



Large Time Behavior on the Linear Self-Interacting Diffusion Driven by Sub-Fractional Brownian Motion With Hurst Index Large Than 0.5 I: Self-Repelling Case

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Let S^H be a sub-fractional Brownian motion with index $\frac{1}{2} < H < 1$. In this paper, we consider the linear self-interacting diffusion driven by S^H , which is the solution to the equation

$$dX_t^H = dS_t^H - \theta \left(\int_0^t (X_t^H - X_s^H) ds \right) dt + \nu dt, \quad X_0^H = 0,$$

where $\theta < 0$ and $\nu \in \mathbb{R}$ are two parameters. Such process X^H is called self-repelling and it is an analogue of the linear self-attracting diffusion [Cranston and Le Jan, *Math. Ann.* 303 (1995), 87–93]. Our main aim is to study the large time behaviors. We show the solution X^H diverges to infinity, as t tends to infinity, and obtain the speed at which the process X^H diverges to infinity as t tends to infinity.

Keywords: the self-repelling diffusion, asymptotic distribution, convergence, sub-fractional Brownian motion, stochastic integral

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1 INTRODUCTION

In 1995, Cranston and Le Jan [1] introduced a linear self-attracting diffusion

$$X_t = B_t - \theta \int_0^t \int_0^s (X_s - X_u) du ds + \nu t, \quad t \geq 0 \quad (1.1)$$

with $\theta > 0$ and $X_0 = 0$, where B is a 1-dimensional standard Brownian motion. They showed that the process X_t converges in L^2 and almost surely, as t tends infinity. This is a special case of path dependent stochastic differential equations. Such path dependent stochastic differential equation was first developed by Durrett and Rogers [2] introduced in 1992 as a model for the shape of a growing polymer (Brownian polymer) as follows

$$X_t = X_0 + B_t + \int_0^t \int_0^s f(X_s - X_u) du ds, \quad (1.2)$$

where B is a d -dimensional standard Brownian motion and f is Lipschitz continuous. X_t corresponds to the location of the end of the polymer at time t . Under some conditions, they established asymptotic behavior of the solution of stochastic differential equation and gave some conjectures and questions. The model is a continuous analogue of the notion of edge (resp. vertex) self-interacting random walk. If $f(x) = g(x)x/\|x\|$ and $g(x) \geq 0$, X_t is a continuous analogue of a process introduced by

Diaconis and studied by Pemantle [3]. Let $\mathcal{L}^X(t, x)$ be the local time of the solution process X . Then, we have

$$X_t = X_0 + B_t + \int_0^t ds \int_{\mathbb{R}} f(-x) \mathcal{L}^X(s, X_s + x) dx$$

for all $t \geq 0$. This formulation makes it clear how the process X interacts with its own occupation density. We may call this solution a Brownian motion interacting with its own passed trajectory, i.e., a *self-interacting motion*. In general, the Eq. 1.2 defines a self-interacting diffusion without any assumption on f . If

$$x \cdot f(x) \geq 0 \quad (x \cdot f(x) \leq 0)$$

for all $x \in \mathbb{R}^d$, we call it self-repelling (resp. self-attracting). In 2002, Benaïm et al [4] also introduced a self-interacting diffusion with dependence on the (convolved) empirical measure. A great difference between these diffusions and Brownian polymers is that the drift term is divided by t . It is noteworthy that the interaction potential is attractive enough to compare the diffusion (a bit modified) to an Ornstein-Uhlenbeck process, in many case of f , which points out an access to its asymptotic behavior. More works can be found in Benaïm et al. [5], Cranston and Mountford [6], Gauthier [7], Herrmann and Roynette [8], Herrmann and Scheutzow [9], Mountford and Tarr [10], Shen et al [11], Sun and Yan [12] and the references therein.

On the other hand, starting from the application of fractional Brownian motion in polymer modeling, Yan et al [13] considered an analogue of the linear self-interacting diffusion:

$$X_t^H = B_t^H - \theta \int_0^t \int_0^s (X_s^H - X_u^H) dud s + \nu t, \quad t \geq 0 \quad (1.3)$$

with $\theta \neq 0$ and $X_0^H = 0$, where B^H is a fractional Brownian motion (fBm, in short) with Hurst parameter $\frac{1}{2} \leq H < 1$. The solution of (1.3) is a Gaussian process. When $\theta > 0$, Yan et al [13] showed that the solution X^H of (1.3) converges in L^2 and almost surely, to the random variable

$$X_\infty^H = \int_0^\infty h_\theta(s) dB_s^H + \nu \int_0^\infty h_\theta(s) ds$$

where the function is defined as follows

$$h_\theta(s) = 1 - \theta s e^{\frac{1}{2}\theta s^2} \int_s^\infty e^{-\frac{1}{2}\theta u^2} du, \quad s \geq 0$$

with $\theta > 0$. Recently, Sun and Yan [14] considered the related parameter estimations with $\theta > 0$ and $\frac{1}{2} \leq H < 1$, and Gan and Yan [15] considered the parameter estimations with $\theta < 0$ and $\frac{1}{2} \leq H < 1$.

Motivated by these results, as a natural extension one can consider the following stochastic differential equation:

$$X_t = G_t - \theta \int_0^t \int_0^s (X_s - X_u) dud s + \nu t, \quad t \geq 0 \quad (1.4)$$

with $\theta > 0$ and $X_0 = 0$, where $G = \{G_t, t \geq 0\}$ is a Gaussian process with some suitable conditions which includes fractional Brownian motion and some related processes. However, for a (general) abstract Gaussian process it is difficult to find some

interesting fine estimates associated with the calculations. So, in this paper we consider the linear self-attracting diffusion driven by a sub-fractional Brownian motion (sub-fBm, in short). We choose this kind of Gaussian process because it is only the generalization of Brownian motion rather than the generalization of fractional Brownian motion. It only has some similar properties of fractional Brownian motion, such as long memory and self similarity, but it has no stationary increment. The so-called sub-fBm with index $H \in (0, 1)$ is a mean zero Gaussian process $S^H = \{S_t^H, t \geq 0\}$ with $S_0^H = 0$ and the covariance

$$R_H(t, s) \equiv E[S_t^H S_s^H] = s^{2H} + t^{2H} - \frac{1}{2} [(s+t)^{2H} + |t-s|^{2H}] \quad (1.5)$$

for all $s, t \geq 0$. For $H = 1/2$, S^H coincides with the standard Brownian motion B . S^H is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with S^H . As a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to S^H (see, for example, Alós et al [16]). The sub-fBm has properties analogous to those of fBm and satisfies the following estimates:

$$\begin{aligned} [(2 - 2^{2H-1}) \wedge 1] (t-s)^{2H} &\leq E[(S_t^H - S_s^H)^2] \\ &\leq [(2 - 2^{2H-1}) \vee 1] (t-s)^{2H}. \end{aligned} \quad (1.6)$$

More works for sub-fBm and related processes can be found in Bojdecki et al. [17–20], Li [21–24], Shen and Yan [25, 26], Sun and Yan [27], Tudor [28–31], Ciprian A. Tudor [32] Yan et al [33–35] and the references therein.

In this present paper, we consider the linear self-interacting diffusion

$$X_t^H = S_t^H - \theta \int_0^t \int_0^s (X_s^H - X_u^H) dud s + \nu t, \quad t \geq 0 \quad (1.7)$$

with $\theta < 0$ and $X_0^H = 0$, where S^H is a sub-fBm with Hurst parameter $\frac{1}{2} \leq H < 1$. Our main aim is to show that the solution of (1.7) diverges to infinity and obtain the speed diverging to infinity, as t tends to infinity. The object of this paper is to expound and prove the following statements:

(I) For $\theta < 0$ and $\frac{1}{2} < H < 1$, the random variable

$$\xi_\infty^H = \int_0^\infty s e^{\frac{1}{2}\theta s^2} dS_s^H$$

exists as an element in L^2 .

(II) For $\theta < 0$ and $\frac{1}{2} < H < 1$, as $t \rightarrow \infty$, we have

$$J_0^H(t; \theta, \nu) := t e^{\frac{1}{2}\theta t^2} X_t^H \rightarrow \xi_\infty^H - \frac{\nu}{\theta}$$

in L^2 and almost surely.

(III) For $\theta < 0$ and $\frac{1}{2} < H < 1$, define the processes $J^H(n, \theta, \nu) = \{J_t^H(n, \theta, \nu), t \geq 0\}, n \geq 1$ by

$$J_n^H(t; \theta, \nu) := \theta t^2 \left(J_{n-1}^H(t; \theta, \nu) - (2n-3)!! \left(\xi_\infty^H - \frac{\nu}{\theta} \right) \right),$$

$$n = 1, 2, \dots,$$

for all $t \geq 0$, where $(-1)!! = 1$. We then have

$$J_n^H(t; \theta, \nu) \rightarrow (2n-1)!! \left(\xi_\infty^H - \frac{\nu}{\theta} \right)$$

holds in L^2 and almost surely for every $n \geq 1$, as $t \rightarrow \infty$.

