



Spacetimes Admitting Conircular Curvature Tensor in $f(R)$ Gravity

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The main object of this paper is to investigate spacetimes admitting concircular curvature tensor in $f(R)$ gravity theory. At first, concircularly flat and concircularly flat perfect fluid spacetimes in $f(R)$ gravity are studied. In this case, the forms of the isotropic pressure p and the energy density σ are obtained. Next, some energy conditions are considered. Finally, perfect fluid spacetimes with divergence free concircular curvature tensor in $f(R)$ gravity are studied; amongst many results, it is proved that if the energy-momentum tensor of such spacetimes is recurrent or bi-recurrent, then the Ricci tensor is semi-symmetric and hence these spacetimes either represent inflation or their isotropic pressure and energy density are constants.

Keywords: perfect fluid, energy-momentum tensor, concircular curvature tensor, $f(R)$ gravity theory, energy conditions in modified gravity

1 INTRODUCTION

A concircular transformation was first coined by Yano in 1940 [1]. Such a transformation preserves geodesic circles. The geometry that deals with a concircular transformation is called concircular geometry. Under concircular transformation the concircular curvature tensor \mathcal{M} remains invariant. Every spacetime M has vanishing concircular curvature tensor is called concircularly flat. A concircularly flat spacetime is of constant curvature. As a result, the deviation of a spacetime from constant curvature is measured by the concircular curvature tensor \mathcal{M} . Researchers have shown the curial role of the concircular curvature tensor in mathematics and physics (for example, see [2–6] and references therein).

In Einstein's theory of gravity, the relation between the matter of spacetimes and the geometry of the spacetimes is given by Einstein's field equations (EFE)

$$\mathcal{R}_{ij} - \frac{R}{2}g_{ij} = \kappa T_{ij},$$

with κ being the Newtonian constant and T_{ij} is the energy-momentum tensor [7]. These equations imply that the energy-momentum tensor T_{ij} is divergence-free. This condition is satisfied whenever $\nabla_l T_{ij} = 0$, where ∇_l denotes the covariant differentiation. There are many modifications of the standard relativity theory. The $f(R)$ gravity theory is the most popular of such modification of the standard theory of gravity. This important modification was first introduced in 1970 [8]. This modified theory can be obtained by replacing the scalar curvature R with a generic function $f(R)$ in the Einstein-Hilbert action. The field equations of $f(R)$ gravity are given as

$$\begin{aligned} \kappa T_{ij} = & f'(R)\mathcal{R}_{ij} - f''(R)f\nabla_i R\nabla_j R - f''(R)\nabla_i\nabla_j R \\ & + g_{ij}\left[f'''(R)\nabla_k R\nabla^k R + f''(R)\nabla^2 R - \frac{1}{2}f'(R)\right], \end{aligned} \tag{1.1}$$

where $f(R)$ is an arbitrary function of the scalar curvature R and $f'(R) = \frac{df}{dR}$ which must be positive to ensure attractive gravity [9]. The $f(R)$ gravity represents a higher order and well-studied theory of gravity. For example, an earlier investigation of quintessence and cosmic acceleration in $f(R)$ gravity theory as a higher order gravity theory are considered in [10]. Also, Capozziello et al. proved that, in a generalized Robertson-Walker spacetime with divergence free conformal curvature tensor, the higher order gravity tensor has the form of perfect fluid [11].

In a series of recent studies, weakly Ricci symmetric spacetimes $(WRS)_4$, almost pseudo-Ricci symmetric spacetimes $(APRS)_4$, and conformally flat generalized Ricci recurrent spacetimes are investigated in $f(R)$ gravity theory [12–14]. Motivated by these studies and many others, the main aim of this paper is to study concircularly flat and concircularly flat perfect fluid spacetimes in $f(R)$ gravity. Also, spacetimes with divergence free concircular curvature tensor in $f(R)$ gravity are considered.

This article is organized as follows. In **Section 2**, concircularly flat spacetimes in $f(R)$ gravity are considered. In **Section 3**, we study concircularly flat perfect fluid spacetimes in $f(R)$ gravity as well as we consider some energy conditions. Finally, spacetimes with divergence free concircular curvature tensor in $f(R)$ gravity are investigated.

