On the fractional weibull process

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Engineering applications of the fractional Weibull distribution (fWd) are quite
limited because a corresponding stochastic process is not yet constituted and
completely analyzed of fundamental properties. In order to fill this gap, the
fractional Weibull process (fWp) is defined in this paper with the realization
algorithm. The self-similarity property as well as long range dependence (LRD)
are proven for the future research. The simulation is conducted by the actual
data. The fWd is utilized to fit the actual probability distribution and the
corresponding process is generated to reflect the stochasticity of the data.
The random walk based on the fWp expands the simulation to the planar space.

1 Introduction

After the discovery of fractal geometry, there has been recently a growing interest for
the application of the fractional dimension processes in several fields. The fractional
process is often used in the modeling of the Internet traffic. An abstract model for
aggregated connectionless traffic, which is based on the fractional Brownian motion, is
presented in [1]. In [2], generalized fractional Gaussian noise is proposed and used in the
traffic modelling. The stochastic process with fractional dimension can also be used in the
field of remaining useful life prediction for the mechanical parts. The remaining useful life
is the remaining time for the mechanical parts before the next failure. The purpose of the
remaining useful life prediction is to schedule the maintenance of the system and improve
the reliability. The multi-modal fractional Lévy stable motion degradation model is
developed to predict the remaining useful life of a blast furnace [3].

The Weibull distribution was originally introduced for modeling the strength data of
material by Weibull in 1939, which is inspired by the works of the extreme value
distribution, and then extended to several fields [4]. The traditional applications of the
Weibull distribution is in the field of mechanical engineering, which is the lifetime
prediction. A new extended model is used in the lifetime prediction in [5]. The Weibull
distribution is combined with the artificial neural network to form a new predicting model
for the prediction of the remaining useful life of the bearing [6]. There are also a lot of

applications of Weibull distribution in the power system. The combining modified Weibull distribution model is proposed for the forecast of the power system reliability [7]. Based on simple and complex Weibull distributions, a hypothesis of the power system reliability has been confirmed [8]. In [9] the upper-truncated Weibull distribution has been used both for modeling the wind speed data and estimating the wind power density. Weibull distribution is combined with the artificial neural network to establish an advanced wind speed prediction model [10].

The versatility of Weibull distribution raises people's interests on the fractional transformation. In [11] the fWd was proposed for modeling the wind speed data. The researchers find out that the actual wind speed data contains a lot of low wind speed values. By discarding the wind speed data that is close to zero, the wind speed is fitted by the fWd with high accuracy for the speed range suitable for wind power production. The fractal parameter is the fraction of data extracted from the wind speed data to make a better fit. Therefore, the essence of the fWd is to improve the fitting results by an additional fractal parameter.

In [12], the power load is modeled by the Weibull distribution. The power load data resembles the wind speed data because the power load is commonly be zero or approximate to zero. In the power break, the load is zero. At night, the factories are closed, therefore the total power load consumption in the area is close to zero. Therefore, this paper proposes the fWd for the modelling of the power load in wish of a better modelling result.

Probability distribution as fWd describes the time series in the statistical sense. In order to express the temporal characteristics of the time series conforming to fWd, the corresponding stochastic process need to be defined and studied. In this paper, the fWp is defined with respect to fWd. At each of the time point, draw a value randomly from the fWd will constitute the corresponding fWp temporally. If we count the values of the fWp in a long enough time period, the frequency distribution of the counted data will be the corresponding fWd. Therefore, the fWd and fWp are strongly connected mathematically.

As a stochastic process is defined, some fundamental and crucial property for the process need to be studied. Random walk characteristics is a common stochastic behavior for a process, which is beneficial in the optimization and graph machine learning. The random walk can be useful for the optimization to avoid the local maximum and the graph machine learning to simplify the graph. A lot of stochastic time series in the application conveys the properties of the self-similarity and LRD, which is useful for the study. Stochastic process with self-similarity and LRD can be employed in the modeling and prediction of the time series with the same properties. Therefore, in this paper, the random walk characteristics, self-similarity and LRD of the fWd are studied and illustrated to facilitate the future engineering application of the proposed process.

A random walk is known as a random behavior, in which a particle in space takes a succession of random steps to create a trajectory of moving [13]. In each of the step, the direction is random and the step length follows a certain distribution. If the distribution is normal distribution, then the random walk is called Rayleigh flight [14]. The random walk with the probability distribution to be Lévy distribution is called the Lévy flight [15]. In this paper, the random walk based on the fWp is proposed, in which the probability for the step length is the fWd.

Self-similarity means that the partial segment of the stochastic process or distribution resembles the whole stochastic process or distribution [16]. In [17], the Weibull distribution is used in the modelling of the self-similar Internet traffic and the formula of the self-similarity parameter is derived in [18]. The Hurst parameter of the Weibull distribution belongs to (0.5,1), which can be confirmed from the formula in [18]. Moreover, the mathematical definition of self-similarity contains the concept of equality in distribution [19]. Two random variables X and Y are said to be equal in distribution, if they have the same probability distribution function. In [20] a proposition is proven for the equivalent condition of the equality in distribution. In this paper, the proposition is employed to prove the self-similarity of the fWp.

LRD means that the value of the stochastic process is strongly influenced by the previous values of the time series [21]. The autocorrelation function of the long range dependent signal cannot be integrated in the infinity range, instead it diverges to infinity. The reason for the divergence is that the autocorrelation decays to zero very slowly as the power functional speed. In order to make the integration to converge, the autocorrelation need to decay exponentially, which corresponds to the short range dependence.

In [22] the following theorem about the connection between the self-similarity and LRD has been proved so that we can prove the LRD based on self-similarity:

If a process is self-similar with self-similarity parameter belonging to the half unit interval of (0.5, 1) and the second moments exist, then it can be shown that its incremental stochastic process is characterized by LRD.