This paper is organized as follows. In **Section 2** we present some preliminaries for sub-fBm and Malliavin calculus. In **Section 3**, we obtain some lemmas. In **Section 4**, we prove the main result. In **Section 5** we give some numerical results.

2 PRELIMINARIES

In this section, we briefly recall the definition and properties of stochastic integral with respect to sub-fBm. We refer to Alós et al [16], Nualart [36], and Tudor [31] for a complete description of stochastic calculus with respect to Gaussian processes. Throughout this paper we assume that $S^H = \{S_t^H, t \geq 0\}$ denotes a sub-fBm defined on the probability space (Ω, \mathcal{F}, P) with index H . As we pointed out before, the sub-fBm S^H is a rather special class of self-similar Gaussian processes such that $S_0^H = 0$, $E[S_t^H] = 0$ and

$$R^H(t, s) := E[S_t^H S_s^H] = s^{2H} + t^{2H} - \frac{1}{2} [(s+t)^{2H} + |t-s|^{2H}] \tag{2.1}$$

for all $s, t \geq 0$. For $H = 1/2$, S^H coincides with the standard Brownian motion B . S^H is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with S^H . As a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to S^H . The sub-fBm appeared in Bojdecki et al [17] in a limit of occupation time fluctuations of a system of independent particles moving in \mathbb{R}^d according a symmetric α -stable Lévy process, and it also appears in Bojdecki et al [18] in a high-density limit of occupation time fluctuations of the above mentioned particle system, where the initial Poisson configuration has finite intensity measure.

The estimate (1.6) and normality imply that the sub-fBm $t \mapsto S_t^H$ admits almost surely a bounded $\frac{1}{H-\vartheta}$ -variation on any finite interval for any sufficiently small $\vartheta \in (0, H)$. That is, the paths of $t \mapsto S_t^H$ admits a bounded p_H -variation on any finite interval with $p_H > \frac{1}{H}$. As an immediate result, one can define the Young integral of a process $u = \{u_t, t \geq 0\}$ with respect to sub-fBm $B^{a,b}$

$$\int_0^t u_s dS_s^H$$

as the limit in probability of a Riemann sum. Clearly, the integral is well-defined and

$$u_t S_t^H = \int_0^t u_s dS_s^H + \int_0^t S_s^H du_s$$

for all $t \geq 0$, provided u is of bounded q_H -variation on any finite interval with $q_H > 1$ and $\frac{1}{p_H} + \frac{1}{q_H} > 1$ (see, for examples, Bertoin [37] and Föllmer [38]).

Let \mathcal{H} be the completion of the linear space \mathcal{E} generated by the indicator functions $1_{[0,t]}$, $t \in [0, T]$ with respect to the inner product

$$\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}} = R^H(t, s)$$

for $s, t \in [0, T]$. When $\frac{1}{2} < H < 1$, we can show that

$$\begin{aligned} \|\varphi\|_{\mathcal{H}}^2 &= \int_0^T \int_0^T \varphi(t)\varphi(s) \frac{\partial^2}{\partial t \partial s} R^H(t, s) ds dt \\ &= \int_0^T \int_0^T \varphi(t)\varphi(s) \psi_H(t, s) ds dt, \quad \forall \varphi \in \mathcal{H}, \end{aligned}$$

where

$$\psi_H(t, s) = \frac{\partial^2}{\partial t \partial s} R^{a,b}(t, s) = H(2H-1)(|t-s|^{2H-2} - |t+s|^{2H-2})$$

for $s, t \in [0, T]$. Define the linear mapping $\mathcal{E} \ni \varphi \mapsto S^H(\varphi)$ by

$$1_{[0,t]} \mapsto S^H(1_{[0,t]}) = \int_0^T 1_{[0,t]}(s) dS_s^H \equiv S_t^H$$

for all $t \in [0, T]$ and it can be continuously extended to \mathcal{H} and we call the mapping Φ is called the Wiener integral with respect to S^H , denoted by

$$S^H(\varphi) = \int_0^T \varphi(s) dS_s^H$$

and

$$\|\varphi\|_{\mathcal{H}}^2 = E \left(\int_0^T \varphi(s) dS_s^H \right)^2 \tag{2.2}$$

for any $\varphi \in \mathcal{H}$.

For simplicity, in this paper we assume that $\frac{1}{2} < H < 1$. Thus, if for every $T > 0$, the integral

$$\int_0^T \varphi(s) dS_s^H$$

exists in L^2 and

$$\int_0^\infty \int_0^\infty \varphi(t)\varphi(s) \psi_H(t, s) ds dt < \infty,$$

we can define the integral

$$\int_0^\infty \varphi(s) dS_s^H$$

and

$$E \left(\int_0^\infty \varphi(s) dS_s^H \right)^2 = \int_0^\infty \int_0^\infty \varphi(t)\varphi(s) \psi_H(t, s) ds dt.$$

Denote by \mathcal{S} the set of smooth functionals of the form

$$F = f(S^H(\varphi_1), S^H(\varphi_2), \dots, S^H(\varphi_n)),$$

where $f \in C_b^\infty(\mathbb{R}^n)$ and $\varphi_i \in \mathcal{H}$. The Malliavin derivative D of a functional F as above is given by

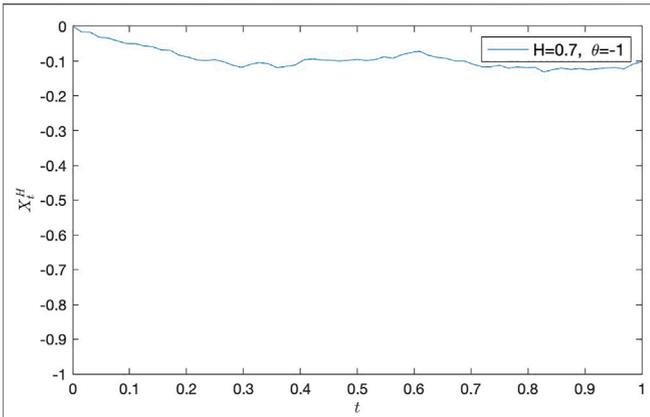


FIGURE 1 | A path of X_t^H with $\theta = -1$ and $H = 0.7$.

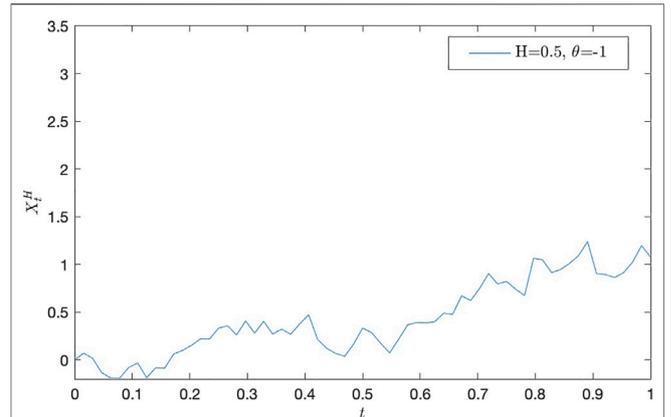


FIGURE 4 | A path of X_t^H with $\theta = -1$ and $H = 0.5$.

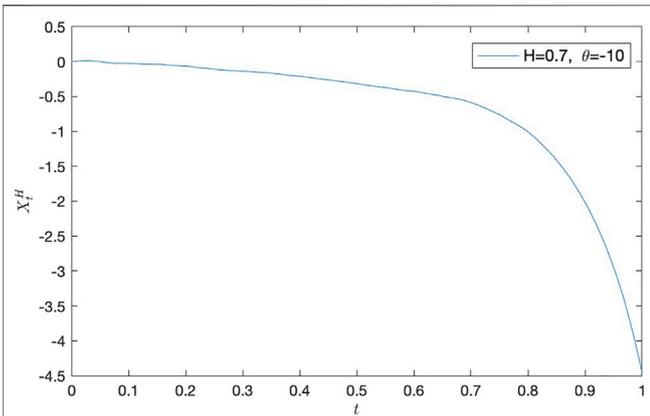


FIGURE 2 | A path of X_t^H with $\theta = -10$ and $H = 0.7$.

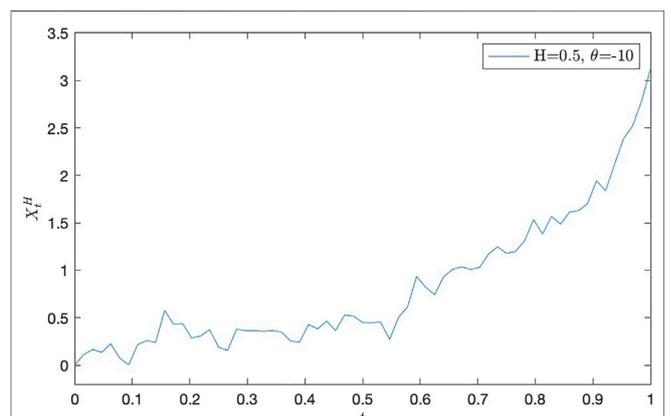


FIGURE 5 | A path of X_t^H with $\theta = -10$ and $H = 0.5$.

