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The Solution Comparison of Time-Fractional Non-Linear Dynamical Systems by Using Different Techniques

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This comparative study of fractional nonlinear fractional Burger's equations and their systems has been done using two efficient analytical techniques. The generalized schemes of the proposed techniques for the suggested problems are obtained in a very sophisticated manner. The numerical examples of Burger's equations and their systems have been solved using Laplace residual power series method and Elzaki transform decomposition method. The obtained results are compared through graphs and tables. The error tables have been constructed to show the associated accuracy of each method. The procedures of both techniques are simple and attractive and, therefore, can be extended to solve other important fractional order problems.

Keywords: Laplace residual power series method, initial value problems, Caputo derivative, Elzaki transform decomposition method, fractional Burger's equations

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1 INTRODUCTION

The branch of mathematics that deals with the study of derivatives and integrals of non-integer orders is known as fractional calculus (FC). It originated on 30 September 1695 due to an important question asked by L'Hospital in a letter to Leibniz. The answer of Leibniz [1] gives motivation to a series of interesting results during the last 325 years [2–4]. In the last decades, FC has been used as a powerful tool by many researchers in various fields of science and engineering, for example, the fractional control theory [2, 5], anomalous diffusion, fractional neutron point kinetic model, fractional filters, soft matter mechanics, non-Fourier heat conduction, notably control theory, Levy statistics, nonlocal phenomena, fractional signal and image processing, porous media, fractional Brownian motion, relaxation, groundwater problems, rheology, acoustic dissipation, creep, fractional phase-locked loops, and fluid dynamics [6–12].

In recent years, Fractional Partial Differential Equations (FPDEs) have gained considerable interest because of their applications in various fields such as finance, biological processes and systems, fluid flow [13, 14], chaotic dynamics, electrochemistry, diffusion processes, material science, electromagnetic, turbulent flow [15–20], elastoplastic indentation problems [21], dynamics of van der Pol equation [22], and statistical mechanics model [23].

Finding the solution to FPDEs is a hard task. However, many mathematicians devoted their sincere work and developed numerical and analytical techniques to solve FPDEs. Some of these techniques include homotopy analysis method (HAM) [24], operational matrix [25], Adomian decomposition method (ADM) [26], homotopy perturbation method (HPM) [27], meshless method [28], variational iteration method (VIM) [29], tau method [30], Bernstein polynomials [31], the Haar wavelet method [32], the Laplace transform method [33], the Legendre base method [34], Laplace variational iteration method [35], G'/G-expansion method [36], Jacobi spectral collocation method [37], Yang–Laplace transform [38], new spectral algorithm [39], fractional complex transform method [40], cylindrical-coordinate method [41], and spectral Legendre–Gauss–Lobatto collocation method [42]. Yusufoglu et al. presented the solutions of the equal-with-wave equation and gBBM equation using various techniques [43, 44]. Bakir et al. used the traveling waves solutions idea for KdV-mKdV and modified Burger–KdV equation in [45]. Kaplan et al. [46, 47] investigated the solutions of conformable equations and Benjamin–Ono equation with the help of generalized Kudryashov techniques and the exp $(-\phi(\xi))$ method, respectively.

Burger's equation was initially introduced by Harry Bateman in 1915 [48]. They have many applications in various fields, especially in equations with non-linear forms. This equation describes many phenomena such as acoustic waves, heat conduction, dispersive water, shock waves [49], continuous stochastic processes [50], and modeling of dynamics [51–53]. The one-dimensional Burger's equations have many applications in plasma physics, gas dynamics, and so on [54]. Various techniques were developed by mathematicians to find the numerical and analytical solutions to Burger's equations. Some of these methods are a direct variational iteration method by [55, 56] solving the equations numerically by the finite difference method. The group explicit method was used by [57]. Singhal and Mittal applied the Galerkin method [58] to solve these equations numerically. A weighted residue method was applied by [59]. The fractional Riccati expansion method was applied by [60], and the variational iteration method was applied by Inc [61] to solve space-time fractional Burger's equation. Esen et al. [62] used HAM to solve the time-fractional Burger equation. The Cubic B-spline finite elements method was applied by Esen et al. to solve these equations [63].

In the present article, we will use the Elzaki transform decomposition method (ETDM) and Laplace residual power series method (LRPSM) to solve Burger's equation with one and two dimensions. The ETDM was introduced by [64] in 2010, which is the combination of ADM [65] and Elzaki transforms (ET). The ET is a modified transformation of Sumudu (ST) and Laplace transformation (LT). Many differential equations with variable coefficients cannot be solved by LT and ST so that equations can be easily managed by ET [66]. Many mathematicians applied ET to solve various kinds of Fisher [67], Navier–Stokes [68], heat-like [69] equations.

LRPSM combines the LT and Residual power series method (RPSM), giving the exact and approximate solution as a power series solution that is rapidly convergent. This technique was developed for the first time in [70] and applied in [71]. LRPSM is a technique that requires fewer computations and time with more accuracy.

This work applies ETDM and LRPSM to Burger's equations of one and two dimensions. Some numerical examples are solved by both methods. The results obtained by these two methods will be compared by making plots and tables for each problem, and then a comparison is made by making combined plots and tables for these two methods. The proposed techniques are more accurate and simple and require fewer calculations than other exciting methods. These techniques need no discretization or extra parameters to obtain the solutions to the problems. The current techniques have the unique capability to utilize the Laplace transformation to reduce the given model into its simple form. The simple series form solutions of the suggested techniques are achieved with easily computable and convergent components. The present techniques provide higher accuracy despite using very few terms of the series solution.

2 PRELIMINARIES

In this section, a few definitions related to our work are considered.

2.1 Definition

The Caputo time-fractional derivative is defined as [2]

$$D_t^{\beta}\mu(\xi,t) = \frac{1}{\Gamma(n-\beta)}\int_0^t (t-\tau)^{n-\beta-1}\frac{\partial^n\mu(\xi,\tau)}{\partial\tau^n}d\tau, \quad n-1<\beta<n,$$
$$= \frac{\partial^n\mu(\xi,\tau)}{\partial t^n}, \quad \beta=n, \quad n\in N.$$

2.2 Definition

A power series is represented as [3]

$$\sum_{n=0}^{\infty} C_n(t-t_0)^{n\beta} = C_0 + C_1(t-t_0)^{\beta} + C_2(t-t_0)^{2\beta} + \ldots,$$

where $0 \le n-1 < \beta \le 1$, $t \in N$, and $t \ge t_0$ is known as fractional power series about t_0.

2.3 Definition

Consider the expansion as follows [3], known as power series of multiple fraction about $t = t_0$:

$$\sum_{n=0}^{\infty} g_m(\xi)(t-t_0)^{n\beta}, \quad n-1<\beta<n.$$

2.4 Lemma

If $\mu(\xi, t)$ is represented as [3] multiple fractions at $t = t_0$, then

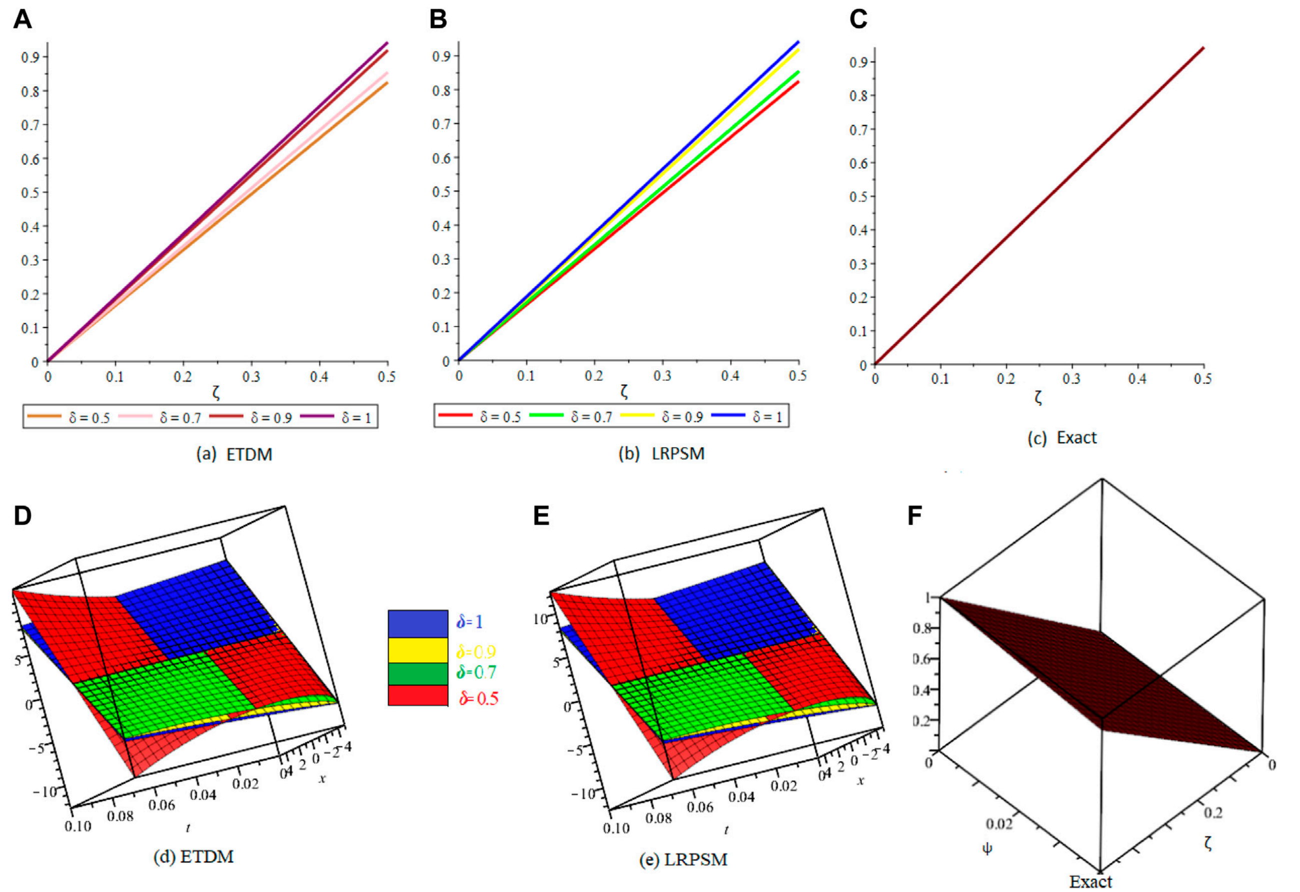

FIGURE 1 | The comparison of **(A,D)** ETDM, **(B,E)** LRPSM, and **(C,F)** exact 2D and 3D plots, at different fractional orders of example 4.1.

$$\mu(\xi,t)=\sum_{n=0}^{\infty}g_n(\xi)(t-t_0)^{n\beta},\quad n-1<\beta<n,$$
$$(\xi,y)\in I_1\times I_2\quad t_0\le t<t_0+R.$$

If $D_t^{n\beta}\mu(\xi,t)$, $n=0,1,\ldots.$ is continuous on $I_1\times I_2\times(t_0,t_0+R)$, then

$$g_m(\xi)=\frac{D^{m\beta}\mu(\xi,t_0)}{\Gamma(1+m\beta)}.$$

2.5 Corollary

If $\mu(\xi,y,t)$ has representation as [3] multiple fractions at $t=t_0$, then

$$\mu(\xi,y,t)=\sum_{n=0}^{\infty}g_n(\xi,y)(t-t_0)^{n\beta},$$
$$(\xi,y)\in I_1\times I_2\quad t_0\le t<t_0+R.$$

If $D_t^{n\beta}\mu(\xi,y,t)$, $m=0,1,2\ldots$ is continuous on $I_1\times I_2\times(t_0,t_0+R)$, then

$$g_m(\xi,y)=\frac{D^{m\beta}\mu(\xi,y,t_0)}{\Gamma(1+n\beta)}.$$

2.6 Definition

The Laplace transform of continuous function f(t) in [0, 1) is defined as [1]

$$F(s)=L[f(t)]=\int_0^{\infty}e^{-st}f(t)dt. \tag{1}$$

2.7 Definition

The Laplace transform of $f(t)=t^{\alpha}$ is defined as [1]

$$L[t^{\alpha}]=\int_0^{\infty}e^{-st}t^{\alpha}dt$$
$$=\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\quad R(s)>0,\quad R(\alpha)>0\quad n-1<\alpha<n. \tag{2}$$

2.8 Definition

The Mittag-leffler function E_{α} with $\alpha>0$ is defined as [1]

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^{\alpha}}{\Gamma(n\alpha+1)}\quad \alpha>0,\quad z\in\mathbf{C}. \tag{3}$$

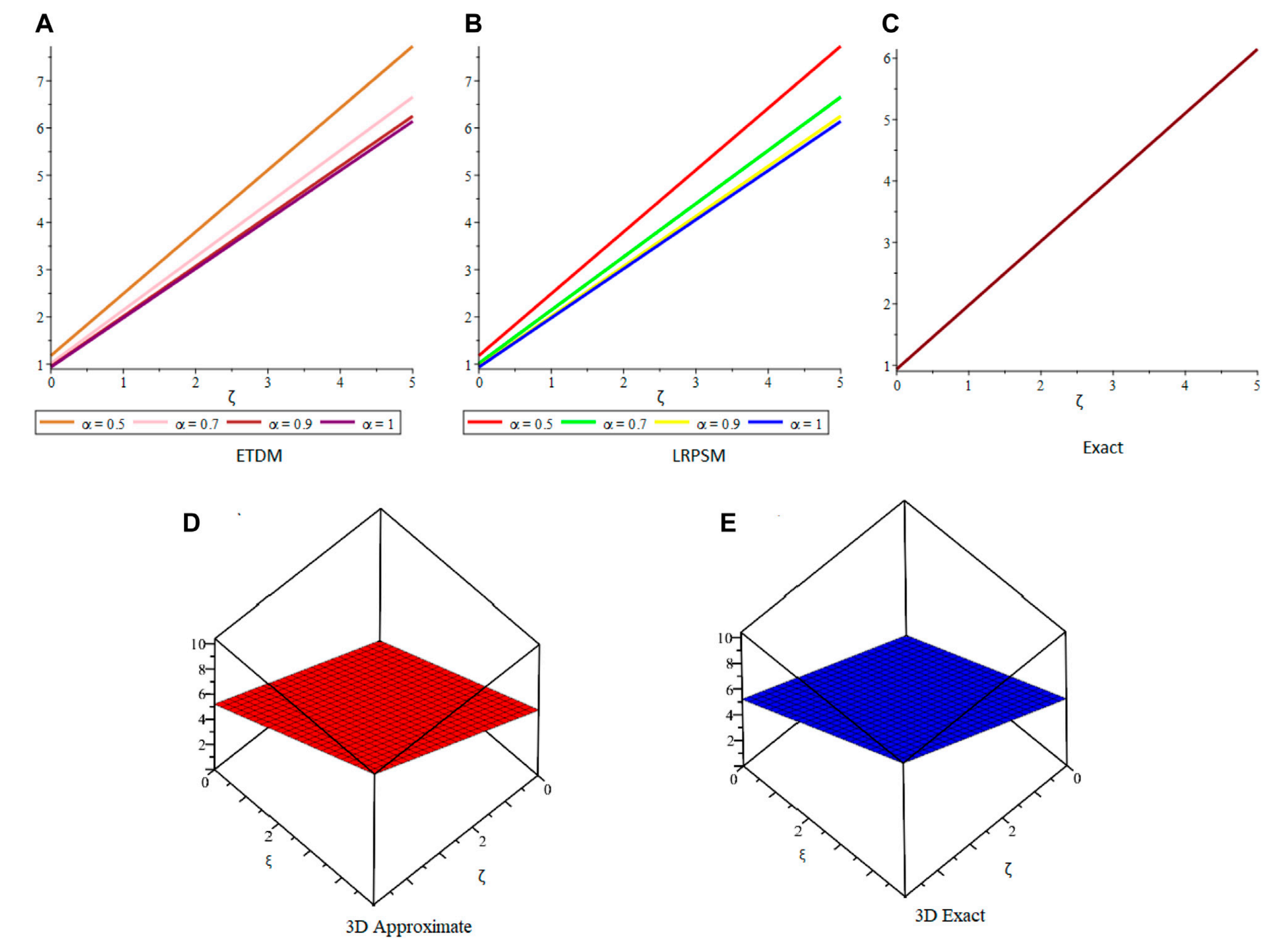

FIGURE 2 | The comparison of **(A)** ETDM, **(B)** LRPSM, and **(C)** exact 2D plots at different-fractional-order. **(D)** Approximate and **(E)** exact 3D plots of example 4.2 at $\delta = 1$.