In this paper, fWp is defined with respect to the fWd and the random walk characteristics is analyzed. The self-similarity property and the existence of the second moments are proved, therefore the incremental process of fWp is characterized by LRD [22]. Furthermore, LRD of the fWp is derived. The simulation is carried out with the real data.

The rest of the paper is arranged as follows: In Section 2, the properties of fWd is elaborated and the corresponding fWp is defined with the realization algorithm. In Section 3 and Section 4, the self-similarity and LRD of the fWp are proven. In Section 5, the simulation of the fWp is carried out with the analysis for the corresponding random walk. The work of this paper is summarized in the conclusion.
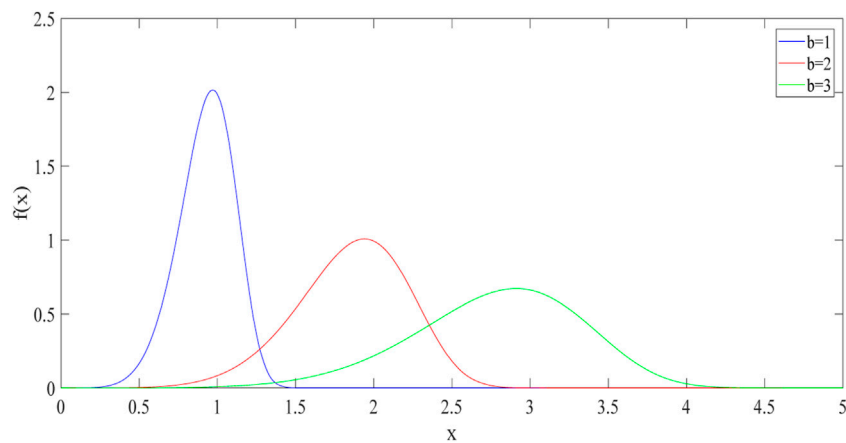
FIGURE 1
The influence of the change of scale parameter.

2 The fWp as well as the corresponding random walk

2.1 Data modeling with the fWd

The density function of the fWd is defined as:

$$f_x(x|a,b,c,\delta) = \frac{c}{b}(1-\delta)\left(\frac{x-a}{b}\right)^{c-1}\exp\left\{-\left(\frac{x-a}{b}\right)^c\right\}, x \geq a,$$
(2.1)

where δ is the fractal parameter, a is the location parameter, b is the scale parameter and c is the shape parameter.

The location parameter a is the origin where the probability of the physical values is above zero. Changing the value of a will cause a shift of curve horizontally. The shape parameter c is the key parameter of the distribution because it can change the shape of the density function dramatically. The density curve of the fWd is L-shaped when the value of parameter c is not larger than 1. Otherwise, the density curve is single-peaked.

Changing the scale parameter b while the other parameters are hold constants will cause the curve to shift both vertically and horizontally (see e. g., Figure 1). With the increase of the scale parameter, the dispersion of the distribution is larger. and the skewness of the data is smaller.

The skewness is very important for the modeling of the real physical quantities. The maximum and minimum of the data is confined by the application background of the data. Therefore, the data area with high probability is relatively small and the probability for other areas are close to zero. This common phenomenon introduces skewness to the actual density function with its admissible range between zero and positive infinity. The skewness of fWd makes it easier to describe this sort of data distribution.

Comparing with the Weibull density function, the probability density function of the fWd has a scaling factor $(1-\delta)$ which is smaller than 1 representing the reserved proportion of the data. Therefore, the fWd can be considered as the generalization of the Weibull distribution with an additional fractal parameter. If fractal parameter δ is zero, then the fWd degenerates to Weibull distribution. Changing δ while the other parameters are kept invariant will cause the peak of the curve to move vertically. In Figure 2, some fWds corresponding to different fractal parameters are depicted.

The physical meaning of δ is the discarding rate of the original data. When the ratio of the low value samples is high, a higher fractal parameter can reduce the modeling error. When the fractal parameter is determined to be smaller, there will be less of the original data to be discarded. The value of δ can influence the accuracy of modeling, thus the fractal parameter should be determined through experiment.

2.2 Definition and realization algorithm of the fWp

Definition of the fWp:

The stochastic process is fWp if the following two conditions are satisfied:

1. $fWp(0) = 0$ (2.2)
2. The increments of the fWp are independent   (2.3)
3. For given $t > s \geq 0$, the increment satisfies

$$fWp(t) - fWp(s) \sim fWd$$
(2.4)

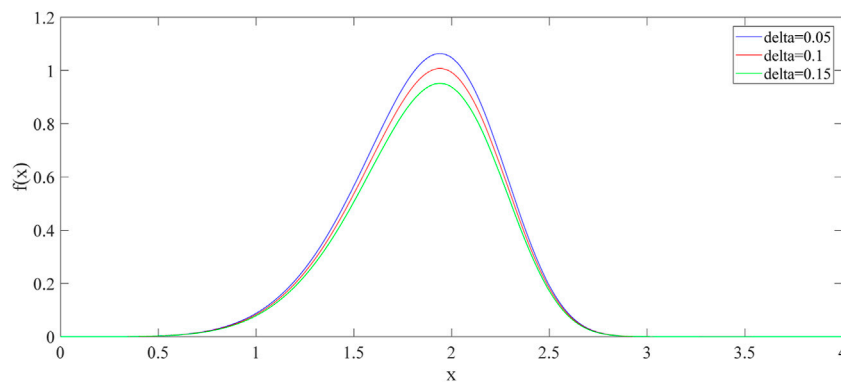Set s to be 0 and combining Eq. 2.2 and Eq. 2.4, Eq. 2.5 can be reached:

FIGURE 2
The influence of the change of fractal parameter.

$$fWp(t) \sim fWd \tag{2.5}$$

Derivation of the realization algorithm for fWp

Select a minimum calculation step lengthΔin the interested definition range of the distribution and then define the discrete argument $x_i$in the definition range.

$$x_i = i\Delta, \tag{2.6}$$

where the argument $i$is the total number of the step lengths pertaining to the value of the physical quantities.