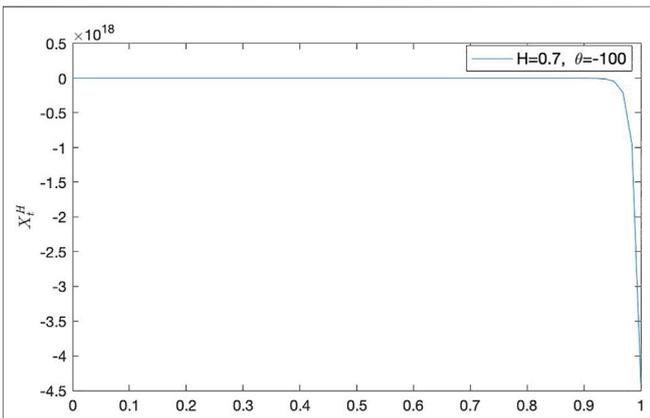


FIGURE 3 | A path of X_t^H with $\theta = -100$ and $H = 0.7$.

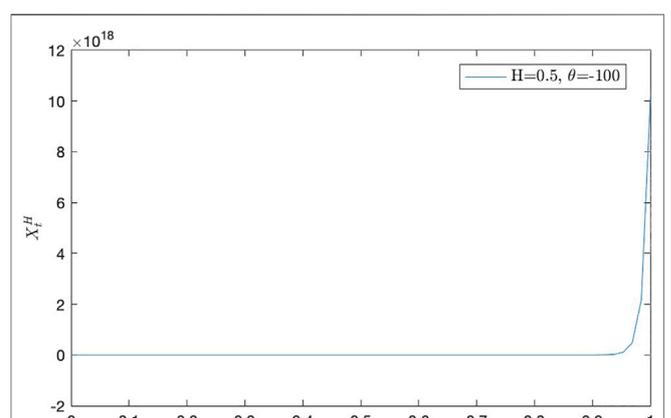


FIGURE 6 | A path of X_t^H with $\theta = -100$ and $H = 0.5$.

TABLE 1 | The data of X_t^H with $\theta = -1$ and $H = 0.7$.

t	X_t^H	T	X_t^H	t	X_t^H
0.0000	0.0000	0.3438	-0.1077	0.6875	-0.0995
0.0156	-0.0167	0.3594	-0.1190	0.7031	-0.1091
0.0313	-0.0178	0.3750	-0.1153	0.7188	-0.1163
0.0469	-0.0320	0.3906	-0.1116	0.7344	-0.1165
0.0625	-0.0338	0.4063	-0.0965	0.7500	-0.1122
0.0781	-0.0420	0.4219	-0.0937	0.7656	-0.1205
0.0938	-0.0492	0.4375	-0.0971	0.7813	-0.1170
0.1094	-0.0496	0.4531	-0.0974	0.7969	-0.1192
0.1250	-0.0564	0.4688	-0.0997	0.8125	-0.1180
0.1406	-0.0590	0.4844	-0.0976	0.8281	-0.1316
0.1563	-0.0682	0.5000	-0.0956	0.8438	-0.1245
0.1719	-0.0692	0.5156	-0.0983	0.8594	-0.1202
0.1875	-0.0834	0.5313	-0.0959	0.8750	-0.1241
0.2031	-0.0886	0.5469	-0.0877	0.8906	-0.1212
0.2188	-0.0969	0.5625	-0.0919	0.9063	-0.1250
0.2344	-0.0983	0.5781	-0.0818	0.9219	-0.1219
0.2500	-0.0961	0.5938	-0.0757	0.9375	-0.1199
0.2656	-0.1022	0.6094	-0.0717	0.9531	-0.1191
0.2813	-0.1120	0.6250	-0.0834	0.9688	-0.1223
0.2969	-0.1182	0.6406	-0.0894	0.9844	-0.1089
0.3125	-0.1094	0.6563	-0.0923	1.0000	-0.1023
0.3281	-0.1042	0.6719	-0.0996	—	—

TABLE 3 | The data of X_t^H with $\theta = -100$ and $H = 0.7$.

t	X_t^H	t	X_t^H	t	X_t^H
0.0000	0.0000	0.3438	-1.0056	0.6875	-2.29E+07
0.0156	0.0132	0.3594	-1.6439	0.7031	-6.63E+07
0.0313	0.0093	0.3750	-2.7733	0.7188	-1.97E+08
0.0469	0.0070	0.3906	-4.8028	0.7344	-5.99E+08
0.0625	0.0103	0.4063	-8.5377	0.7500	-1.87E+09
0.0781	0.0116	0.4219	-15.5941	0.7656	-5.98E+09
0.0938	0.0092	0.4375	-29.2598	0.7813	-1.96E+10
0.1094	0.0066	0.4531	-56.3669	0.7969	-6.59E+10
0.1250	0.0081	0.4688	-111.4786	0.8125	-2.27E+11
0.1406	0.0049	0.4844	-226.2866	0.8281	-8.02E+11
0.1563	0.0094	0.5000	-471.3711	0.8438	-2.91E+12
0.1719	-0.0029	0.5156	-1.01E+03	0.8594	-1.08E+13
0.1875	-0.0114	0.5313	-2.21E+03	0.8750	-4.10E+13
0.2031	-0.0279	0.5469	-4.97E+03	0.8906	-1.60E+14
0.2188	-0.0484	0.5625	-1.15E+04	0.9063	-6.40E+14
0.2344	-0.0557	0.5781	-2.72E+04	0.9219	-2.62E+15
0.2500	-0.0837	0.5938	-6.59E+04	0.9375	-1.10E+16
0.2656	-0.1240	0.6094	-1.64E+05	0.9531	-4.75E+16
0.2813	-0.1834	0.6250	-4.19E+05	0.9688	-2.10E+17
0.2969	-0.2706	0.6406	-1.10E+06	0.9844	-9.48E+17
0.3125	-0.4085	0.6563	-2.95E+06	1.0000	-4.40E+18
0.3281	-0.6332	0.6719	-8.12E+06	—	—

TABLE 2 | The data of X_t^H with $\theta = -10$ and $H = 0.7$.

t	X_t^H	t	X_t^H	t	X_t^H
0.0000	0.0000	0.3438	-0.1597	0.6875	-0.5552
0.0156	0.0087	0.3594	-0.1729	0.7031	-0.5943
0.0313	0.0113	0.3750	-0.1912	0.7188	-0.6439
0.0469	0.0040	0.3906	-0.2051	0.7344	-0.7019
0.0625	-0.0153	0.4063	-0.2130	0.7500	-0.7595
0.0781	-0.0239	0.4219	-0.2342	0.7656	-0.8345
0.0938	-0.0234	0.4375	-0.2494	0.7813	-0.9066
0.1094	-0.0279	0.4531	-0.2654	0.7969	-0.9868
0.1250	-0.0348	0.4688	-0.2820	0.8125	-1.0919
0.1406	-0.0372	0.4844	-0.2980	0.8281	-1.2177
0.1563	-0.0395	0.5000	-0.3156	0.8438	-1.3507
0.1719	-0.0530	0.5156	-0.3363	0.8594	-1.5050
0.1875	-0.0587	0.5313	-0.3543	0.8750	-1.6776
0.2031	-0.0648	0.5469	-0.3694	0.8906	-1.8811
0.2188	-0.0835	0.5625	-0.3865	0.9063	-2.1081
0.2344	-0.0942	0.5781	-0.4093	0.9219	-2.3699
0.2500	-0.1100	0.5938	-0.4204	0.9375	-2.6701
0.2656	-0.1213	0.6094	-0.4368	0.9531	-3.0170
0.2813	-0.1317	0.6250	-0.4620	0.9688	-3.4144
0.2969	-0.1365	0.6406	-0.4810	0.9844	-3.8989
0.3125	-0.1418	0.6563	-0.5086	1.0000	-4.4510
0.3281	-0.1541	0.6719	-0.5258	—	—

TABLE 4 | The data of X_t^H with $\theta = -1$ and $H = 0.5$.

