $|E(1; m)| = m(m + 1)/2$. According to known conditions, we straightforwardly calculate the value of $|V(t; m)|$ and $|E(t; m)|$ from Eq. 1 and arrive at

$$\begin{cases} |V(t; m)| = \frac{m^{t+1} - 1}{m - 1} \\ |E(t; m)| = |E(1; m)|m^{t+1} + \frac{(t - 1)m^{t+1} - m^t - m}{m - 1} \end{cases} \quad (2)$$

Until now, we have finished the construction of our networks and computed certain fundamental quantities. It is an extremely evident fact to discover the hierarchy of our network related to the step. It is worth noticing that the involved network construction with hierarchy has been discussed in [1]. In [1], Ravasz *et al.* constructed a hierarchical network that combines the scale-free property with a high clustering. However, it should be noted that the approach adopted in this article is slightly different from the approach in [1]. While Ravasz *et al.* applied an iterative process for all vertices, our method is based on an idea that the next network $N(t; m)$ is created by connecting a part of vertices of each replica $N(t - 1; m)$ of the preceding model to an active vertex instead of all vertices in network $N(t - 1; m)$. What calls for special attention is that for different m , our networks are different due to the number of vertices and edges being different, but the topological structure of our networks is the same. Additionally, as will be shown later, our network $N(t; m)$ are proven to have some distinguished topological properties, such as the density feature and mixture degree distribution.

B Topological Properties

As has been mentioned above, we mainly focus on the investigation of certain topological properties related to the potential structure of the proposed network. In this section, we will calculate several topological properties, including the density feature, mixture degree distribution, small-world property, and mean first-passage time. What we present here will be significant constituents in the coming subsections.

Density feature: A key topological structure of a network is its average degree. Average degree $\langle k(t; m) \rangle$ can be defined as the average value of all such vertex degrees over the entire network, which is applied to judge whether the network is sparse or dense. One can call a network sparse if $|E(t; m)| \ll |V(t; m)|(|V(t; m)| - 1)/2$; in other words, the value of $\langle k(t; m) \rangle$ tends to a finite and positive real value in the large number of vertices [15]. According to the definition of sparsity, the majority of real networks with scale-free properties, both stochastic and deterministic, are sparse. Yet, for our networks, we have

$$\langle k(t; m) \rangle = \frac{2|E(t; m)|}{|V(t; m)|} \sim 2[t - 1 + |E(1; m)|(m - 1)]. \quad (3)$$

We give a detailed calculation process of the average degree in the Appendix. From Eq. 3, it goes without saying that the average degree $\langle k(t; m) \rangle$ increases linearly with t and does not tend to a

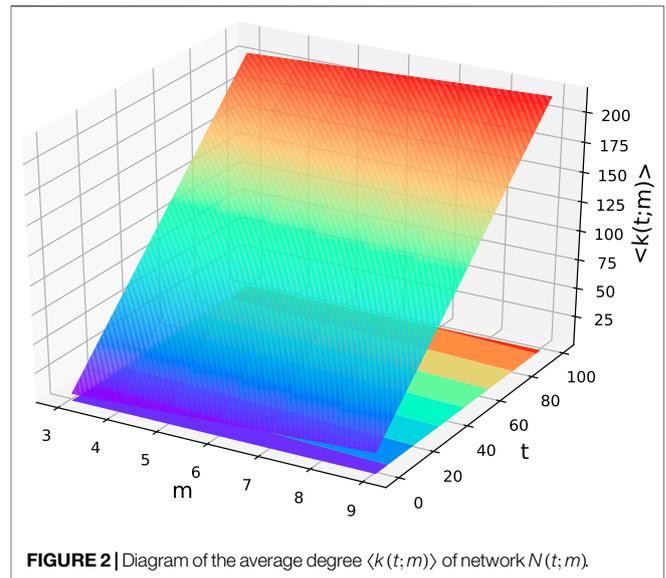


FIGURE 2 | Diagram of the average degree $\langle k(t; m) \rangle$ of network $N(t; m)$.

real positive constant. In the large number of vertices, the average degree $\langle k(t; m) \rangle$ of Eq. 3 approaches to infinity, as shown in Figure 2. Consequently, we have the result that our network $N(t; m)$ turns out to be of density. It can be said with certainty that our networks are different from some real-world networks with sparse networks, such as hierarchical networks [1] and deterministic networks [16]. Therefore, our network $N(t; m)$ can be selected as an underlying network to reveal some remarkable properties behind those dense networks in real life.