2.9 Definition

If $\phi \in H[0, T]$, $T > 0$, $r \in (0, 1)$, then Caputo–Fabrizio of fractional derivative is expressed as [1]

$$ {}_0^{CF}D_\tau^r[\phi(\tau)] = \frac{M(r)}{1-r}\int_0^\tau \phi'(\zeta)K(\tau,\zeta)d\zeta, $$

where $M(r)$ is the normalization function with $M(1) = M(0) = 1$. If $\phi \in H[0, T]$,

$$ {}_0^{CF}D_\tau^r[\phi(\tau)] = \frac{M(r)}{1-r}\int_0^\tau (\phi(\tau)-\phi(\zeta))(\zeta)K(\tau,\zeta)d\zeta. $$

2.10 Definition

The Caputo–Fabrizio of fractional integral due to $r \in (0, 1)$ is defined as [1]

$$ {}_0^{CF}I_\tau^r[\phi(\tau)] = \frac{1-r}{M(r)}\phi(\tau) + \frac{r}{M(r)}\int_0^\tau \phi(\zeta)d\zeta, \quad \tau \geq 0. $$

2.11 Definition

The Laplace transformation of relation with Caputo–Fabrizio as [29]

$$ \pounds\left[{}_0^{CF}D_\tau^{r+M}[\phi(\tau)]\right] = \frac{1}{1-r}\left[\phi^{(m+r)(\tau)}\right]\pounds\left\{e^{\frac{-r\tau}{1-r}}\right\} = \frac{1}{s+r(1-r)}\left\{s^{m+1}\pounds(\phi(\tau)) + \sum_{k=1}^{m} s^{m-k}g^k(0)\right\}. $$

If $m = 0, 1,$

$$ \pounds\left[{}_0^{CF}D_\tau^{r}[\phi(\tau)]\right] = \frac{s\pounds\{\phi(\zeta)\}}{s+r(1-s)}, $$
$$ \pounds\left[{}_0^{CF}D_\tau^{r+1}[\phi(\tau)]\right] = \frac{s\pounds\{\phi(\zeta)\} + s\phi(0) - \phi'(0)}{s+r(1-s)}. $$

2.12 Definition

The Laplace transformation of the Caputo operators is provided by [29]

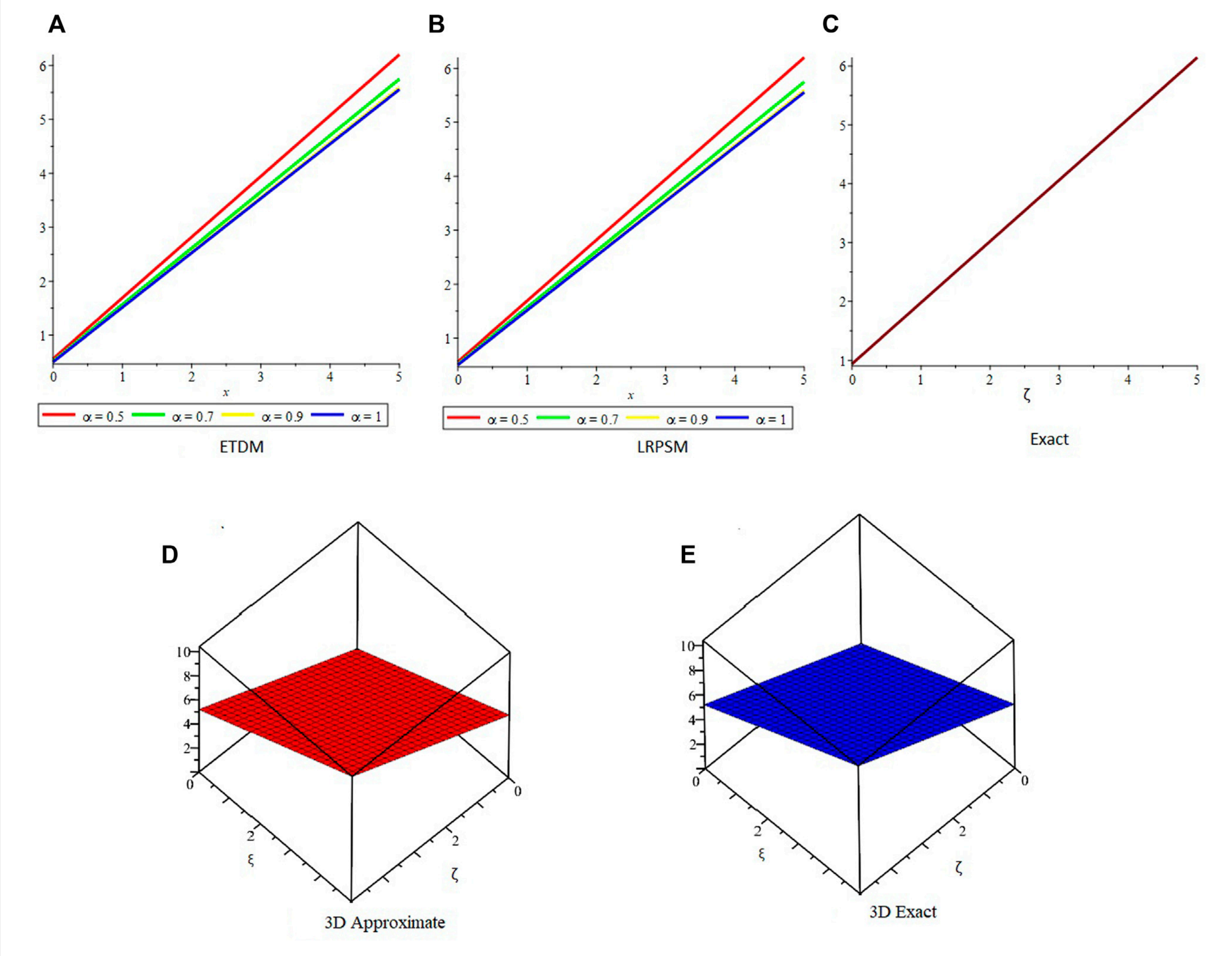

FIGURE 3 | The comparison of **(A)** ETDM, **(B)** LRPSM, and **(C)** exact 2D plots at different-fractional-order δ. **(D)** Approximate and **(E)** exact 3D plots of example 4.3 at δ = 1.

$$£\left[{}_{0}^{CF}D_{\tau}^{r}\left[\phi(\tau)\right]\right] = s£\{\phi(\zeta)\} - \sum_{k=0}^{\rho-1} s^{\rho-k-1}\phi^{k}(0).$$

2.13 Elzaki Transform

ET is the generalized form of the Sumudu transformation, which can be define as [72]

$$\varepsilon[f(\vartheta)] = F(q) = q\int_{0}^{\infty} f(\vartheta)e^{\frac{-\vartheta}{q}}d\vartheta, \quad \vartheta > 0.$$

The following are the results of ET for certain partial differential equations:

i. $\varepsilon\left[\frac{\partial f(\zeta,\vartheta)}{\partial\vartheta}\right] = \frac{1}{q}F(\zeta,q) - qf(\zeta,0).$

ii. $\varepsilon\left[\frac{\partial^2 f(\zeta,\vartheta)}{\partial\vartheta^2}\right] = \frac{1}{q^2}F(\zeta,q) - f(\zeta,0) - q\frac{\partial f(\zeta,0)}{\partial\vartheta}$

iii. $\varepsilon\left[\frac{\partial f(\zeta,\vartheta)}{\partial\zeta}\right] = \frac{d}{d\zeta}F(\zeta,q).$

iv. $\varepsilon\left[\frac{\partial^2 f(\zeta,\vartheta)}{\partial\zeta^2}\right] = \frac{d^2}{d\zeta^2}F(\zeta,q).$

3 METHODOLOGY OF LRPSM AND ETDM

3.1 Implementation of LRPSM

To understand the basic concept of this algorithm [70], we consider a particular non-linear FPDEs:

$$D_{\tau}^{\beta}\mu(\xi,\tau) = N(\mu) + L(\mu) + q(\xi,\tau), \quad \xi,\tau \geq 0, \quad m-1 < \beta < m, \tag{4}$$

where $D^{\beta}\mu(\xi,\tau)$ in **Eq. 4** is Caputo–Fabrizio derivatives, while $\mathcal{L}$ and $\mathcal{N}$ are the linear and non-linear operators, respectively, and $g(\zeta,\tau)$ is the source term.

The initial condition is given as follows:

$$\mu(\xi,0) = f(\xi).$$

Using the idea of RPSM, **Eq. 4** is written as the fractional power series at the initial point $\tau = 0$:

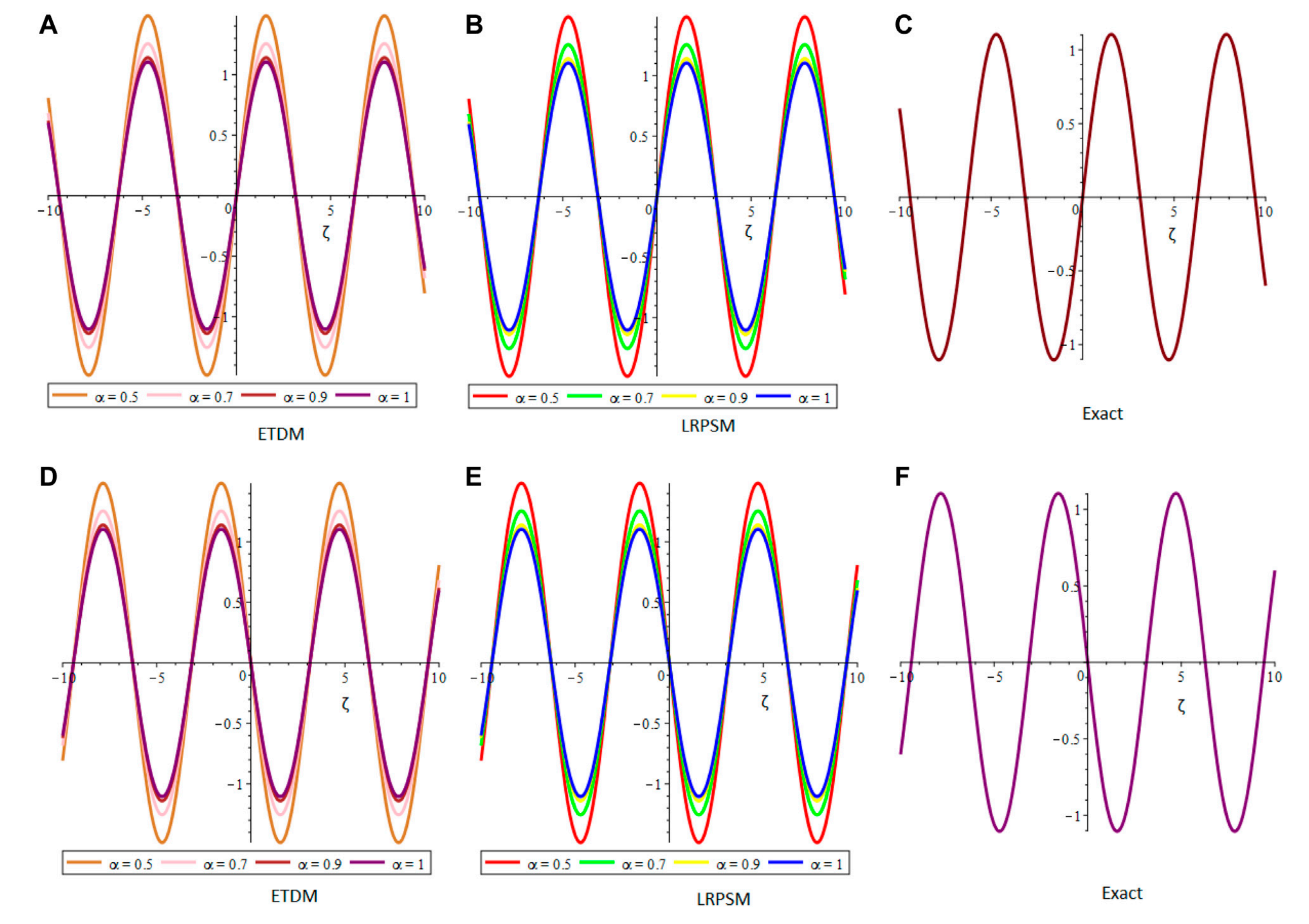

FIGURE 4 | The comparison of **(A,D)** ETDM, **(B,E)** LRPSM, and **(C,F)** exact 2D plots of μ and ν-solution, at different fractional orders of example 4.4.

$$\mu(\xi,\tau)=\sum_{n=0}^{\infty}\frac{f_n(\xi)\tau^{n\beta}}{\Gamma(1+n\beta)},$$

where $0<\beta\leq 1, -\infty<\xi<\infty$, and $0\leq\tau<R$.Let $\mu_k(\xi,\tau)$ be the kth truncated series of $\mu(\xi,\tau)$:

$$\mu_k(\xi,\tau)=\sum_{n=0}^{\infty}\frac{f_n(\xi)\tau^{n\beta}}{\Gamma(1+n\beta)}. \tag{5}$$

The 0*th* RPS approximation is

$$\mu_0(\xi,\tau)=\mu(\xi,0)=f(\xi).$$

Eq. 5 is written as

$$\mu_k(\xi,\tau)=f(\xi)+\sum_{n=0}^{\infty}\frac{f_n(\xi)\tau^{n\beta}}{\Gamma(1+n\beta)},\quad k=1,2,\cdots. \tag{6}$$

The residual function for **Eq. 4** is defined as

$$Res_{\mu}(\xi,\tau)=D_{\tau}^{\beta}\mu(\xi,\tau)-N(\mu)-L(\mu)+q(\xi,\tau).$$

Therefore, the *kth* residual function $Res_{\mu,k}$ is

$$Res_{\mu,k}(\xi,\tau)=D_{\tau}^{\beta}\mu_k(\xi,\tau)-L(\mu_k)-R(\mu_k). \tag{7}$$

If $Res\mu(\xi,\tau)=0$ and $\lim_{k\to\infty}Res_k(\xi,\tau)=Res(\xi,t)$, then $D_{\tau}^{n\beta}=0$ because using the Caputo sense, the fractional derivative of a constant is zero and the fractional derivatives $D_t^{n\beta}$ of $Res(\xi,t)$ and $Res_k(\xi,\tau)$ are same at $\tau=0$ for each $k=0,1,\cdot,k$; that is $D_{\tau}^{n\beta}Res(\xi,0)=D_t^{n\beta}Res_k(\xi,0),\ n=0,1,\ldots,k$. To find out $f_1(\xi)$, $f_2(\xi), f_3(\xi)\ldots$, let $k=0,1,\cdots$ in **Eq. 6** and substitute it into **Eq. 7**, applying fractional derivative $D_{\tau}^{K-1\beta}$ in both sides, $k=1,2,\cdots$, and finally we solve

$$D_{\tau}^{K-1\beta}Res_{\mu,k}(\xi,0)=0,\quad k=0,1,\cdots.$$

3.2 Implementation of ETDM

To understand the basic concept of this algorithm [30] , we consider particular non-linear FPDEs:

$$D^{\beta}\mu(\xi,\tau)+L\mu(\xi,\tau)+N\mu(\xi,\tau)=q(\xi,\tau),\quad \xi,\tau\geq 0,$$

$$m-1<\beta<m, \tag{8}$$

where $D^{\beta}\mu(\xi,\tau)$ in **Eq. 8** is Caputo–Fabrizio derivatives, while $\mathcal{L}$ and $\mathcal{N}$ are the linear and non-linear operators, respectively, and $g(\zeta,\tau)$ is the source term.The initial condition is as follows:

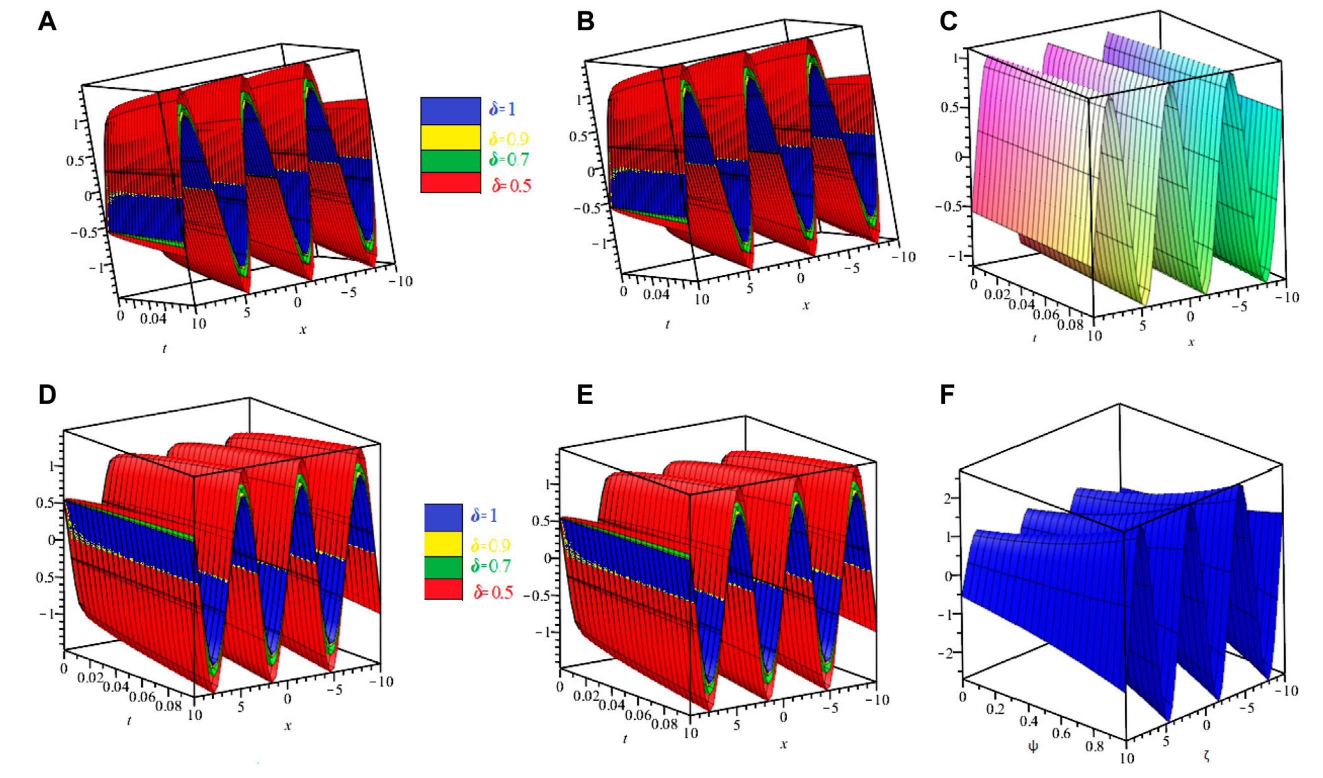

FIGURE 5 | The comparison of **(A,D)** ETDM, **(B,E)** LRPSM, and **(C,F)** exact 3D plots of μ and ν-solution, at different fractional orders of example 4.4.

$$\mu(\xi, 0) = f(\xi).$$

Taking ET to **Eq. 8**, we have

$$\mathcal{E}\left[D^{\beta}\mu(\xi,\tau)\right] + \mathcal{E}\left[L u(\xi,\tau) + N\mu(\xi,\tau)\right] = \mathcal{E}\left[q(\xi,\tau)\right].$$

With the help of fractional differential property of ET, we have

$$\frac{\mathcal{E}[\mu(\xi,\tau)]}{s^{\beta}} - \sum_{j=0}^{N-1} s^{2+j-\beta}\frac{\partial^k \mu(\xi,\tau)}{\partial^k \tau}\Big|_{\tau=0} = \mathcal{E}[q(\xi,\tau)] - \mathcal{E}[L\mu(\xi,\tau) + N\mu(\xi,\tau)],$$

$$\mathcal{E}\mu(\xi,\tau) = s^2\mu(\xi,0) + s^{\beta}[\mathcal{E}[q(\xi,\tau)]] - s^{\beta}[\mathcal{E}[L\mu(\xi,\tau) + N\mu(\xi,\tau)]],$$

and

$$\mathcal{E}[\mu(\xi,\tau)] = s^2 k(\xi) + s^{\beta}\mathcal{E}[q(\xi,\tau)] - s^{\beta}\mathcal{E}[L\mu(\xi,\tau) + N\mu(\xi,\tau)]. \tag{9}$$

Using ETDM procedure, the solution is expressed as

$$\mu(\xi,\tau) = \sum_{j=0}^{\infty}\mu_j(\xi,\tau). \tag{10}$$

The nonlinear term can be decompose as

$$N\mu(\xi,\tau) = \sum_{j=0}^{\infty} A_j, \tag{11}$$

$$A_j = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j}\left[N\sum_{j=0}^{\infty}\left(\lambda^j\mu_j\right)\right]\right]_{\lambda=0}, \qquad j = 0, 1, \cdots$$

