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# Stability of Woven Frames

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This article studies the stability of woven frames by introducing some special limits. We show that there exist certain relations among the different types of convergence of frames and obtain some new and more general stability results. As an application of these results, we provide a method for constructing woven frames.

**Keywords: frames, woven frames, convergence of frames, stability, weaving frames**



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## 1 INTRODUCTION

Woven frames with some applications in coding and decoding [6, 9], distributed signal processing [6], and wireless sensor networks [1, 4] were first introduced in 2015 by Bemrose, Casazza, Grochenig, Lammers, and Lynch [1, 4]. Right now, it has been generalized to g-frames [10], K-frames [5, 11], fusion frames [8], etc.

**Definition 1.1** *A family $\{f_i\}_{i=1}^{\infty}$ for separable Hilbert space H is said to be a frame if there exist $0 < A \le B < \infty$ such that*

$$A\|f\|^2 \le \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \quad f \in H,$$

*where A, B are the lower frame bound and upper frame bound, respectively.*

If only the second inequality is required, it is called a *Bessel sequence*, and the $B$ is called the Bessel sequence bound. For a Bessel sequence $\{f_i\}_{i=1}^{\infty}$, the *synthesis operator* is defined by

$$T: l^2(\mathbb{N}) \to H, \quad T(c) = \sum_{i=1}^{\infty} c_i f_i, \quad c = \{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$$

is bounded. Its adjoint operator $T^*$ is called the *analysis operator*. The composite operator $S = TT^*$ is bounded, positive, and self-adjoint, and it is called the *frame operator* while $\{f_i\}_{i=1}^{\infty}$ is a frame for $H$.

**Definition 1.2** *The frames family $\{\{f_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ for separable Hilbert space H is woven, if there are universal constants $0 < A \le B < \infty$ such that*

$$A\|f\|^2 \le \sum_{j=1}^{m} \sum_{i\in\sigma_j} |\langle f, f_{ij}\rangle|^2 \le B\|f\|^2, \quad f \in H,$$

*for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{N}$. The family $\{f_{ij}\}_{i\in\sigma_j, j\in[m]}$ is called a weaving for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{N}$.*

We note that $i, j, k, m \in \mathbb{N}$ and $m \ge 2$, $[m] = \{1, 2, \ldots, m\}$, $\sigma = \{\sigma_j\}_{j\in[m]}$ is a partition of $\mathbb{N}$, $\Omega = \left\{\sigma | \sigma = \{\sigma_j\}_{j\in[m]} \text{ is any partition of } \mathbb{N}\right\}$. $H$ is a separable Hilbert space, $\{f_{ij}\}_{i=1}^{\infty}$ and $\{f_{ij}^{(k)}\}_{i=1}^{\infty}$ are the Bessel sequences for $H$, $\mathcal{F} = \{\{f_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ and $\mathcal{F}_k = \{\{f_{ij}^{(k)}\}_{i=1}^{\infty} | j \in [m]\}$ are the families of Bessel sequences. The synthesis operators of $\{f_{ij}^{(k)}\}_{i=1}^{\infty}$, $\{f_{ij}\}_{i=1}^{\infty}$, $\{f_{ij}^{(k)}\}_{i\in\sigma_j}$, $\{f_{ij}\}_{i\in\sigma_j}$ are listed as follows:

$$T_j^{(k)}(c)=\sum_{i=1}^{\infty}c_i f_{ij}^{(k)},\qquad T_j(c)=\sum_{i=1}^{\infty}c_i f_{ij},\quad c=\{c_i\}_{i=1}^{\infty}\in l^2(\mathbb{N});$$
$$T_{\sigma_j}^{(k)}(c)=\sum_{i\in\sigma_j}c_i f_{ij}^{(k)},\quad T_{\sigma_j}(c)=\sum_{i\in\sigma_j}c_i f_{ij},\quad c=\{c_i\}_{i\in\sigma_j}\in l^2\left(\sigma_j\right).$$

Moreover, $S_{\sigma_j}^{(k)}=T_{\sigma_j}^{(k)}T_{\sigma_j}^{(k)*}$, $S_{\sigma_j}=T_{\sigma_j}T_{\sigma_j}^{*}$, $S_{\sigma}^{(k)}=\sum_{j=1}^{m}S_{\sigma_j}^{(k)}$ and $S_\sigma=\sum_{j=1}^{m}S_{\sigma_j}$ for any $f\in H$ and $\sigma=\{\sigma_j\}_{j\in[m]}\in\Omega$.

This article focuses on the stability of woven frames, i.e., answers the following question: Suppose that the frames family $\{\{f_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ is woven for $H$ with universal lower and upper bounds $A$ and $B$. We want to find some conditions about $\{\{f_{ij}-g_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ such that the family $\{\{g_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ is woven for $H$. In $m=2$ case, this question was first considered by Bemrose, Casazza, Grochenig, Lammers, and Lynch [1, 4], after that it was reconsidered by Ghobadzadeh, Najati, Anastassiou, and Park [7], right now their results have been generalized to K-frames, fusion frames, g-frames, and so on. Analyzing the existing results, it is not difficult to find that they are all based on sufficiently small perturbation. In order to explore the relations among them and generalize them from $m=2$ to $2\le m<\infty$, we introduce four types of convergence of frames $\mathcal{F}_k\to\mathcal{F}$.

### 1.1 Strong Convergence

In this case, the limit $\mathcal{F}$ is unique. Naturally, if the frames family $\mathcal{F}$ is woven for $H$, then all $\mathcal{F}_k$ in a sufficiently small neighborhood of $\mathcal{F}$ is woven for $H$.

### 1.2 Convergence in Terms of Synthesis Operator or Analysis Operator

Note that the two types of convergence are equivalent. In both cases, the limit $\mathcal{F}$ is not necessarily unique, but the synthesis operators of frames in $\mathcal{F}$ are unique, this means that the corresponding analysis operators are also unique. Similarly to the first case, if the frames family $\mathcal{F}$ is woven for $H$, then $\mathcal{F}_k$ is woven for all sufficiently big $k$.