Mixture degree distribution: The concept of degree is the most fundamental character and measure of a vertex in a network. Since in a network every vertex has a degree value, some large and some small, the distribution of vertex in the network is a key topological feature, which may be of great concern in application. The degree distribution is one of the most important topological features of a network. Degree distribution can be applied to determine whether a given network is scalefree or not. Combined with the process of network construction, it is straightforward to find that the degree spectrum of network $N(t; m)$ is a series of discrete real values. Armed with the above constructions and the method proposed by Dorogovtsev, the cumulative degree distribution of our networks can be calculated in a discrete form, as follows in [17]:

$$P_{cum}(k \geq k_i) = \frac{\sum_{k \geq k_i} N_k}{|V(t; m)|} \quad (4)$$

where notation N_k denotes the total number of vertices with a degree exactly equal to k in network $N(t; m)$. As stated in Eq. 4, according to the vertex degree of our networks, we need to classify all the vertices of networks to determine the degree distribution of our networks.

For a network $N(t; m)$ with $t + 1$ layers, it has to be noticed that the largest degree vertex is that active vertex which joins at time step t and has degree m^t , and the next largest vertices are m vertices, that is, the active vertex of the m units added to the

TABLE 1 | Degree spectrum.

L	$k_{L,t,m}$	$N_{L,t,m}$
0	m^t	1
1	$m^{t-1} + 2$	m
...
$t - t_i$	$m^{t_i} + 2(t - t_i)$	m^{t-t_i}
...
$t - 2$	$m^2 + 2(t - 2)$	m^{t-2}
$t - 1$	$m^1 + 2(t - 1)$	m^{t-1}
T	$t + 2$	m^t

network in the last layer (namely, $L = 1$). Based on this classification, there is no denying the fact that all vertices at the layer $L = t$ have the degree $t + 2$. This degree value must be within the interval $[m^{t_i} + 2(t - t_i), m^{t_i+1} + 2(t - t_i - 1)]$. Hence, for the sake of simplicity, we will only adjust the initial layer of vertices with respect to the vertex degree. Then, we have **Table 1**.

On the basis of the aforementioned **Table 1** and **Eq. 4**, the dependence of the cumulative distribution $P_{cum}(k \geq k_{L,t,m})$ on the degree $k_{L,t,m}$ and the step t is captured by the following equation:

$$P_{cum}(k \geq k_{L,t,m}) = \begin{cases} k_{L,t,m}^{-\gamma_\alpha} & m^{t_i} > 2(t - t_i) \\ e^{-\lambda k_{L,t,m}} & m^{t_i} \leq 2(t - t_i) \end{cases}, \quad (5)$$

where power-law exponent $\gamma_\alpha = 1$. Taking the derivative of both sides of k in **Eq. 5** yields

$$P(k) = k^{-\gamma}, \gamma = \gamma_\alpha + 1 = 2 \quad (6)$$

The detailed calculation process of cumulative degree distribution is presented in Appendix. The cumulative degree distribution of our networks is composed of two parts, exponential distribution and power-law distribution. Besides, this result does not match the statement in the previous work. Genio believes that there is no network whose power-law exponent belongs to $0 < \gamma \leq 2$ in the limitation of the large number of vertices in [18]. Timar *et al.* explore two models of scale-free networks that have the degree distribution exponent $\gamma = 1$ [8]. Courtney *et al.* present a modeling framework which produces networks that are both dense and have scale-free properties in [9]. In addition, in [19], Ma *et al.* propose a framework for producing scale-free networks with the dense feature using two operations, that is, first-order subdivision and line operation. However, our approach is different from those in the above methods for constructing scale-free networks with dense properties. Meanwhile, the lights shed by them may be helpful to construct some novel networks with certain constraints. Next, we will discuss the small-world property of our networks.