By substituting **Eqs 10–11** in **Eq. 9**, we get

$$\mathcal{E}\left[\sum_{j=0}^{\infty}\mu(\xi,\tau)\right] = s^2 k(\xi) + s^{\beta}[\mathcal{E}[q(\xi,\tau)]] - s^{\beta}\left[\mathcal{E}\sum_{j=0}^{\infty}\mu_j(\xi,\tau) + \sum_{j=0}^{\infty}A_j\right],$$

$$\mathcal{E}[\mu_0(\xi,\tau)] = s^2\mu(\xi,0) + s^{\beta}\mathcal{E}[q(\xi,\tau)],$$

$$\mathcal{E}[\mu_1(\xi,\tau)] = -s^{\beta}[\mathcal{E}[\mu_0(\xi,\tau) + A_0]].$$

Generally, we can write

$$\mathcal{E}\left[\mu_{j+1}(\xi,\tau)\right] = -s^{\beta}\left[\mathcal{E}\left[\mu_j(\xi,\tau) + A_j\right]\right], \quad j \geq 1. \tag{12}$$

Taking the inverse ET of **Eq. 12**, we have

$$\mu_0(\xi,\tau) = f(\xi,\tau) + \mathcal{E}^{-1}\left[s^{\beta}\mathcal{E}[q(\xi,\tau)]\right],$$

$$\mu_{j+1}(\xi,\tau) = -\mathcal{E}^{-1}\left[s^{\beta}\mathcal{E}\left[\mu_j(\xi,\tau) + A_j\right]\right].$$

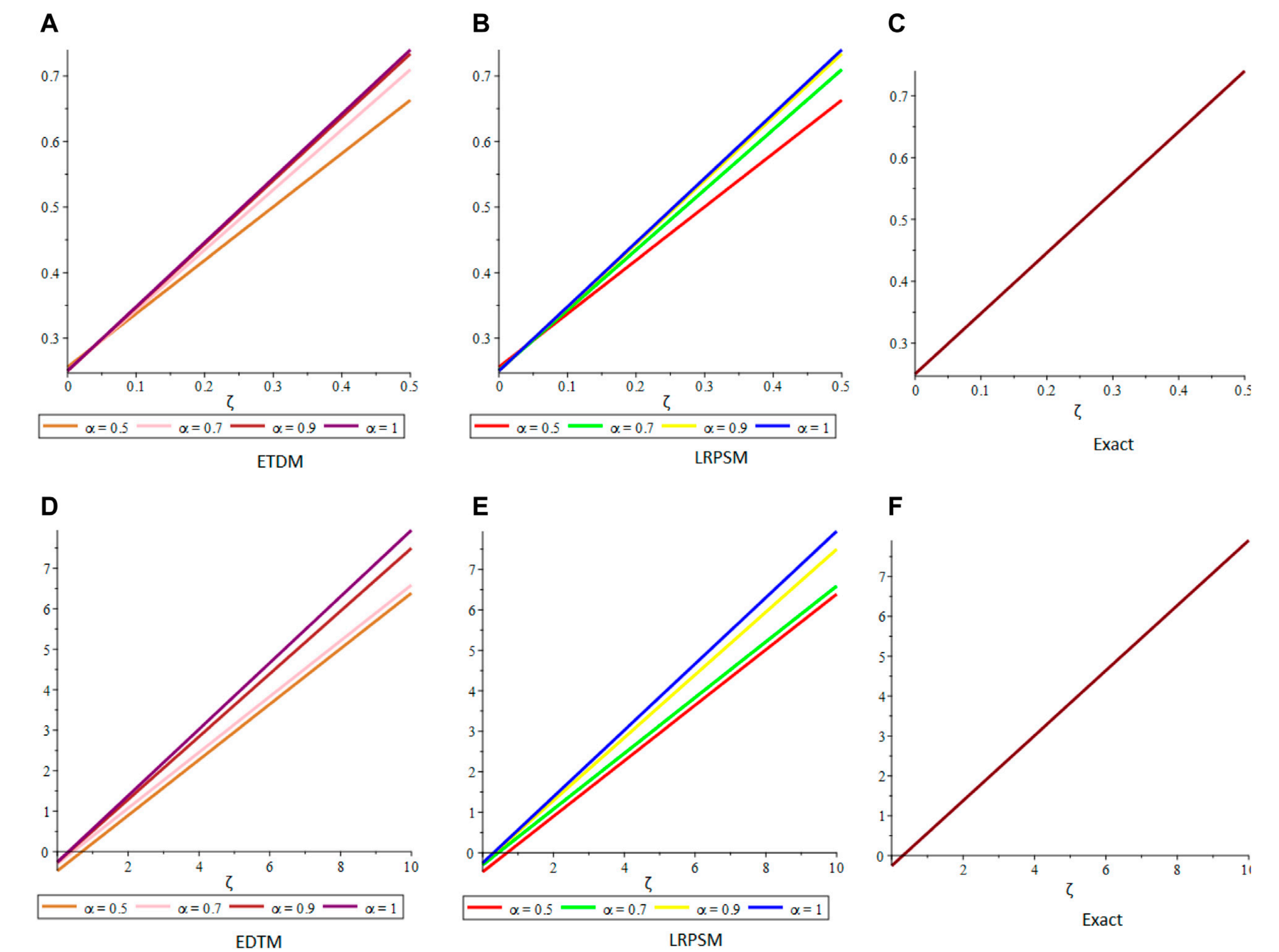

FIGURE 6 | The comparison of **(A,D)** ETDM, **(B,E)** LRPSM, and **(C,F)** exact 2D plots of μ and ν-solution, at different fractional orders of example 4.5.

4 NUMERICAL RESULTS

4.1 Example

Consider the following one-dimensional time-fractional-order Burger's equation:

$$D_\tau^\delta \mu = \mu_{\zeta\zeta} - \mu\mu_\zeta, \quad 0 < \delta \leq 1. \tag{13}$$

Subject to the initial condition,

$$\mu(\zeta, 0) = 2\zeta = f_0(\zeta). \tag{14}$$

The exact solution of **Eq. 13** is

$$\mu(\zeta, \tau) = \frac{2\zeta}{1 + 2\tau}. \tag{15}$$

4.1.1 Solution by LRPSM

Applying LT to **Eq. 13** and using the initial condition **Eq. 14**, we get

$$\mu(\zeta, s) = \frac{2\zeta}{s} - \frac{1}{s^\delta}\mathcal{L}_\tau\left[\left[\mathcal{L}_\tau^{-1}\mu(\zeta, s)\right]\left[\mathcal{L}_\tau^{-1}\mu_\zeta(\zeta, s)\right]\right] + \frac{\mu_{\zeta\zeta}(\zeta, s)}{s^\delta}. \tag{16}$$

The kth truncated term series of **Eq. 16** is

$$\mu_k(\zeta, s) = \frac{2\zeta}{s} + \sum_{n=1}^{k} \frac{f_n(\zeta)}{s^{n\delta+1}} \tag{17}$$

and the kth Laplace residual function is

$$\mathcal{L}_\tau Res_k(\zeta, s) = \mu(\zeta, s) - \frac{2\zeta}{s} + \frac{1}{s^\delta}\mathcal{L}_\tau\left[\left(\mathcal{L}_\tau^{-1}\mu(\zeta, s)\right)\left(\mathcal{L}_\tau^{-1}\mu_\zeta(\zeta, s)\right)\right] - \frac{\mu_{\zeta\zeta}(\zeta, s)}{s^\delta}. \tag{18}$$

Now, to determine $f_k(\zeta)$, $k = 1, 2, 3, \cdots$, we substitute the kth-truncated series **Eq. 17** into the kth-Laplace residual function **Eq. 18**, multiply the resulting equation by $s^{k\delta+1}$, and then solve recursively the relation $\lim_{s\to\infty}[s^{k\delta+1}Res_k$ $(\zeta,$

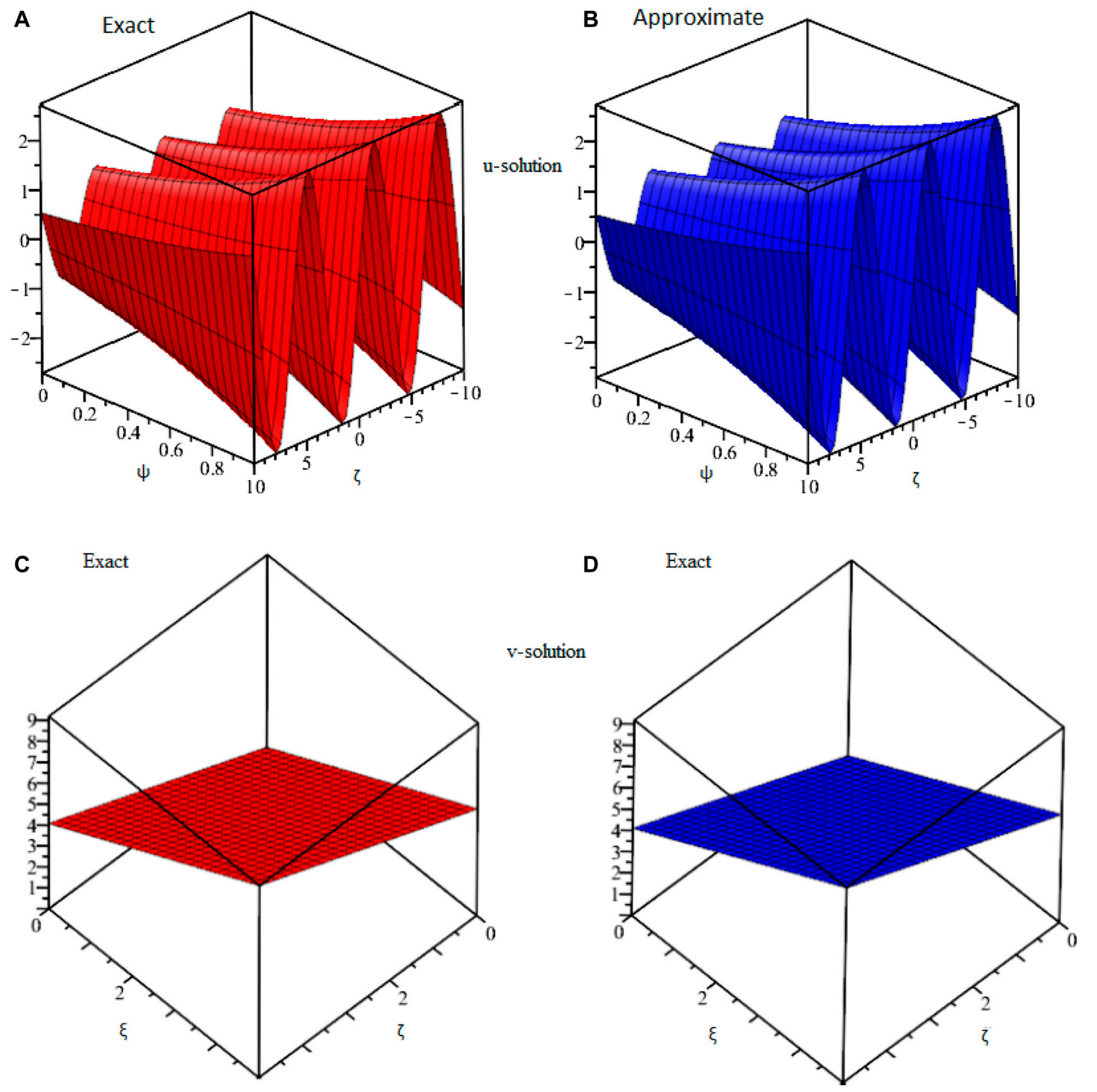

FIGURE 7 | The comparison of **(A,C)** exact and **(B,D)** approximate 3D plots of μ and ν-solution, respectively, at fractional-order $\delta = 1$ of example 4.5.

$s)] = 0$, $k = 1, 2, 3, \cdots$, for $f_k(\zeta)$. The following are the first few elements of the sequences $f_k(\zeta)$:

$$\begin{aligned} f_1(\zeta) &= -4\zeta, \\ f_2(\zeta) &= 16\zeta, \\ f_3(\zeta) &= 16\zeta\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2} - 64\zeta, \\ f_4(\zeta) &= \left[64\zeta\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2} + 128\zeta\frac{\Gamma(3\delta+1)}{\Gamma(\delta+1)\Gamma(2\delta+1)}\right], \\ &\vdots \end{aligned} \tag{19}$$

Putting the values of $f_n(\zeta)$ $(n \geq 1)$ in **Eq. 17**, we have

$$\begin{aligned} \mu(\zeta, s) = {} & \frac{2\zeta}{s} - \frac{4\zeta}{s^{\delta+1}} + \frac{16\zeta}{s^{2\delta+1}} + \left(16\zeta\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2} - 64\zeta\right)\frac{1}{s^{3\delta+1}} \\ & + \left(64\zeta\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2} + 128\zeta\frac{\Gamma(3\delta+1)}{\Gamma(\delta+1)\Gamma(2\delta+1)}\right)\frac{1}{s^{4\delta+1}} + \cdots, \end{aligned}$$

TABLE 1 | Comparison of LRPSM and ETDM errors at different time levels and spaces for example 4.1

τ	ζ	LRPSM error at $\delta = 0.7$	ETDM error at $\delta = 0.7$	LRPSM error at $\delta = 0.9$	ETDM error at $\delta = 0.9$	LRPSM error at $\delta = 1$	ETDM error at $\delta = 1$
0.01	0.2	$2.33171186 \times 10^{-2}$	$2.33171186 \times 10^{-2}$	4.8668611×10^{-3}	4.8668611×10^{-3}	2.1132×10^{-6}	2.1132×10^{-6}
	0.4	$4.66342370 \times 10^{-2}$	$4.66342370 \times 10^{-2}$	9.7337221×10^{-3}	9.7337221×10^{-3}	4.2266×10^{-6}	4.2266×10^{-6}
	0.6	6.9951355×10^{-2}	6.9951355×10^{-2}	1.4600583×10^{-2}	1.4600583×10^{-2}	6.340×10^{-6}	6.340×10^{-6}
	0.8	9.3268473×10^{-2}	9.3268473×10^{-2}	1.9467445×10^{-2}	1.9467445×10^{-2}	8.453×10^{-6}	8.453×10^{-6}
	1	0.116585593	0.116585593	2.4334306×10^{-2}	2.4334306×10^{-2}	1.0566×10^{-5}	1.0566×10^{-5}
0.2		$3.548017035 \times 10^{-2}$	$3.548017035 \times 10^{-2}$	$9.437110666 \times 10^{-3}$	$9.437110666 \times 10^{-3}$	$5.616540000 \times 10^{-5}$	$5.616540000 \times 10^{-5}$
0.4		$7.096034069 \times 10^{-2}$	$7.096034069 \times 10^{-2}$	$1.887422131 \times 10^{-2}$	$1.887422131 \times 10^{-2}$	$1.123308000 \times 10^{-4}$	$1.123308000 \times 10^{-4}$
0.03	0.6	0.1064405113	0.1064405113	$2.831133224 \times 10^{-2}$	$2.831133224 \times 10^{-2}$	$1.684960000 \times 10^{-4}$	$1.684960000 \times 10^{-4}$
	0.8	0.1419206810	0.1419206810	$3.774844227 \times 10^{-2}$	$3.774844227 \times 10^{-2}$	$2.246620000 \times 10^{-4}$	$2.246620000 \times 10^{-4}$
	1	0.1774008517	0.1774008517	$4.718555328 \times 10^{-2}$	$4.718555328 \times 10^{-2}$	$2.808270000 \times 10^{-4}$	$2.808270000 \times 10^{-4}$
0.2		$3.629466234 \times 10^{-2}$	$3.629466234 \times 10^{-2}$	$1.144271401 \times 10^{-2}$	$1.144271401 \times 10^{-2}$	$2.569697334 \times 10^{-4}$	$2.569697334 \times 10^{-4}$
0.4	$7.258932478 \times 10^{-2}$	$7.258932478 \times 10^{-2}$	$2.288542800 \times 10^{-2}$	$2.288542800 \times 10^{-2}$	$5.139394666 \times 10^{-4}$	$5.139394666 \times 10^{-4}$	
0.05	0.6	0.1088839873	0.1088839873	$3.432814224 \times 10^{-2}$	$3.432814224 \times 10^{-2}$	$7.709090000 \times 10^{-4}$	$7.709090000 \times 10^{-4}$
	0.8	0.1451786502	0.1451786502	$4.577085659 \times 10^{-2}$	$4.577085659 \times 10^{-2}$	$1.027878333 \times 10^{-3}$	$1.027878333 \times 10^{-3}$
	1	0.1814733119	0.1814733119	$5.721356993 \times 10^{-2}$	$5.721356993 \times 10^{-2}$	$1.284848666 \times 10^{-3}$	$1.284848666 \times 10^{-3}$

TABLE 2 | Comparison of LRPSM and ETDM errors of $\mu(\zeta, \tau)$ solution at different time levels and spaces of example 4.2

τ	ζ	LRPSM error at $\delta = 0.7$	ETDM error at $\delta = 0.7$	LRPSM error at $\delta = 0.9$	ETDM error at $\delta = 0.9$	LRPSM error at $\delta = 1$	ETDM error at $\delta = 1$
0.01	0.2	0.04218789287	0.04218789287	0.007840157192	0.007840157192	0.000060000000	0.000060000000
	0.4	0.04921920834	0.04921920834	0.009146850061	0.009146850061	0.000070000000	0.000070000000
	0.6	0.05625052383	0.05625052383	0.01045354292	0.01045354292	0.000080000000	0.000080000000
	0.8	0.06328183930	0.06328183930	0.01176023579	0.01176023579	0.000090000000	0.000090000000
	1	0.07031315477	0.07031315477	0.01306692865	0.01306692865	0.00010000000	0.00010000000
0.2		0.08562354719	0.08562354719	0.01747351782	0.01747351782	0.00054003000	0.00054003000
0.4		0.09989413839	0.09989413839	0.02038577080	0.02038577080	0.00063003500	0.00063003500
0.03	0.6	0.1141647296	0.1141647296	0.02329802378	0.02329802378	0.00072004000	0.00072004000
	0.8	0.1284353208	0.1284353208	0.02621027674	0.02621027674	0.00081004500	0.00081004500
	1	0.1427059120	0.1427059120	0.02912252971	0.02912252971	0.00090005000	0.00090005000
0.2	0.1202848422	0.1202848422	0.02482959536	0.02482959536	0.0015003950	0.0015003950	
0.4	0.1403323159	0.1403323159	0.02896786109	0.02896786109	0.0017504610	0.0017504610	
0.05	0.6	0.1603797904	0.1603797904	0.03310612781	0.03310612781	0.0020005260	0.0020005260
	0.8	0.1804272639	0.1804272639	0.03724439354	0.03724439354	0.0022505920	0.0022505920
	1	0.2004747375	0.2004747375	0.04138265927	0.04138265927	0.0025006580	0.0025006580

$$\mu(\zeta, s) = 2\zeta\left[\frac{1}{s} - \frac{2}{s^{\delta+1}} + \frac{8}{s^{2\delta+1}} + \left(8\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2} - 32\right)\frac{1}{s^{3\delta+1}} + \left(32\zeta\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2} + 64\zeta\frac{\Gamma(3\delta+1)}{\Gamma(\delta+1)\Gamma(2\delta+1)}\right)\frac{1}{s^{4\delta+1}} + \cdots\right].$$