Substitute the discrete argument to the density function of the fWd:

$$p(x_i) = p(i\Delta)$$
$$= (1 - \delta)\frac{c}{b}\left(\frac{i\Delta - a}{b}\right)^{c-1}\exp\left\{-\left(\frac{i\Delta - a}{b}\right)^c\right\}, i\Delta \geq a \tag{2.7}$$

The definition range of the stochastic time series is limited by the physical law and there is only a proportion of the definition range that can raise people's research interests. Therefore, the maximum value of the argument i, which is i^*, can be defined:

$$i^* = \max i = \left[\frac{\max x}{\Delta}\right] \tag{2.8}$$

The maximum value of $i$is determined based on the types of the physical values and the applications

Therefore, the finite valued and discretized density function of the fWd:

$$pdf(i\Delta) \sim \begin{bmatrix} 0 & \Delta & i\Delta & i^*\Delta \\ 0 & \frac{p(\Delta)}{\theta} \cdots & \frac{p(i\Delta)}{\theta} \cdots & \frac{p(i^*\Delta)}{\theta} \end{bmatrix}, i = 0, 1, 2, ..., i^*, \tag{2.9}$$

where

$$\theta = \sum_{i=1}^{i^*} p(i\Delta), \tag{2.10}$$

The parameter $\theta$is introduced to normalize the values in the random variable Such that it can meet the basic requirements of a discrete random variable.

Define another discrete random variable X_{i^*}

$$X_{i^*} \sim \left\{ \begin{array}{cccc} 0 & 1 & i & i^* \\ \frac{1}{i^*} & \frac{1}{i^*} \cdots & \frac{1}{i^*} \cdots & \frac{1}{i^*} \end{array} \right\} \tag{2.11}$$

The discrete random variable X_{i^*} means that the integer value i is equally distributed in the range of $[0, i^*]$.

The constitution of the stochastic process can be separated into three steps. The.

First step is to draw an integer value of $i$from the discrete random variable X_{i^*} with the uniform probability. The second step is to calculate the probability value of $pdf(i\Delta)$. The third step is to multiply the probability values of the two steps.

Therefore, we can define the fWp as follows:

$$fWp(t) \sim \left\{ \begin{array}{cccc} 0 & \Delta & i\Delta & i^*\Delta \\ 0 & \frac{1}{i^*}\frac{p(\Delta)}{\theta\sigma} \cdots & \frac{1}{i^*}\frac{p(i\Delta)}{\theta\sigma} \cdots & \frac{1}{i^*}\frac{p(i^*\Delta)}{\theta\sigma} \end{array} \right\}, \tag{2.12}$$

where $\sigma$is introduced to meet the basic requirement of the discretized random variable:

$$\sigma = \sum_{i=1}^{i^*}\frac{1}{i^*\theta}p(i\Delta) = \frac{1}{i^*}, \tag{2.13}$$

Substitute the value of $\sigma$to the definition of the discrete random variable:
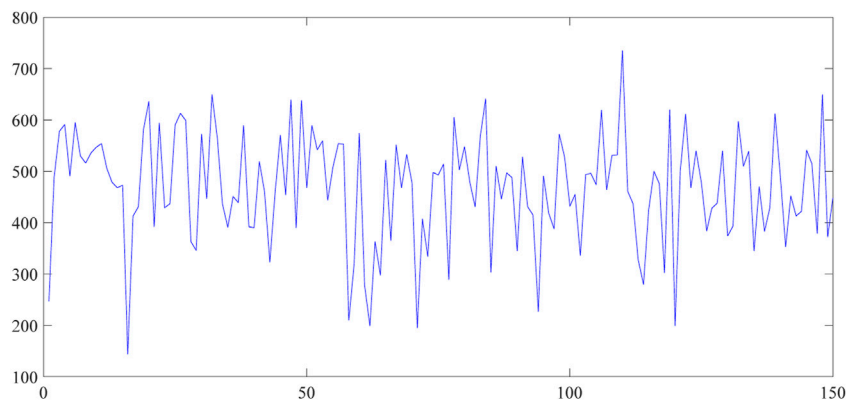
FIGURE 3
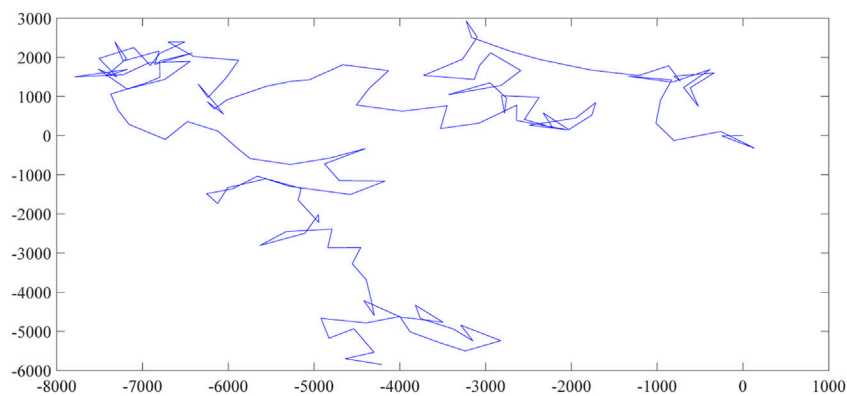An exemplary trajectory of the fWp.