2 CONCIRCULARLY FLAT SPACETIMES IN $f(R)$ GRAVITY

The concircular curvature tensor of type $(0, 4)$ is defined locally as

$$\mathcal{M}_{jklm} = \mathcal{R}_{jklm} - \frac{R}{n(n-1)}[g_{kl}g_{jm} - g_{km}g_{jl}], \tag{2.1}$$

where \mathcal{R}_{jklm} , R , and g_{kl} are the Riemann curvature tensor, the scalar curvature tensor, and the metric tensor [1].

Here, we will consider $\mathcal{M}_{jklm} = 0$, thus it follows from **Eq. 2.1** that

$$\mathcal{R}_{jklm} = \frac{R}{n(n-1)}[g_{kl}g_{jm} - g_{km}g_{jl}]. \tag{2.2}$$

This equation leads us to state the following theorem:

Theorem 1. *A concircularly flat spacetime is of constant curvature.*
Corollary 1. *A concircularly flat spacetime is of constant scalar curvature.*

Contracting **Eq. 2.2** with g^{jm} , we get

$$\mathcal{R}_{kl} = \frac{R}{n}g_{lk}. \tag{2.3}$$

In view of **Eq. 2.3** we can state the following corollary:

Corollary 2. *A concircularly flat spacetime is Einstein.*

In view of corollary 1, the field **Eq. 1.1** in $f(R)$ gravity become

$$\mathcal{R}_{ij} - \frac{f}{2f'}g_{ij} = \frac{\kappa}{f'}T_{ij}. \tag{2.4}$$

In vacuum case, we have

$$\mathcal{R}_{ij} - \frac{f}{2f'}g_{ij} = 0.$$

Contracting with g^{ij} and integrating the result, one gets

$$f = \lambda R^{\frac{n}{2}}, \tag{2.5}$$

where λ is a constant.

Conversely, if **Eq. 2.5** holds, then

$$T_{ij} = 0.$$

We can thus state following theorem:

Theorem 2. *A concircularly flat spacetime in $f(R)$ gravity is vacuum if and only if $f = \lambda R^{\frac{n}{2}}$.*

The vector field ξ is called Killing if

$$\mathcal{L}_\xi g_{ij} = 0, \tag{2.6}$$

whereas ξ is called conformal Killing if

$$\mathcal{L}_\xi g_{ij} = 2\varphi g_{ij}, \tag{2.7}$$

where \mathcal{L}_ξ is the Lie derivative with respect to the vector field ξ and φ is a scalar function [15–17]. The symmetry of a spacetime is measured by the number of independent Killing vector fields the spacetime admits. A spacetime of maximum symmetry has a constant curvature.

A spacetime M is said to admit a matter collineation with respect to a vector field ξ if the Lie derivative of the energy-momentum tensor T with respect to ξ satisfies

$$\mathcal{L}_\xi T_{ij} = 0. \tag{2.8}$$

It is clear that every Killing vector field is a matter collineation, but the converse is not generally true. The energy-momentum tensor T_{ij} has the Lie inheritance property along the flow lines of the vector field ξ if the Lie derivative of T_{ij} with respect to ξ satisfies [15–17].

$$\mathcal{L}_\xi T_{ij} = 2\varphi T_{ij}. \tag{2.9}$$

Now using **Eq. 2.3** in **Eq. 2.4**, one gets

$$\left(\frac{R}{n} - \frac{f}{2f'}\right)g_{ij} = \frac{\kappa}{f'}T_{ij}. \tag{2.10}$$

In a concircularly flat spacetime the scalar curvature R is constant, and hence f and f' are also constants. Now, we consider a non-vacuum concircularly flat spacetime M . Therefore the Lie derivative \mathcal{L}_ξ of **Eq. 2.10** implies that

$$\left(\frac{R}{n} - \frac{f}{2f'}\right)\mathcal{L}_\xi g_{ij} = \frac{\kappa}{f'}\mathcal{L}_\xi T_{ij}. \tag{2.11}$$

Assume that the vector field ξ is Killing on M , that is, Eq. 2.6 holds, thus we have

$$\mathcal{L}_\xi \mathcal{T}_{ij} = 0.$$

Conversely, if Eq. 2.8 holds, then from Eq. 2.11 it follows that

$$\mathcal{L}_\xi g_{ij} = 0.$$

We thus motivate to state the following theorem:

Theorem 3. *Let M be a concircularly flat spacetime satisfying $f(R)$ gravity, then the vector field ξ is Killing if and only if M admits matter collineation with respect to ξ .*

The isometry of spacetimes prescribed by Killing vector fields represents a very important type of spacetime symmetry. Spacetimes of constant curvature are known to have maximum such symmetry, that is, they admit the maximum number of linearly independent Killing vector fields. The maximum number of linearly independent Killing vector fields in an n – dimensional spacetime is $\frac{n(n+1)}{2}$ (The reader is referred to [18–22] and references therein for a more discussion on this topic). This fact with the above theorem leads to the following corollary.