### 1.3 Convergence in Terms of Frame Operator $S_\sigma$

In this case, the limit $\mathcal{F}$, the corresponding synthesis operators, and the analysis operators of frames in limit $\mathcal{F}$ are not necessarily unique, but the universal infimum and supremum of $\mathcal{F}$ are unique. This implies that the judgment theorem about $\mathcal{F}_k$ still holds.

It can be proved that type 1 implies type 2 and 3, type 2 and 3 imply type 4, but the reverse is not true. More generally, we conjecture that there probably exist other types of convergence and some different results about the stability of woven frames can be obtained from the new type of convergence.

This article is organized as follows: In **Section 2**, we introduce some special limits for woven frames and show the relations among different types of convergence of frames. In **Section 3**, we show some new results about the stability of woven frames.

## 2 CONVERGENCE FOR WOVEN FRAMES

In this section, we introduce four types of convergence for woven frames and discuss the relations among them.

**Definition 2.1** *We say a point sequence $\{\mathcal{F}_k\}_{k=1}^{\infty}$ strongly converges to the point $\mathcal{F}$ if $\lim_{k\to\infty}\max_{j\in[m]}\sum_{i=1}^{\infty}\|f_{ij}^{(k)}-f_{ij}\|^2=0$.*

**Definition 2.2** *We say a point sequence $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to the point $\mathcal{F}$ in terms of synthesis operator if $\lim_{k\to\infty}\max_{j\in[m]}\|T_j^{(k)}-T_j\|=0$.*

**Definition 2.3** *We say a point sequence $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to the point $\mathcal{F}$ in terms of analysis operator if $\lim_{k\to\infty}\max_{j\in[m]}\|T_j^{(k)*}-T_j^{*}\|=0$.*

It is known that $\|T_j^{(k)}-T_j\|=\|T_j^{(k)*}-T_j^{*}\|$, thus $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to $\mathcal{F}$ in terms of synthesis operator if and only if $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to $\mathcal{F}$ in terms of analysis operator.

**Definition 2.4** *We say a point sequence $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to the point $\mathcal{F}$ in terms of frame operator $S_\sigma$ if $\lim_{k\to\infty}\sup_{\sigma\in\Omega}\|S_\sigma^{(k)}-S_\sigma\|=0$.*

Next, we show the relations among the four types of convergence in Theorem 2.5 and Theorem 2.6.

**Theorem 2.5** *While $\{\mathcal{F}_k\}_{k=1}^{\infty}$ strongly converges to $\mathcal{F}$ or $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to $\mathcal{F}$ in terms of analysis operator or synthesis operator, we have $\lim_{k\to\infty}\|f_{ij}^{(k)}-f_{ij}\|=0$ for all $i\in\mathbb{N}$, $j\in[m]$.*

Proof. From

$$\|f_{ij}^{(k)}-f_{ij}\|^2\le\sum_{i=1}^{\infty}\|f_{ij}^{(k)}-f_{ij}\|^2,\quad \|f_{ij}^{(k)}-f_{ij}\|\le\|T_j^{(k)}-T_j\|$$
$$=\|T_j^{(k)*}-T_j^{*}\|$$

for all $i,k\in\mathbb{N}$ and $j\in[m]$, we can obtain this theorem.

**Theorem 2.6** If $\{\mathcal{F}_k\}_{k=1}^{\infty}$ strongly converges to $\mathcal{F}$ then $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to $\mathcal{F}$ in terms of analysis operator; $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to $\mathcal{F}$ in terms of analysis operator if and only if $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to $\mathcal{F}$ in terms of synthesis operator; If $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to $\mathcal{F}$ in terms of synthesis operator then $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to $\mathcal{F}$ in terms of frame operator $S_\sigma$.

**Proof**. It is obvious that $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to $\mathcal{F}$ in terms of synthesis operator if and only if $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to $\mathcal{F}$ in terms of analysis operator. For all $f\in H$ and $\|f\|=1$, we compute

$$\|T_j^{(k)*}f-T_j^{*}f\|^2=\sum_{i=1}^{\infty}|\langle f,f_{ij}^{(k)}-f_{ij}\rangle|^2\le\sum_{i=1}^{\infty}\|f\|^2\|f_{ij}^{(k)}-f_{ij}\|^2$$
$$=\sum_{i=1}^{\infty}\|f_{ij}^{(k)}-f_{ij}\|^2.$$

From $\lim_{k\to\infty}\max_{j\in[m]}\sum_{i=1}^{\infty}\|f_{ij}^{(k)}-f_{ij}\|^2=0$, we have $\lim_{k\to\infty}\max_{j\in[m]}\|T_j^{(k)}-T_j\|=0$. This means that if $\{\mathcal{F}_k\}_{k=1}^{\infty}$ strongly converges to $\mathcal{F}$ then $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to $\mathcal{F}$ in terms of analysis operator. We compute

$$\begin{aligned}\|S_{\sigma_j}^{(k)} - S_{\sigma_j}\| &= \|T_{\sigma_j}^{(k)}T_{\sigma_j}^{(k)*} - T_{\sigma_j}T_{\sigma_j}^*\| \\ &= \|\left(T_{\sigma_j}^{(k)}T_{\sigma_j}^{(k)*} - T_{\sigma_j}^{(k)}T_{\sigma_j}^*\right) \\ &\quad + \left(T_{\sigma_j}^{(k)}T_{\sigma_j}^* - T_{\sigma_j}T_{\sigma_j}^*\right)\| \le \|T_{\sigma_j}^{(k)}\left(T_{\sigma_j}^{(k)*} - T_{\sigma_j}^*\right)\| \\ &\quad + \|\left(T_{\sigma_j}^{(k)} - T_{\sigma_j}\right)T_{\sigma_j}^*\| \le \|T_{\sigma_j}^{(k)}\|\|T_{\sigma_j}^{(k)*} - T_{\sigma_j}^*\| + \|T_{\sigma_j}^{(k)} \\ &\quad - T_{\sigma_j}\|\|T_{\sigma_j}^*\| \\ &= \left(\|T_{\sigma_j}^{(k)}\| + \|T_{\sigma_j}\|\right)\|T_{\sigma_j}^{(k)} - T_{\sigma_j}\|.\end{aligned}$$