FIGURE 4
An exemplary random walk based on the fWp.

$$fWp(t) \sim \left\{ \begin{matrix} 0 & \Delta & i\Delta & i^*\Delta \\ 0 & \dfrac{p(\Delta)}{\theta} & \cdots & \dfrac{p(i\Delta)}{\theta} & \cdots & \dfrac{p(i^*\Delta)}{\theta} \end{matrix} \right\}, i = 0, ..., i^* \quad (2.14)$$

There are six different parameters in the fWp. Four of them are the same as the fWd and are estimated by the actual data. These four parameters can reflect the statistical characteristics of the data. The other two parameters are determined based on the physical background of the data. In this section, a trajectory of the fWp and the corresponding random walk are depicted separately in Figure 3 and Figure 4. The parameters are chosen randomly in this section and in Section 5 the whole procedure for the construction of the fWp as well as the corresponding random walk is provided.

3 Self-similarity property

3.1 Self-similarity criterion for a general stochastic process

Let X_k be a stochastic process, and k is the argument. X_k is self-similar with self-similarity parameter $H \in (0, 1)$ if and only if

$$X(\lambda k) \overset{(d)}{=} \lambda^H X(k) \; \forall \lambda > 0, \qquad (3.1)$$

where $\overset{(d)}{=}$ denotes equality in distribution, i. e., they have the same probability distribution function.

In [20], the following proposition expressed by Eq. 3.2 and Eq. 3.3 are proved. Equations 3.2 is the general form of the equivalent condition for equality in distribution with respect to

two different random variables. If the random variable is Weibull distribution, then the specialized form is presented as Eq. 3.3.

Given two cumulative distribution functions $F_1(k)$ and $F_2(k)$, and a fixed number m, we have that $F_1(k)$ is equal to $F_2(k)$ in distribution if and only if $g_4(F_2(k), m)$ is equal to $g_4(F_1(k), m)$, where the function $g_4(F(k), m)$ is called the discriminant function and is defined as:

$$g_4(F(k), m) = F^{-1}((F(k) - F(m)) \bmod 1), \qquad (3.2)$$

where F^{-1} is the inverse function of the cumulative function and mod is the modular operation.

For the Weibull distribution, we can get the specialized definition for $g_4(F(k), m)$:

$$g_4'(F(k), m) = \left(-\log\left[\exp\left\{ -\left(\frac{k-a}{b}\right)^c \right\} - \exp\left\{ -\left(\frac{m-a}{b}\right)^c \right\} \right] \right)^{\frac{1}{c}} \tag{3.3}$$

By combining the self-similarity definition (3.1) with Eq. 3.2, we can get the criterion of the self-similarity for a general stochastic process:

$$g_4(F(\lambda k), m) = g_4(F'(k), m) \tag{3.4}$$

where $F(k)$ is the cumulative distribution function of X_k, $F'(k)$ is the cumulative distribution function of $\lambda^H X_k$.

$$F'(k) = P(\lambda^H X_k \leq k) = P\left(X_k \leq \frac{k}{\lambda^H} \right) = F\left(\frac{k}{\lambda^H} \right) \tag{3.5}$$

Substituting (3.5) into (3.4) we get

$$g_4(F(\lambda k), m) = g_4\left(F\left(\frac{k}{\lambda^H}\right), m \right), \tag{3.6}$$

which is a practical criterion for the self-similarity of a stochastic process

On the left side of Eq. 3.6, it is the discriminant function $g_4(F(k), m)$ with argument λk, while on the right hand side, it is the discriminant function $g_4(F(k), m)$ with argument $\frac{k}{\lambda^H}$.

3.2 The discriminant function for the fWp

In the following, we will derive the discriminant function for the fWp.

The following equations can be derived from the definitions:

$$f^*(k) = (1 - \delta) f(k), \tag{3.7}$$

$$F^*(k) = \int_{-\infty}^{k} f^*(t) dt = \int_{-\infty}^{k} (1 - \delta) f(t) dt = (1 - \delta) F(k), \tag{3.8}$$

$$F^{-1*}(k) = \frac{1}{1 - \delta} F^{-1}(k), \tag{3.9}$$

where $f(x)$ and $f^*(x)$ are the density functions of the Weibull distribution and the fWp. $F(x)$ and $F^*(x)$ are the cumulative

density functions of the Weibull distribution and the fWp. $F^{-1}(x)$ and $F^{-1*}(x)$ are the inverse cumulative density functions of the Weibull distribution and the fWp.

From Eq. 3.2, we have:

$$g_4^*(F(k), m) = F^{-1*}((F^*(k) - F^*(m)) \bmod 1)$$
$$= \frac{1}{1 - \delta} F^{-1}(((1 - \delta)[F(k) - F(m)]) \bmod 1) \tag{3.10}$$

Besides the additional coefficient $\frac{1}{1-\delta}$, the only change between (3.2) and (3.10) is the difference $F(k) - F(m)$, which is added by a scaling factor $(1 - \delta)$. $F(k) - F(m)$ in Eq. 3.2 results in a $\exp\{-(\frac{k-a}{b})^c\} - \exp\{-(\frac{m-a}{b})^c\}$ proportion of Eq. 3.3 specialized for the Weibull distribution. Thus we can conclude that the discriminant function specialized for the fWp is:

$$g_4^*(F(k), m) = \left(-\log\left[(1 - \delta)\left\{ \exp\left\{ -\left(\frac{k-a}{b}\right)^c \right\} \right. \right. \right.$$
$$\left. \left. \left. -\exp\left\{ -\left(\frac{m-a}{b}\right)^c \right\} \right\} \right] \right)^{\frac{1}{c}} \tag{3.11}$$

3.3 Self-similarity for the fWp

In this subsection we will show the self-similarity of the fWp.