Corollary 3. *A non-vacuum concircularly flat spacetime satisfying $f(R)$ gravity admits the maximum number of matter collineations $\frac{n(n+1)}{2}$.*

Let ξ be a conformal Killing vector field, that is, Eq. 2.7 holds. Eq. 2.11 implies

$$\mathcal{L}_\xi \mathcal{T}_{ij} = 2\phi \mathcal{T}_{ij}.$$

Conversely, assume that Eq. 2.9 holds, then from Eq. 2.11 we obtain

$$\mathcal{L}_\xi g_{ij} = 2\phi g_{ij}.$$

Hence, we can state the following theorem:

Theorem 4. *Let M be a concircularly flat spacetime satisfying $f(R)$ gravity, then M has a conformal Killing vector field ξ if and only if the energy-momentum tensor T_{ij} has the Lie inheritance property along ξ .*

The covariant derivative of both sides of Eq. 2.10 implies that

$$\nabla_k \mathcal{T}_{ij} = 0. \tag{2.12}$$

Since in a concircularly flat spacetime R is constant, then f and f' are constant. Inserting Eq. 2.4 in Eq. 2.12, we get

$$\nabla_k \mathcal{R}_{ij} = 0.$$

Thus, we have:

Theorem 5. *Let M be a concircularly flat spacetime satisfying $f(R)$ gravity, then M is Ricci symmetric.*

3 CONCIRCULARLY FLAT PERFECT FLUID SPACETIMES IN $f(R)$ GRAVITY

In a perfect fluid 4 – dimensional spacetime, the energy-momentum tensor T_{ij} obeys

$$\mathcal{T}_{ij} = (p + \sigma)u_i u_j + p g_{ij}, \tag{3.1}$$

where p is the isotropic pressure, σ is the energy density, and u_i is a unit timelike vector field [7, 23].

Making use of Eq. 3.1 in Eq. 2.4, we get

$$\mathcal{R}_{ij} = \frac{\kappa}{f'} [(p + \sigma)u_i u_j + p g_{ij}] + \frac{f}{2f'} g_{ij}. \tag{3.2}$$

The use of Eq. 2.3 implies that

$$\frac{R}{4} g_{ij} = \frac{\kappa}{f'} [(p + \sigma)u_i u_j + p g_{ij}] + \frac{f}{2f'} g_{ij}. \tag{3.3}$$

Contracting Eq. 3.3 with u^i , we get

$$\sigma = \frac{2f - Rf'}{4\kappa}. \tag{3.4}$$

Transvecting Eq. 3.3 with g^{ij} and using Eq. 3.4, one obtains

$$p = -\frac{2f - Rf'}{4\kappa}. \tag{3.5}$$

In consequence of the above we can state the following theorem:

Theorem 6. *In a concircularly flat perfect fluid spacetime obeying $f(R)$ gravity, the isotropic pressure p and the energy density σ are constants and $p = -\frac{2f - Rf'}{4\kappa}$ and $\sigma = \frac{2f - Rf'}{4\kappa}$.*

Combining Eq. 3.4 and Eq. 3.5, one easily gets

$$p + \sigma = 0, \tag{3.6}$$

which means that the spacetime represents dark matter era or alternatively the perfect fluid behaves as a cosmological constant [[24]]. Thus we can state the following theorem:

Theorem 7. *A concircularly flat perfect fluid spacetime obeying $f(R)$ gravity represents dark matter era.*

In radiation era $\sigma = 3p$, therefor the energy-momentum tensor T_{ij} takes the form

$$\mathcal{T}_{ij} = 4pu_j u_k + p g_{jk}. \tag{3.7}$$

Eq. 3.6 implies that $p = 0$. It follows that

$$\mathcal{T}_{ij} = 0.$$

which means that the spacetime is devoid of matter.

Thus we motivate to state the following corollary:

Corollary 4. *Let M be a concircularly flat spacetime obeying $f(R)$ gravity, then the Radiation era in M is vacuum.*

In pressureless fluid spacetime $p = 0$, the energy-momentum tensor is expressed as [25].

$$\mathcal{T}_{ij} = \sigma u_i u_j. \tag{3.8}$$

From Eq. 3.6 it follows that $\sigma = 0$. And consequently from Eq. 3.8 we infer

$$\mathcal{T}_{ij} = 0,$$

which means that the spacetime is vacuum.