Note that

$$\|T_{\sigma_j}^{(k)}\| \le \|T_j^{(k)}\|,\ \|T_{\sigma_j}\| \le \|T_j\|,\ \|T_{\sigma_j}^{(k)} - T_{\sigma_j}\| \le \|T_j^{(k)} - T_j\|,$$

so

$$\|S_{\sigma_j}^{(k)} - S_{\sigma_j}\| \le \left(\|T_j^{(k)}\| + \|T_j\|\right)\|T_j^{(k)} - T_j\| \le M_j\|T_j^{(k)} - T_j\|$$

for some positive $M_j$. Furthermore,

$$\begin{aligned}\|S_\sigma^{(k)} - S_\sigma\| &= \left\|\sum_{j\in[m]} S_{\sigma_j}^{(k)} - \sum_{j\in[m]} S_{\sigma_j}\right\| \le \sum_{j\in[m]} \|S_{\sigma_j}^{(k)} \\ &\quad - S_{\sigma_j}\| \le \sum_{j\in[m]} M_j\|T_j^{(k)} - T_j\|.\end{aligned}$$

From $\lim_{k\to\infty}\max_{j\in[m]}\|T_j^{(k)*} - T_j^*\| = 0$, we have $\lim_{k\to\infty}\sup_{\sigma\in\Omega}\|S_{\sigma_j}^{(k)} - S_{\sigma_j}\| = 0$. It means that if $\{\mathcal{F}_k\}_{k=1}^\infty$ converges to $\mathcal{F}$ in terms of synthesis operator then $\{\mathcal{F}_k\}_{k=1}^\infty$ converges to $\mathcal{F}$ in terms of frame operator $S_\sigma$.

The following Example 2.7 and Example 2.8 show that the inverse proposition of Theorem 2.6 is untenable.

**Example 2.7** Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis for $H$. If $f_{ij}^{(k)} = (1 + \frac{1}{\sqrt{k+i}})e_i$, $f_{ij} = e_i$ for all $i, k \in \mathbb{N}$, $j \in [m]$, then $\{\mathcal{F}_k\}_{k=1}^\infty$ converges to $\mathcal{F}$ in terms of analysis operator, but $\{\mathcal{F}_k\}_{k=1}^\infty$ does not strongly converge to $\mathcal{F}$.

**Proof**. From

$$\|\left(T_j^{(k)*} - T_j^*\right)f\|^2 = \sum_{i=1}^\infty \left|\left\langle f, \frac{1}{\sqrt{k+i}}e_i\right\rangle\right|^2 \le \frac{1}{k+1}\|f\|^2, \quad f \in H,$$

we have $\lim_{k\to\infty}\max_{j\in[m]}\|T_j^{(k)*} - T_j^*\| = 0$, i.e., $\{\mathcal{F}_k\}_{k=1}^\infty$ converges to $\mathcal{F}$ in terms of analysis operator. From

$$\sum_{i=1}^\infty \|f_{ij}^{(k)} - f_{ij}\|^2 = \sum_{i=1}^\infty \left\|\frac{e_i}{\sqrt{k+i}}\right\|^2 = \sum_{i=1}^\infty \frac{\|e_i\|^2}{k+i} = \sum_{i=1}^\infty \frac{1}{k+i} = \infty,$$

we have that $\{\mathcal{F}_k\}_{k=1}^\infty$ does not strongly converge to $\mathcal{F}$.

**Example 2.8** Let $\{f_i\}_{i=1}^\infty$ be a Parseval frame for $H$. If $f_{ij}^{(k)} = -f_{ij} = f_i$ for all $i, k \in \mathbb{N}$, $j \in [m]$, then $\{\mathcal{F}_k\}_{k=1}^\infty$ converges to $\mathcal{F}$ in terms of frame operator $S_\sigma$, but $\{\mathcal{F}_k\}_{k=1}^\infty$does not converge to $\mathcal{F}$ in terms of synthesis operator.

**Proof**. We compute

$$\lim_{k\to\infty}\|f_{ij}^{(k)} - f_{ij}\| = \lim_{k\to\infty}\|2f_i\| > 0$$

for all $i \in \mathbb{N}$, $j \in [m]$ and

$$\lim_{k\to\infty}\sup_{\sigma\in\Omega}\|S_\sigma^{(k)} - S_\sigma\| = \lim_{k\to\infty}\sup_{\sigma\in\Omega}\|\mathbf{0}\| = 0.$$

From Definition 2.4 and Theorem 2.5, we complete the proof.

# 3 STABILITY OF WOVEN FRAMES

This section discusses the stability of woven frames by limits in **Section 2**. We generalize the existing results from $m = 2$ to $2 \le m < \infty$. Moreover, many new woven frames can be obtained by using the limits.

In Theorem 3.1, Corollary 3.2, and Theorem 3.4, we discuss the stability of woven frames in terms of frame operator $S_\sigma$.

**Theorem 3.1** Suppose that the frames family $\mathcal{F}$ is woven for $H$ with universal bounds $A$ and $B$. If $\{\mathcal{F}_k\}_{k=1}^\infty$ converges to $\mathcal{F}$ in terms of frame operator $S_\sigma$ then for any non-negative number $\varepsilon < A$ there exists a natural number $N$ such that $\sup_{\sigma\in\Omega}\|S_\sigma^{(k)} - S_\sigma\| \le \varepsilon$ for every $k > N$, this implies that $\mathcal{F}_k$ is woven for $H$ with universal bounds $A - \varepsilon$ and $B + \varepsilon$ for every $k > N$.