Small-world property: Watts and Strogatz proposed the small-world property of complex networks by using two features: a relative smaller diameter and a higher clustering coefficient [6]. The small-world property of our networks is still not well discussed; despite that, they well describe these two important parameters of topological structures. In the coming discussion, we focus on the diameter and clustering coefficient of our networks.

Diameter: The distance between two vertices is the smallest number of edges to get from u to v . The longest shortest path between all pairs of vertices is called the diameter. The diameter is itself a feature of network structure and can be applied to characterize a communication delay over a network. In general, the larger the diameter is, the lower the communication efficiency is. The diameter of our network is denoted as $D(t; m)$. Fortunately, for our networks, it can be calculated easily. Here, we will introduce the main computation of the diameter. With the help of the construction process of network $N(t; m)$, we have

$$D(t; m) = 4 = O(1). \quad (7)$$

We give an exhaustive calculation process of diameter in Appendix. The reason comprises two main cases: 1) all vertices of the layer $L = t$ are connected to that active vertex of the highest layer, that is, $L = 0$; 2) each vertex in the middle layers, $L = 1, \dots, L = t - 1$, always attaches to a vertex of the lowest layer, that is, $L = t$. In fact, the diameters $D(t; m)$ are equal to the distance between two vertices, both of which are in the middle layers and in different branches of network $N(t; m)$. This forces the network into an active vertex and unit in which all vertices are close to each other because they all connect to the active vertex. In this regime, the diameter is independent of t and m . Note that this $D(t; m)$ actually is not very large, implying that our networks also have an ultrasmall diameter—two randomly chosen vertices are connected by a fairly short path length. Hence, formally, $D(t; m) = 4$; for any $t \geq 0$, the diameter is small and far less than the number of vertices. So, we have a surprising result of an ultrasmall diameter in this sense.

Clustering coefficient: The clustering coefficient is another vital property of a network, which provides the measure of the local structure within the network. The most immediate measure of clustering is the clustering coefficient C_i for every vertex i . By definition, the clustering coefficient of a vertex i is the ratio of the total number E_i of edges that actually exist between all k_i (its nearest neighbors) and the number of $k_i(k_i - 1)/2$ of all possible edges between them, that is, $C_i = 2E_i/[k_i(k_i - 1)]$. The clustering coefficient $\langle C \rangle$ of the entire network can be defined as the average of all vertex C 's as follows:

$$\langle C \rangle = \frac{\sum_{v \in V} C_v}{|V|}. \quad (8)$$

The clustering coefficient of a vertex i in our networks is as follows:

$$C_i(k) \propto k^{-1}. \quad (9)$$

The detailed process of the clustering coefficient will be given in Appendix. The degree of the clustering coefficient of a whole network is captured by the average value of the clustering coefficient, $\langle C(t; m) \rangle$, representing the average of C_i over all vertices $i = 1, 2, \dots, |V(t; m)|$.

$$\langle C(t; m) \rangle = \frac{\sum_{i=1}^t \frac{2(m^{t-i+2i})m^i}{(m^{t-i+2i})(m^{t-i+2i-1})} + \frac{2(m-1)m^t}{(t+2)(t+1)}}{|V(t; m)|}. \quad (10)$$