t	X_t^H	t	X_t^H	t	X_t^H
0.0000	0.0000	0.3438	0.2713	0.6875	0.6225
0.0156	0.0711	0.3594	0.3234	0.7031	0.7483
0.0313	0.0168	0.3750	0.2698	0.7188	0.9047
0.0469	-0.1326	0.3906	0.3765	0.7344	0.7963
0.0625	-0.1887	0.4063	0.4725	0.7500	0.8221
0.0781	-0.1911	0.4219	0.2156	0.7656	0.7416
0.0938	-0.0792	0.4375	0.1224	0.7813	0.6743
0.1094	-0.0320	0.4531	0.0691	0.7969	1.0655
0.1250	-0.1853	0.4688	0.0377	0.8125	1.0480
0.1406	-0.0827	0.4844	0.1668	0.8281	0.9146
0.1563	-0.0861	0.5000	0.3344	0.8438	0.9478
0.1719	0.0616	0.5156	0.2866	0.8594	1.0125
0.1875	0.1014	0.5313	0.1759	0.8750	1.0931
0.2031	0.1542	0.5469	0.0739	0.8906	1.2403
0.2188	0.2224	0.5625	0.2168	0.9063	0.9036
0.2344	0.2205	0.5781	0.3676	0.9219	0.8949
0.2500	0.3345	0.5938	0.3904	0.9375	0.8626
0.2656	0.3581	0.6094	0.3878	0.9531	0.9140
0.2813	0.2635	0.6250	0.3985	0.9688	1.0247
0.2969	0.4084	0.6406	0.4900	0.9844	1.1976
0.3125	0.2820	0.6563	0.4769	1.0000	1.0780
0.3281	0.4043	0.6719	0.6713	—	—

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j} (S^H(\varphi_1), S^H(\varphi_2), \dots, S^H(\varphi_n)) \varphi_j.$$

The derivative operator D is then a closable operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$. We denote by $\mathbb{D}^{1,2}$ the closure of \mathcal{S} with respect to the norm

$$\|F\|_{1,2} := \sqrt{E|F|^2 + E\|DF\|_{\mathcal{H}}^2}.$$

The divergence integral δ is the adjoint of derivative operator D^H . That is, we say that a random variable u in $L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator δ , denoted by $\text{Dom}(\delta^S)$, if

$$E|\langle DF, u \rangle_{\mathcal{H}}| \leq c\|F\|_{L^2(\Omega)}$$

for every $F \in \mathbb{D}^{1,2}$, where c is a constant depending only on u . In this case $\delta(u)$ is defined by the duality relationship

$$E[F\delta(u)] = E\langle DF, u \rangle_{\mathcal{H}} \tag{2.3}$$

TABLE 5 | The data of X_t^H with $\theta = -10$ and $H = 0.5$.

t	X_t^H	t	X_t^H	t	X_t^H
0.0000	0.0000	0.3438	0.3643	0.6875	1.0084
0.0156	0.1112	0.3594	0.3489	0.7031	1.0312
0.0313	0.1668	0.3750	0.2532	0.7188	1.1722
0.0469	0.1353	0.3906	0.2453	0.7344	1.2474
0.0625	0.2259	0.4063	0.4297	0.7500	1.1783
0.0781	0.0764	0.4219	0.3837	0.7656	1.1997
0.0938	0.0025	0.4375	0.4639	0.7813	1.3114
0.1094	0.2166	0.4531	0.3663	0.7969	1.5335
0.1250	0.2593	0.4688	0.5287	0.8125	1.3820
0.1406	0.2412	0.4844	0.5164	0.8281	1.5679
0.1563	0.5773	0.5000	0.4502	0.8438	1.4858
0.1719	0.4322	0.5156	0.4488	0.8594	1.6145
0.1875	0.4384	0.5313	0.4538	0.8750	1.6282
0.2031	0.2872	0.5469	0.2729	0.8906	1.7043
0.2188	0.3078	0.5625	0.5069	0.9063	1.9432
0.2344	0.3761	0.5781	0.6164	0.9219	1.8384
0.2500	0.1896	0.5938	0.9359	0.9375	2.1171
0.2656	0.1558	0.6094	0.8222	0.9531	2.3878
0.2813	0.3807	0.6250	0.7422	0.9688	2.5204
0.2969	0.3637	0.6406	0.9326	0.9844	2.7823
0.3125	0.3641	0.6563	1.0095	1.0000	3.1237
0.3281	0.3580	0.6719	1.0371	—	—

TABLE 6 | The data of X_t^H with $\theta = -100$ and $H = 0.5$.

t	X_t^H	t	X_t^H	t	X_t^H
0.0000	0.0000	0.3438	2.1870	0.6875	5.26E+07
0.0156	-0.1749	0.3594	3.5867	0.7031	1.52E+08
0.0313	-0.3397	0.3750	6.3084	0.7188	4.52E+08
0.0469	-0.4106	0.3906	11.0159	0.7344	1.37E+09
0.0625	-0.3348	0.4063	19.5047	0.7500	4.29E+09
0.0781	-0.3567	0.4219	35.6469	0.7656	1.37E+10
0.0938	-0.3936	0.4375	66.9024	0.7813	4.50E+10
0.1094	-0.3411	0.4531	129.1499	0.7969	1.51E+11
0.1250	-0.2522	0.4688	255.5964	0.8125	5.21E+11
0.1406	-0.1583	0.4844	518.9528	0.8281	1.84E+12
0.1563	-0.1543	0.5000	1.08E+03	0.8438	6.66E+12
0.1719	0.0877	0.5156	2.31E+03	0.8594	2.47E+13
0.1875	-0.1242	0.5313	5.07E+03	0.8750	9.42E+13
0.2031	-0.0522	0.5469	1.14E+04	0.8906	3.67E+14
0.2188	0.1336	0.5625	2.63E+04	0.9063	1.47E+15
0.2344	0.0243	0.5781	6.23E+04	0.9219	6.02E+15
0.2500	0.1665	0.5938	1.51E+05	0.9375	2.53E+16
0.2656	0.2096	0.6094	3.77E+05	0.9531	1.09E+17
0.2813	0.4085	0.6250	9.62E+05	0.9688	4.81E+17
0.2969	0.5852	0.6406	2.52E+06	0.9844	2.18E+18
0.3125	0.8397	0.6563	6.76E+06	1.0000	1.01E+19
0.3281	1.3366	0.6719	1.86E+07	—	—

for any $F \in \mathbb{D}^{1,2}$. We have $\mathbb{D}^{1,2} \subset \text{Dom}(\delta)$ and for any $u \in \mathbb{D}^{1,2}$

$$E[\delta(u)^2] = E\|u\|_{\mathcal{H}}^2 + E\langle Du, (Du)^* \rangle_{\mathcal{H} \otimes \mathcal{H}} \\ = E\|u\|_{\mathcal{H}}^2 + E \int_{[0,T]^4} D_{\xi} u_r D_{\eta} u_s \phi_H(\eta, r) \phi_H(\xi, s) ds dr d\xi d\eta,$$

where $(D^S u)^*$ is the adjoint of Du in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$. We will denote

$$\delta(u) = \int_0^T u_s \delta S_s^H$$

for an adapted process u , and it is called Skorohod integral. Alós et al [16], we can obtain the relationship between the Skorohod and Young integral as follows

$$\int_0^T u_s dS_s^H = \int_0^T u_s \delta S_s^H + \int_0^T \int_0^T D_s u_t \psi(t, s) ds dt,$$

provided u has a bounded q -variation with $1 \leq q < \frac{1}{H}$ and $u \in \mathbb{D}^{1,2}(\mathcal{H})$ such that

$$\int_0^T \int_0^T D_s u_t \psi(t, s) ds dt < \infty.$$

Theorem 2.1. (Alós et al [16]). Let $0 < H < 1$ and let $f \in C^2(\mathbb{R})$ such that

$$\max\{|f(x)|, |f'(x)|, |f''(x)|\} \leq \kappa e^{\beta x^2}, \tag{2.4}$$

where κ and β are two positive constants with $\beta < \frac{1}{4}T^{-2H}$. Then we have

$$f(S_t^H) = f(0) + \int_0^t f'(S_s^H) dS_s^H + H(2 - 2^{2H-1}) \int_0^t f''(S_s^H) s^{2H-1} ds$$

for all $t \in [0, T]$.

3 SOME BASIC ESTIMATES

Throughout this paper we assume that $\theta < 0$ and $\frac{1}{2} < H < 1$. Recall that the linear self-interacting diffusion with sub-fBm S^H defined by the stochastic differential equation

$$X_t^H = S_t^H - \theta \int_0^t \int_0^s (X_s^H - X_u^H) du ds + \nu t, \quad t \geq 0 \tag{3.1}$$

with $\theta < 0$. Define the kernel $(t, s) \mapsto h_{\theta}(t, s)$ as follows

$$h_{\theta}(t, s) = \begin{cases} 1 - \theta s e^{\frac{1}{2}\theta s^2} \int_s^t e^{-\frac{1}{2}\theta u^2} du, & t \geq s, \\ 0, & t < s \end{cases} \tag{3.2}$$

for $s, t \geq 0$. By the variation of constants method (see, Cranston and Le Jan [1]) or Itô's formula we may introduce the following representation:

$$X_t^H = \int_0^t h_{\theta}(t, s) dS_s^H + \nu \int_0^t h_{\theta}(t, s) ds \tag{3.3}$$

for $t \geq 0$.