We thus can state the following:

The spacetime	Equation of state(EoS)	$f(R)$
represents quintessence era	$\sigma + 3p = 0$	a constant multiple of R
represents dust matter era	$p = 0$	a constant multiple of R^3
represents radiation era	$\sigma - 3p = 0$	a constant multiple of R^2
represents stiff matter era	$p - \sigma = 0$	a constant multiple of R^3
determines an EoS in quintessence era	$5\sigma + 3p = 0$	a constant multiple of R^4

Corollary 5. Let M be a concircularly flat dust fluid spacetime obeying $f(R)$ gravity, then M is vacuum.

3.1 Energy Conditions in Concircularly Flat Spacetime

In this subsection, some energy conditions in concircularly flat spacetimes obeying $f(R)$ gravity are considered. Indeed, energy conditions serve as a filtration system of the energy-momentum tensor in standard theory of gravity and the modified theories of gravity.[12–14]. In [26], the authors studied weak energy condition (WEC), dominant energy conditions (DEC), null energy conditions (NEC), and strong energy conditions in two extended theories of gravity. As a starting point, we need to determine the effective isotropic pressure p^{eff} and the effective energy density σ^{eff} to state some of these energy conditions.

Eq. 2.4 may be rewritten as

$$\mathcal{R}_{ij} - \frac{1}{2}Rg_{ij} = \frac{\kappa}{f'}T_{ij}^{eff}, \tag{3.9}$$

where

$$T_{ij}^{eff} = T_{ij} + \frac{f - Rf'}{2\kappa}g_{ij}.$$

This leads us to rewrite Eq. 3.1 in the following form

$$T_{ij}^{eff} = (p^{eff} + \sigma^{eff})u_i u_j + p^{eff}g_{ij},$$

where

$$p^{eff} = p + \frac{f - Rf'}{2\kappa} \text{ and } \sigma^{eff} = \sigma - \frac{f - Rf'}{2\kappa}.$$

The use of Eq. 3.4 and Eq. 3.5 entails that

$$p^{eff} = -\frac{Rf'}{4\kappa},$$

$$\sigma^{eff} = \frac{Rf'}{4\kappa}.$$

Let us investigate certain energy conditions of a perfect fluid type effective matter in $f(R)$ gravity theory [12, 26, 27]:

- 1) Null energy condition (NEC): it says that $p^{eff} + \sigma^{eff} \geq 0$.
- 2) Weak energy condition (WEC): it states that $\sigma^{eff} \geq 0$ and $p^{eff} + \sigma^{eff} \geq 0$.
- 3) Dominant energy condition (DEC): it states that $\sigma^{eff} \geq 0$ and $p^{eff} \pm \sigma^{eff} \geq 0$.
- 4) Strong energy condition (SEC): it states that $\sigma^{eff} + 3p^{eff} \geq 0$ and $p^{eff} + \sigma^{eff} \geq 0$.

In this context, all mentioned energy conditions are consistently satisfied if $Rf' \geq 0$. As mentioned earlier, f' must be positive to ensure attractive gravity. Therefore, the previous energy conditions are always satisfied if $R \geq 0$.

4 SPACETIMES WITH DIVERGENCE FREE CONCIRCULAR CURVATURE TENSOR IN $F(R)$ GRAVITY

The divergence of the concircular curvature tensor, for $n = 4$, is given by [28].

$$\nabla_h \mathcal{M}^h_{klm} = \nabla_h \mathcal{R}^h_{klm} - \frac{1}{12} [g_{lm} \nabla_k R - g_{km} \nabla_l R]. \tag{4.1}$$

It is well-known that

$$\nabla_h \mathcal{R}^h_{klm} = \nabla_k \mathcal{R}_{lm} - \nabla_l \mathcal{R}_{km}. \tag{4.2}$$

The use of Eq. 4.2 in Eq. 4.1 implies that

$$\nabla_h \mathcal{M}^h_{klm} = \nabla_k \mathcal{R}_{lm} - \nabla_l \mathcal{R}_{km} - \frac{1}{12} [g_{lm} \nabla_k R - g_{km} \nabla_l R]. \tag{4.3}$$