Applying the inverse Laplace transform, we get

$$\mu(\zeta, \tau) = 2\zeta\left[1 - \frac{2\tau^{\delta}}{\Gamma(\delta+1)} + \frac{8\tau^{\delta}}{\Gamma(2\delta+1)} + \left(8\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2} - 32\right)\frac{\tau^{3\delta}}{\Gamma(3\delta+1)} + \left(32\zeta\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2} + 64\zeta\frac{\Gamma(3\delta+1)}{\Gamma(\delta+1)\Gamma(2\delta+1)}\right)\frac{\tau^{4\delta}}{\Gamma(4\delta+1)} + \cdots\right]. \tag{20}$$

Now, if we substitute $\delta = 1$ in **Eq. 20**, we have

$$\mu(\zeta, \tau) = 2\zeta\left[1 - 2\tau + 8\tau^2 - 16\tau^3 + \cdots\right]. \tag{21}$$

The results given in **Eq. 21** agree with the Maclaurin series of

$$\mu(\zeta, \tau) = \frac{2\zeta}{1 + 2\tau}. \tag{22}$$

4.1.2 Solution by ETDM

Applying ET to **Eq. 13** and using the initial condition **Eq. 14**, we get

$$\mathcal{E}\left[\frac{\partial^{\delta}\mu}{\partial\tau^{\delta}}\right] = \mathcal{E}\left[\mu_{\zeta\zeta} - \mu\mu_{\zeta}\right]. \tag{23}$$

With the help of fractional differential property of ET, we have

$$\mathcal{E}\left[\mu(\zeta, \tau)\right] = s^2\mu(\zeta, 0) + s^{\delta}\left\{\mathcal{E}\left(\frac{\partial^2\mu}{\partial\zeta^2} - \frac{\partial(\mu\nu)}{\partial\zeta}\right)\right\}. \tag{24}$$

Applying inverse ET to **Eq. 24**, we obtain

TABLE 3 | Comparison of LRPSM and ETDM errors of $\mu(\zeta, \tau)$ solution at different time levels and spaces of example 4.3

τ	ζ	LRPSM at $\delta = 0.7$	ETDM error at $\delta = 0.7$	LRPSM error at $\delta = 0.9$	ETDM error at $\delta = 0.9$	LRPSM error at $\delta = 1$	ETDM error at $\delta = 1$
0.01	0.2	$7.734447024 \times 10^{-2}$	$7.734447024 \times 10^{-2}$	$1.437362152 \times 10^{-2}$	$1.437362152 \times 10^{-2}$	$1.100000000 \times 10^{-4}$	$1.100000000 \times 10^{-4}$
	0.4	$8.437578569 \times 10^{-2}$	$8.437578569 \times 10^{-2}$	$1.568031438 \times 10^{-2}$	$1.568031438 \times 10^{-2}$	$1.200000000 \times 10^{-4}$	$1.200000000 \times 10^{-4}$
	0.6	$9.140710129 \times 10^{-2}$	$9.140710129 \times 10^{-2}$	$1.698700724 \times 10^{-2}$	$1.698700724 \times 10^{-2}$	$1.300000000 \times 10^{-4}$	$1.300000000 \times 10^{-4}$
	0.8	$9.843841670 \times 10^{-2}$	$9.843841670 \times 10^{-2}$	$1.829370012 \times 10^{-2}$	$1.829370012 \times 10^{-2}$	$1.400000000 \times 10^{-4}$	$1.400000000 \times 10^{-4}$
		0.1054697322	0.1054697322	$1.960039298 \times 10^{-2}$	$1.960039298 \times 10^{-2}$	$1.500000000 \times 10^{-4}$	$1.500000000 \times 10^{-4}$
1							
		0.1569765032	0.1569765032	$3.203478268 \times 10^{-2}$	$3.203478268 \times 10^{-2}$	$9.90055000 \times 10^{-4}$	$9.90055000 \times 10^{-4}$
0.2							
		0.1712470945	0.1712470945	$3.494703571 \times 10^{-2}$	$3.494703571 \times 10^{-2}$	$1.080060000 \times 10^{-3}$	$1.080060000 \times 10^{-3}$
0.4							
0.03	0.6	0.1855176856	0.1855176856	$3.785928864 \times 10^{-2}$	$3.785928864 \times 10^{-2}$	$1.170065000 \times 10^{-3}$	$1.170065000 \times 10^{-3}$
	0.8	0.1997882769	0.1997882769	$4.077154157 \times 10^{-2}$	$4.077154157 \times 10^{-2}$	$1.260070000 \times 10^{-3}$	$1.260070000 \times 10^{-3}$
	1	0.2140588680	0.2140588680	$4.368379451 \times 10^{-2}$	$4.368379451 \times 10^{-2}$	$1.350075000 \times 10^{-3}$	$1.350075000 \times 10^{-3}$
		0.2205222110	0.2205222110	$4.552092492 \times 10^{-2}$	$4.552092492 \times 10^{-2}$	$2.750724000 \times 10^{-3}$	$2.750724000 \times 10^{-3}$
0.2							
		0.2405696855	0.2405696855	$4.965919166 \times 10^{-2}$	$4.965919166 \times 10^{-2}$	$3.000789000 \times 10^{-3}$	$3.000789000 \times 10^{-3}$
0.4							
0.05	0.6	0.2606171592	0.2606171592	$5.379745739 \times 10^{-2}$	$5.379745739 \times 10^{-2}$	$3.250855000 \times 10^{-3}$	$3.250855000 \times 10^{-3}$
	0.8	0.2806646326	0.2806646326	$5.793572323 \times 10^{-2}$	$5.793572323 \times 10^{-2}$	$3.500921000 \times 10^{-3}$	$3.500921000 \times 10^{-3}$
	1	0.3007121061	0.3007121061	$6.207398887 \times 10^{-2}$	$6.207398887 \times 10^{-2}$	$3.750987000 \times 10^{-3}$	$3.750987000 \times 10^{-3}$

TABLE 4 | Comparison of LRPSM and ETDM errors of $\mu(\zeta, \tau)$ solution at different time levels and spaces of example 4.4

τ	ζ	ETDM error at $\delta = 0.7$	LRPSM error at $\delta = 0.7$	ETDM error at $\delta = 0.9$	LRPSM error at $\delta = 0.9$	ETDM error at $\delta = 1$	ETDM error at $\delta = 1$
0.01	0.2	$6.967048873 \times 10^{-3}$	$6.967048873 \times 10^{-3}$	$1.307167615 \times 10^{-3}$	$1.307167615 \times 10^{-3}$	$3.112598217 \times 10^{-11}$	$3.112598217 \times 10^{-11}$
		$1.365634355 \times 10^{-2}$	$1.365634355 \times 10^{-2}$	$2.562222648 \times 10^{-3}$	$2.562222648 \times 10^{-3}$	$5.4346926 \times 10^{-12}$	$5.4346926 \times 10^{-12}$
0.4							
		$1.980120287 \times 10^{-2}$	$1.980120287 \times 10^{-2}$	$3.715129907 \times 10^{-3}$	$3.715129907 \times 10^{-3}$	1.65972×10^{-14}	1.65972×10^{-14}
0.6							
		$8.148448015 \times 10^{-2}$	$8.148448015 \times 10^{-2}$	$2.515665080 \times 10^{-2}$	$2.515665080 \times 10^{-2}$	$4.719926745 \times 10^{-13}$	$4.719926745 \times 10^{-13}$
0.8							
		$2.950918241 \times 10^{-2}$	$2.950918241 \times 10^{-2}$	$5.536554941 \times 10^{-3}$	$5.536554941 \times 10^{-3}$	$9.30170103 \times 10^{-11}$	$9.30170103 \times 10^{-11}$
1							
		$1.397057061 \times 10^{-2}$	$1.397057061 \times 10^{-2}$	$2.968159461 \times 10^{-3}$	$2.968159461 \times 10^{-3}$	$6.0071388 \times 10^{-11}$	$6.0071388 \times 10^{-11}$
0.2							
		$2.738417871 \times 10^{-2}$	$2.738417871 \times 10^{-2}$	$5.817987829 \times 10^{-3}$	$5.817987829 \times 10^{-3}$	5.059095×10^{-10}	5.059095×10^{-10}
0.4							
		$3.970606595 \times 10^{-2}$	$3.970606595 \times 10^{-2}$	$8.435871292 \times 10^{-3}$	$8.435871292 \times 10^{-3}$	$1.3918652 \times 10^{-10}$	$1.3918652 \times 10^{-10}$
0.6							
0.03	0.8	$5.044499763 \times 10^{-2}$	$5.044499763 \times 10^{-2}$	$1.071744320 \times 10^{-2}$	$1.071744320 \times 10^{-2}$	$2.1592393 \times 10^{-10}$	$2.1592393 \times 10^{-10}$
		$5.917284653 \times 10^{-2}$	$5.917284653 \times 10^{-2}$	$1.257174454 \times 10^{-2}$	$1.257174454 \times 10^{-2}$	$1.9772226 \times 10^{-10}$	$1.9772226 \times 10^{-10}$
1							
		$5.179361068 \times 10^{-2}$	$5.179361068 \times 10^{-2}$	$1.925845704 \times 10^{-2}$	$1.925845704 \times 10^{-2}$	$4.1530308 \times 10^{-10}$	$4.1530308 \times 10^{-10}$
0.2							
		$3.774914013 \times 10^{-2}$	$3.774914013 \times 10^{-2}$	$8.436763988 \times 10^{-3}$	$8.436763988 \times 10^{-3}$	8.589414×10^{-10}	8.589414×10^{-10}
0.4							
0.05	0.6	$5.473488408 \times 10^{-2}$	$5.473488408 \times 10^{-2}$	$1.223300175 \times 10^{-2}$	$1.223300175 \times 10^{-2}$	1.3108292×10^{-9}	1.3108292×10^{-9}
	0.8	$6.953852103 \times 10^{-2}$	$6.953852103 \times 10^{-2}$	$1.554154844 \times 10^{-2}$	$1.554154844 \times 10^{-2}$	1.6164580×10^{-9}	1.6164580×10^{-9}
	1	$8.156987661 \times 10^{-2}$	$8.156987661 \times 10^{-2}$	$1.823050263 \times 10^{-2}$	$1.823050263 \times 10^{-2}$	1.8504110×10^{-9}	1.8504110×10^{-9}

TABLE 5 | Comparison of LRPSM and ETDM errors of $\nu(\zeta, \tau)$ solution at different time levels and spaces of example 4.4

τ	ζ	ETDM error at δ = 0.7	LRPSM error at δ = 0.7	ETDM error at δ = 0.9	LRPSM error at δ = 0.9	ETDM error at δ = 1	ETDM error at δ = 1
0.01	0.2	$6.967048873 \times 10^{-3}$	$6.967048873 \times 10^{-3}$	$1.307167615 \times 10^{-3}$	$1.307167615 \times 10^{-3}$	$3.112598217 \times 10^{-11}$	$3.112598217 \times 10^{-11}$
	0.4	$1.365634355 \times 10^{-2}$	$1.365634355 \times 10^{-2}$	$2.562222648 \times 10^{-3}$	$2.562222648 \times 10^{-3}$	$5.4346926 \times 10^{-12}$	$5.4346926 \times 10^{-12}$
	0.6	$1.980120287 \times 10^{-2}$	$1.980120287 \times 10^{-2}$	$3.715129907 \times 10^{-3}$	$3.715129907 \times 10^{-3}$	1.65972×10^{-14}	1.65972×10^{-14}
	0.8	$2.515665080 \times 10^{-2}$	$2.515665080 \times 10^{-2}$	$4.719926745 \times 10^{-3}$	$4.719926745 \times 10^{-3}$	$7.17868712 \times 10^{-11}$	$7.17868712 \times 10^{-11}$
1		$2.950918241 \times 10^{-2}$	$2.950918241 \times 10^{-2}$	$5.536554941 \times 10^{-3}$	$5.536554941 \times 10^{-3}$	$9.30170103 \times 10^{-11}$	$9.30170103 \times 10^{-11}$
0.2		$1.397057061 \times 10^{-2}$	$1.397057061 \times 10^{-2}$	$2.968159461 \times 10^{-3}$	$2.968159461 \times 10^{-3}$	$6.0071388 \times 10^{-11}$	$6.0071388 \times 10^{-11}$
0.4		$2.738417871 \times 10^{-2}$	$2.738417871 \times 10^{-2}$	$5.817987829 \times 10^{-3}$	$5.817987829 \times 10^{-3}$	5.059095×10^{-11}	5.059095×10^{-11}
0.6	$0.03970606595 \times 10^{-2}$	$0.03970606595 \times 10^{-2}$	$8.435871292 \times 10^{-3}$	$8.435871292 \times 10^{-3}$	$1.3918652 \times 10^{-10}$	$1.3918652 \times 10^{-10}$	
0.03	0.8	$5.044499763 \times 10^{-2}$	$5.044499763 \times 10^{-2}$	$1.071744320 \times 10^{-2}$	$1.071744320 \times 10^{-2}$	$2.1592393 \times 10^{-10}$	$2.1592393 \times 10^{-10}$
	1	$5.917284653 \times 10^{-2}$	$5.917284653 \times 10^{-2}$	$1.257174454 \times 10^{-2}$	$1.257174454 \times 10^{-2}$	$1.9772226 \times 10^{-10}$	$1.9772226 \times 10^{-10}$
0.2		$1.925845704 \times 10^{-2}$	$1.925845704 \times 10^{-2}$	$4.304179040 \times 10^{-3}$	$4.304179040 \times 10^{-3}$	$4.1530308 \times 10^{-10}$	$4.1530308 \times 10^{-10}$
0.4		$3.774914013 \times 10^{-2}$	$3.774914013 \times 10^{-2}$	$8.436763988 \times 10^{-3}$	$8.436763988 \times 10^{-3}$	8.589414×10^{-10}	8.589414×10^{-10}
0.05	0.6	$5.473488408 \times 10^{-2}$	$5.473488408 \times 10^{-2}$	$1.223300175 \times 10^{-2}$	$1.223300175 \times 10^{-2}$	1.3108292×10^{-9}	1.3108292×10^{-9}
	0.8	$6.953852103 \times 10^{-2}$	$6.953852103 \times 10^{-2}$	$1.554154844 \times 10^{-2}$	$1.554154844 \times 10^{-2}$	1.6164580×10^{-9}	1.6164580×10^{-9}
	1	$8.156987661 \times 10^{-2}$	$8.156987661 \times 10^{-2}$	$1.823050263 \times 10^{-2}$	$1.823050263 \times 10^{-2}$	1.8504110×10^{-9}	1.8504110×10^{-9}

TABLE 6 | Comparison of LRPSM and ETDM errors of $\mu(\zeta, \tau)$ solution at different time levels and spaces of example 4.5

τ	ζ	ETDM error at δ = 0.7	LRPSM error at δ = 0.7	ETDM error at δ = 0.9	LRPSM error at δ = 0.9	ETDM error at δ = 1	LRPSM error at δ = 1
0.01	0.2	$8.272484336 \times 10^{-3}$	$8.272484336 \times 10^{-3}$	$2.133853821 \times 10^{-3}$	$2.133853821 \times 10^{-3}$	$3.248000000 \times 10^{-6}$	$3.248000000 \times 10^{-6}$
	0.4	$2.081639886 \times 10^{-2}$	$2.081639886 \times 10^{-2}$	$4.644794801 \times 10^{-3}$	$4.644794801 \times 10^{-3}$	$2.456000000 \times 10^{-6}$	$2.456000000 \times 10^{-6}$
	0.6	$3.336031344 \times 10^{-2}$	$3.336031344 \times 10^{-2}$	$7.155735788 \times 10^{-3}$	$7.155735788 \times 10^{-3}$	$1.664000000 \times 10^{-6}$	$1.664000000 \times 10^{-6}$
	0.8	$4.590422697 \times 10^{-2}$	$4.590422697 \times 10^{-2}$	$9.666675784 \times 10^{-3}$	$9.666675784 \times 10^{-3}$	$8.71000000 \times 10^{-7}$	$8.71000000 \times 10^{-7}$
		$5.844814150 \times 10^{-2}$	$5.844814150 \times 10^{-2}$	$1.217761675 \times 10^{-2}$	$1.217761675 \times 10^{-2}$	7.9000000×10^{-8}	7.9000000×10^{-8}
1							
		$5.794293003 \times 10^{-3}$	$5.794293003 \times 10^{-3}$	$3.094741281 \times 10^{-3}$	$3.094741281 \times 10^{-3}$	$9.02560000 \times 10^{-5}$	$9.02560000 \times 10^{-5}$
0.2							
		$2.719536302 \times 10^{-2}$	$2.719536302 \times 10^{-2}$	$8.285164133 \times 10^{-3}$	$8.285164133 \times 10^{-3}$	$6.92660000 \times 10^{-5}$	$6.92660000 \times 10^{-5}$
0.4							
0.03	0.6	$4.859643309 \times 10^{-2}$	$4.859643309 \times 10^{-2}$	$1.347558698 \times 10^{-2}$	$1.347558698 \times 10^{-2}$	$4.82760000 \times 10^{-5}$	$4.82760000 \times 10^{-5}$
	0.8	$6.999750410 \times 10^{-2}$	$6.999750410 \times 10^{-2}$	$1.866601083 \times 10^{-2}$	$1.866601083 \times 10^{-2}$	$2.72870000 \times 10^{-5}$	$2.72870000 \times 10^{-5}$
		$9.139857406 \times 10^{-2}$	$9.139857406 \times 10^{-2}$	$2.385643368 \times 10^{-2}$	$2.385643368 \times 10^{-2}$	6.2970000×10^{-6}	6.2970000×10^{-6}
1							
		2.6897470×10^{-4}	2.6897470×10^{-4}	$2.668568330 \times 10^{-3}$	$2.668568330 \times 10^{-3}$	$4.296480000 \times 10^{-4}$	$4.296480000 \times 10^{-4}$
0.2							
		$2.552687900 \times 10^{-2}$	$2.552687900 \times 10^{-2}$	$9.459664710 \times 10^{-3}$	$9.459664710 \times 10^{-3}$	$3.341710000 \times 10^{-4}$	$3.341710000 \times 10^{-4}$
0.4							
0.05	0.6	$5.078478230 \times 10^{-2}$	$5.078478230 \times 10^{-2}$	$1.625076012 \times 10^{-2}$	$1.625076012 \times 10^{-2}$	$2.386930000 \times 10^{-4}$	$2.386930000 \times 10^{-4}$
	0.8	$7.604268660 \times 10^{-2}$	$7.604268660 \times 10^{-2}$	$2.304185648 \times 10^{-2}$	$2.304185648 \times 10^{-2}$	$1.432160000 \times 10^{-4}$	$1.432160000 \times 10^{-4}$
	1	0.1013005908	0.1013005908	$0.02983295289 \times 10^{-2}$	$0.02983295289 \times 10^{-2}$	$4.77390000 \times 10^{-5}$	$4.77390000 \times 10^{-5}$

$$\mu(\zeta,\tau) = \mu(\zeta,0) + \mathcal{E}^{-}\left[s^{\delta}\left\{\mathcal{E}\left(\frac{\partial^2\mu}{\partial\zeta^2} - \frac{\partial(\mu\nu)}{\partial\zeta}\right)\right\}\right]. \tag{25}$$