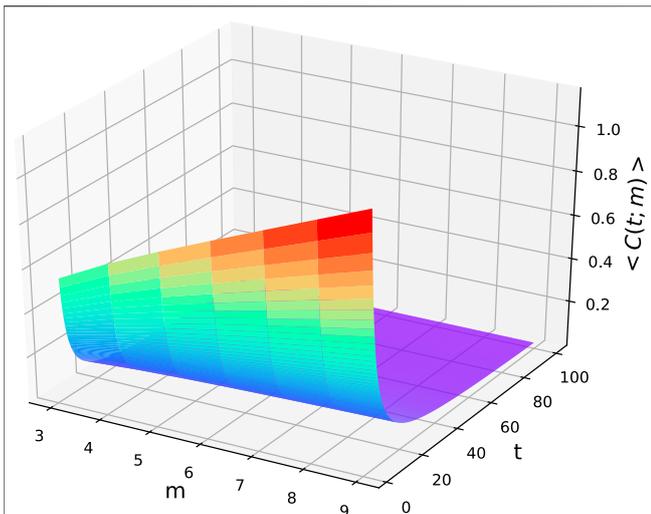


FIGURE 3 | Diagram of the clustering coefficient $\langle C(t; m) \rangle$ of network $N(t; m)$.

Taking the limit on the number of vertices, the clustering coefficient $\langle C(t; m) \rangle$ of Eq. 10 approaches to a nonzero value, as shown in Figure 3. So we can say that our networks have a high clustering coefficient.

Consequently, our networks, with a smaller diameter and a higher clustering coefficient, can be considered as small-world networks.

Mean first-passage time: In this subsection, we have an attempt to study the trapping problem on our network $N(t; m)$. Indeed, what is considered here is a simple unbiased discrete-time Markovian random walk with a single trap, namely, an ideal absorber located at a specified vertex on a network. As previously discussed, that active vertex of network $N(t; m)$ at the step t has the largest degree, referred to as a_t . For the sake of convenience, let the active vertex a_t be the trap vertex. At each time step, a walker starting from its current position v moves to one of its neighbors N_v with the transition probability $1/d_v(t; m)$ before visiting the absorber, where $d_v(t; m)$ is the number of its existing edges of v in network $N(t; m)$.

Here we think a walker starts from the vertex v at the initial time. Obviously, we can obtain the fact that the transition probability P_{vu} of starting out from v to reach u holds the following equation:

$$P_{vu}(l + 1) = \sum_{i \in V(t; m)} \frac{a_{iu}}{d_i(t; m)} P_{vi}(l), \quad (11)$$

where a_{iu} is the entry of the adjacency matrix of network $N(t; m)$: $a_{iu} = 1$ if there is an edge connecting vertices i and u , and $a_{iu} = 0$ otherwise.

Armed with the rule mentioned above, we want to discuss the most important quantity for the trapping problem, generally named the first-passage time FPT . The first-passage time is the expected number of steps for a walker, starting from the vertex v to first arrive at the trap in the trapping problem. For our network $N(t; m)$, we represent the first-passage time for a walker

located in the vertex v by FPT_v and let $P(FPT_v = l)$ be the probability for that vertex v to first drop into the trap, namely, the active vertex a_t . Similar to Eq. 11, we have

$$P(FPT_v = l) = \sum_{\substack{i \in V(t; m), \\ i \neq a_t}} \frac{a_{iu}}{d_i(t; m)} P(FPT_v = l - 1). \quad (12)$$

The classical method to solve the aforementioned equation is the generating function. Without loss of generality, we can write down the corresponding generating function of $P(FPT_v = l)$ as follows:

$$\mathcal{P}_v(x) = \sum_{l=0}^{\infty} P(FPT_v = l) x^l. \quad (13)$$

A trial yet helpful fact associated with $\mathcal{P}_v(x)$, the expected time FPT_v , is exactly equal to the value $\mathcal{P}_v(1)$, as we will explain shortly in the coming discussion.