Theorem 1. The fWp is self-similar
Proof Combining Eq. 3.3 and Eq. 3.6, we can reach Eq. 3.12:

$$g_4'(F(\lambda k), m) = \left(-\log\left[\exp\left\{ -\left(\frac{\lambda k - a}{b}\right)^c \right\} - \exp\left\{ -\left(\frac{m-a}{b}\right)^c \right\} \right] \right)^{\frac{1}{c}}$$
$$= g_4'\left(F\left(\frac{k}{\lambda^H}\right), m \right) = \left(-\log\left[\exp\left\{ -\left(\frac{\frac{k}{\lambda^H} - a}{b}\right)^c \right\} \right. \right.$$
$$\left. \left. -\exp\left\{ -\left(\frac{m-a}{b}\right)^c \right\} \right] \right)^{\frac{1}{c}} - \exp\left\{ -\left(\frac{m-a}{b}\right)^c \right\} \right] \tag{3.12}$$

Eq. 3.12 can be simplified to be Eq. 3.13 by removing the minus sign and the power.

$$\log\left[\exp\left\{ -\left(\frac{\lambda k - a}{b}\right)^c \right\} - \exp\left\{ -\left(\frac{m-a}{b}\right)^c \right\} \right]$$
$$= \log\left[\exp\left\{ -\left(\frac{\frac{k}{\lambda^H} - a}{b}\right)^c \right\} - \exp\left\{ -\left(\frac{m-a}{b}\right)^c \right\} \right] \tag{3.13}$$

By considering the self-similarity condition (3.6) and Eq. 3.11, we have to show that the following Eq. 3.14 and Eq. 3.15 are equal to prove the self-similarity of the fWp.

$$g_4^*(F(\lambda k), m) = \left(-\log\left[(1 - \delta)\left\{ \exp\left\{ -\left(\frac{\lambda k - a}{b}\right)^c \right\} \right. \right. \right.$$
$$\left. \left. \left. -\exp\left\{ -\left(\frac{m-a}{b}\right)^c \right\} \right\} \right] \right)^{\frac{1}{c}} \tag{3.14}$$

$$g_4^*\left(F\left(\frac{k}{\lambda^H}\right), m\right) = \left(-\log\left[(1-\delta)\left\{\exp\left\{-\left(\frac{\frac{k}{\lambda^H}-a}{b}\right)^c\right\}-\exp\left\{-\left(\frac{m-a}{b}\right)^c\right\}\right\}\right]\right) \quad (3.15)$$

In fact (3.14) and (3.15) can be transformed into (3.16) and (3.17), respectively, as follows.

$$\log\left[(1-\delta)\left\{\exp\left\{-\left(\frac{\lambda k-a}{b}\right)^c\right\}-\exp\left\{-\left(\frac{m-a}{b}\right)^c\right\}\right\}\right]$$
$$= \log(1-\delta) + \log\left\{\exp\left\{-\left(\frac{\lambda k-a}{b}\right)^c\right\}-\exp\left\{-\left(\frac{m-a}{b}\right)^c\right\}\right\} \quad (3.16)$$

$$\log\left[(1-\delta)\left\{\exp\left\{-\left(\frac{\frac{k}{\lambda^H}-a}{b}\right)^c\right\}-\exp\left\{-\left(\frac{m-a}{b}\right)^c\right\}\right\}\right] \quad (3.17)$$

Therefore, the equality of (3.16) and 3.17 can be transformed into the equality of (3.18) and 3.19:

$$\log\left\{\exp\left\{-\left(\frac{\lambda k-a}{b}\right)^c\right\}-\exp\left\{-\left(\frac{m-a}{b}\right)^c\right\}\right\} \quad (3.18)$$

$$\log\left\{\exp\left\{-\left(\frac{\frac{k}{\lambda^H}-a}{b}\right)^c\right\}-\exp\left\{-\left(\frac{m-a}{b}\right)^c\right\}\right\} \quad (3.19)$$

According to Eq. 3.13, the two functions (3.18) and (3.19) are equal if we select the same self-similar parameter with the Weibull distribution. Therefore, the self-similarity of the fWp is proven because there exists a self-similar parameter to make Eq. 3.1 to hold, which is within the interval of (0.5,1).

4 The second moments and LRD property

4.1 The second moments of the fWp exists

For simplicity we give here only one kind of second moment, since the derivation of other moments are similar. The second moment about zero of the fWp can be easily obtained by a direct computation as follows:

$$E\left(fWb(t)^2\right) = \int_0^\infty x^2 f_X^*(x \mid 0,1,c)dx = (1-\delta)\int_0^\infty x^2 f_X(x \mid 0,1,c)dx$$
$$\xrightarrow{u=x^c} (1-\delta)\int_0^\infty x^2 cx^{c-1}\exp\{-x^c\}dx$$
$$= (1-\delta)\int_0^\infty u^{\frac{2}{c}}e^{-u}du = (1-\delta)\Gamma\left(\frac{2}{c}+1\right) \quad (4.1)$$

Therefore, we have proven that the second moments of the fWp exists.

4.2 The LRD of the fWp

The LRD means the slow decay of the auto-correlation function $\rho(k)$, which also means that the auto-correlation in the process remains strong as the time lag goes to infinity. The strong auto-correlation means that the value of the process is influenced by other values of the process. With the slow decay of the auto-correlation function, it cannot be integrated in the infinite range.

$$\rho(k) \sim \beta k^{-\alpha} \text{ as } k \to \infty, \quad (4.2)$$

where

$$\rho(k) = E(X(k)X(k-\tau)) \quad (4.3)$$

$0 < \alpha < 1$ and β is a given non-zero constant.