Using the ADM procedure, we get

$$\sum_{j=0}^{\infty}\mu_j(\zeta,\tau) = 2(\zeta) - \mathcal{E}^{-1}\left[s^{\delta}\mathcal{E}\left\{\sum_{j=0}^{\infty}\left(\mu_{\zeta\zeta}\right)_j - \sum_{j=0}^{\infty}A_j(\mu\nu)_{\zeta}\right\}\right]. \tag{26}$$

The nonlinear term is represented by the Adomian polynomial $A_j(\mu\nu)_\zeta$, where

$$\begin{aligned}
A_0(\mu\nu)_\zeta &= \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta},\\
A_1(\mu\nu)_\zeta &= \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_1}{\partial\zeta} + \frac{\partial\mu_1}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta},\\
A_2(\mu\nu)_\zeta &= \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_2}{\partial\zeta} + \frac{\partial\mu_1}{\partial\zeta}\frac{\partial\nu_1}{\partial\zeta} + \frac{\partial\mu_2}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta}.
\end{aligned} \tag{27}$$

By applying decomposition procedure,

$$\mu_0(\zeta,\tau) = 2\zeta. \tag{28}$$

A general solution of ETDM is given by

TABLE 7 | Comparison of LRPSM and ETDM errors of $\nu(\zeta, \tau)$ solution at different time levels and spaces of example 4.5

τ	ζ	ETDM error at $\delta = 0.7$	LRPSM error at $\delta = 0.7$	ETDM error at $\delta = 0.9$	LRPSM error at $\delta = 0.9$	ETDM error at $\delta = 1$	LRPSM error at $\delta = 1$
	0.2	$1.807966439 \times 10^{-2}$	$1.807966439 \times 10^{-2}$	$2.932413956 \times 10^{-3}$	$2.932413956 \times 10^{-3}$	$3.168000000 \times 10^{-6}$	$3.168000000 \times 10^{-6}$
	0.4	$3.062357892 \times 10^{-2}$	$3.062357892 \times 10^{-2}$	$5.443354937 \times 10^{-3}$	$5.443354937 \times 10^{-3}$	$2.376000000 \times 10^{-6}$	$2.376000000 \times 10^{-6}$
	0.6	$4.316749349 \times 10^{-2}$	$4.316749349 \times 10^{-2}$	$7.954295923 \times 10^{-3}$	$7.954295923 \times 10^{-3}$	$1.584000000 \times 10^{-6}$	$1.584000000 \times 10^{-6}$
	0.8	$5.571140703 \times 10^{-2}$	$5.571140703 \times 10^{-2}$	$1.046523592 \times 10^{-2}$	$1.046523592 \times 10^{-2}$	$7.91000000 \times 10^{-7}$	$7.91000000 \times 10^{-7}$
	1	$6.825532156 \times 10^{-2}$	$6.825532156 \times 10^{-2}$	$1.297617689 \times 10^{-2}$	$1.297617689 \times 10^{-2}$	1.000000×10^{-9}	1.000000×10^{-9}
	0.2	$4.970807503 \times 10^{-2}$	$4.970807503 \times 10^{-2}$	$8.148033567 \times 10^{-3}$	$8.148033567 \times 10^{-3}$	$8.37640000 \times 10^{-5}$	$8.37640000 \times 10^{-5}$
	0.4	$7.110914605 \times 10^{-2}$	$7.110914605 \times 10^{-2}$	$1.333845742 \times 10^{-2}$	$1.333845742 \times 10^{-2}$	$6.27750000 \times 10^{-5}$	$6.27750000 \times 10^{-5}$
	0.6	$9.251021610 \times 10^{-2}$	$9.251021610 \times 10^{-2}$	$1.852888027 \times 10^{-2}$	$1.852888027 \times 10^{-2}$	$4.17850000 \times 10^{-5}$	$4.17850000 \times 10^{-5}$
	0.8	0.1139112861	0.1139112861	$2.371930312 \times 10^{-2}$	$2.371930312 \times 10^{-2}$	$2.07950000 \times 10^{-5}$	$2.07950000 \times 10^{-5}$
	1	0.1353123561	0.1353123561	$2.890972597 \times 10^{-2}$	$2.890972597 \times 10^{-2}$	1.950000×10^{-7}	1.950000×10^{-7}
	0.2	$0.08737417775 \times 10^{-2}$	$0.08737417775 \times 10^{-2}$	$1.433706719 \times 10^{-3}$	$1.433706719 \times 10^{-3}$	$3.793970000 \times 10^{-4}$	$3.793970000 \times 10^{-4}$
	0.4	0.1126320821	0.1126320821	$2.112816357 \times 10^{-3}$	$2.112816357 \times 10^{-3}$	$2.112816357 \times 10^{-3}$	$2.112816357 \times 10^{-3}$
	0.6	0.1378899853	0.1378899853	$2.791925898 \times 10^{-3}$	$2.791925898 \times 10^{-3}$	$1.884420000 \times 10^{-4}$	$1.884420000 \times 10^{-4}$
	0.8	0.1631478896	0.1631478896	$3.471035534 \times 10^{-3}$	$3.471035534 \times 10^{-3}$	$9.29650000 \times 10^{-5}$	$9.29650000 \times 10^{-5}$
	1	0.1884057928	0.1884057928	$4.150145075 \times 10^{-3}$	$4.150145075 \times 10^{-3}$	2.5130000×10^{-6}	2.5130000×10^{-6}

$$\mu_{j+1}(\zeta,\tau) = -\mathcal{E}^{-1}\left[s^{\delta}\mathcal{E}\left\{\sum_{j=0}^{\infty}\left(\mu_{\zeta\zeta}\right)_j - \sum_{j=0}^{\infty}A_j\left(\mu\nu\right)_{\zeta}\right\}\right], \quad j = 0, 1\cdots, \tag{29}$$

$$\mu(\zeta,\tau) = 2\zeta\left[1 - \frac{2\tau^{\delta}}{\Gamma(\delta+1)} + \frac{8\tau^{\delta}}{\Gamma(2\delta+1)} + \left(8\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2} - 32\right)\frac{\tau^{3\delta}}{\Gamma(3\delta+1)} + \left(32\zeta\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2} + 64\zeta\frac{\Gamma(3\delta+1)}{\Gamma(\delta+1)\Gamma(2\delta+1)}\right)\frac{\tau^{4\delta}}{\Gamma(4\delta+1)} + \cdots\right]. \tag{30}$$

If $\delta = 1$, then **Eq. 30** gives

$$\mu(\zeta,\tau) = 2\zeta\left[1 - 2\tau + 8\tau^2 - 16\tau^3 + \cdots\right], \tag{31}$$

$$\mu(\zeta,\tau) = \frac{2\zeta}{1+2\tau},$$

which is an exact solution.

4.2 Example

Consider the two-dimensional time-fractional-order Burger's equation:

$$D_{\tau}^{\delta}\mu = \mu\mu_{\zeta} + \mu_{\zeta\zeta} + \mu_{\xi\xi}, \quad 0 < \delta \leq 1, \tag{32}$$

subject to the IC

$$\mu(\zeta,\xi,0) = \zeta + \xi = f_0(\zeta). \tag{33}$$

The exact solution of **Eq. 32** is

$$\mu(\zeta,\xi,t) = \frac{\zeta+\xi}{1-\tau}. \tag{34}$$

4.2.1 Solution by LRPSM

Applying LT to **Eq. 32** and using the IC of **Eq. 33**, we get

$$\mu(\zeta,s) = \frac{\zeta+\xi}{s} + \frac{1}{s^{\delta}}\mathcal{L}_{\tau}\left[\left(\mathcal{L}_{\tau}^{-1}\left\{\mu_{\zeta}(\zeta,\xi,s)\right\}\right)\left(\mathcal{L}_{\tau}^{-1}\left\{\mu_{\zeta}(\zeta,\xi,s)\right\}\right)\right] + \frac{\mu_{\zeta\zeta}(\zeta,\xi,s)}{s^{\delta}} + \frac{\mu_{\xi\xi}(\zeta,\xi,s)}{s^{\delta}}. \tag{35}$$

The k-th truncated term series of **Eq. 35** is

$$\mu_k(\zeta,\xi,s) = \frac{\zeta+\xi}{s} + \sum_{n=1}^{k}\frac{f_n(\zeta)}{s^{n\delta+1}} \tag{36}$$

and the kth Laplace residual function is

$$\mathcal{L}_{\tau}Res_k(\zeta,\xi,s) = \mu_k(\zeta,\xi,s) - \frac{\zeta+\xi}{s} - \frac{1}{s^{\delta}}\mathcal{L}_{\tau}\left[\left(\mathcal{L}_{\tau}^{-1}\left\{\mu_{\zeta}(\zeta,\xi,s)\right\}\right)\left(\mathcal{L}_{\tau}^{-1}\left\{\mu_{\zeta}(\zeta,\xi,s)\right\}\right)\right] - \frac{\mu_{\zeta\zeta}(\zeta,\xi,s)}{s^{\delta}} - \frac{\mu_{\xi\xi}(\zeta,\xi,s)}{s^{\delta}}. \tag{37}$$

Now, to determine $f_k(\zeta,\xi)$, $k = 1, 2, \cdots$, we substitute the kth-truncated series (**Eq. 36**) into the kth-Laplace residual function (**Eq. 37**), multiply the resulting equation by $s^{k\delta+1}$, and then solve recursively the relation $\lim_{s\to\infty}[s^{k\delta+1}Res_k(\zeta,\xi,s)] = 0$, $k = 1, 2, \cdots$, for f_k. The following are the first few elements of the sequences $f_k(\zeta,\xi)$

$$\begin{aligned} f_1(\zeta,\xi) &= (\zeta+\xi), \\ f_2(\zeta,\xi) &= 2(\zeta+\xi), \\ f_3(\zeta,\xi) &= (\zeta+\xi)\left[4 + \frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}\right], \\ f_4(\zeta,\xi) &= 2(\zeta+\xi)\left[4 + \frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2} + \frac{2\Gamma(1+3\delta)}{\Gamma(1+\delta)\Gamma(1+2\delta)}\right], \\ &\vdots \end{aligned} \tag{38}$$

Putting the values of $f_n(\zeta,\xi)$ $(n \geq 1)$ in **Eq. 36**, we have

$$\mu(\zeta,\xi,s) = \frac{\zeta+\xi}{s} + \frac{(\zeta+\xi)}{s^{\delta+1}} + \frac{2(\zeta+\xi)}{s^{2\delta+1}} + \frac{(\zeta+\xi)\left[4 + \frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}\right]}{s^{3\delta+1}} + \frac{(\zeta+\xi)\left[4 + \frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2} + \frac{2\Gamma(1+3\delta)}{\Gamma(1+\delta)\Gamma(1+2\delta)}\right]}{s^{4\delta+1}} + \cdots,$$

$$\mu(\zeta,\xi,s)=(\zeta+\xi)\left[\frac{1}{s}+\frac{1}{s^{\delta+1}}+\frac{2}{s^{2\delta+1}}+\frac{\left[4+\frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}\right]}{s^{3\delta+1}}+\frac{\left[4+\frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}+\frac{2\Gamma(1+3\delta)}{\Gamma(1+\delta)\Gamma(1+2\delta)}\right]}{s^{4\delta+1}}+\cdots\right].$$

Applying the inverse Laplace transform, we get

$$\mu(\zeta,\xi,\tau)=(\zeta+\xi)\left[1+\frac{\tau^\delta}{\Gamma(\delta+1)}+\frac{2\tau^\delta}{\Gamma(2\delta+1)}+\frac{\left[4+\frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}\right]}{\Gamma(3\delta+1)}\tau^{3\delta}+\frac{\left[4+\frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}+\frac{2\Gamma(1+3\delta)}{\Gamma(1+\delta)\Gamma(1+2\delta)}\right]}{\Gamma(4\delta+1)}\tau^{4\delta}+\cdots\right]. \tag{39}$$

Now, if we substitute $\delta = 1$ in **Eq. 39**, we have

$$\mu(\zeta,\xi,\tau)=(\zeta+\xi)\left[1+\tau+\frac{\tau^2}{2!}+\frac{\tau^3}{3!}+\frac{\tau^4}{4!}+\cdots\right]. \tag{40}$$

The results given in **Eq. 40** agree with the Maclaurin series of

$$\mu(\zeta,\xi,t)=\frac{\zeta+\xi}{1-\tau}. \tag{41}$$

4.2.2 Solution by ETDM

Taking ET of **Eq. 32**,

$$\mathcal{E}\left[\frac{\partial^\delta\mu}{\partial\tau^\delta}\right]=-\mathcal{E}\left[\mu\frac{\partial\mu}{\partial\zeta}+\frac{\partial^2\mu}{\partial\zeta^2}+\frac{\partial^2\mu}{\partial\xi^2}\right],$$

$$\frac{\mathcal{E}[\mu(\zeta,\xi,\tau)]-s^2\mu(\zeta,\xi,0)}{s^\delta}=-\mathcal{E}\left[\mu\frac{\partial\mu}{\partial\zeta}+\frac{\partial^2\mu}{\partial\zeta^2}+\frac{\partial^2\mu}{\partial\xi^2}\right].$$

Applying the inverse ET,

$$\mu(\zeta,\xi,\tau)=\mathcal{E}^{-1}\left[s^2\mu(\zeta,\xi,0)-s^\delta\mathcal{E}\left\{\mu\frac{\partial\mu}{\partial\zeta}+\frac{\partial^2\mu}{\partial\zeta^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}\right],$$

$$\mu(\zeta,\xi,\tau)=\zeta+\xi-\mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\mu\frac{\partial\mu}{\partial\zeta}+\frac{\partial^2\mu}{\partial\zeta^2}+\frac{\partial^2\mu}{\partial\xi^2}\right\}\right].$$

Using the ADM procedure, we get

$$\sum_{j=0}^{\infty}\mu_j(\zeta,\xi,\tau)=\zeta+\xi-\mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\sum_{j=0}^{\infty}A_j(\mu\mu_\zeta)+\sum_{j=0}^{\infty}\mu_{\zeta\zeta}+\sum_{j=0}^{\infty}\mu_{\xi\xi}\right\}\right],$$

where A_j $(\mu\mu_\zeta)$, and the Adomian polynomials are given as follows:

$$A_0(\mu\mu_\zeta)=\mu_0\frac{\partial\mu_0}{\partial\zeta},$$
$$A_1(\mu\mu_\zeta)=\mu_0\frac{\partial\mu_1}{\partial\zeta}+\mu_1\frac{\partial\mu_0}{\partial\zeta},$$
$$A_2(\mu\mu_\zeta)=\mu_0\frac{\partial\mu_2}{\partial\zeta}+\mu_1\frac{\partial\mu_1}{\partial\zeta}+\mu_2\frac{\partial\mu_0}{\partial\zeta}.$$

$$\mu_0(\zeta,\xi,\tau)=\zeta+\xi,\quad \nu_0(\zeta,\xi,\tau)=\zeta-\xi, \tag{42}$$

$$\mu_{j+1}(\zeta,\xi,\tau)=-\mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\sum_{j=0}^{\infty}A_j(\mu\mu_\zeta)+\sum_{j=0}^{\infty}\mu_{\zeta\zeta}+\sum_{j=0}^{\infty}\mu_{\xi\xi}\right\}\right],$$

for $j = 0, 1\cdots$,

$$\mu_1(\zeta,\xi,\tau)=-\mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left[\mu_0\frac{\partial\mu_0}{\partial\zeta}+\frac{\partial^2\mu_0}{\partial\zeta^2}+\frac{\partial^2\mu_0}{\partial\xi^2}\right]\right],\quad \mu_1(\zeta,\xi,\tau)=(\zeta+\xi)\left\{\frac{\tau^\delta}{\Gamma(\delta+1)}\right\}. \tag{43}$$

The subsequent terms are

$$\mu_2(\zeta,\xi,\tau)=-\mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left[\mu_0\frac{\partial\mu_1}{\partial\zeta}+\mu_1\frac{\partial\mu_0}{\partial\zeta}+\frac{\partial^2\mu_1}{\partial\zeta^2}+\frac{\partial^2\mu_1}{\partial\xi^2}\right]\right],\quad \mu_2(\zeta,\xi,\tau)=(\zeta+\xi)\left\{\frac{2\tau^\delta}{\Gamma(2\delta+1)}\right\}. \tag{44}$$

The ETDM solution for example **Eq. 32** is

$$\mu(\zeta,\xi,\tau)=\mu_0(\zeta,\xi,\tau)+\mu_1(\zeta,\xi,\tau)+\mu_2(\zeta,\xi,\tau)+\mu_3(\zeta,\xi,\tau)+\cdots,$$

$$\mu(\zeta,\xi,\tau)=(\zeta+\xi)\left[1+\frac{\tau^\delta}{\Gamma(\delta+1)}+\frac{2\tau^\delta}{\Gamma(2\delta+1)}+\cdots\right].$$

When $\delta = 1$, then the ETDM solution is

$$\mu(\zeta,\xi,\tau)=(\zeta+\xi)\left[1+\tau+\frac{\tau^2}{2!}+\frac{\tau^3}{3!}+\frac{\tau^4}{4!}+\cdots\right].$$