**Proof**. If $\{\mathcal{F}_k\}_{k=1}^\infty$ converges to $\mathcal{F}$ in terms of frame operator $S_\sigma$, from the Definition 2.4, for any $\varepsilon < A$ there exists a natural number $N$ such that $\sup_{\sigma\in\Omega}\|S_\sigma^{(k)} - S_\sigma\| \le \varepsilon$ for all $k > N$. Hence,

$$\begin{aligned}\|S_\sigma^{(k)}f\| &\ge \|S_\sigma f\| - \|S_\sigma^{(k)}f - S_\sigma f\| \ge \|S_\sigma f\| - \|S_\sigma^{(k)} \\ &\quad - S_\sigma\|\|f\| \ge (A - \varepsilon)\|f\|\end{aligned}$$

and

$$\begin{aligned}\|S_\sigma^{(k)}f\| &\le \|S_\sigma f\| + \|S_\sigma^{(k)}f - S_\sigma f\| \le \|S_\sigma f\| + \|S_\sigma^{(k)} \\ &\quad - S_\sigma\|\|f\| \le (B + \varepsilon)\|f\|,\end{aligned}$$

i.e., $(A - \varepsilon)\|f\| \le \|S_\sigma^{(k)}f\| \le (B + \varepsilon)\|f\|$ for all $f \in H$ and $\sigma \in \Omega$. This implies that the bounded linear operator $S_\sigma^{(k)}$ is an injection. It is known that the operator $S_\sigma^{(k)}$ is self-adjoint, thus $S_\sigma^{(k)}$ is also a surjection. From $S_\sigma^{(k)} = T_\sigma^{(k)}T_\sigma^{(k)*}$ we have $T_\sigma^{(k)}$ is a surjection, this implies that $\{f_{ij}^{(k)}\}_{i\in\sigma_j, j\in[m]}$ is a frame for $H$ with the frame operator $S_\sigma^{(k)}$. Hence $\mathcal{F}_k$ is woven with universal bounds $A-\ \varepsilon$ and $B+\ \varepsilon$ for every $k > N$.

**Corollary 3.2** Suppose that the frames family $\{\{f_{ij}\}_{i=1}^\infty | j \in [m]\}$ is woven for $H$ with universal bounds $A, B$ and $\{\{g_{ij}\}_{i=1}^\infty | j \in [m]\}$ is a family of Bessel sequences for $H$. If there exist non-negative numbers $\alpha_j$, $\mu_j$ satisfied $\varepsilon = \sum_{j=1}^m (\alpha_j B + \mu_j) < A$ such that

$$\begin{aligned}&\|\sum_{i\in\sigma_j}\langle f, f_{ij}\rangle f_{ij} - \sum_{i\in\sigma_j}\langle f, g_{ij}\rangle g_{ij}\| \le \alpha_j\|\sum_{i=1}^\infty\langle f, f_{ij}\rangle f_{ij}\| \\ &\quad + \mu_j\|f\|, \quad f \in H,\end{aligned}$$

for any $j \in [m]$ and $\sigma \in \Omega$, then $\{\{g_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ is woven for $H$ with the universal lower and upper bounds $A - \varepsilon$ and $B + \varepsilon$.

**Proof**. Since

$$\|\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle f, f_{ij}\rangle f_{ij} - \sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle f, g_{ij}\rangle g_{ij}\| \le \sum_{j=1}^{m}\|\sum_{i\in\sigma_j}\langle f, f_{ij}\rangle f_{ij} - \sum_{i\in\sigma_j}\langle f, g_{ij}\rangle g_{ij}\|$$

and

$$\sum_{j=1}^{m}\left(\alpha_j\|\sum_{i=1}^{\infty}\langle f, f_{ij}\rangle f_{ij}\| + \mu_j\|f\|\right) \le \sum_{j=1}^{m}\left(\alpha_j B + \mu_j\right)\|f\|, \quad f \in H,$$

combing with the inequality in Corollary 3.2, we have

$$\|\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle f, f_{ij}\rangle f_{ij} - \sum_{j=1}^{m} \times \sum_{i\in\sigma_j}\langle f, g_{ij}\rangle g_{ij}\| \le \sum_{j=1}^{m}\left(\alpha_j B + \mu_j\right)\|f\| < A\|f\|, \quad f \in H.$$

Take $\varepsilon = \sum_{j=1}^{m}(\alpha_j B + \mu_j) < A$ and $\{\{g_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ can be regarded as $\mathcal{F}_k$ for some $k > N$, from Theorem 3.1, we obtain Corollary 3.2.

**Example 3.3** Suppose that the frames family $\{\{f_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ is woven for $H$ with the universal bounds $A$, $B$ and $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $H$. If the number $\lambda$, $\delta$ satisfied

$$\sqrt{1 - m^{-1}AB^{-1}} < \lambda \le 1, \quad 0 \le \delta < \sqrt{m^{-1}A + (2\lambda^2 - 1)B} - \lambda\sqrt{B},$$

then the family $\{\{\delta e_i - \lambda f_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ is woven for $H$ with the universal lower and upper bounds $A - m[(1 - \lambda^2)B + 2\lambda\delta\sqrt{B} + \delta^2]$, $B + m[(1 - \lambda^2)B + 2\lambda\delta\sqrt{B} + \delta^2]$.

**Proof**. Let $g_{ij} = \delta e_i - \lambda f_{ij}$ for all $i \in \mathbb{N}, j \in [m]$. Then $\{g_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a Bessel sequence for $H$ with bound $(\delta + \lambda\sqrt{B})^2$ and $\lambda f_{ij} + g_{ij} = \delta e_i$ for $i \in \mathbb{N}, j \in [m]$.