Eq. 4.2 summarizes the physical meaning of the LRD. The slow decaying rate of the auto-correlation function does not specify the exact decaying rate. By changing the values of constants α and β, the decaying rate of the auto-correlation changes accordingly. The slower the decaying is, the stronger is the LRD. Therefore, the proof of the LRD focuses on the existence of the slow decaying auto-correlation.

Let $X(k)$ be the fWp, then its incremental process $Y(k)$ fulfills:

$$\begin{cases} Y(k) = X(k) - X(k-1) & , \quad k = 1, ..., n \\ Y(0) \xrightarrow{def} 0 & , \qquad X(0) = 0 \end{cases} \quad (4.4)$$

Based on the theorem in [22], we can show the incremental process $Y(k)$ is of LRD, which means the auto-correlation function of $Y(k)$ decays as a power function. In the following, we can show the fWp is of LRD.

Theorem 2. The fWp is of LRD

Proof: According to the definition (4-4), we can express $X(k)$ by the linear combinations of $Y(k), k = 0, 1, ..., n$. Moreover, the auto-correlation function of $X(k)$ can be expressed as the linear combination of auto-correlation functions of $Y(k)$, k = 0, 1, ..., n.

$$X(k) = Y(k) + Y(k-1) + ... + Y(1) + Y(0) \quad (4.5)$$

$$\rho(X(k)) = E(X(k)X(k-\tau)) = E\left(\left(\sum_{i=0}^k Y(k)\right)\left(\sum_{j=0}^{k-\tau} Y(j)\right)\right)$$
$$= \sum_{i=0}^k \sum_{j=0}^{k-\tau} E(Y(i)Y(j)) = \sum \rho(Y(k)) \quad (4.6)$$

From Eq. 4.6 and the law of infinitesimal substitution, there follows that the auto-correlation function of $X(k)$ decays in a multinomial power law way. It also means that the auto-correlation function of $X(k)$ decays at the same speed as the slowest decaying rate among auto-correlation functions of $Y(k)$, $k = 0, 1, ..., n$. The auto-correlation function of $Y(k)$ that processes the slowest decaying rate is the biggest one, in other words, the one with the smallest power.

$$\rho(X(k)) = \sum \rho(Y(k)) \sim \sum \beta_i k^{\alpha i} \sim \max(\beta_i k^{\alpha i}) = \beta_* k^{-\min(\alpha i)} \quad (4.7)$$

Therefore, we have proven that the auto-correlation function of $X(k)$ is decaying as the power function, which proves the LRD of the fWp.
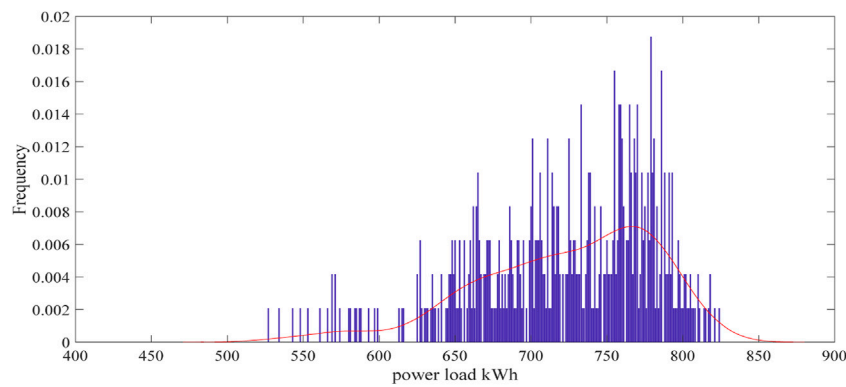
FIGURE 5
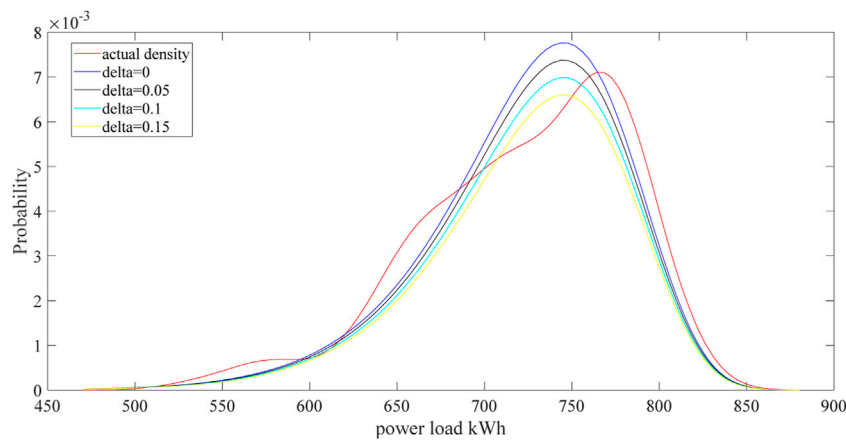The actual power load frequency distribution and density function.

FIGURE 6
The fitting results of fWd with different fractal parameters.

5 Simulation on the fWp

5.1 Time series fitted with the fWd

In this paper, the power load data from the European Network on Intelligent.

Technologies prediction contest is used in the validation of the proposed process. In Figure 5, the actual power load frequency distribution is plotted and the corresponding probability density function is provided numerically

The power load data cannot be negative values, therefore the Gaussian fit is not appropriate. Select fWd to fit the actual power load distribution and can achieve a better fitting result than the Weibull distribution because of the fractal parameter. The location parameter a is set to be zero for the power load are positive and the scale fractal parameter is determined later through experiment. The scale parameter

and shape parameter are estimated by the maximum similarity estimation.