The exact solutions are

$$\mu(\zeta,\xi,t)=\frac{\zeta+\xi}{1-\tau}. \tag{45}$$

4.3 Example

Consider the two-dimensional time-fractional-order Burger's equation:

$$D_\psi^\alpha\mu=\mu\mu_\zeta+\mu_{\zeta\zeta}+\mu_{\xi\xi}+\mu_{\eta\eta},\qquad 0<\delta\le 1, \tag{46}$$

subject to the initial condition

$$\mu(\zeta,\xi,\eta,0)=\zeta+\xi+\eta=f_0(\zeta,\xi,\eta). \tag{47}$$

The exact solution of **Eq. 46** is

$$\mu(\zeta,\xi,\eta,t)=\frac{\zeta+\xi+\eta}{1-t}. \tag{48}$$

4.3.1 Solution by LRPSM

Applying LT to **Eq. 46** and using the IC of **Eq. 47**, we get

$$\mu(\zeta,\xi,\eta,s)=\frac{\zeta+\xi+\eta}{s}+\frac{1}{s^{\delta}}\mathcal{L}_{\tau}\left[\left(\mathcal{L}_{\tau}^{-1}\mu(\zeta,\xi,\eta,s)\right)\left(\mathcal{L}_{\tau}^{-1}\mu_{\zeta}(\zeta,\xi,\eta,s)\right)\right] +\frac{\mu_{\zeta\zeta}(\zeta,\xi,\eta,s)}{s^{\delta}}+\frac{\mu_{\xi\xi}(\zeta,\xi,\eta,s)}{s^{\delta}}+\frac{\mu_{\eta\eta}(\zeta,\xi,\eta,s)}{s^{\delta}}. \tag{49}$$

The k-th truncated term series of **Eq. 49** is

$$\mu_k(\zeta,\xi,\eta,s)=\frac{\zeta+\xi+\eta}{s}+\sum_{n=1}^{k}\frac{f_n(\zeta,\xi,\eta)}{s^{n\delta+1}} \tag{50}$$

and the k-th Laplace Residual function is

$$\mathcal{L}_{\tau}Res_k(\zeta,\xi,\eta,s)=\mu_k(\zeta,\xi,\eta,s)-\frac{\zeta+\xi+\eta}{s} -\frac{1}{s^{\delta}}\mathcal{L}_{\tau}\left[\left(\mathcal{L}_{\tau}^{-1}\mu_k(\zeta,\xi,\eta,s)\right)\left(\mathcal{L}_{\tau}^{-1}\mu_{k,\zeta}(\zeta,\xi,\eta,s)\right)\right] -\frac{\mu_{k,\zeta\zeta}(\zeta,\xi,\eta,s)}{s^{\delta}}-\frac{\mu_{k,\xi\xi}(\zeta,\xi,\eta,s)}{s^{\delta}}-\frac{\mu_{k,\eta\eta}(\zeta,\xi,\eta,s)}{s^{\delta}}. \tag{51}$$

Now, to determine $f_k(\zeta,\xi,\eta)$, $k=1,2,3,\cdots$, we substitute the kth-truncated series **Eq. 50** into the kth-Laplace residual function **Eq. 51**, multiply the resulting equation by $s^{k\delta+1}$, and then solve recursively the relation $\lim_{s\to\infty}[s^{k\delta+1}Res_k(\zeta,\xi,\eta,s)]=0$, $k=1,2,3,\cdots$, for f_k. The following are the first few elements of the sequences $f_k(\zeta,\xi,\eta)$:

$$\begin{aligned}
f_1(\zeta,\xi,\eta)&=(\zeta+\xi+\eta),\\
f_2(\zeta,\xi,\eta)&=2(\zeta+\xi+\eta),\\
f_3(\zeta,\xi,\eta)&=(\zeta+\xi+\eta)\left[4+\frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}\right],\\
f_4(\zeta,\xi,\eta)&=2(\zeta+\xi+\eta)\left[4+\frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}+\frac{2\Gamma(1+3\delta)}{\Gamma(1+\delta)\Gamma(1+2\delta)}\right].\\
&\vdots
\end{aligned} \tag{52}$$

Putting the values of $f_n(\zeta,\xi,\eta)$, $(n\geq 1)$ in **Eq. 50**, we have

$$\mu(\zeta,\xi,\eta,s)=\frac{\zeta+\xi+\eta}{s}+\frac{(\zeta+\xi+\eta)}{s^{\delta+1}}+\frac{2(\zeta+\xi+\eta)}{s^{2\delta+1}}+\frac{(\zeta+\xi+\eta)\left[4+\frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}\right]}{s^{3\delta+1}} +\frac{2(\zeta+\xi+\eta)\left[4+\frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}+\frac{2\Gamma(1+3\delta)}{\Gamma(1+\delta)\Gamma(1+2\delta)}\right]}{s^{4\delta+1}}+\cdots.$$

$$\mu(\zeta,\xi,\eta,s)=(\zeta+\xi+\eta)\left[\frac{1}{s}+\frac{1}{s^{\delta+1}}+\frac{2}{s^{2\delta+1}}+\frac{\left[4+\frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}\right]}{s^{3\delta+1}} +\frac{\left[4+\frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}+\frac{2\Gamma(1+3\delta)}{\Gamma(1+\delta)\Gamma(1+2\delta)}\right]}{s^{4\delta+1}}+\cdots\right].$$

Applying the inverse LT, we get

$$\mu(\zeta,\xi,\eta,\tau)=(\zeta+\xi+\eta)\left[1+\frac{\tau^{\delta}}{\Gamma(\delta+1)}+\frac{2\tau^{\delta}}{\Gamma(2\delta+1)}+\frac{\left[4+\frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}\right]}{\Gamma(3\delta+1)}\tau^{3\delta} +\frac{\left[4+\frac{\Gamma(1+2\delta)}{\Gamma(1+\delta)^2}+\frac{2\Gamma(1+3\delta)}{\Gamma(1+\delta)\Gamma(1+2\delta)}\right]}{\Gamma(4\delta+1)}\tau^{4\delta}+\cdots\right]. \tag{53}$$

Now, if we substitute $\delta=1$ in **Eq. 53**, we have

$$\mu(\zeta,\xi,\eta,\tau)=(\zeta+\xi+\eta)\left[1+\tau+\tau^2+\tau^3+\tau^4+\cdots\right]. \tag{54}$$

The results given in **Eq. 54** agree with the Maclaurin series of

$$\mu(\zeta,\xi,\eta,\tau)=\frac{\zeta+\xi+\eta}{1-\tau}. \tag{55}$$

4.3.2 Solution by ETDM

Taking ET of **Eq. 46**,

$$\mathcal{E}\left[\frac{\partial^{\delta}\mu}{\partial\tau^{\delta}}\right]=-\mathcal{E}\left[\mu\frac{\partial\mu}{\partial\zeta}+\frac{\partial^2\mu}{\partial\zeta^2}+\frac{\partial^2\mu}{\partial\xi^2}+\frac{\partial^2\mu}{\partial\eta^2}\right],$$

$$\frac{\mathcal{E}[\mu(\zeta,\xi,\tau)]-s^2\mu(\zeta,\xi,0)}{s^{\delta}}=-\mathcal{E}\left[\mu\frac{\partial\mu}{\partial\zeta}+\frac{\partial^2\mu}{\partial\zeta^2}+\frac{\partial^2\mu}{\partial\xi^2}+\frac{\partial^2\mu}{\partial\eta^2}\right].$$

Applying the inverse ET,

$$\mu(\zeta,\xi,\tau)=\mathcal{E}^{-1}\left[s^2\mu(\zeta,\xi,0)-s^{\delta}\mathcal{E}\left\{\mu\frac{\partial\mu}{\partial\zeta}+\frac{\partial^2\mu}{\partial\zeta^2}+\frac{\partial^2\mu}{\partial\xi^2}+\frac{\partial^2\mu}{\partial\eta^2}\right\}\right],$$

$$\mu(\zeta,\xi,\tau)=\zeta+\xi+\eta-\mathcal{E}^{-1}\left[s^{\delta}\mathcal{E}\left\{\mu\frac{\partial\mu}{\partial\zeta}+\frac{\partial^2\mu}{\partial\zeta^2}+\frac{\partial^2\mu}{\partial\xi^2}+\frac{\partial^2\mu}{\partial\eta^2}\right\}\right].$$

Using the ADM procedure, we get

$$\sum_{j=0}^{\infty}\mu_j(\zeta,\xi,\tau)=\zeta+\xi+\eta-\mathcal{E}^{-1} \times\left[s^{\delta}\mathcal{E}\left\{\sum_{j=0}^{\infty}A_j(\mu\mu_{\zeta})+\sum_{j=0}^{\infty}\mu_{\zeta\zeta}+\sum_{j=0}^{\infty}\mu_{\xi\xi}+\sum_{j=0}^{\infty}\mu_{\zeta\zeta}+\sum_{j=0}^{\infty}\mu_{\eta\eta}\right\}\right],$$

where A_j $(\mu\mu_{\zeta})$, and the Adomian polynomials are given as follows:

$$\begin{aligned}
A_0(\mu\mu_{\zeta})&=\mu_0\frac{\partial\mu_0}{\partial\zeta},\\
A_1(\mu\mu_{\zeta})&=\mu_0\frac{\partial\mu_1}{\partial\zeta}+\mu_1\frac{\partial\mu_0}{\partial\zeta},\\
A_2(\mu\mu_{\zeta})&=\mu_0\frac{\partial\mu_2}{\partial\zeta}+\mu_1\frac{\partial\mu_1}{\partial\zeta}+\mu_2\frac{\partial\mu_0}{\partial\zeta}.
\end{aligned}$$

$$\mu_0(\zeta,\xi,\tau)=\zeta+\xi+\eta,$$

$$\mu_{j+1}(\zeta,\xi,\tau)=-\mathcal{E}^{-1}\left[s^{\delta}\mathcal{E}\left\{\sum_{j=0}^{\infty}A_j(\mu\mu_{\zeta})+\sum_{j=0}^{\infty}\mu_{\zeta\zeta}+\sum_{j=0}^{\infty}\mu_{\xi\xi}+\sum_{j=0}^{\infty}\mu_{\eta\eta}\right\}\right],$$

for $j=0,1\cdots$:

$$\mu_1(\zeta,\xi,\tau) = -\mathcal{E}^{-1}\left[s^{\delta}\mathcal{E}\left[\mu_0\frac{\partial\mu_0}{\partial\zeta}+\frac{\partial^2\mu_0}{\partial\zeta^2}+\frac{\partial^2\mu_0}{\partial\xi^2}+\frac{\partial^2\mu_0}{\partial\eta^2}\right]\right],$$
$$\mu_1(\zeta,\xi,\tau) = (\zeta+\xi)\left\{\frac{\tau^{\delta}}{\Gamma(\delta+1)}\right\}.$$

The subsequent terms are

$$\mu_2(\zeta,\xi,\tau) = -\mathcal{E}^{-1}\left[s^{\delta}\mathcal{E}\left[\mu_0\frac{\partial\mu_1}{\partial\zeta}+\mu_1\frac{\partial\mu_0}{\partial\zeta}+\frac{\partial^2\mu_1}{\partial\zeta^2}+\frac{\partial^2\mu_1}{\partial\xi^2}+\frac{\partial^2\mu_1}{\partial\zeta^2}+\frac{\partial^2\mu_1}{\partial\eta^2}\right]\right],$$
$$\mu_2(\zeta,\xi,\tau) = (\zeta+\xi+\eta)\left\{\frac{2\tau^{\delta}}{\Gamma(2\delta+1)}\right\}.$$

The ETDM solution for example (4.3) is

$$\mu(\zeta,\xi,\tau) = \mu_0(\zeta,\xi,\tau)+\mu_1(\zeta,\xi,\tau)+\mu_2(\zeta,\xi,\tau)+\mu_3(\zeta,\xi,\tau)+\cdots,$$
$$\mu(\zeta,\xi,\tau) = (\zeta+\xi)\left[1+\frac{\tau^{\delta}}{\Gamma(\delta+1)}+\frac{2\tau^{\delta}}{\Gamma(2\delta+1)}+\cdots\right].$$

When $\delta = 1$, the ETDM solution is

$$\mu(\zeta,\xi,\tau) = (\zeta+\xi)\left[1+\tau+\frac{\tau^2}{2!}+\frac{\tau^3}{3!}+\frac{\tau^4}{4!}+\cdots\right].$$

The exact solutions are

$$\mu(\zeta,\xi,t) = \frac{\zeta+\xi}{1-\tau}.$$

4.4 Example

Let us consider the following system of 1D-order Burger's equation:

$$\begin{aligned} D_{\tau}^{\delta}\mu &= 2\mu\mu_{\zeta}-\mu_{\zeta\zeta}+(\mu\nu)_{\zeta}, \\ D_{\tau}^{\delta}\nu &= -2\nu\nu_{\zeta}-\nu_{\zeta\zeta}-(\mu\nu)_{\zeta}, \quad 0<\delta\le 1 \end{aligned}. \tag{56}$$

Subject to the initial condition,

$$\mu(\zeta,0) = \sin(\zeta), \quad \nu(\zeta,0) = -\sin(\zeta). \tag{57}$$

4.4.1 Solution by LRPSM

Applying LT to **Eq. 56**, we get

$$\begin{aligned}\mu(\zeta,s) &= \frac{\sin(\zeta)}{s}+2\frac{1}{s^{\delta}}\mathcal{L}_{\tau}\left[\left(\mathcal{L}_{\tau}^{-1}\mu(\zeta,s)\right)\left(\frac{\partial}{\partial\zeta}\mu(\zeta,s)\right)\right] \\ &+\frac{1}{s^{\delta}}\mathcal{L}_{\tau}\left[\left\{\left(\mathcal{L}_{\tau}^{-1}\mu(\zeta,s)\right)\left(\mathcal{L}_{\tau}^{-1}\nu(\zeta,s)\right)\right\}_{\zeta}\right]-\frac{1}{s^{\delta}}\frac{\partial^2\mu(\zeta,s)}{\partial\zeta^2}, \\ \nu(\zeta,s) &= \frac{-\sin(\zeta)}{s}-2\frac{1}{s^{\delta}}\mathcal{L}_{\tau}\left[\left(\mathcal{L}_{t}^{-1}\nu(\zeta,s)\frac{\partial}{\partial\zeta}\nu(\zeta,s)\right)\right] \\ &-\frac{1}{s^{\delta}}\mathcal{L}_{\tau}\left[\left\{\left(\mathcal{L}_{\tau}^{-1}\mu(\zeta,s)\right)\left(\mathcal{L}_{\tau}^{-1}\nu(\zeta,s)\right)\right\}_{\zeta}\right]-\frac{1}{s^{\delta}}\frac{\partial^2\nu(\zeta,s)}{\partial\zeta^2}\nu(\zeta,s).\end{aligned} \tag{58}$$

The k-th truncated terms series of **Eq. 58** is

$$\begin{aligned}\mu_k(\zeta,s) &= \frac{\sin(\zeta)}{s}+\sum_{n=0}^{\infty}\frac{f_n(\zeta)}{s^{n\delta+1}}, \\ \nu_k(\zeta,s) &= \frac{-\sin(\zeta)}{s}+\sum_{n=0}^{\infty}\frac{g_n(\zeta)}{s^{n\delta+1}}\end{aligned} \tag{59}$$

and the k-th Laplace residual function is

$$\begin{aligned}\mathcal{L}_{\tau}Res_k(\zeta,s) &= \mu_k(\zeta,s)-\frac{\sin(\zeta)}{s}-2\frac{1}{s^{\delta}}\mathcal{L}_{\tau}\left[\mathcal{L}_{\tau}^{-1}\left\{\mu_k(\zeta,s)\frac{\partial}{\partial\zeta}\mu_k(\zeta,s)\right\}\right] \\ &-\frac{1}{s^{\delta}}\mathcal{L}_{\tau}\left(\left[\mathcal{L}_{\tau}^{-1}\mu_k(\zeta,s)\right]\left[\mathcal{L}_{\tau}^{-1}\nu_k(\zeta,s)\right]\right)_{\zeta}+\frac{1}{s^{\delta}}\left\{\mathcal{L}_{\tau}^{-1}\left[\frac{\partial^2}{\partial\zeta^2}\mu_k(\zeta,s)\right]\right\}, \\ \mathcal{L}_{\tau}Res_k(\zeta,s) &= \nu(\zeta,s)-\frac{-\sin(\zeta)}{s}+2\frac{1}{s^{\delta}}\mathcal{L}_{\tau}\left[\mathcal{L}_{\tau}^{-1}\left\{\nu_k(\zeta,s)\frac{\partial}{\partial\zeta}\nu_k(\zeta,s)\right\}\right] \\ &-\frac{1}{s^{\delta}}\mathcal{L}_{\tau}\left(\left[\mathcal{L}_{\tau}^{-1}\mu_k(\zeta,s)\right]\left[\mathcal{L}_{\tau}^{-1}\nu_k(\zeta,s)\right]\right)_{\zeta}-\frac{1}{s^{\delta}}\left\{\mathcal{L}_{\tau}^{-1}\left[\frac{\partial^2}{\partial\zeta^2}\nu_k(\zeta,s)\right]\right\}.\end{aligned} \tag{60}$$

Now, to determine $f_k(\zeta)$ and $g_k(\zeta)$, $k = 1, 2, 3, \cdots$, we substitute the kth-truncated series of **Eq. 59** into the kth-Laplace residual function of **Eq. 60**, multiply the resulting equation by $s^{k\delta+1}$, and then solve recursively the relation $\lim_{s\to\infty}[s^{k\delta+1}Res_k(\zeta,s)] = 0$, $k = 1, 2, 3, \cdots$, for $f_k(\zeta)$ and $g_k(\zeta)$. The following are the first few elements of the sequences $f_k(\zeta)$ and $g_k(\zeta)$:

$$\begin{aligned} f_1(\zeta) &= \sin(\zeta), \\ g_1(\zeta) &= -\sin(\zeta). \\ f_2(\zeta) &= \sin(\zeta), \\ g_2(\zeta) &= -\sin(\zeta). \\ f_3(\zeta) &= \sin(\zeta), \\ g_3(\zeta) &= -\sin(\zeta). \\ f_4(\zeta) &= \sin(\zeta), \\ g_4(\zeta) &= -\sin(\zeta). \\ &\vdots \end{aligned} \tag{61}$$