Before proceeding further, we first define two notations, $P_t(s)$ and $Q_L(s)$. Let $P_t(s)$ denote the probability for a walker on any vertex at the layer $L = t + 1$ of network $N(t; m)$ to first visit the active a_t after s steps and $Q_L(s)$ represent the probability that a walker starting from any vertex w at the layer L ($L = 1, 2, \dots, t$) hits one at a randomly chosen vertex at the layer $L = t + 1$ of network $N(t; m)$, which connects to vertex w , after s steps. Combined with the structure of network $N(t; m)$ and the statement above, we arrive at the explicit equation for $P_t(s)$ on our networks as follows:

$$P_t(s) = \frac{\delta_{s,1} + \sum_{L=1}^t \sum_{i=1}^{s-1} Q_L(s) P_t(s - 1 - i)}{k_{L=t+1}}, \quad (14)$$

where $\delta_{s,1}$ is the Kronecker delta function defined as $\delta_{s,1} = 1$ if s is equal to one and $\delta_{s,1} = 0$; otherwise, $k_{L=t+1}$ is the number of edges associated with the vertex at the layer $L = t + 1$ and equals $t + 1$.

Then, together with Eq. 13, the generating function $\mathcal{P}_t(x)$ corresponding to $P_t(s)$ can be written as

$$\mathcal{P}_t(x) = \frac{x}{k_{L=t+1}} + \frac{tx^2}{k_{L=t+1}} \mathcal{P}_t(x) \quad (15)$$

in which we apply a result $Q_L(i) = 1$ only for both $L = 1, \dots, t$ and $i = 1$ and $Q_L(i) = 0$ otherwise.

Meanwhile, let FPT_{t+1}^t represent the first-passage time for a walker originally placed on any vertex at the layer $L = t + 1$ which can be expressed as

$$FPT_{t+1}^t = \frac{d}{dx} \mathcal{P}_t(x) \Big|_{x=1}. \quad (16)$$

On the basis of Eq. 16, doing the derivative of both sides of Eq. 15 evolves the exact solution of FPT_{t+1}^t . It is represented as

$$FPT_{t+1}^t = 2t + 1. \quad (17)$$

For each vertex at the layer $L = 1, \dots, t$, together with the definition of $Q_L(s)$ and the hierarchical structure of networks $N(t; m)$, the first-passage time FPT_L^t for a walker formerly set on an arbitrary vertex at the layer L can be expressed in terms of FPT_{t+1}^t .

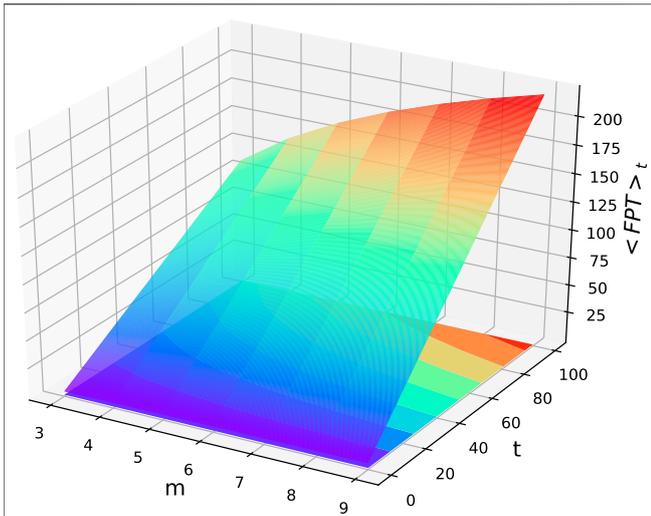


FIGURE 4 | Diagram of the mean first-passage time $\langle FPT \rangle_t$ of network $N(t; m)$.

$$FPT_L^t = FPT_{t+1}^t + 1. \tag{18}$$

Then, the first-passage time FPT_L^t for a walker at the layer $L = 1, \dots, t + 1$ can be computed in a sharp mode. Our next purpose is to deduce the mean first-passage time $\langle FPT \rangle_t$, which depicts the trapping process with a random walk on average.

$$\langle FPT \rangle_t = \frac{1}{|V(t; m) - 1|} \sum_{L=1}^{t+1} FPT_L^t |N_t(L)|. \tag{19}$$

Here, we again apply the hierarchical structure of network $N(t; m)$, and $|N_t(L)|$ represents the total number of vertices at the layer $L (L = 1, \dots, t + 1)$. As previously stated, we find that $|N_t(L)|$ is equal to $N_{L,t;m}$.