The maximum likelihood function of the fWd:

$$L(x|b,c) = \prod_{i=1}^{n} f_X(x_i|b,c)$$

$$= \prod_{i=1}^{n}\left[(1-\delta)\left(\frac{c}{b}\right)\left(\frac{x_i}{b}\right)^{c-1} \exp\left\{ -\left(\frac{x_i}{b}\right)^{c}\right\}\right]$$

$$= (1-\delta)^{n}\left(\frac{c^{n}}{b^{n}}\right)\left(\frac{1}{b}\right)^{n(c-1)}\prod_{i=1}^{n} x_i^{c-1} \exp\left\{ -\left(\frac{1}{b}\right)^{c}\sum_{i=1}^{n} x_i^{c}\right\}$$

$$(5.1)$$

The logarithmic maximum likelihood function of the fWd:

$$\ln L(x|b,c) = \ln\left[\left(\frac{c}{b}\right)^{n}\left(\frac{1}{b}\right)^{n(c-1)}\prod_{i=1}^{n}(x_i)^{c-1} \exp\left\{ -\left(\frac{1}{b}\right)^{c}\sum_{i=1}^{n} x_i^{c}\right\}\right]$$

$$= n\ln(1-\delta) + n\ln\left(\frac{c}{b}\right) + n(c-1)\ln\left(\frac{1}{b}\right) + (c-1)\sum_{i=1}^{n}\ln x_i - \left(\frac{1}{b}\right)^{c}\sum_{i=1}^{n} x_i^{c}$$

$$(5.2)$$

	delta = 0	delta = 0.05	delta = 0.1	delta = 0.15
SSE	0.3812E-0.4	0.3309E-0.4	0.3455E-0.4	0.4250E-0.4
RMSE	0.6174E-03	0.5753E-03	0.5878E-03	0.6520E-03

Take the partial derivative of the logarithmic maximum likelihood function with respect to the scale parameter b and then make it equal to zero.

$$\frac{\partial \ln L(x|b,c)}{\partial b} = \left(\frac{-n}{b}\right) - n(c-1)\left(\frac{1}{b}\right) + c\sum_{i=1}^{n} x_i^c \left(\frac{1}{b}\right)^{c+1} = 0 \quad (5.3)$$

Separate the scale parameter b and the shape parameter c:

$$\hat{b} = \left\{ \frac{\sum_{i=1}^{n} x_i^c}{nc} \right\}^{\frac{1}{c}} \quad (5.4)$$

Take the partial derivative of the logarithmic maximum likelihood function with respect to the shape parameter c and then make it equal to zero.

$$\frac{\partial \ln L(x|b,c)}{\partial c} = n\left(\frac{1}{c}\right) + n\ln\left(\frac{1}{b}\right) + \sum_{i=1}^{n}\ln x_i - \left[\left(\frac{1}{b}\right)^c \sum_{i=1}^{n} \frac{\partial x_i^c}{\partial c}\right]$$
$$- \left[\sum_{i=1}^{n} x_i^c \left(\frac{1}{b}\right)^c \ln\left(\frac{1}{b}\right)\right]$$
$$= n\left(\frac{1}{c}\right) + n\ln\left(\frac{1}{b}\right) + \sum_{i=1}^{n}\ln x_i - \left[\left(\frac{1}{b}\right)^c \sum_{i=1}^{n} x_i^c \ln x_i\right]$$
$$- \left[\sum_{i=1}^{n} x_i^c \left(\frac{1}{b}\right)^c \ln\left(\frac{1}{b}\right)\right] = 0 \quad (5.5)$$

After performing some algebraic calculation, the equivalent equation can be derived

$$\ln\left(\frac{1}{b}\right) + \frac{1}{n}\sum_{i=1}^{n}\ln x_i + \frac{c}{n^2} - \left(\frac{1}{b}\right)^c \frac{\sum_{i=1}^{n} x_i^c \ln\left(\frac{x_i}{b}\right)}{n} = 0 \quad (5.6)$$

Substituting Eq. 5.4 to Eq. 5.6:

$$n^{\frac{1}{c}} c^{\left(\frac{1}{c}+n\right)} \left[\sum_{i=1}^{n} x_i^c\right]^{\left(n-\frac{1}{c}\right)} \left[\prod_{i=1}^{n} x_i\right]^{-c} - \frac{e^{\frac{1}{n^3}}}{n\left[\prod_{i=1}^{n} x_i\right]^{\frac{1}{n^2}}} = 0 \quad (5.7)$$

Therefore, the estimation of the scale parameter b and the shape parameter c is reached. The scale parameter b is estimated to be 748.4289 and the shape parameter c is estimated to be 15.7742.

The fractal parameter needs to be evaluated chosen by the fitting results. The density function of the fWd with

different fractal parameters are depicted in Figure 6 as well as the actual power load density function. The criterions of goodness of fitting (GoF) are the sum of squared error (SSE) and the root mean square error (RMSE) [23]. The formulas are listed below. For both criterions, the smaller value means better fitting results.

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(p_i - f_i)^2} \quad (5.8)$$

$$SSE = \sum_{i=1}^{n}(p_i - f_i)^2 \quad (5.9)$$

The calculations of the SSE and RMSE are listed in Table 1. The density function of fWd with zero fractal parameter is Weibull distribution. As we can see from Table 1, the delta value of 0.05 is chosen in the end because both the values of SSE and RMSE are the smallest.

After the parameter estimation and the experiment, the power load is fitted with the fractional Weibull distribution in Figure 7.

5.2 The construction of the fWp

On the purpose of generating the fWp conveying the stochasticity of the stochastic time series, the other parameters in the fWp need to be calculated.

The step length Δ of the modeling can set to be 1 kWh because the power load data are all positive integers. The maximum value of the power load concerned in the research is 900 kWh. Therefore, the maximum value of the argument i, which is i^*, can be calculated:

$$i^* = \max i = \left[\frac{\max x}{\Delta}\right] = 900 \quad (5.10)$$

where x is the value of the power load series for research and the square bracket represents the integer valued function.