The kernel function $(t, s) \mapsto h_{\theta}(t, s)$ with $\theta < 0$ admits the following properties (these properties are proved partly in Sun and Yan [12]):

- For all $s \geq 0$, the limit

$$\lim_{t \rightarrow \infty} (t e^{\frac{1}{2}\theta t^2} h_{\theta}(t, s)) = s e^{\frac{1}{2}\theta s^2} \tag{3.4}$$

for all $s \geq 0$.

- For all $t \geq s \geq 0$, we have

$$1 \leq h_\theta(t, s) \leq e^{-\frac{1}{2}\theta(t^2-s^2)}.$$

- For all $t \geq s, r \geq 0$, we have

$$h_\theta(t, 0) = h_\theta(t, t) = 1, \quad \int_s^t h_\theta(t, u) du = e^{\frac{1}{2}\theta s^2} \int_s^t e^{-\frac{1}{2}\theta u^2} du.$$

Lemma 3.1. Let $\theta < 0$ and define function

$$I_\theta(t) = -\theta t e^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} du - 1.$$

We then have $\lim_{t \rightarrow \infty} t^2 I_\theta(t) = -\frac{1}{\theta}$ and

$$\lim_{t \rightarrow \infty} t^2 \left(1 + \theta t e^{-\frac{1}{2}\theta t^2} \int_t^\infty e^{\frac{1}{2}\theta u^2} du \right) = -\frac{1}{\theta}$$

Proof. This is simple calculus exercise.

Lemma 3.2. (Sun and Yan [12]). Let $\theta < 0$ and define the functions $t \mapsto I_\theta(t, n)$, $n = 1, 2, \dots$ as follows

$$I_\theta(t, 1) = -\theta t^2 I_\theta(t), \quad I_\theta(t, n+1) = -\theta t^2 [I_\theta(t, n) - (2n-1)!].$$

Then we have

$$\lim_{t \rightarrow \infty} I_\theta(t, n) = (2n-1)!. \tag{3.5}$$

for every $n \geq 0$, where $(-1)! = 1$.

Lemma 3.3. Let $\theta < 0$. Then the integral

$$\Delta(H) = \int_0^\infty \int_0^\infty x y e^{\frac{1}{2}\theta(x^2+y^2)} \psi_H(x, y) dx dy \tag{3.6}$$

converges and as $t \rightarrow \infty$,

$$\lim_{t \rightarrow \infty} t^2 e^{-\theta t^2} E(X_t^H)^2 = \Delta(H).$$

Proof. An elementary may show that (3.6) converges for all $\theta < 0$. It follows from L'Hôpital's rule that

$$\begin{aligned} \lim_{t \rightarrow \infty} t^2 e^{\theta t^2} E(X_t^H)^2 &= \lim_{t \rightarrow \infty} t^2 e^{\theta t^2} \int_0^t \int_0^t h_\theta(t, x) h_\theta(t, y) \psi_H(x, y) dx dy \\ &= \lim_{t \rightarrow \infty} \frac{\theta^2}{t^{-2} e^{-\theta t^2}} \int_0^t dx \int_0^t x y e^{\frac{1}{2}\theta(x^2+y^2)} \psi_H(x, y) dy \int_x^t du \int_y^t e^{-\frac{1}{2}\theta(u^2+v^2)} dv \\ &= 2 \lim_{t \rightarrow \infty} \frac{\theta^2}{t^{-2} e^{-\theta t^2}} \int_0^t du \int_0^u dx \int_0^u dv \int_0^v d y x y e^{\frac{1}{2}\theta(x^2+y^2-i^2-v^2)} \psi_H(x, y) \\ &= \lim_{t \rightarrow \infty} \frac{-\theta}{t^{-1} e^{-\frac{1}{2}\theta t^2}} \int_0^t dx \int_0^t e^{-\frac{1}{2}\theta v^2} dv \int_0^v x y e^{\frac{1}{2}\theta(x^2+y^2)} \psi_H(x, y) dy \\ &= \lim_{t \rightarrow \infty} \frac{-\theta}{t^{-1} e^{-\frac{1}{2}\theta t^2}} \int_0^t e^{-\frac{1}{2}\theta v^2} dv \int_0^t dx \int_0^v x y e^{\frac{1}{2}\theta(x^2+y^2)} \psi_H(x, y) dy \\ &= \lim_{t \rightarrow \infty} \frac{-\theta}{t^{-1} e^{-\frac{1}{2}\theta t^2}} \int_0^t e^{-\frac{1}{2}\theta v^2} dv \int_0^v dx \int_0^v x y e^{\frac{1}{2}\theta(x^2+y^2)} \psi_H(x, y) dy \\ &= \int_0^\infty dx \int_0^\infty x y e^{\frac{1}{2}\theta(x^2+y^2)} \psi_H(x, y) dy, \end{aligned}$$

where we have used the following fact:

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t^{-1} e^{-\frac{1}{2}\theta t^2}} \int_0^t e^{-\frac{1}{2}\theta v^2} dv \int_v^t dx \int_0^v x y e^{\frac{1}{2}\theta(x^2+y^2)} \psi_H(x, y) dy \\ = \lim_{t \rightarrow \infty} \frac{1}{t^{-1} e^{-\frac{1}{2}\theta t^2}} \int_0^t dx \int_0^x e^{-\frac{1}{2}\theta v^2} dv \\ \int_0^v x y e^{\frac{1}{2}\theta(x^2+y^2)} \psi_H(x, y) dy = 0. \end{aligned}$$

This completes the proof.

Lemma 3.4. Let $\theta < 0$. Then, convergence

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t^{2-2H} e^{-\theta t^2}} \int_t^\infty \int_s^\infty s r e^{\frac{1}{2}\theta(s^2+r^2)} \psi_H(s, r) ds dr \\ = \frac{1}{4} (-\theta)^{-2H} \Gamma(2H+1). \end{aligned} \tag{3.7}$$

holds.

Proof. It follows from L'Hôpital's rule that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t^{2-2H} e^{\theta t^2}} \int_t^\infty u e^{\frac{1}{2}\theta u^2} \left(\int_u^\infty v e^{\frac{1}{2}\theta v^2} \psi_H(u, v) dv \right) du \\ = -\frac{1}{2\theta} \lim_{t \rightarrow \infty} \frac{1}{t^{2-2H} e^{\frac{1}{2}\theta t^2}} \int_t^\infty v e^{\frac{1}{2}\theta v^2} \psi_H(t, v) dv \\ = -\lim_{t \rightarrow \infty} \frac{H(2H-1)}{2\theta t^{2-2H}} \int_t^\infty v e^{\frac{1}{2}\theta(v^2-t^2)} ((v-t)^{2H-2} - (v+t)^{2H-2}) dv \end{aligned}$$

for all $\theta < 0$ and $\frac{1}{2} < H < 1$. By making the change of variable $\frac{1}{2}\theta(v^2 - t^2) = x$, we see that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{2\theta t^{2-2H}} \int_t^\infty v e^{\frac{1}{2}\theta(v^2-t^2)} ((v-t)^{2H-2} - (v+t)^{2H-2}) dv \\ = \lim_{t \rightarrow \infty} \frac{1}{2\theta^2 t^{2-2H}} \int_0^\infty e^{-x} \left\{ \left(\sqrt{t^2 + \frac{2x}{-\theta}} - t \right)^{2H-2} \right. \\ \left. - \left(\sqrt{t^2 + x} + t \right)^{2H-2} \right\} dx \\ = \lim_{t \rightarrow \infty} \frac{1}{2\theta^2 t^{2-2H}} \int_0^\infty e^{-x} \left(\frac{2x}{-\theta} \right)^{2H-2} \left(\sqrt{t^2 + \frac{2x}{-\theta}} + t \right)^{2-2H} dx \\ - \lim_{t \rightarrow \infty} \frac{1}{2\theta^2 t^{2-2H}} \int_0^\infty e^{-x} \left(\sqrt{t^2 + x} + t \right)^{2H-2} dx \\ = \frac{1}{2} (-\theta)^{-2H-1} \Gamma(2H-1) \end{aligned}$$

for all $\theta < 0$ and $\frac{1}{2} < H < 1$. This completes the proof.