Let

$$M_1 = \sum_{i\in\sigma_j}\langle f, \lambda f_{ij}\rangle\lambda f_{ij} - \sum_{i\in\sigma_j}\langle f, g_{ij}\rangle g_{ij},$$
$$M = \sum_{i\in\sigma_j}\langle f, f_{ij}\rangle f_{ij} - \sum_{i\in\sigma_j}\langle f, g_{ij}\rangle g_{ij}, \quad f \in H.$$

Then,

$$\begin{aligned}
\|M_1\| &= \|\sum_{i\in\sigma_j}\langle f, \lambda f_{ij}\rangle\left(\lambda f_{ij} + g_{ij}\right) - \sum_{i\in\sigma_j}\langle f, \lambda f_{ij} + g_{ij}\rangle g_{ij}\| \\
&= \|\sum_{i\in\sigma_j}\langle f, \lambda f_{ij}\rangle\delta e_i - \sum_{i\in\sigma_j}\langle f, \delta e_i\rangle g_{ij}\| \\
&\le \|\sum_{i\in\sigma_j}\langle f, \lambda f_{ij}\rangle\delta e_i\| + \|\sum_{i\in\sigma_j}\langle f, \delta e_i\rangle g_{ij}\| \\
&\le \lambda\delta\sqrt{B}\|f\| + \delta(\delta + \lambda\sqrt{B})\|f\| \\
&= (2\lambda\delta\sqrt{B} + \delta^2)\|f\|,
\end{aligned}$$

and furthermore,

$$\|M\| = \|(1 - \lambda^2)\sum_{i\in\sigma_j}\langle f, f_{ij}\rangle f_{ij} + M_1\| \le (1 - \lambda^2)\|\sum_{i\in\sigma_j}\langle f, f_{ij}\rangle f_{ij}\| + \|M_1\|,$$

where

$$\sqrt{1 - m^{-1}AB^{-1}} < \lambda \le 1, \quad 0 \le \delta < \sqrt{m^{-1}A + (2\lambda^2 - 1)B} - \lambda\sqrt{B},$$

i.e.,

$$\varepsilon = m(1 - \lambda^2)B + m\left(2\lambda\delta\sqrt{B} + \delta^2\right) < A,$$

and by Corrolary 3.2, we have $\{\{\delta e_i - \lambda f_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ is woven for $H$ with the universal lower and upper bounds

$$A - m\left[(1 - \lambda^2)B + 2\lambda\delta\sqrt{B} + \delta^2\right],$$
$$B + m\left[(1 - \lambda^2)B + 2\lambda\delta\sqrt{B} + \delta^2\right].$$

The proof is completed.

Similarly to the classical perturbations of frames, we have the following Theorem 3.4.

**Theorem 3.4** Suppose that the frames family $\{\{f_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ is woven for $H$ with the universal bounds $A, B$ and $\{\{g_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ is a family of Bessel sequences for $H$. If there exist non-negative numbers $\alpha$, $\beta$, $\mu$ satisfied $\max\left\{\alpha + \frac{\mu}{A}, \beta\right\} < 1$ such that

$$\|\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle f, f_{ij}\rangle f_{ij} - \sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle f, g_{ij}\rangle g_{ij}\| \le \alpha\|\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle f, f_{ij}\rangle f_{ij}\| + \beta\|\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle f, g_{ij}\rangle g_{ij}\| + \mu\|f\|$$

for any $f \in H$ and $\sigma \in \Omega$, then $\{\{g_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ is woven for $H$ with the universal lower and upper bounds $((1 - \alpha)A - \mu)(1 + \beta)^{-1}$ and $((1 + \alpha)B + \mu)(1 - \beta)^{-1}$

**Proof**. Let

$$S_\sigma f = \sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle f, f_{ij}\rangle f_{ij},$$
$$S'_\sigma f = \sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle f, g_{ij}\rangle g_{ij}, \quad f \in H, \sigma \in \Omega.$$

Then,

$$\begin{aligned}
\|S'_\sigma f\| &\le \|S_\sigma f\| + \|S'_\sigma f - S_\sigma f\| \le (1 + \alpha)\|S_\sigma f\| + \beta\|S'_\sigma f\| + \mu\|f\| \\
&\le \left((1 + \alpha)B + \mu\right)\|f\| + \beta\|S'_\sigma f\|, \quad f \in H, \sigma \in \Omega,
\end{aligned}$$

and this implies that $\|S'_\sigma f\| \le ((1 + \alpha)B + \mu)(1 - \beta)^{-1}\|f\|$ for any $f \in H$, $\sigma \in \Omega$. On the other hand,

$$\begin{aligned}
\|S'_\sigma f\| &\ge \|S_\sigma f\| - \|S_\sigma f - S'_\sigma f\| \ge (1 - \alpha)\|S_\sigma f\| - \beta\|S'_\sigma f\| - \mu\|f\| \\
&\ge \left((1 - \alpha)A - \mu\right)\|f\| - \beta\|S'_\sigma f\|, \quad f \in H, \sigma \in \Omega,
\end{aligned}$$

and this implies that $\|S'_\sigma f\| \ge ((1 - \alpha)A - \mu)(1 + \beta)^{-1}\|f\|$ for any $f \in H$, $\sigma \in \Omega$.

Since $S'_\sigma = T'_\sigma T'^{*}_\sigma$ is self-adjoint and $\max\left\{\alpha + \frac{\mu}{A}, \beta\right\} < 1$, we have $S'_\sigma$ is a surjection, i.e., the synthesis operator $T'_\sigma$ is a surjection. This means that $\{\{g_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ is woven for $H$. Furthermore, we can obtain the universal lower and upper bounds $((1-\alpha)A - \mu)(1+\beta)^{-1}$ and $((1+\alpha)B + \mu)(1-\beta)^{-1}$ by the frame operator $S'_\sigma$.

**Example 3.5** Suppose that the frames family $\{\{f_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ is woven for $H$ with the universal bounds $A$, $B$ and $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $H$. If the number $\lambda$, $\eta$, $\delta$ satisfied

$$0 < \lambda^2 \le 1, \quad 0 < \eta^2 \le 1, \quad 0 \le \delta^2 < \lambda^2 \left(\sqrt{A+B} - \sqrt{B}\right)^2,$$

then the family $\{\{\delta\eta^{-1}e_i - \lambda\eta^{-1}f_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ is woven for $H$ with the universal lower and upper bounds

$$\frac{\lambda^2 A - \left(2|\lambda\|\delta|\sqrt{B} + \delta^2\right)}{2-\eta^2}, \quad \frac{(2-\lambda^2)B + \left(2|\lambda\|\delta|\sqrt{B} + \delta^2\right)}{\eta^2}.$$