Putting the values of $f_n(\zeta)$ and $g_n(\zeta)$ for $(n \ge 1)$ in **Eq. 59**, we get

$$\begin{aligned}\mu(\zeta,s) &= \frac{\sin(\zeta)}{s}+\frac{\sin(\zeta)}{s^{\delta+1}}+\frac{\sin(\zeta)}{s^{2\delta+1}}+\frac{\sin(\zeta)}{s^{3\delta+1}}+\cdots, \\ \nu(\zeta,s) &= \frac{-\sin(\zeta)}{s}+\frac{-\sin(\zeta)}{s^{\delta+1}}+\frac{-\sin(\zeta)}{s^{2\delta+1}}+\frac{-\sin(\zeta)}{s^{3\delta+1}}+\cdots. \\ \mu(\zeta,s) &= \sin(\zeta)\left[\frac{1}{s}+\frac{1}{s^{\delta+1}}+\frac{1}{s^{2\delta+1}}+\frac{1}{s^{3\delta+1}}+\cdots\right], \\ \nu(\zeta,s) &= -\sin(\zeta)\left[\frac{1}{s}+\frac{1}{s^{\delta+1}}+\frac{1}{s^{2\delta+1}}+\frac{1}{s^{3\delta+1}}+\cdots\right].\end{aligned}$$

Applying the inverse LT, we get

$$\begin{aligned}\mu(\zeta,\tau) &= \sin(\zeta)\left[1+\frac{\tau^{\delta}}{\Gamma(\delta+1)}+\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}+\frac{\tau^{3\delta}}{\Gamma(3\delta+1)}+\cdots\right], \\ \nu(\zeta,\tau) &= -\sin(\zeta)\left[1+\frac{\tau^{\delta}}{\Gamma(\delta+1)}+\frac{\tau^{2\delta}}{\Gamma(2\delta+1)}+\frac{\tau^{3\delta}}{\Gamma(3\delta+1)}+\cdots\right].\end{aligned} \tag{62}$$

Putting $\delta = 1$, we get the solution of **Eq. 62** in a closed form:

$$\begin{aligned}\mu(\zeta,\tau) &= e^{\tau}\sin(\zeta), \\ \nu(\zeta,\tau) &= -e^{\tau}\sin(\zeta).\end{aligned} \tag{63}$$

4.4.2 Solution by ETDM

Taking ET of **Eq. 56**,

$$\mathcal{E}\left[\frac{\partial^{\delta}\mu}{\partial\tau^{\delta}}\right] = -\mathcal{E}\left[\frac{\partial^2\mu}{\partial\zeta^2}-2\mu\frac{\partial\mu}{\partial\zeta}-\frac{\partial(\mu\nu)}{\partial\zeta}\right],$$

$$\mathcal{E}\left[\frac{\partial^\delta \nu}{\partial \tau^\delta}\right] = -\mathcal{E}\left[\frac{\partial^2 \nu}{\partial \zeta^2} - 2\nu\frac{\partial \nu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right],$$

$$\frac{\mathcal{E}[\mu(\zeta,\tau)] - \mu(\zeta,0)}{s^\delta} = -\mathcal{E}\left[\frac{\partial^2 \mu}{\partial \zeta^2} - 2\mu\frac{\partial \mu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right],$$

$$\frac{\mathcal{E}[\nu(\zeta,\tau)] - \nu(\zeta,0)}{s^\delta} = -\mathcal{E}\left[\frac{\partial^2 \nu}{\partial \zeta^2} - 2\nu\frac{\partial \nu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right].$$

Applying the inverse ET,

$$\mu(\zeta,\tau) = \mathcal{E}^{-1}\left[\frac{\mu(\zeta,0)}{s} - s^\delta\mathcal{E}\left\{\frac{\partial^2 \mu}{\partial \zeta^2} - 2\mu\frac{\partial \mu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right\}\right],$$

$$\nu(\zeta,\tau) = \mathcal{E}^{-1}\left[\frac{\mu(\zeta,0)}{s} - s^\delta\mathcal{E}\left\{\frac{\partial^2 \nu}{\partial \zeta^2} - 2\nu\frac{\partial \nu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right\}\right],$$

$$\mu(\zeta,\tau) = \sin(\zeta) - \mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\frac{\partial^2 \mu}{\partial \zeta^2} - 2\mu\frac{\partial \mu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right\}\right],$$

$$\nu(\zeta,\tau) = -\sin(\zeta) - \mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\frac{\partial^2 \nu}{\partial \zeta^2} - 2\nu\frac{\partial \nu}{\partial \zeta} - \frac{\partial(\mu\nu)}{\partial \zeta}\right\}\right].$$

Using the ADM procedure, we get

$$\sum_{j=0}^{\infty}\mu_j(\zeta,\tau) = \sin(\zeta) - \mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\sum_{j=0}^{\infty}(\mu_{\zeta\zeta})_j - 2\sum_{j=0}^{\infty}A_j(\mu\mu_\zeta) - \sum_{j=0}^{\infty}B_j(\mu\nu)_\zeta\right\}\right],$$

$$\sum_{j=0}^{\infty}\nu_j(\zeta,\tau) = -\sin(\zeta) - \mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\sum_{j=0}^{\infty}(\nu_{\zeta\zeta})_j - 2\sum_{j=0}^{\infty}C_j(\nu\nu_\zeta) - \sum_{j=0}^{\infty}D_j(\mu\nu)_\zeta\right\}\right],$$

where A_j $(\mu\mu_\zeta)$, B_j $(\mu\nu)_\zeta$, C_j $(\nu\nu_\zeta)$, and D_j $(\mu\nu)_\zeta$ are Adomian polynomials given as follows:

$$A_0(\mu\mu_\zeta) = \mu_0\frac{\partial\mu_0}{\partial\zeta}, \quad B_0(\mu\nu)_\zeta = \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta},$$

$$A_1(\mu\mu_\zeta) = \mu_0\frac{\partial\mu_1}{\partial\zeta} + \mu_1\frac{\partial\mu_0}{\partial\zeta}, \quad B_1(\mu\nu)_\zeta = \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_1}{\partial\zeta} + \frac{\partial\mu_1}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta},$$

$$A_2(\mu\mu_\zeta) = \mu_0\frac{\partial\mu_2}{\partial\zeta} + \mu_1\frac{\partial\mu_1}{\partial\zeta} + \mu_2\frac{\partial\mu_0}{\partial\zeta}. \quad B_2(\mu\nu)_\zeta = \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_2}{\partial\zeta} + \frac{\partial\mu_1}{\partial\zeta}\frac{\partial\nu_1}{\partial\zeta} + \frac{\partial\mu_2}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta}.$$

$$C_0(\nu\nu_\zeta) = \nu_0\frac{\partial\nu_0}{\partial\zeta}, \quad D_0(\mu\nu)_\zeta = \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta},$$

$$C_1(\nu\nu_\zeta) = \nu_0\frac{\partial\nu_1}{\partial\zeta} + \nu_1\frac{\partial\nu_0}{\partial\zeta}, \quad D_1(\mu\nu)_\zeta = \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_1}{\partial\zeta} + \frac{\partial\mu_1}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta},$$

$$C_2(\nu\nu_\zeta) = \nu_0\frac{\partial\nu_2}{\partial\zeta} + \nu_1\frac{\partial\nu_1}{\partial\zeta} + \nu_2\frac{\partial\nu_0}{\partial\zeta}. \quad D_2(\mu\nu)_\zeta = \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_2}{\partial\zeta} + \frac{\partial\mu_1}{\partial\zeta}\frac{\partial\nu_1}{\partial\zeta} + \frac{\partial\mu_2}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta}.$$

$$\mu_0(\zeta,\tau) = \sin\zeta,$$
$$\nu_0(\zeta,\tau) = -\sin(\zeta),$$

$$\mu_{j+1}(\zeta,\tau) = -\mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\sum_{j=0}^{\infty}(\mu_{\zeta\zeta})_j - 2\sum_{j=0}^{\infty}A_j(\mu\mu_\zeta) - \sum_{j=0}^{\infty}B_j(\mu\nu)_\zeta\right\}\right],$$

$$\nu_{j+1}(\zeta,\tau) = -\mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\sum_{j=0}^{\infty}(\nu_{\zeta\zeta})_j - 2\sum_{j=0}^{\infty}C_j(\nu\nu_\zeta) - \sum_{j=0}^{\infty}D_j(\mu\nu)_\zeta\right\}\right],$$

for $j = 0, 1, \cdots$:

$$\mu_1(\zeta,\xi,\tau) = -\mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\frac{\partial^2\mu_0}{\partial\zeta^2} - 2\mu_0\frac{\partial\mu_0}{\partial\zeta} - \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta}\right\}\right],$$

$$\mu_1(\zeta,\tau) = -\mathcal{E}^{-1}\left[s^\delta \times \frac{-\sin\zeta}{s}\right] = \sin(\zeta)\{\delta\tau + (1-\delta)\},$$

$$\nu_1(\zeta,\tau) = -\mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\frac{\partial^2\nu_0}{\partial\zeta^2} - 2\nu_0\frac{\partial\nu_0}{\partial\zeta} - \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta}\right\}\right]$$

$$\nu_1(\zeta,\tau) = -\mathcal{E}^{-1}\left[s^\delta \times \frac{\sin(\zeta)}{s}\right] = -\sin(\zeta)\{\delta\tau + (1-\delta)\}.$$

The subsequent terms are

$$\mu_2(\zeta,\xi,\tau) = -\mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\frac{\partial^2\mu_1}{\partial\zeta^2} - 2\mu_0\frac{\partial\mu_1}{\partial\zeta} - 2\mu_1\frac{\partial\mu_0}{\partial\zeta} - \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_1}{\partial\zeta} - \frac{\partial\mu_1}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta}\right\}\right],$$

$$\mu_2(\zeta,\tau) = \sin(\zeta)\left\{(1-\delta)^2 + 2\delta(1-\delta)\tau + \frac{\delta^2\tau^2}{2}\right\},$$

$$\nu_2(\zeta,\tau) = -\mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\frac{\partial^2\nu_1}{\partial\zeta^2} - 2\nu_0\frac{\partial\nu_1}{\partial\zeta} - 2\nu_1\frac{\partial\nu_0}{\partial\zeta} - \frac{\partial\mu_0}{\partial\zeta}\frac{\partial\nu_1}{\partial\zeta} - \frac{\partial\mu_1}{\partial\zeta}\frac{\partial\nu_0}{\partial\zeta}\right\}\right]$$

$$\nu_2(\zeta,\tau) = -\sin(\zeta)\left\{(1-\delta)^2 + 2\delta(1-\delta)\tau + \frac{\delta^2\tau^2}{2}\right\}.$$

The ETDM solution for example (4.4) is

$$\mu(\zeta,\tau) = \mu_0(\zeta,\tau) + \mu_1(\zeta,\tau) + \mu_2(\zeta,\tau) + \mu_3(\zeta,\tau) + \cdots,$$
$$\nu(\zeta,\tau) = \nu_0(\zeta,\tau) + \nu_1(\zeta,\tau) + \nu_2(\zeta,\tau) + \nu_3(\zeta,\tau) + \cdots,$$
$$\mu(\zeta,\tau) = \sin(\zeta) + \sin(\zeta)\{\delta\tau + (1-\delta)\} + \sin(\zeta)\left\{(1-\delta)^2 + 2\delta(1-\delta)\tau + \frac{\delta^2\tau^2}{2}\right\} + \cdots$$
$$\nu(\zeta,\tau) = -\sin(\zeta) - \sin(\zeta)\{\delta\tau + (1-\delta)\} - \sin(\zeta)\left\{(1-\delta)^2 + 2\delta(1-\delta)\tau + \frac{\delta^2\tau^2}{2}\right\} - \cdots.$$

When $\delta = 1$, the ETDM solution is

$$\mu(\zeta,\tau) = \sin(\zeta) + \sin(\zeta)\tau + \sin(\zeta)\frac{\tau^2}{2} + \sin(\zeta)\frac{\tau^3}{6} + \sin(\zeta)\frac{\tau^4}{24} + \cdots$$

$$\nu(\zeta,\tau) = -\sin(\zeta) - \sin(\zeta)\tau - \sin(\zeta)\frac{\tau^2}{2} - \sin(\zeta)\frac{\tau^3}{6} - \sin(\zeta)\frac{\tau^4}{24} - \cdots.$$

The exact solutions are

$$\mu(\zeta,\tau) = e^\tau\sin(\zeta),$$
$$\nu(\zeta,\tau) = -e^\tau\sin(\zeta).$$

4.5 Example

Let us consider the following system of 2D-order Burger's equation:

$$D_\tau^\delta\mu = \mu_{\zeta\zeta} + \mu_{\xi\xi} - u u_\zeta - \nu\mu_\xi,$$
$$D_\tau^\delta\nu = \nu_{\zeta\zeta} + \mu_{\xi\xi} - \mu\nu_\zeta - \nu\nu_\xi, \quad 0 < \delta \le 1. \tag{64}$$

Subject to the initial condition,

$$\mu(\zeta,\xi,0) = \zeta + \xi, \quad \nu(\zeta,\xi,0) = \zeta - \xi. \tag{65}$$

4.5.1 Solution by LRPSM

Applying LT to **Eq. 64**, we get

$$\begin{aligned}\mu(\zeta,\xi,s) &= \frac{\zeta+\xi}{s} + \frac{1}{s^\delta}\frac{\partial^2}{\partial\zeta^2}\mu(\zeta,\xi,s) + \frac{1}{s^\delta}\frac{\partial^2}{\partial\xi^2}\mu(\zeta,\xi,s)\\ &-\frac{1}{s^\delta}\mathcal{L}_\tau\left[\left(\mathcal{L}_\tau^{-1}\mu(\zeta,\xi,s)\right)\left(\mathcal{L}_\tau^{-1}\frac{\partial}{\partial\zeta}\mu(\zeta,\xi,s)\right)\right]\\ &-\frac{1}{s^\delta}\mathcal{L}_\tau\left[\left(\mathcal{L}_\tau^{-1}\nu(\zeta,\xi,s)\right)\left(\mathcal{L}_\tau^{-1}\frac{\partial}{\partial\zeta}\mu(\zeta,\xi,s)\right)\right],\\ \nu(\zeta,\xi,s) &= \frac{\zeta-\xi}{s} + \frac{1}{s^\delta}\frac{\partial^2}{\partial\zeta^2}\nu(\zeta,\xi,s) + \frac{1}{s^\delta}\frac{\partial^2}{\partial\xi^2}\nu(\zeta,\xi,s)\\ &-\frac{1}{s^\delta}\mathcal{L}_\tau\left[\left(\mathcal{L}_\tau^{-1}\mu(\zeta,\xi,s)\right)\left(\mathcal{L}_\tau^{-1}\frac{\partial}{\partial\zeta}\nu(\zeta,\xi,s)\right)\right]\\ &-\frac{1}{s^\delta}\mathcal{L}_\tau\left[\left(\mathcal{L}_\tau^{-1}\nu(\zeta,\xi,s)\right)\left(\mathcal{L}_\tau^{-1}\frac{\partial}{\partial\zeta}\nu(\zeta,\xi,s)\right)\right].\end{aligned} \tag{66}$$

The kth truncated terms series of **Eq. 66** is

$$\begin{aligned}\mu_k(\zeta,\xi,s) &= \frac{\zeta+\xi}{s} + \sum_{n=0}^{\infty}\frac{f_n(\zeta,\xi)}{s^{n\delta+1}},\\ \nu_k(\zeta,\xi,s) &= \frac{\zeta-\xi}{s} + \sum_{n=0}^{\infty}\frac{g_n(\zeta,\xi)}{s^{n\delta+1}}.\end{aligned} \tag{67}$$

The kth Laplace residual function is

$$\begin{aligned}\mathcal{L}_\tau Res_k(\zeta,\xi,s) &= \mu_k(\zeta,\xi,s) + \frac{\zeta+\xi}{s} - \frac{1}{s^\delta}\frac{\partial^2}{\partial\zeta^2}\mu_k(\zeta,\xi,s) - \frac{1}{s^\delta}\frac{\partial^2}{\partial\xi^2}\mu_k(\zeta,\xi,s)\\ &+\frac{1}{s^\delta}\mathcal{L}_\tau\left[\left(\mathcal{L}_\tau^{-1}\mu_k(\zeta,\xi,s)\right)\left(\mathcal{L}_\tau^{-1}\frac{\partial}{\partial\zeta}\mu_k(\zeta,\xi,s)\right)\right]\\ &+\frac{1}{s^\delta}\mathcal{L}_\tau\left[\left(\mathcal{L}_\tau^{-1}\nu_k(\zeta,\xi,s)\right)\left(\mathcal{L}_\tau^{-1}\frac{\partial}{\partial\zeta}\mu_k(\zeta,\xi,s)\right)\right],\\ \mathcal{L}_\tau Res_k(\zeta,\xi,s) &= \nu(\zeta,\xi,s) + \frac{\zeta-\xi}{s} - \frac{1}{s^\delta}\frac{\partial^2}{\partial\zeta^2}\nu_k(\zeta,\xi,s) - \frac{1}{s^\delta}\frac{\partial^2}{\partial\xi^2}\nu_k(\zeta,\xi,s)\\ &+\frac{1}{s^\delta}\mathcal{L}_\tau\left[\left(\mathcal{L}_\tau^{-1}\mu_k(\zeta,\xi,s)\right)\left(\mathcal{L}_\tau^{-1}\frac{\partial}{\partial\zeta}\nu_k(\zeta,\xi,s)\right)\right]\\ &+\frac{1}{s^\delta}\mathcal{L}_\tau\left[\left(\mathcal{L}_\tau^{-1}\nu_k(\zeta,\xi,s)\right)\left(\mathcal{L}_\tau^{-1}\frac{\partial}{\partial\zeta}\nu_k(\zeta,\xi,s)\right)\right].\end{aligned} \tag{68}$$

Now, to determine $f_k(\zeta,\xi)$ and $g_k(\zeta,\xi)$, $k = 1, 2, 3, \cdots$, we substitute the kth-truncated series (**Eq. 67**) into the kth-Laplace residual function (**Eq. 68**), multiply the resulting equation by $s^{k\delta+1}$, and then solve recursively the relation $\lim_{s\to\infty}[s^{k\delta+1}Res_k(\zeta, s)] = 0$, $k = 1, 2, 3, \cdots$, for f_k and g_k. The following are the first few elements of the sequences $f_k(\zeta,\xi)$ and $g_k(\zeta,\xi)$:

$$\begin{aligned}f_1(\zeta,\xi) &= -2\zeta,\\ g_1(\zeta,\xi) &= -2\xi.\\ f_2(\zeta,\xi) &= 4(\zeta+\xi),\\ g_2(\zeta,\xi) &= 4(\zeta-\xi).\\ f_3(\zeta,\xi) &= -16\zeta - 4\zeta\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2},\\ g_3(\zeta,\xi) &= -16\xi - 4\xi\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2}.\\ &\vdots\end{aligned} \tag{69}$$