Lemma 3.5. Let $\theta < 0$ and $0 \leq s < t \leq T$. We then have

$$c(t-s)^{2H} \leq E[(X_t^H - X_s^H)^2] \leq C(t-s)^{2H} \tag{3.8}$$

Proof. Given $0 \leq s < t \leq T$ and denote

$$\hat{X}_t^H = \int_0^t h_\theta(t, r) dS_r^H, \quad t \geq 0.$$

It follows that

$$E\left[\left(\hat{X}_t^H - \hat{X}_s^H\right)^2\right] = E\left(\int_0^s [h_\theta(t, x) - h_\theta(s, x)]dS_x^H\right)^2 + E\left(\int_s^t h_\theta(t, x)dS_x^H\right)^2 + 2E\left(\int_s^t h_\theta(t, y)dS_y^H \int_0^s [h_\theta(t, x) - h_\theta(s, x)]dS_x^H\right). \tag{3.9}$$

Now, we estimate the three terms. For the first term, we have

$$\begin{aligned} 0 &\leq E\left(\int_0^s [h_\theta(t, x) - h_\theta(s, x)]dS_x^H\right)^2 \\ &= \int_0^s \int_0^s (h_\theta(t, x) - h_\theta(s, x)) \\ &\quad (h_\theta(t, y) - h_\theta(s, y))\psi_H(x, y)dxdy \\ &= \theta^2 \left(\int_s^t e^{-\frac{1}{2}\theta u^2} du\right)^2 \int_0^s \int_0^s xye^{\frac{1}{2}\theta(x^2+y^2)}\psi_H(x, y)dxdy \\ &\leq \theta^2 s^2 (t-s)^2 e^{-\theta t^2} \int_0^s \int_0^s \psi_H(x, y)dxdy \\ &= \theta^2 s^2 (t-s)^2 e^{-\theta t^2} E(S_s^H)^2 \leq C_{H,T}(t-s)^2 \end{aligned}$$

for all $\theta < 0$ and $0 < s < t \leq T$. For the second term, we have

$$\begin{aligned} E\left(\int_s^t h_\theta(t, x)dS_x^H\right)^2 &= \int_s^t \int_s^t h_\theta(t, x)h_\theta(t, y)\psi_H(x, y)dxdy \\ &\leq e^{-\theta t^2} \int_s^t \int_s^t xye^{\frac{1}{2}\theta(x^2+y^2)}\psi_H(x, y)dxdy \\ &\leq t^2 e^{-\theta t^2} \int_s^t \int_s^t \psi_H(x, y)dxdy \\ &\leq C_{H,T}(t-s)^{2H}. \end{aligned}$$

for all $\theta < 0$ and $0 < s < t \leq T$. Similarly, for the third term, we also prove

$$\begin{aligned} 0 &\leq E\left(\int_s^t h_\theta(t, y)dS_y^H \int_0^s [h_\theta(t, x) - h_\theta(s, x)]dS_x^H\right) \\ &= \int_s^t \int_0^s h_\theta(t, y)[h_\theta(t, x) - h_\theta(s, x)]\psi_H(x, y)dxdy \\ &\leq \theta^2 e^{-\frac{1}{2}\theta t^2} \left(\int_s^t e^{-\frac{1}{2}\theta u^2} du\right) \int_s^t ye^{\frac{1}{2}\theta y^2} dy \int_0^s xe^{\frac{1}{2}\theta x^2} \psi_H(x, y)dx \\ &\leq \theta^2 e^{-\theta t^2} (t-s) \int_s^t ye^{\frac{1}{2}\theta y^2} dy \int_0^s xe^{\frac{1}{2}\theta x^2} \psi_H(x, y)dx \\ &\leq C_{H,T}(t-s)^2 \end{aligned}$$

for all $\theta < 0$ and $0 < s < t \leq T$. Thus, we have obtained the following estimate:

$$E\left[\left(\hat{X}_t^H - \hat{X}_s^H\right)^2\right] \leq C_{H,T}|t-s|^{2H}$$

for all $\theta < 0$ and $0 < s < t \leq T$.

On the other hand, elementary calculations may show that

$$\int_0^s [h_\theta(t, r) - h_\theta(s, r)]dr = \theta \int_s^t e^{-\frac{1}{2}\theta u^2} du \int_0^s re^{\frac{1}{2}\theta r^2} dr \leq C_{H,T}(t-s)$$

and

$$\int_s^t h_\theta(t, r)dr = e^{-\frac{1}{2}\theta s^2} \int_s^t e^{\frac{1}{2}\theta r^2} dr \leq C_{H,T}(t-s)$$

for all $\theta < 0$ and $0 < s < t \leq T$. It follows that

$$\begin{aligned} \left(\int_0^t h_\theta(t, r)dr - \int_0^s h_\theta(s, r)dr\right)^2 &= \left(\int_0^s [h_\theta(t, r) - h_\theta(s, r)]dr\right)^2 \\ &\quad + \left(\int_s^t h_\theta(t, r)dr\right)^2 + 2 \int_s^t h_\theta(t, r)dr \int_0^s [h_\theta(t, r) - h_\theta(s, r)]dr \\ &\leq C_{H,T}(t-s)^2 \end{aligned}$$

for all $\theta < 0$ and $0 < s < t \leq T$, which implies that

$$\begin{aligned} E\left[\left(X_t^{a,b} - X_s^{a,b}\right)^2\right] &= E\left[\left(\hat{X}_t^{a,b} - \hat{X}_s^{a,b}\right)^2\right] \\ &\quad + \nu^2 \left(\int_0^t h_\theta(t, r)dr - \int_0^s h_\theta(s, r)dr\right)^2 \\ &\leq C_{H,T}(t-s)^{2H} \end{aligned}$$

for all $\theta < 0$ and $0 < s < t \leq T$. Noting that the above calculations are invertible for all $\theta < 0$ and $0 < s < t \leq T$, one can obtain the left hand side in (3.8) and the lemma follows.

4 CONVERGENCE

In this section, we obtain the large time behaviors associated with the solution X^H to Eq. 3.1. From Lemma 3.5 and Guassinness, we find that the self-repelling diffusion $\{X_t^H, t \geq 0\}$ is H -Hölder continuous. So, the integral

$$\int_0^t s dX_s^H$$

exists with $t \geq 0$ as a Young integral and

$$tX_t^H = \int_0^t s dX_s^H + \int_0^t X_s^H ds$$

for all $t \geq 0$. Define the process $Y = \{Y_t, t \geq 0\}$ by

$$\begin{aligned} Y_t &:= \int_0^t (X_t^H - X_s^H)ds = tX_t^H - \int_0^t X_s^H ds = \int_0^t s dX_s^H \\ &= \int_0^t s dS_s^H - \int_0^t \theta s Y_s ds + \frac{1}{2} \nu t^2. \end{aligned}$$

By the variation of constants method, one can prove

$$Y_t = e^{-\frac{1}{2}\theta t^2} \int_0^t se^{\frac{1}{2}\theta s^2} dS_s^H - \frac{\nu}{\theta}(e^{-\frac{1}{2}\theta t^2} - 1)$$

for all $t \geq 0$. Define Gaussian process $\xi^H = \{\xi_t^H, t \geq 0\}$ as follows

$$\xi_t^H := \int_0^t se^{\frac{1}{2}\theta s^2} dS_s^H, \quad t \geq 0.$$

Lemma 4.1. Let $\theta < 0$ and $\frac{1}{2} < H < 1$. Then, the random variable

$$\xi_\infty^H := \int_0^\infty se^{\frac{1}{2}\theta s^2} dS_s^H$$

exists as an element in L^2 . Moreover, ξ^H is H -Hölder continuous and $\xi_t^H \rightarrow \xi_\infty^H$ in L^2 and almost surely, as t tends to infinity.

Proof. This is simple calculus exercise. In fact, we have

$$\begin{aligned} E\left(\int_0^\infty xe^{\frac{1}{2}\theta x^2} dS_x^H\right)^2 &= \int_0^\infty \int_0^\infty xy e^{\frac{1}{2}\theta(x^2+y^2)} \psi_H(x,y) dx dy \\ &= 2 \int_0^\infty xe^{\frac{1}{2}\theta x^2} dx \int_0^x ye^{\frac{1}{2}\theta y^2} \psi_H(x,y) dy \\ &= 2H(2H-1) \int_0^\infty xe^{\frac{1}{2}\theta x^2} dx \\ &\quad \int_0^x ye^{\frac{1}{2}\theta y^2} ((x-y)^{2H-2} - (x+y)^{2H-2}) dy \\ &\leq 2H(2H-1) \int_0^\infty xe^{\frac{1}{2}\theta x^2} dx \\ &\quad \int_0^x ((x-y)^{2H-2} - (x+y)^{2H-2}) y dy \\ &= 2H(2H-1) C_H \int_0^\infty x^{2H+1} e^{\frac{1}{2}\theta x^2} dx \\ &= C_{\theta,H} \Gamma(2H+2) \end{aligned}$$

for all $\theta < 0$ and $\frac{1}{2} < H < 1$, which shows that the random variable ξ_∞^H exists as an element in L^2 .