Assume that the concircular curvature tensor is divergence free, that is $\nabla_h \mathcal{M}^h_{klm} = 0$, then

$$\nabla_k \mathcal{R}_{lm} - \nabla_l \mathcal{R}_{km} - \frac{1}{12} [g_{lm} \nabla_k R - g_{km} \nabla_l R] = 0. \tag{4.4}$$

Contracting with g^{lm} and using $\nabla_j \mathcal{R}_i^j = \frac{1}{2} \nabla_i R$, we obtain

$$\nabla_k R = 0. \tag{4.5}$$

Utilizing (4.5) in Eq. 4.4, we have

$$\nabla_k \mathcal{R}_{lm} = \nabla_l \mathcal{R}_{km}, \tag{4.6}$$

which means that the Ricci tensor is of Codazzi type [29]. The converse is trivial. Thus we can state the following theorem: Theorem 8. Let M be a spacetime with concircular curvature tensor, then M has Codazzi type of Ricci tensor if and only if the concircular curvature tensor is divergence free.

In view of Eq. 4.5, the field Eq. 1.1 in $f(R)$ gravity are

$$\mathcal{R}_{ij} - \frac{f}{2f'}g_{ij} = \frac{\kappa}{f'}T_{ij}. \tag{4.7}$$

Using Eq. 4.7 in Eq. 4.6, we get

$$\nabla_k T_{lm} = \nabla_l T_{km}.$$

Hence, we have the following corollary:

Corollary 6. *The energy-momentum tensor of a spacetime with divergence free concircular curvature tensor obeying $f(R)$ gravity is of Codazzi type.*

The spacetime M is called Ricci semi-symmetric [30] if

$$(\nabla_h \nabla_k - \nabla_k \nabla_h) \mathcal{R}_{ij} = 0, \tag{4.8}$$

whileas the energy-momentum tensor is called semi-symmetric if

$$(\nabla_h \nabla_k - \nabla_k \nabla_h) \mathcal{T}_{ij} = 0. \tag{4.9}$$

Now, Eq. 4.7 implies

$$(\nabla_h \nabla_k - \nabla_k \nabla_h) \mathcal{R}_{ij} = \frac{\kappa}{f'} (\nabla_h \nabla_k - \nabla_k \nabla_h) \mathcal{T}_{ij}.$$

Thus, we can state the following theorem:

Theorem 9. *Let M be a spacetime with divergence free concircular curvature tensor satisfying $f(R)$ gravity, then M is Ricci semi-symmetric if and only if the energy-momentum tensor of M is semi-symmetric.*

The energy-momentum tensor \mathcal{T}_{ij} is called recurrent if there exists a non-zero 1 – form λ_k such that

$$\nabla_k \mathcal{T}_{ij} = \lambda_k \mathcal{T}_{ij},$$

whereas \mathcal{T}_{ij} is called bi-recurrent if there exists a non-zero tensor ε_{hk} such that

$$\nabla_h \nabla_k \mathcal{T}_{ij} = \varepsilon_{hk} \mathcal{T}_{ij}.$$

In view of the above definition, it is clear that every recurrent tensor field is bi-recurrent.

Now assume that \mathcal{T}_{ij} is any $(0, 2)$ symmetric recurrent tensor, that is,

$$\nabla_k \mathcal{T}_{ij} = \lambda_k \mathcal{T}_{ij}. \tag{4.10}$$

Contracting with g^{ij} , we obtain

$$\lambda_k = \frac{1}{T} \nabla_k T, \tag{4.11}$$

where $T = g^{ij} \mathcal{T}_{ij}$.

Applying the covariant derivative on both sides and using Eq. 4.11, we find

$$\nabla_l \lambda_k + \lambda_k \lambda_l = \frac{1}{T} \nabla_l \nabla_k T. \tag{4.12}$$

Taking the covariant derivative of Eq. 4.10 and utilizing Eq. 4.12, we get

$$\nabla_l \nabla_k \mathcal{T}_{ij} = \left(\frac{1}{T} \nabla_l \nabla_k T \right) \mathcal{T}_{ij}.$$

It follows that

$$(\nabla_l \nabla_k - \nabla_k \nabla_l) \mathcal{T}_{ij} = 0.$$

Similarly, the same result holds for a bi-recurrent $(0, 2)$ symmetric tensor. In view of the above discussion, we have the following:

Lemma 1. *A (bi)-recurrent $(0, 2)$ symmetric tensor is semi-symmetric.*

Assume that the energy-momentum tensor \mathcal{T}_{ij} is recurrent or bi-recurrent, it follows from Lemma one that \mathcal{T}_{ij} is semi-symmetric. Consequently, M is Ricci semi-symmetric.