Plugging Eqs 17 and 18 and the value of $|V(t; m)|$ into Eq. 19, we obtain the subsequent equation by taking advantage of several simple mathematics

$$\langle FPT \rangle_t = O\left(2t + \frac{1}{m}\right). \tag{20}$$

Moreover, we think the logarithm of the number of vertices of network $N(t; m)$, that is, $\ln|V(t; m)| \sim (t + 1)\ln m$. It is straightforward to discover that for the entire network $N(t; m)$, the mean first-passage time $\langle FPT \rangle_t$ is closely related to the number of vertices of network $N(t; m)$.

$$\langle FPT \rangle_t \sim \ln|V(t; m)|. \tag{21}$$

Thus, in the large limit of the number of vertices of network $N(t; m)$, the mean first-passage time increases logarithmically with the number of vertices of our networks. Equation 21 reveals the relationship of the mean first-passage time on the number of edges of our networks. Figure 4 describes the fact that the mean first-passage time is a function of step t .

This is a bit different from some previous results in the existing literature; for instance, for a complete graph with N vertices, the

mean first-passage time is exactly equal to $N - 1$, which scales linearly with the number of vertices. One can see that the mean first-passage time of the Apollonian network increases as a fractional power of the number of vertices, which implies that the Apollonian network has a faster transmit time than any other analytically soluble media [20]. In contrast to our networks, the complete graph has no scale-free feature and hierarchical structure, but has the ultrasmall diameter and dense feature. On the other hand, the Apollonian network has a larger diameter than our networks, which indicates the scale-free feature and hierarchical structure. All things considered, it may be safely said that our network $N(t; m)$ outperforms those networks in [20] with respect to the mean first-passage time in the trapping problem.

3 CONCLUSION AND DISCUSSION

In our article, we present a class of scale-free networks with the dense feature. On the basis of our analysis, we deduce some striking results. 1) The average degree of our networks is not approximately a fixed constant value, and its value diverges with the step t , and then we reveal the fact that our networks are dense. 2) The cumulative degree distribution of our networks $N(t; m)$ contains two parts: exponential degree distribution and power-law distribution with the exponent $\gamma = 2$. 3) Combined with the diameter and clustering coefficient, we hold the opinion that our networks have the small-world effect. 4) The mean first-passage time in network $N(t; m)$ is approximately related to the logarithm number of vertices of networks. Our findings are insightful for the study of the random walks on various deterministically growing networks. Our work creates a broader perspective on previous research studies of trapping on diverse networks and sheds light on some aspects related to the trapping problem, providing some relationship information between efficiency and underlying the network size. In the future, we hope that the current study can provide some inspiration for trapping problems in real networks with topologies similar to ours.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary Material; further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

XW provided this topic and wrote the paper. FM discussed and modified it, and BY guided the manuscript. All authors contributed to manuscript and approved the submitted version.

FUNDING

This research was supported in part by the National Key Research and Development Program of China under Grant

No. 2019YFA0706401; in part by the National Natural Science Foundation of China under Grant No. 61632002; in part by the General Program of the National Natural Science Foundation of China under Grant No. 61872399, Grant No. 61872166, and Grant No. 61672264; in part by the

Young Scientists Fund of the National Natural Science Foundation of China under Grant No. 61802009, Grant No. 61902005, and Grant No. 62002002; and in part by the National Natural Science Foundation of China under Grant No. 61662066.