Now, we show that the process $\xi^{a,b}$ is Hölder continuous. For all $0 < s < t$ by the inequality $e^{-x^2} x \leq C$ for all $x \geq 0$, we have

$$\begin{aligned} E(\xi_t^H - \xi_s^H)^2 &= E\left(\int_s^t xe^{\frac{1}{2}\theta x^2} dS_x^H\right)^2 \\ &= \int_s^t \int_s^t xy e^{\frac{1}{2}\theta(x^2+y^2)} \psi_H(x,y) dx dy \\ &= 2 \int_s^t xe^{\frac{1}{2}\theta x^2} dx \int_s^x ye^{\frac{1}{2}\theta y^2} \psi_H(x,y) dy \\ &= 2H(2H-1) \int_s^t xe^{\frac{1}{2}\theta x^2} dx \int_s^x ye^{\frac{1}{2}\theta y^2} ((x-y)^{2H-2} \\ &\quad - (x+y)^{2H-2}) dy \\ &\leq 2HC_\theta(2H-1) \int_s^t dx \int_s^x (x-y)^{2H-2} dy \\ &= C_{\theta,H} (t-s)^{2H}. \end{aligned}$$

Thus, the normality of ξ^H implies that

$$E(\xi_t^H - \xi_s^H)^{2n} \leq C_{\theta,H,n} (t-s)^{2nH}$$

for all $0 \leq s < t$, $\frac{1}{2} < H < 1$ and integer numbers $n \geq 1$, and the Hölder continuity follows.

Nextly, we check the $\xi_t^{a,b}$ converges to ξ_∞^H in L^2 . This follows from the next estimate:

$$\begin{aligned} E(\xi_t^H - \xi_\infty^H)^2 &= \int_t^\infty \int_t^\infty xy e^{\frac{1}{2}\theta(x^2+y^2)} \psi_H(x,y) dx dy \\ &= 2 \int_t^\infty \int_t^x xy e^{\frac{1}{2}\theta(x^2+y^2)} \psi_H(x,y) dx dy \\ &\leq 2e^{\frac{1}{2}\theta t^2} \int_t^\infty xe^{\frac{1}{2}\theta x^2} dx \int_t^x y \psi_H(x,y) dy \\ &\leq 2e^{\frac{1}{2}\theta t^2} \int_t^\infty xe^{\frac{1}{2}\theta x^2} dx \int_0^x y \psi_H(x,y) dy \\ &\leq 2H(2H-1)e^{\frac{1}{2}\theta t^2} \\ &\quad \cdot \int_t^\infty xe^{\frac{1}{2}\theta x^2} dx \int_0^x y((x-y)^{2H-2} \\ &\quad - (x+y)^{2H-2}) dy \\ &\leq 2H(2H-1)e^{\frac{1}{2}\theta t^2} \cdot \int_t^\infty xe^{\frac{1}{2}\theta x^2} dx \int_0^x y(x-y)^{2H-2} dy \\ &= 2H(2H-1) \left(\int_0^1 u(1-u)^{2H-2} du \right) e^{\frac{1}{2}\theta t^2} \\ &\quad \int_t^\infty x^{2H+1} e^{\frac{1}{2}\theta x^2} dx \rightarrow 0, \end{aligned} \tag{4.1}$$

as t tends to infinity.

Finally, we check the $\xi_t^{a,b}$ converges to ξ_∞^H almost surely. By integration by parts we see that

$$\xi_t^H - \xi_\infty^H = \int_t^\infty se^{\frac{1}{2}\theta s^2} dS_s^H = -te^{\frac{1}{2}\theta t^2} S_t^H - \int_t^\infty (1+\theta s^2)e^{\frac{1}{2}\theta s^2} S_s^H ds \tag{4.2}$$

for all $t \geq 0$. Elementary may check that the convergence

$$\eta_t^H := \int_t^\infty (1+\theta s^2)e^{\frac{1}{2}\theta s^2} S_s^H ds \xrightarrow{a.s.} 0$$

holds almost surely, as t tends to infinity. In fact, by inequality

$$\int_t^\infty s^\alpha e^{\frac{1}{2}\theta s^2} ds \leq Ct^{\alpha-1} e^{\frac{1}{2}\theta t^2}, \quad \alpha > -1,$$

with $t \geq 0$, we may show that

$$\begin{aligned} E\left(\sup_{n \leq t < n+1} |\eta_t^H|^2\right) &\leq \int_n^\infty \int_n^\infty (1+\theta s^2) \\ &\quad (1+\theta r^2)e^{\frac{1}{2}\theta(s^2+r^2)} E|S_r^H| |S_s^H| dr ds \\ &\leq C \left(\int_n^\infty s^{2+H} e^{\frac{1}{2}\theta s^2} ds \right)^2 \\ &\leq Cn^{2+2H} e^{\theta n^2}, \end{aligned}$$

for all integer numbers $n \geq 1$, and hence

$$\sum_{n=0}^{\infty} P\left(\sup_{n \leq t < n+1} |\eta_t^H|^2 \geq \varepsilon\right) \leq C\varepsilon^{-2} \sum_{n=0}^{\infty} n^{2+2H} e^{-\theta n^2} < \infty.$$

Thus, Borel-Cantelli's lemma implies that η_t^H converges to zero almost surely as t tends to infinity, and the lemma follows from (4.2).

Corollary 4.1. For all $\gamma > 0$, we have

$$t^\gamma (\xi_t^H - \xi_\infty^H) = t^\gamma \int_t^\infty se^{\frac{1}{2}\theta s^2} dS_s^H \rightarrow 0,$$

in L^2 and almost surely, as t tends to infinity.

Lemma 4.2. Let $\theta < 0$ and $\frac{1}{2} < H < 1$. Then, we have

$$\Lambda_\gamma(t, \theta) := t^{\gamma+1} e^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} (\xi_\infty^{a,b} - \xi_u^{a,b}) du \rightarrow 0$$

in L^2 and almost surely for every $\gamma \geq 0$, as t tends to infinity.

Proof. Given $0 < s \leq t$, $\theta < 0$ and denote

$$\begin{aligned} Y_\theta(s, t) &:= \int_0^t e^{-\frac{1}{2}\theta v^2} dv \int_v^\infty re^{\frac{1}{2}\theta r^2} \psi_H(s, r) dr \\ &= \int_0^t re^{\frac{1}{2}\theta r^2} \psi_H(s, r) dr \int_0^r e^{-\frac{1}{2}\theta v^2} dv \\ &\quad + \left(\int_0^t e^{-\frac{1}{2}\theta v^2} dv\right) \int_t^\infty re^{\frac{1}{2}\theta r^2} \psi_H(s, r) dr \\ &\leq C \int_0^t r \psi_H(s, r) dr + \frac{C}{t} e^{-\frac{1}{2}\theta t^2} \int_t^\infty re^{\frac{1}{2}\theta r^2} \psi_H(s, r) dr \\ &\leq C \left(\int_0^t r \psi_H(s, r) dr + (t-s)^{2H-2} t^{-1}\right), \end{aligned}$$

where we have used the fact

$$\int_0^x e^{-\frac{1}{2}\theta v^2} dv \leq \frac{C}{x} e^{-\frac{1}{2}\theta x^2}, \quad \forall x \geq 0$$

and estimates

$$\begin{aligned} \int_t^\infty re^{\frac{1}{2}\theta r^2} \psi_H(s, r) dr &= H(2H-1) \int_t^\infty r((r-s)^{2H-2} \\ &\quad - (s+r)^{2H-2}) e^{\frac{1}{2}\theta r^2} dr \\ &\leq H(2H-1) \int_t^\infty r(r-s)^{2H-2} e^{\frac{1}{2}\theta r^2} dr \\ &\leq H(2H-1)(t-s)^{2H-2} \int_t^\infty re^{\frac{1}{2}\theta r^2} dr \\ &= \frac{H(2H-1)}{-\theta} (t-s)^{2H-2} e^{\frac{1}{2}\theta t^2}. \end{aligned}$$

It follows that

$$\begin{aligned} E|\Lambda_\gamma(t, \theta)|^2 &= t^{2\gamma+2} e^{\theta t^2} \int_0^t \int_0^t e^{-\frac{1}{2}\theta(u^2+v^2)} \\ &\quad \cdot E\left(\int_u^\infty se^{\frac{1}{2}\theta s^2} dS_s^H\right) \left(\int_v^\infty re^{\frac{1}{2}\theta r^2} dS_r^H\right) dudv \\ &= t^{2\gamma+2} e^{\theta t^2} \int_0^t \int_0^t e^{-\frac{1}{2}\theta(u^2+v^2)} dudv \\ &\quad \int_u^\infty \int_v^\infty rse^{\frac{1}{2}\theta(r^2+s^2)} \psi_H(s, r) dr ds \\ &= t^{2\gamma+2} e^{\theta t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} du \int_u^\infty se^{\frac{1}{2}\theta s^2} Y_\theta(s, t) ds \\ &= t^{2\gamma+2} e^{\theta t^2} \int_0^t se^{\frac{1}{2}\theta s^2} \psi_H(s, \theta) ds \int_0^s e^{-\frac{1}{2}\theta u^2} du \\ &\quad + t^{2\gamma+2} e^{\theta t^2} \int_t^\infty se^{\frac{1}{2}\theta s^2} Y_\theta(s, t) ds \int_0^t e^{-\frac{1}{2}\theta u^2} du \\ &\leq t^{2\gamma+2} e^{\theta t^2} \int_0^t s^2 Y_\theta(s, t) ds \\ &\quad + t^{2\gamma+1} e^{\frac{1}{2}\theta t^2} \int_t^\infty se^{\frac{1}{2}\theta s^2} Y_\theta(s, t) ds \\ &\rightarrow 0 \quad (t \rightarrow \infty), \end{aligned}$$

which shows that $\Lambda_\gamma(t, \theta)$ converges to zero in L^2 .