Therefore, the fWp with the stochasticity of the time series is depicted in Figure 8.

5.3 The simulated path for the random walk of the fWp

After the construction of the fWp, the corresponding random walk path can be simulated. As depicted in Figure 9, the particle in the random walk starts from the origin. At each of the iteration, the jump direction is random and the jump length follows the fWd with parameters estimated from the actual data. The simulation path of the random walk expands the dimension of the fWp to a planar space with the same stochasticity coming from the actual data.
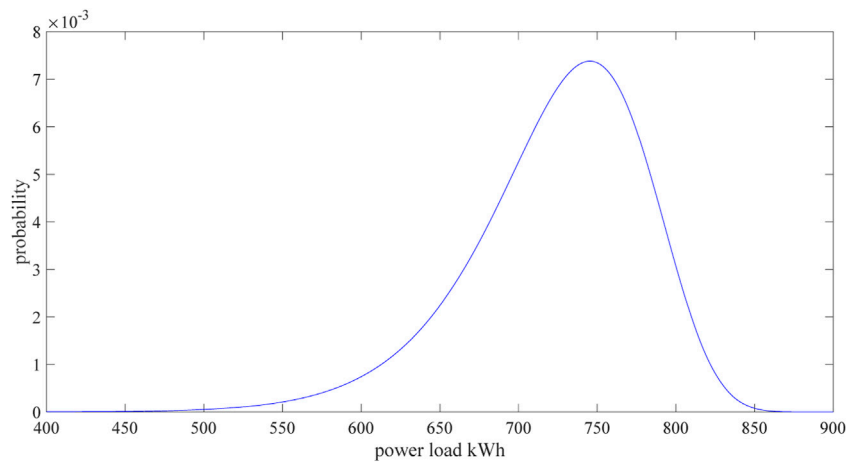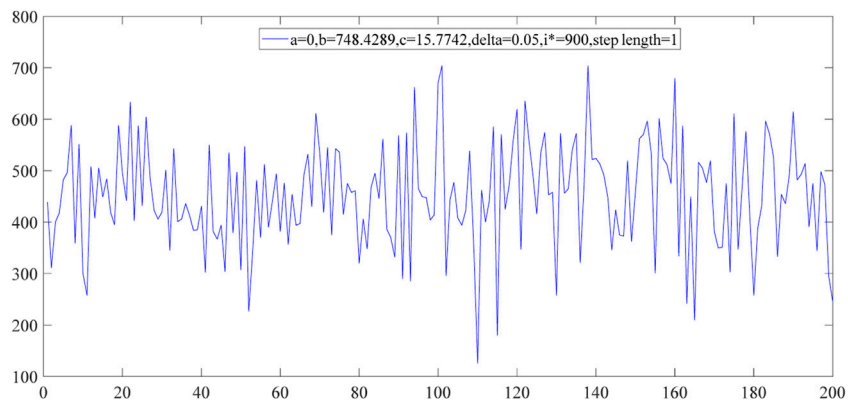
FIGURE 7
The fractional Weibull fit of the data.

FIGURE 8
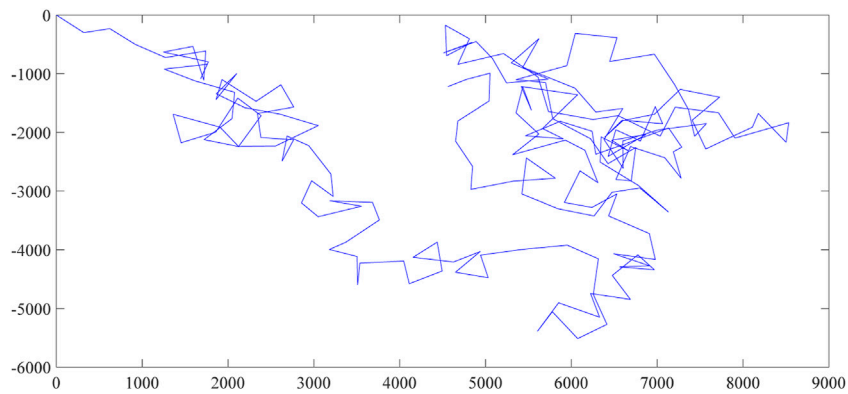The fWp based on the power load series.

FIGURE 9
The random walk characteristics of the fWp.

6 The potential engineering applications of the fWp

Wind power generation is very unreliable because it relies solely on the wind speed. The historical wind speed data can be used to construct the fWp for representing its stochasticity and the statistical wind speed prediction model can be constructed. Once the wind speed variation tendency is predicted in advance, the wind power generation can be predicted with accuracy and the power system reliability can be improved [24].

The power load prediction is very important for the reliability of the power system. If there is surplus electricity according to the power load prediction results, the maintenance of the power facilities can be scheduled. If the power consumption is difficult to guarantee, the dispatching of electricity need to be carried out [25]. With the fWp conveying the stochasticity of the power load, the power load statistical prediction model can be established and the blackouts caused by the high temperature can be reduced.

7 Conclusion

The fWd can be considered to be the generalization of the Weibull distribution with a non-zero fractal parameter. In this paper, the definition of the fWp is provided and the algorithm of realization is derived. In order to facilitate the future research, the self-similarity and LRD are proven. The actual time series is employed for the simulation of the fWp. The random walk characteristics for the fWp is analyzed in the planar space. The future research can focus on the prediction model for the potential engineering application areas of the fWp.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

Author contributions

WD: initial draft, software; WS: concept, validation, revision; CC: revision, language editing; JC: validation, revision; XC: validation, revision.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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