**Proof**. Let $g_{ij} = \delta\eta^{-1}e_i - \lambda\eta^{-1}f_{ij}$ for all $i \in \mathbb{N}$, $j \in [m]$. Then, $\lambda f_{ij} + \eta g_{ij} = \delta e_i$ for all $i \in \mathbb{N}$, $j \in [m]$ and $\{g_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a Bessel sequence for $H$ with the bound $(|\delta||\eta^{-1}| + |\lambda||\eta^{-1}|\sqrt{B})^2$ from

$$\|\sum_{j=1}^{m}\sum_{i\in\sigma_j} c_i g_{ij}\| \le \|\sum_{j=1}^{m}\sum_{i\in\sigma_j} c_i \delta\eta^{-1} e_i\| + \|\sum_{j=1}^{m} \times \sum_{i\in\sigma_j} c_i \lambda\eta^{-1} f_{ij}\| \le \left(|\delta||\eta^{-1}| + |\lambda||\eta^{-1}|\sqrt{B}\right)\|c\|$$

for all $c = \{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$ and $\|c\| = 1$. Let

$$M_1 = \sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle f, \lambda f_{ij}\rangle \lambda f_{ij} - \sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle f, \eta g_{ij}\rangle \eta g_{ij}, \quad f \in H$$

and

$$M = \sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle f, f_{ij}\rangle f_{ij} - \sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle f, g_{ij}\rangle g_{ij}, \quad f \in H.$$

Then,

$$\begin{aligned}
\|M_1\| &= \|\sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle f, \lambda f_{ij}\rangle \left(\lambda f_{ij} + \eta g_{ij}\right) - \sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle f, \lambda f_{ij} \\
&\quad + \eta g_{ij}\rangle \eta g_{ij}\| \\
&= \|\sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle f, \lambda f_{ij}\rangle \delta e_i - \sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle f, \delta e_i\rangle \eta g_{ij}\| \le \|\sum_{j=1}^{m} \\
&\quad \times \sum_{i\in\sigma_j} \langle f, \lambda f_{ij}\rangle \delta e_i\| + \|\sum_{j=1}^{m} \\
&\quad \times \sum_{i\in\sigma_j} \langle f, \delta e_i\rangle \eta g_{ij}\| \le |\lambda\|\delta|\sqrt{B}\|f\| \\
&\quad + |\delta\|\eta|\left(|\delta||\eta^{-1}| + |\lambda||\eta^{-1}|\sqrt{B}\right)\|f\| \\
&= \left(2|\lambda\|\delta|\sqrt{B} + \delta^2\right)\|f\|,
\end{aligned}$$

and further more,

$$\begin{aligned}
\|M\| &= \|\left(1-\lambda^2\right)\sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle f, f_{ij}\rangle f_{ij} - \left(1-\eta^2\right)\sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle f, g_{ij}\rangle g_{ij} \\
&\quad + M_1\| \le \left(1-\lambda^2\right)\|\sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle f, f_{ij}\rangle f_{ij}\| + \left(1-\eta^2\right)\|\sum_{j=1}^{m} \\
&\quad \times \sum_{i\in\sigma_j} \langle f, g_{ij}\rangle g_{ij}\| + \|M_1\|,
\end{aligned}$$

where

$$0 < \lambda^2, \eta^2 \le 1, \quad 0 \le \delta^2 < \lambda^2\left(\sqrt{A+B} - \sqrt{B}\right)^2,$$

i.e.,

$$\max\left\{\left(1-\lambda^2\right) + \frac{2|\lambda\|\delta|\sqrt{B} + \delta^2}{A}, 1-\eta^2\right\} < 1,$$

and by Theorem 3.4, we have $\{\{\delta\eta^{-1}e_i - \lambda\eta^{-1}f_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ is woven for $H$ with the universal lower and upper bounds

$$\frac{\lambda^2 A - \left(2|\lambda\|\delta|\sqrt{B} + \delta^2\right)}{2-\eta^2}, \quad \frac{(2-\lambda^2)B + \left(2|\lambda\|\delta|\sqrt{B} + \delta^2\right)}{\eta^2}.$$

The proof is completed.

Lemma 3.6 is a remarkable result on the perturbation of frames.

**Lemma 3.6** [2] Suppose that $\{f_i\}_{i=1}^{\infty}$ is a frame for $H$ with the bounds $A$ and $B$, $\{g_i\}_{i=1}^{\infty} \subset H$. If there exist non-negative numbers $\alpha$, $\beta$, $\mu$ satisfied $\max\left\{\alpha + \frac{\mu}{\sqrt{A}}, \beta\right\} < 1$ such that for any $c_1, c_2, \ldots, c_n (n \in \mathbb{N})$, we have

$$\|\sum_{i=1}^{n} c_i f_i - \sum_{i=1}^{n} c_i g_i\| \le \alpha\|\sum_{i=1}^{n} c_i f_i\| + \beta\|\sum_{i=1}^{n} c_i g_i\| + \mu\left(\sum_{i=1}^{n} |c_i|^2\right)^{\frac{1}{2}}$$

then $\{g_i\}_{i=1}^{\infty}$ is a frame for $H$ with bounds $A(1 - \frac{\alpha+\beta+\mu/\sqrt{A}}{1+\beta})^2$ and $B(1 + \frac{\alpha+\beta+\mu/\sqrt{B}}{1-\beta})^2$.

From this lemma, we can obtain the following theorem.

**Theorem 3.7** Suppose that the frames family $\mathcal{F}$ is woven for $H$ with the universal lower and upper bounds $A$ and $B$. If $\{\mathcal{F}_k\}_{k=1}^{\infty}$ converges to $\mathcal{F}$ in terms of analysis operator or synthesis operator then for any non-negative number $\varepsilon < \sqrt{A}$ there exists a natural number $N$ such that $\sum_{j=1}^{m} \|T_j^{(k)^*} - T_j^*\| = \sum_{j=1}^{m} \|T_j^{(k)} - T_j\| \le \varepsilon$ for every $k > N$, this implies that $\mathcal{F}_k$ is woven for $H$ with the universal lower and upper bounds $(\sqrt{A} - \varepsilon)^2$ and $(\sqrt{B} + \varepsilon)^2$ for every $k > N$.