Putting the values of $f_n(\zeta,\xi)$ and $g_n(\zeta,\xi)$, for $n \ge 1$ in **Eq. 67**, we have

$$\begin{aligned}\mu(\zeta,\xi,s) &= \frac{\zeta+\xi}{s} + \frac{-2\zeta}{s^{\delta+1}} + \frac{4(\zeta+\xi)}{s^{2\delta+1}} + \frac{-16\zeta - 4\zeta\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2}}{s^{\delta+1}} + \cdots,\\ \nu(\zeta,\xi,s) &= \frac{\zeta-\xi}{s} + \frac{-2\xi}{s^{\delta+1}} + \frac{4(\zeta-\xi)}{s^{\delta+1}} + \frac{-16\xi - 4\xi\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2}}{s^{\delta+1}} + \cdots.\end{aligned}$$

Applying the inverse LT, we have

$$\begin{aligned}\mu(\zeta,\xi,\tau) &= \zeta+\xi + \frac{-2\zeta}{\Gamma(\delta+1)}\tau^\delta + \frac{4(\zeta+\xi)}{\Gamma(2\delta+1)}\tau^{2\delta} + \frac{-16\zeta - 4\zeta\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2}}{\Gamma(3\delta+1)}\tau^{3\delta} + \cdots,\\ \nu(\zeta,\xi,\tau) &= \zeta-\xi + \frac{-2\xi}{\Gamma(\delta+1)}\tau^\delta + \frac{4(\zeta-\xi)}{\Gamma(2\delta+1)}\tau^{2\delta} + \frac{-16\xi - 4\xi\frac{\Gamma(2\delta+1)}{\Gamma(\delta+1)^2}}{\Gamma(3\delta+1)}\tau^{3\delta} + \cdots.\end{aligned} \tag{70}$$

Putting $\delta = 1$, we get the solution of **Eq. 70** in closed form:

$$\begin{aligned}\mu(\zeta,\xi,\tau) &= \zeta+\xi - 2\zeta\tau + 2(\zeta+\xi)\tau^2 - 4\zeta\tau^3 + 4(\zeta+\xi)\tau^4 + \cdots,\\ \nu(\zeta,\xi,\tau) &= \zeta-\xi - 2\xi\tau + 2(\zeta-\xi)\tau^2 - 4\xi\tau^3 + 4(\zeta-\xi)\tau^4 + \cdots.\\ \mu(\zeta,\xi,\tau) &= \frac{\zeta+\xi-2\zeta\tau}{1-2\tau^2},\\ \nu(\zeta,\xi,\tau) &= \frac{\zeta-\xi-2\xi\tau}{1-2\tau^2}.\end{aligned} \tag{71}$$

4.5.2 Solution by ETDM

Taking ET of **Eq. 64**,

$$\begin{aligned}\mathcal{E}\left[\frac{\partial^\delta\mu}{\partial\tau^\delta}\right] &= -\mathcal{E}\left[\mu\frac{\partial\mu}{\partial\zeta} + \nu\frac{\partial\mu}{\partial\xi} - \frac{\partial^2\mu}{\partial\zeta^2} - \frac{\partial^2\mu}{\partial\xi^2}\right],\\ \mathcal{E}\left[\frac{\partial^\delta\nu}{\partial\tau^\delta}\right] &= -\mathcal{E}\left[\mu\frac{\partial\nu}{\partial\zeta} + \nu\frac{\partial\nu}{\partial\xi} - \frac{\partial^2\nu}{\partial\zeta^2} - \frac{\partial^2\nu}{\partial\xi^2}\right],\\ \frac{\mathcal{E}[\mu(\zeta,\xi,\tau)] - \mu(\zeta,\xi,0)}{s^\delta} &= -\mathcal{E}\left[\mu\frac{\partial\mu}{\partial\zeta} + \nu\frac{\partial\mu}{\partial\xi} - \frac{\partial^2\mu}{\partial\zeta^2} - \frac{\partial^2\mu}{\partial\xi^2}\right],\\ \frac{\mathcal{E}[\nu(\zeta,\xi,\tau)] - \nu(\zeta,\xi,0)}{s^\delta} &= -\mathcal{E}\left[\mu\frac{\partial\nu}{\partial\zeta} + \nu\frac{\partial\nu}{\partial\xi} - \frac{\partial^2\nu}{\partial\zeta^2} - \frac{\partial^2\nu}{\partial\xi^2}\right].\end{aligned}$$

Applying the inverse ET,

$$\begin{aligned}\mu(\zeta,\xi,\tau) &= \mathcal{E}^{-1}\left[s^2\mu(\zeta,\xi,0) - s^\delta\mathcal{E}\left\{\mu\frac{\partial\mu}{\partial\zeta} + \nu\frac{\partial\mu}{\partial\xi} - \frac{\partial^2\mu}{\partial\zeta^2} - \frac{\partial^2\mu}{\partial\xi^2}\right\}\right],\\ \nu(\zeta,\xi,\tau) &= \mathcal{E}^{-1}\left[s^2\mu(\zeta,\xi,0) - s^\delta\mathcal{E}\left\{\mu\frac{\partial\nu}{\partial\zeta} + \nu\frac{\partial\nu}{\partial\xi} - \frac{\partial^2\nu}{\partial\zeta^2} - \frac{\partial^2\nu}{\partial\xi^2}\right\}\right],\\ \mu(\zeta,\xi,\tau) &= \zeta+\xi - \mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\mu\frac{\partial\mu}{\partial\zeta} + \nu\frac{\partial\mu}{\partial\xi} - \frac{\partial^2\mu}{\partial\zeta^2} - \frac{\partial^2\mu}{\partial\xi^2}\right\}\right],\\ \nu(\zeta,\xi,\tau) &= \zeta-\xi - \mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\mu\frac{\partial\nu}{\partial\zeta} + \nu\frac{\partial\nu}{\partial\xi} - \frac{\partial^2\nu}{\partial\zeta^2} - \frac{\partial^2\nu}{\partial\xi^2}\right\}\right].\end{aligned}$$

Using the ADM procedure, we get

$$\sum_{j=0}^{\infty}\mu_j(\zeta,\xi,\tau) = \zeta+\xi - \mathcal{E}^{-1}\left[s^\delta\mathcal{E}\left\{\sum_{j=0}^{\infty}A_j(\mu\mu_\zeta) + \sum_{j=0}^{\infty}B_j(\nu\mu_\xi) - \sum_{j=0}^{\infty}\mu_{\zeta\zeta} - \sum_{j=0}^{\infty}\mu_{\xi\xi}\right\}\right],$$

$$\sum_{j=0}^{\infty} \nu_j(\zeta,\xi,\tau) = \zeta - \xi - \mathcal{E}^{-1}\left[s^{\delta}\mathcal{E}\left\{\sum_{j=0}^{\infty} C_j(\mu\nu_\zeta) + \sum_{j=0}^{\infty} D_j(\nu\nu_\xi) - \sum_{j=0}^{\infty}\nu_{\zeta\zeta} - \sum_{j=0}^{\infty}\nu_{\xi\xi}\right\}\right],$$

where A_j $(\mu\mu_\zeta)$, B_j $(\nu\mu_\xi)$, C_j $(\mu\nu_\zeta)$, and D_j $(\nu\nu_\xi)$ are the Adomian polynomials given as follows:

$$A_0(\mu\mu_\zeta) = \mu_0\frac{\partial\mu_0}{\partial\zeta}, \quad B_0(\nu\mu_\xi) = \nu_0\frac{\partial\mu_0}{\partial\xi},$$
$$A_1(\mu\mu_\zeta) = \mu_0\frac{\partial\mu_1}{\partial\zeta} + \mu_1\frac{\partial\mu_0}{\partial\zeta}, \quad B_1(\nu\mu_\xi) = \nu_0\frac{\partial\mu_1}{\partial\xi} + \nu_1\frac{\partial\mu_0}{\partial\xi},$$
$$A_2(\mu\mu_\zeta) = \mu_0\frac{\partial\mu_2}{\partial\zeta} + \mu_1\frac{\partial\mu_1}{\partial\zeta} + \mu_2\frac{\partial\mu_0}{\partial\zeta}. \quad B_2(\nu\mu_\xi) = \nu_0\frac{\partial\mu_2}{\partial\xi} + \nu_1\frac{\partial\mu_1}{\partial\xi} + \nu_2\frac{\partial\mu_0}{\partial\xi}.$$

$$C_0(\mu\nu_\zeta) = \mu_0\frac{\partial\nu_0}{\partial\zeta}, \quad D_0(\nu\nu_\xi) = \nu_0\frac{\partial\nu_0}{\partial\xi},$$
$$C_1(\mu\nu_\zeta) = \mu_0\frac{\partial\nu_1}{\partial\zeta} + \mu_1\frac{\partial\nu_0}{\partial\zeta}, \quad D_1(\nu\nu_\xi) = \nu_0\frac{\partial\nu_1}{\partial\xi} + \nu_1\frac{\partial\nu_0}{\partial\xi},$$
$$C_2(\mu\nu_\zeta) = \mu_0\frac{\partial\nu_2}{\partial\zeta} + \mu_1\frac{\partial\nu_1}{\partial\zeta} + \mu_2\frac{\partial\nu_0}{\partial\zeta}. \quad D_2(\nu\nu_\xi) = \nu_0\frac{\partial\nu_2}{\partial\xi} + \nu_1\frac{\partial\nu_1}{\partial\xi} + \nu_2\frac{\partial\nu_0}{\partial\xi}.$$

$$\mu_0(\zeta,\xi,\tau) = \zeta + \xi, \quad \nu_0(\zeta,\xi,\tau) = \zeta - \xi, \tag{72}$$

$$\mu_{j+1}(\zeta,\xi,\tau) = -\mathcal{E}^{-1}\left[s^{\delta}\mathcal{E}\left\{\sum_{j=0}^{\infty} A_j(\mu\mu_\zeta) + \sum_{j=0}^{\infty} B_j(\nu\mu_\xi) - \sum_{j=0}^{\infty}\mu_{\zeta\zeta} - \sum_{j=0}^{\infty}\mu_{\xi\xi}\right\}\right],$$
$$\nu_{j+1}(\zeta,\xi,\tau) = -\mathcal{E}^{-1}\left[s^{\delta}\mathcal{E}\left\{\sum_{j=0}^{\infty} C_j(\mu\nu_\zeta) + \sum_{j=0}^{\infty} D_j(\nu\nu_\xi) - \sum_{j=0}^{\infty}\nu_{\zeta\zeta} - \sum_{j=0}^{\infty}\nu_{\xi\xi}\right\}\right],$$

for $j = 0, 1\cdots$:

$$\mu_1(\zeta,\xi,\tau) = -\mathcal{E}^{-1}\left[s^{\delta}\mathcal{E}\left[\mu_0\frac{\partial\mu_0}{\partial\zeta} + \nu_0\frac{\partial\mu_0}{\partial\xi} - \frac{\partial^2\mu_0}{\partial\zeta^2} - \frac{\partial^2\mu_0}{\partial\xi^2}\right]\right],$$
$$\mu_1(\zeta,\xi,\tau) = \mathcal{E}^{-1}\left[s^{\delta}\times s^2 2\zeta\right] = -2\zeta\{\delta\tau + (1-\delta)\},$$
$$\nu_1(\zeta,\xi,\tau) = -\mathcal{E}^{-1}\left[s^{\delta}\mathcal{E}\left[\mu_0\frac{\partial\nu_0}{\partial\zeta} + \nu_0\frac{\partial\nu_0}{\partial\xi} - \frac{\partial^2\nu_0}{\partial\zeta^2} - \frac{\partial^2\nu_0}{\partial\xi^2}\right]\right],$$
$$\nu_1(\zeta,\xi,\tau) = \mathcal{E}^{-1}\left[s^{\delta}\times s^2 2\xi\right] = -2\xi\{\delta\tau + (1-\delta)\}.$$

The subsequent terms are

$$\mu_2(\zeta,\xi,\tau) = -\mathcal{E}^{-1}\left[s^{\delta}\mathcal{E}\left[\mu_0\frac{\partial\mu_1}{\partial\zeta} + \mu_1\frac{\partial\mu_0}{\partial\zeta} + \nu_0\frac{\partial\mu_1}{\partial\xi} + \nu_1\frac{\partial\mu_0}{\partial\xi} - \frac{\partial^2\mu_1}{\partial\zeta^2} - \frac{\partial^2\mu_1}{\partial\xi^2}\right]\right],$$
$$\mu_2(\zeta,\xi,\tau) = 2(\zeta+\xi)\left\{(1-\delta)^2 + 2\delta(1-\delta)\tau + \frac{\delta^2\tau^2}{2}\right\},$$
$$\nu_2(\zeta,\xi,\tau) = -\mathcal{E}^{-1}\left[s^{\delta}\mathcal{E}\left[\mu_0\frac{\partial\nu_1}{\partial\zeta} + \mu_1\frac{\partial\nu_0}{\partial\zeta} + \nu_0\frac{\partial\nu_1}{\partial\xi} + \nu_1\frac{\partial\nu_0}{\partial\xi} - \frac{\partial^2\nu_0}{\partial\zeta^2} - \frac{\partial^2\nu_0}{\partial\xi^2}\right]\right],$$
$$\nu_2(\zeta,\xi,\tau) = 2(\zeta-\xi)\left\{(1-\delta)^2 + 2\delta(1-\delta)\tau + \frac{\delta^2\tau^2}{2}\right\}.$$

The ETDM solution for example (5.5) is

$$\mu(\zeta,\xi,\tau) = \mu_0(\zeta,\xi,\tau) + \mu_1(\zeta,\xi,\tau) + \mu_2(\zeta,\xi,\tau) + \mu_3(\zeta,\xi,\tau) + \cdots,$$
$$\nu(\zeta,\xi,\tau) = \nu_0(\zeta,\xi,\tau) + \nu_1(\zeta,\xi,\tau) + \nu_2(\zeta,\xi,\tau) + \nu_3(\zeta,\xi,\tau) + \cdots,$$
$$\mu(\zeta,\xi,\tau) = \zeta + \xi - 2\zeta\{\delta\tau + (1-\delta)\} + 2(\zeta+\xi)\left\{(1-\delta)^2 + 2\delta(1-\delta)\tau + \frac{\delta^2\tau^2}{2}\right\} + \cdots$$
$$\nu(\zeta,\xi,\tau) = \zeta - \xi - 2\xi\{\delta\tau + (1-\delta)\} + 2(\zeta-\xi)\left\{(1-\delta)^2 + 2\delta(1-\delta)\tau + \frac{\delta^2\tau^2}{2}\right\} + \cdots.$$

When $\delta = 1$, the ETDM solution is

$$\mu(\zeta,\xi,\tau) = \zeta + \xi - 2\zeta\tau + 2(\zeta+\xi)\tau^2 - 4\tau^3\zeta + 4(\zeta+\xi)\tau^4 + \cdots$$
$$\nu(\zeta,\xi,\tau) = \zeta - \xi - 2\xi\tau + 2(\zeta-\xi)\tau^2 - 4\tau^3\xi + 4(\zeta-\xi)\tau^4 + \cdots.$$

The exact solutions are

$$\mu(\zeta,\xi,\tau) = \frac{\zeta - 2\zeta\tau + \xi}{1 - 2\tau^2},$$
$$\nu(\zeta,\xi,\tau) = \frac{\zeta - 2\xi\tau - \xi}{1 - 2\tau^2}.$$

5 RESULTS AND DISCUSSION

Figure 1 shows the comparison of ETDM, LRPSM, and exact 2D and 3D plots of fractional order solutions of example 4.1. The 2D and 3D plots have confirmed the closed contact between the ETDM, LRPSM, and exact solutions of example 4.1. **Figure 2** is dealing with ETDM, LRPSM, and exact 2D and 3D plots of example 4.2 solutions at different fractional-order and also at integer order of the derivative. **Figure 3**, represents the 2D and 3D plots of ETDM, LRPSM, and exact solutions at different fractional-order derivatives and integer-order derivatives of Example 4.3. **Figure 4**, represents the 2D plots of fractional and integer-order solutions by ETDM, LRPSM, and solutions of example 4.4. **Figure 5** confirms the clear relation among ETDM, LRPSM, and exact solutions of Example 4.4, using 3D plots of Example 4.4. **Figure 6** shows the 2D- plots of ETDM, LRPSM, and exact solutions of Example 4.5 at different fractional orders. In **Figure 7**, the 3D plot of u and w solutions of Example 4.5 at integer order 1 of ETDM, LRPSM, and Exact are presented.

Table 1, confirm the higher accuracy of LRPSM, and ETDM at different values of space and time variables, of Example 4.1. In **Table 2**, the μ corresponding errors associated with ETDM and LRPSM for μ-variable at various fractional order of example 4.2 are shown. In **Table 3**, the error associated with LRPSM and ETDM for the μ-solution of example 4.3 at different times and spaces is calculated. **Table 4**, displays the absolute error for μ-solution associated with ETDM and LRPSM at different times levels and spaces of example 4.4. **Table 5**, displays the absolute error for ν-solution associated with ETDM and LRPSM at different times levels and spaces of example 4.4. **Table 6**, displays the absolute error for μ-solution associated with ETDM and LRPSM at different times levels and spaces of example 4.5. **Table 7**, displays the absolute error for ν-solution associated with ETDM and LRPSM at different times levels and spaces of example 4.5.

6 CONCLUSION

The current article, presents the analytical solutions of one- and two-dimensional fractional Burger's equations and their systems using efficient techniques. In this regard, the solutions of the two innovative techniques, the Laplace residual power series method (LRPS) and the Elzaki transform decomposition method (ETDM), are compared within the Caputo operator. The comparison has confirmed that the suggested techniques provide identical solutions to both fractional- and integer-order solutions of the targeted problems. For the validity and applicability of the proposed techniques, the solutions of some illustrative examples are presented. The ETDM and LRPS algorithms are developed in a very simple and straightforward manner. The calculations in each algorithm are up to the limit. The tables and graphs are presented for the best display of the obtained results and errors associated with ETDM and LRPSM. The fractional-order solutions are calculated and are represented by graphs and tables. The accuracy of the suggested techniques is calculated in terms of absolute error associated with suggested techniques. The error analysis has confirmed the higher degree of accuracy and convergence rates. The present modifications to the existing techniques have brought significant change in the field of computational mathematics. It is, therefore, suggested to implement the current techniques in various areas of science and engineering.

DATA AVAILABILITY STATEMENT

The raw data supporting the conclusion of this article will be made available by the authors without undue reservation.

AUTHOR CONTRIBUTIONS

HK: supervision. QK: methodology, Hajira: draft writing. SK: analytic calculations, MA: draft writing. PK: funding.

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