Now, we obtain the convergence with probability one. Noting that

$$\xi_\infty^H - \xi_u^H = \int_u^\infty se^{\frac{1}{2}\theta s^2} dS_s^H$$

for all $u \geq 0$, we get

$$\begin{aligned} |\Lambda_\gamma(t, \theta)| &\leq t^{\gamma+1} e^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} \left| \int_u^\infty se^{\frac{1}{2}\theta s^2} dS_s^H \right| du \\ &\leq t^{\gamma+1} e^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} \left(u |S_u^H| e^{\frac{1}{2}\theta u^2} + \int_u^\infty |S_s^H| (1-\theta s^2) |e^{\frac{1}{2}\theta s^2} ds \right) du \\ &= t^{\gamma+1} e^{\frac{1}{2}\theta t^2} \int_0^t u |S_u^H| du + t^{\gamma+1} e^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} du \\ &\quad \int_u^\infty |S_s^H| (1-\theta s^2) |e^{\frac{1}{2}\theta s^2} ds \\ &= t^{\gamma+1} e^{\frac{1}{2}\theta t^2} \int_0^t u |S_u^H| du \\ &\quad + t^{\gamma+1} e^{\frac{1}{2}\theta t^2} \int_0^t |S_s^H| (1-\theta s^2) |e^{\frac{1}{2}\theta s^2} ds \int_0^s e^{-\frac{1}{2}\theta u^2} du \\ &\quad + t^{\gamma+1} e^{\frac{1}{2}\theta t^2} \int_t^\infty |S_s^H| (1-\theta s^2) |e^{\frac{1}{2}\theta s^2} ds \int_0^t e^{-\frac{1}{2}\theta u^2} du \\ &\leq t^{\gamma+1} e^{\frac{1}{2}\theta t^2} \int_0^t u |S_u^H| du + t^{\gamma+1} e^{\frac{1}{2}\theta t^2} \int_0^t |S_s^H| (1-\theta s^2) |s ds \\ &\quad + C_\theta t^\gamma \int_t^\infty |S_s^H| (1-\theta s^2) |e^{\frac{1}{2}\theta s^2} ds \\ &\rightarrow 0 \end{aligned}$$

almost surely for all $\gamma \geq 0$, $\theta < 0$ and $\frac{1}{2} < H < 1$, as t tends to infinity. This completes the proof.

The objects of this paper are to prove the following theorems which give the long time behaviors for X^H with $\frac{1}{2} < H < 1$.

Theorem 4.1. *Let $\theta < 0$ and $\frac{1}{2} < H < 1$. Then, as $t \rightarrow \infty$, the convergence*

$$J_0^H(t; \theta, \nu) := te^{\frac{1}{2}\theta t^2} X_t^H \rightarrow \xi_\infty^H - \frac{\nu}{\theta}$$

holds in L^2 and almost surely.

Proof. Given $t > 0$ and $\theta < 0$. Simple calculations may prove

$$\begin{aligned} J_0^H(t; \theta, \nu) &= te^{\frac{1}{2}\theta t^2} X_t^H \\ &= te^{\frac{1}{2}\theta t^2} \int_0^t h_\theta(t, s) dS_s^H + \nu te^{\frac{1}{2}\theta t^2} \int_0^t h_\theta(t, s) ds \\ &= te^{\frac{1}{2}\theta t^2} S_t^H - \theta t^2 e^{\frac{1}{2}\theta t^2} \int_0^t se^{\frac{1}{2}\theta s^2} \left(\int_s^t e^{-\frac{1}{2}\theta u^2} du \right) dS_s^H \\ &\quad + \nu te^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta s^2} ds \\ &= te^{-\frac{1}{2}\theta t^2} S_t^H - \theta te^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} \left(\int_0^u se^{\frac{1}{2}\theta s^2} dS_s^H \right) du \\ &\quad + \nu te^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta s^2} ds \\ &= te^{\frac{1}{2}\theta t^2} S_t^H - \theta te^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} \xi_u^H du \\ &\quad + \nu te^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta s^2} ds. \end{aligned} \tag{4.3}$$

It follows from Lemma 4.1, Corollary 4.1, and Lemma 4.2 that

$$\begin{aligned} J_0^H(t; \theta, \nu) - \left(\xi_\infty^H - \frac{\nu}{\theta} \right) &= te^{\frac{1}{2}\theta t^2} X_t^H - \left(\xi_\infty^H - \frac{\nu}{\theta} \right) \\ &= te^{\frac{1}{2}\theta t^2} S_t^H - \theta te^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} (\xi_u^H - \xi_\infty^H) du \\ &\quad + \left(\xi_\infty^H - \frac{\nu}{\theta} \right) \left(-\theta te^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} du - 1 \right) \rightarrow 0 \quad (t \rightarrow \infty) \end{aligned} \tag{4.4}$$

in L^2 and almost surely for all $\theta < 0$ and $\frac{1}{2} < H < 1$, as t tends to infinity.

Theorem 4.2. *Define the processes $J^H(n, \theta, \nu) = \{J_t^H(n, \theta, \nu), t \geq 0\}, n \geq 1$ by*

$$J_n^H(t; \theta, \nu) := \theta t^2 \left(J_{n-1}^H(t; \theta, \nu) - (2n-3)!! \left(\xi_\infty^H - \frac{\nu}{\theta} \right) \right), \quad n = 1, 2, \dots,$$

for all $t \geq 0$, where $(-1)!! = 1$. Then, the convergence

$$J_n^H(t; \theta, \nu) \rightarrow (2n-1)!! \left(\xi_\infty^H - \frac{\nu}{\theta} \right)$$

holds in L^2 and almost surely for every $n \geq 1$, as $t \rightarrow \infty$.

Proof. From the proof of Theorem 4.1, we find that the identities

$$\begin{aligned} J_0^H(t; \theta, \nu) - \left(\xi_\infty^H - \frac{\nu}{\theta} \right) &= te^{\frac{1}{2}\theta t^2} S_t^H + \theta te^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} (\xi_u^H - \xi_\infty^H) du \\ &\quad + \left(\xi_\infty^H - \frac{\nu}{\theta} \right) \left(\theta te^{\frac{1}{2}\theta t^2} \int_0^t e^{-\frac{1}{2}\theta u^2} du - 1 \right), \\ J_n^H(t; \theta, \nu) &= \left(\xi_\infty^H - \frac{\nu}{\theta} \right) I_n(t, \theta) + t(\theta t^2)^n e^{-\frac{1}{2}\theta t^2} S_t^H \\ &\quad + \theta t(\theta t^2)^n e^{\frac{1}{2}\theta t^2} \int_0^t e^{\frac{1}{2}\theta u^2} (\xi_u^H - \xi_\infty^H) du. \end{aligned}$$

holds for all $t > 0, n \geq 1$ and $\theta < 0$, where $I_n(t, \theta)$ is given in Lemma 3.2. Thus, the theorem follows from Lemma 4.1, Corollary 4.1, Lemma 4.2 and Theorem 4.1.

5 SIMULATION

We have applied our results to the following linear self-repelling diffusion driven by a sub-fBm S^H with $\frac{1}{2} < H < 1$:

$$dX_t^H = dS_t^H - \theta \left(\int_0^t (X_s^H - X_s^H) ds \right) dt + \nu dt, \quad X_0^H = 0,$$

where $\theta < 0$ and $\nu \in \mathbb{R}$ are two parameters. We will simulate the process with $\nu = 0$ in the following cases:

- $H = 0.7$ and $\theta = -1, \theta = -10$, and $\theta = -100$, respectively (see, **Figure 1, Figure 2, Figure 3**, and **Table 1, Table 2, Table 3**);
- $H = 0.5$ and $\theta = -1, \theta = -10$, and $\theta = -100$, respectively (see, **Figure 4, Figure 5, Figure 6**, and **Table 4, Table 5, Table 6**);

Remark 1. From the following numerical results, we can find that it is important to study the estimates of parameters θ and ν .

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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