**Proof**. For any $\sigma = \{\sigma_j\}_{j=1}^{m} \in \Omega$ and $c = \{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$, we compute

$$\begin{aligned}
\|\sum_{j=1}^{m}\sum_{i\in\sigma_j} c_i f_{ij} - \sum_{j=1}^{m}\sum_{i\in\sigma_j} c_i f_{ij}^{(k)}\| &\le \sum_{j=1}^{m}\|\sum_{i\in\sigma_j} c_i f_{ij} - \sum_{i\in\sigma_j} c_i f_{ij}^{(k)}\| \\
&\le \sum_{j=1}^{m}\|T_{\sigma_j} - T_{\sigma_j}^{(k)}\|\left(\sum_{i\in\sigma_j}|c_i|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Combining with

$$\left(T_{\sigma_j} - T_{\sigma_j}^{(k)}\right)^* f = \left\{\langle f, f_{ij} - f_{ij}^{(k)}\rangle\right\}_{i\in\sigma_j}, \ \left(T_j - T_j^{(k)}\right)^* f = \left\{\langle f, f_{ij} - f_{ij}^{(k)}\rangle\right\}_{i=1}^{\infty}$$

for $f \in H$ and

$$\left\|\left(T_{\sigma_j} - T_{\sigma_j}^{(k)}\right)^*\right\| = \sup_{\|f\|=1}\left(\sum_{i\in\sigma_j}|\langle f, f_{ij} - f_{ij}^{(k)}\rangle|^2\right)^{\frac{1}{2}} \le \sup_{\|f\|=1}\left(\sum_{i=1}^{\infty}|\langle f, f_{ij} - f_{ij}^{(k)}\rangle|^2\right)^{\frac{1}{2}} = \left\|\left(T_j - T_j^{(k)}\right)^*\right\|,$$

we have

$$\sum_{j=1}^{m}\|T_{\sigma_j} - T_{\sigma_j}^{(k)}\|\left(\sum_{i\in\sigma_j}|c_i|^2\right)^{\frac{1}{2}} \le \sum_{j=1}^{m}\|T_j - T_j^{(k)}\|\left(\sum_{i=1}^{\infty}|c_i|^2\right)^{\frac{1}{2}} \le \varepsilon\left(\sum_{i=1}^{\infty}|c_i|^2\right)^{\frac{1}{2}}$$

for $c = \{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$. Let $\{g_i\}_{i=1}^{\infty} = \{f_{ij}^{(k)}\}_{i\in\sigma_j, j\in[m]}$ and $\{f_i\}_{i=1}^{\infty} = \{f_{ij}\}_{i\in\sigma_j, j\in[m]}$, then $\|\sum_{i=1}^{\infty} c_i f_i - \sum_{i=1}^{\infty} c_i g_i\| \le \varepsilon(\sum_{i=1}^{\infty}|c_i|^2)^{\frac{1}{2}}$, this implies that $\|\sum_{i=1}^{n} c_i f_i - \sum_{i=1}^{n} c_i g_i\| \le \varepsilon(\sum_{i=1}^{n}|c_i|^2)^{\frac{1}{2}}$. From Lemma 3.6, the frames family $\mathcal{F}_k$ is woven for $H$ with the universal lower and upper bounds $(\sqrt{A} - \varepsilon)^2$ and $(\sqrt{B} + \varepsilon)^2$ for every $k > N$.

**Corollary 3.8** Suppose that $\mathcal{F}$ is woven for $H$ with the universal lower and upper bounds $A$ and $B$. If $\{\mathcal{F}_k\}_{k=1}^{\infty}$ strongly converges to $\mathcal{F}$ then for any non-negative number $\varepsilon < \sqrt{A}$ there exists a natural number $N$ such that $\sum_{j=1}^{m}(\sum_{i=1}^{\infty}\|f_{ij}^{(k)} - f_{ij}\|^2)^{\frac{1}{2}} \le \varepsilon$ for every $k > N$, this implies that $\mathcal{F}_k$ is woven for $H$ with the universal lower and upper bounds $(\sqrt{A} - \varepsilon)^2$ and $(\sqrt{B} + \varepsilon)^2$ for every $k > N$.

**Proof**. From the proof of Theorem 2.6, we have

$$\sum_{j=1}^{m}\|T_j^{(k)*} - T_j^*\| = \sum_{j=1}^{m}\|T_j^{(k)} - T_j\| \le \sum_{j=1}^{m}\left(\sum_{i=1}^{\infty}\|f_{ij}^{(k)} - f_{ij}\|^2\right)^{\frac{1}{2}} \le \varepsilon < \sqrt{A}.$$

By Theorem 3.7, we have that $\mathcal{F}_k$ is woven for $H$ with the universal lower and upper bounds $(\sqrt{A} - \varepsilon)^2$ and $(\sqrt{B} + \varepsilon)^2$ for every $k > N$.

**Example 3.9** Suppose that $\mathcal{F}$ is woven for $H$ with the universal lower and upper bounds $A$, $B$ and $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $H$. If

$$f_{ij}^{(k)} = f_{ij} + 2^{-(k+i)}m^{-1}\sqrt{A}e_i \quad i \in \mathbb{N}, j \in [m]$$

then $\mathcal{F}_k$ is woven for $H$ for every $k \in \mathbb{N}$.

**Proof**. From

$$\sum_{j=1}^{m}\left(\sum_{i=1}^{\infty}\|f_{ij}^{(k)} - f_{ij}\|^2\right)^{\frac{1}{2}} < \sum_{j=1}^{m} m^{-1}\sqrt{A} = \sqrt{A},$$

we complete the proof.