Theorem 10. *Let M be a spacetime with divergence free concircular curvature tensor obeying $f(R)$ gravity. If the energy-momentum (Ricci) tensor is recurrent or bi-recurrent, then the Ricci (energy-momentum) tensor is semi-symmetric.*

Let us now consider that M be a perfect fluid spacetime with divergence free concircular curvature tensor and whose energy-momentum tensor is recurrent or bi-recurrent. Thus the use of Eq. 3.1 in Eq. 4.7 implies that

$$\mathcal{R}_{ij} = a g_{ij} + b u_i u_j, \tag{4.13}$$

where

$$a = \frac{1}{2f'} (2\kappa p + f) \text{ and } b = \frac{\kappa}{f'} (p + \sigma). \tag{4.14}$$

Making a contraction of Eq. 4.13 with g^{ij} , we get

$$R = \frac{1}{f'} (3\kappa p - \kappa \sigma + 2f). \tag{4.15}$$

Since the \mathcal{T}_{ij} is recurrent or bi-recurrent, then $(\nabla_h \nabla_k - \nabla_k \nabla_h) \mathcal{T}_{ij} = 0$ it follows that $(\nabla_h \nabla_k - \nabla_k \nabla_h) \mathcal{R}_{ij} = 0$ Thus Eq. 4.13 implies that

$$b u_i (\nabla_h \nabla_k - \nabla_k \nabla_h) u_j + b u_j (\nabla_h \nabla_k - \nabla_k \nabla_h) u_i = 0.$$

Contracting with u^i , we obtain

$$-b (\nabla_h \nabla_k - \nabla_k \nabla_h) u_j + b u_j u^i (\nabla_h \nabla_k - \nabla_k \nabla_h) u_i = 0.$$

But $u^i (\nabla_h \nabla_k - \nabla_k \nabla_h) u_i = 0$, thus we have

$$b (\nabla_h \nabla_k - \nabla_k \nabla_h) u_j = 0.$$

Equivalently, it is

$$b \mathcal{R}_{hkj}^m u_m = 0.$$

This equation implies the following cases:

Case 1. *If $\mathcal{R}_{hkj}^m u_m = 0 \neq 0$, then $b = 0$ and hence we get $p + \sigma = 0$ which means that the spacetime represents inflation and the fluid behaves as a cosmological constant.*

Case 2. *If $b \neq 0$, then $\mathcal{R}_{hkj}^m u_m = 0$, hence a contraction with g^{hj} implies that $\mathcal{R}_{mk} u^m = 0$. Contracting equation (4.13) with u^i and using $\mathcal{R}_{mk} u^m = 0$, one gets*

$$(a - b) u_i = 0.$$

Then $a - b = 0$. With the help of equation (4.14), we have

$$\sigma = \frac{f}{2\kappa}. \tag{4.16}$$

Using (4.16) in (4.15), we get

$$p = \frac{2f'R - 3f}{6\kappa}. \tag{4.17}$$

Theorem 11. *Let the energy-momentum tensor of a perfect fluid spacetime with divergence free concircular curvature tensor obeying $f(R)$ gravity be recurrent or bi-recurrent. Then,*

- 1) The spacetime represents inflation, or
- 2) The isotropic pressure p and the energy density σ are constants. Moreover, they are given by Eq. 4.16 and Eq. 4.17.

In virtue of Eq. 4.16 and Eq. 4.17 we have

$$\frac{p}{\sigma} = \frac{2f'R - 3f}{3f}.$$

In view of the previous theorem we can state the following:

Remark 1. According to the different states of cosmic evolution of the Universe we can conclude that the spacetime with $\nabla_h \mathcal{M}_{klm}^h = 0$ obeying $f(R)$ gravity either represents inflation or

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary material, further inquiries can be directed to the corresponding authors.

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AUTHOR CONTRIBUTIONS

Conceptualization and methodology, SS, UD, AS, NT, HA-D, and SA; formal analysis, SS, UD and AS; writing original draft preparation, SS, AS and NT; writing-review and editing, SS, UD, HA-D, and SA; supervision, SS and UD; project administration, NT and AS; and funding acquisition, NT and SA. All authors have read and agreed to the published version of the manuscript.

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