REFERENCES

- Ravasz E, and Barabási AL. Hierarchical organization in complex networks. *Phys Rev E* (2003) 67:026112. doi:10.1103/physreve.67.026112
- Fornito A, Zalesky A, and Breakspear M. The connectomics of brain disorders. *Nat Rev Neurosci* (2015) 16:159–172. doi:10.1038/nrn3901
- Iranzo J, Buldú JM, and Aguirre J. Competition among networks highlights the power of the weak. *Nat Commun* (2016) 7:13273. doi:10.1038/ncomms13273
- Mocanu DC, Exarchakos G, and Liotta A. 2004 IEEE international conference on systems, man, and cybernetics; 2004 Oct 10–13; The Hague, Netherlands. (2014). p. 19–24.
- Albert R, Jeong H, and Barabási A-L. Error and attack tolerance of complex networks. *Nature* (2000) 406:378–382. doi:10.1038/35019019
- Watts DJ, and Strogatz SH. Collective dynamics of “small-world” networks. *Nature* (1998) 393:440–2. doi:10.1038/30918
- Barabási A-L, and Albert R. Emergence of scaling in random networks. *Science* (1999) 286:509–12. doi:10.1126/science.286.5439.509
- Timár G, Dorogovtsev SN, and Mendes JFF. Scale-free networks with exponent one. *Phys Rev E* (2016) 94:022302. doi:10.1103/physreve.94.022302
- Courtney OT, and Bianconi G. Dense power-law networks and simplicial complexes. *Phys Rev E* (2018) 97:052303. doi:10.1103/physreve.97.052303
- Barabási A-L, Albert R, and Jeong H. Mean-field theory for scale-free random networks. *Physica A: Stat Mech its Appl* (1999) 272:173–87. doi:10.1016/s0378-4371(99)00291-5
- Newman MEJ, Moore C, and Watts DJ. Mean-field solution of the small-world network model. *Phys Rev Lett* (2000) 84(14):3201–4. doi:10.1103/physrevlett.84.3201
- Newman MEJ, Strogatz SH, and Watts DJ. *Phys Rev E* (2000) 64:026118.
- Krapivsky PL, Redner S, and Leyvraz F. Connectivity of growing random networks. *Phys Rev Lett* (2000) 85(21):4629–32. doi:10.1103/physrevlett.85.4629
- Krapivsky PL, Rodgers GJ, and Redner S. Degree distributions of growing networks. *Phys Rev Lett* (2001) 86(23):5401. doi:10.1103/physrevlett.86.5401
- Barabási AL. *Network science*. Cambridge, United Kingdom: Cambridge University Press (2016). p. 10–3.
- Barabási A-L, Ravasz E, and Vicsek T. Deterministic scale-free networks. *Physica A: Stat Mech its Appl* (2001) 299:559–64. doi:10.1016/s0378-4371(01)00369-7
- Dorogovtsev SN, Goltsev AV, and Mendes JFF. Pseudofractal scale-free web. *Phys Rev E* (2002) 65:066122. doi:10.1103/physreve.65.066122
- Del Genio CI, Gross T, and Bassler KE. All scale-free networks are sparse. *Phys Rev Lett* (2011) 107:178701. doi:10.1103/physrevlett.107.178701
- Ma F, Wang XM, Wang P, and Luo XD. Dense networks with scale-free feature. *Phys Rev E* (2020) 101:052317. doi:10.1103/physreve.101.052317
- Zhang Z, Guan J, Xie W, Qi Y, and Zhou S. Random walks on the Apollonian network with a single trap. *Europhys Lett* (2009) 86:10006. doi:10.1209/0295-5075/86/10006

Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Copyright © 2021 Wang, Ma and Yao. This is an open-access article distributed under the terms of the Creative Commons Attribution License (CC BY). The use, distribution or reproduction in other forums is permitted, provided the original author(s) and the copyright owner(s) are credited and that the original publication in this journal is cited, in accordance with accepted academic practice. No use, distribution or reproduction is permitted which does not comply with these terms.