**Corollary 3.10** Suppose that the frames family $\{\{f_{ij}\}_{i=1}^{\infty}|j \in [m]\}$ is woven for $H$ with the universal lower and upper bounds $A$, $B$ and $\{\{g_{ij}\}_{i=1}^{\infty}|j \in [m]\}$ is a family of sequences for $H$. If there exist non-negative numbers $\alpha_j$, $\beta_j$, $\mu_j$ satisfied $\varepsilon = \sum_{j=1}^{m}[(\alpha_j + \beta_j)\sqrt{B} + \mu_j](1 - \beta_j)^{-1} < \sqrt{A}$ such that for any $c_1, c_2, \ldots, c_n (n \in \mathbb{N})$ and $j \in [m]$, we have

$$\|\sum_{i=1}^{n} c_i f_{ij} - \sum_{i=1}^{n} c_i g_{ij}\| \le \alpha_j\|\sum_{i=1}^{n} c_i f_{ij}\| + \beta_j\|\sum_{i=1}^{n} c_i g_{ij}\| + \mu_j\left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac{1}{2}},$$

then $\{\{g_{ij}\}_{i=1}^{\infty}|j \in [m]\}$ is woven for $H$ with the universal lower and upper bounds $(\sqrt{A} - \varepsilon)^2$ and $(\sqrt{B} + \varepsilon)^2$.

**Proof**. By Lemma 3.6, we have that for any $j \in [m]$, $\{g_{ij}\}_{i=1}^{\infty}$ is a frame for $H$ with bounds $A(1 - \frac{\alpha_j + \beta_j + \mu_j/\sqrt{A}}{1+\beta_j})^2$ and $B(1 + \frac{\alpha_j + \beta_j + \mu_j/\sqrt{B}}{1-\beta_j})^2$. Hence,

$$\begin{aligned}\|\sum_{i=1}^{n} c_i f_{ij} - \sum_{i=1}^{n} c_i g_{ij}\| &\le \alpha_j\|\sum_{i=1}^{n} c_i f_{ij}\| + \beta_j\|\sum_{i=1}^{n} c_i g_{ij}\| + \mu_j\left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac{1}{2}} \le \left[\alpha_j\sqrt{B} + \beta_j\sqrt{B}\left(1 + \frac{\alpha_j + \beta_j + \mu_j/\sqrt{B}}{1 - \beta_j}\right) + \mu_j\right]\left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac{1}{2}} \\ &= \left[\left(\alpha_j + \beta_j\right)\sqrt{B} + \mu_j\right]\left(1 - \beta_j\right)^{-1}\left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac{1}{2}}, \\ &\le \left[\left(\alpha_j + \beta_j\right)\sqrt{B} + \mu_j\right]\left(1 - \beta_j\right)^{-1}\left(\sum_{i=1}^{\infty}|c_i|^2\right)^{\frac{1}{2}},\end{aligned}$$

and let $n \to \infty$; then, we have

$$\|\sum_{i=1}^{\infty} c_i f_{ij} - \sum_{i=1}^{\infty} c_i g_{ij}\| \le \left[\left(\alpha_j + \beta_j\right)\sqrt{B} + \mu_j\right]\left(1 - \beta_j\right)^{-1}\left(\sum_{i=1}^{\infty}|c_i|^2\right)^{\frac{1}{2}},$$

i.e., $\|T_j^{(k)} - T_j\| \le [(\alpha_j + \beta_j)\sqrt{B} + \mu_j](1 - \beta_j)^{-1}$, where

$$T_j\left(\{c_i\}_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} c_i f_{ij}, \quad T_j^{(k)}\left(\{c_i\}_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} c_i g_{ij}, \quad \{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N}).$$

Computing

$$\sum_{j=1}^{m}\|T_j^{(k)} - T_j\| \le \sum_{j=1}^{m}\left[\left(\alpha_j + \beta_j\right)\sqrt{B} + \mu_j\right]\left(1 - \beta_j\right)^{-1} = \varepsilon < \sqrt{A},$$

from Theorem 3.7, the frames family $\{\{g_{ij}\}_{i=1}^{\infty}|j \in [m]\}$ is woven for $H$ with the universal lower and upper bounds $(\sqrt{A} - \varepsilon)^2$ and $(\sqrt{B} + \varepsilon)^2$.

**Example 3.11** Suppose that the frames family $\{\{f_{ij}\}_{i=1}^{\infty}|j \in [m]\}$ is woven for $H$ with the universal bounds $A$, $B$ and $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $H$. If the number $\lambda$, $\eta$, $\delta$ satisfied

$$1-\frac{\sqrt{A}}{m\sqrt{B}}<\lambda\leq 1,\quad \frac{m(2-\lambda)\sqrt{B}}{m\sqrt{B}+\sqrt{A}}<\eta\leq 1,\quad 0\leq\delta<m^{-1}\eta\sqrt{A}-(2-\lambda-\eta)\sqrt{B},$$

then the family $\{\{\lambda\eta^{-1}f_{ij}-\delta\eta^{-1}e_i\}_{i=1}^{\infty}|j\in[m]\}$ is woven for $H$ with the universal lower and upper bounds

$$A\left[1-\frac{m(2-\lambda-\eta)\sqrt{B}+m\delta}{\eta\sqrt{A}}\right]^2,\quad B\left[1+\frac{m(2-\lambda-\eta)\sqrt{B}+m\delta}{\eta\sqrt{B}}\right]^2.$$

**Proof**. Let $g_{ij}=\lambda\eta^{-1}f_{ij}-\delta\eta^{-1}e_i$, i.e. $\lambda f_{ij}-\eta g_{ij}=\delta e_{ij}$ for $i\in\mathbb{N}, j\in[m]$. Then

$$\|\sum_{i=1}^{n}c_if_{ij}-\sum_{i=1}^{n}c_ig_{ij}\|=\|(1-\lambda)\sum_{i=1}^{n}c_if_{ij}-(1-\eta)\sum_{i=1}^{n}c_ig_{ij}+\delta\sum_{i=1}^{n}c_ie_i\|\leq(1-\lambda)\|\sum_{i=1}^{n}c_if_{ij}\|+(1-\eta)\|\sum_{i=1}^{n}c_ig_{ij}\|+\delta\left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac{1}{2}},$$

where

$$1-\frac{\sqrt{A}}{m\sqrt{B}}\leq\lambda\leq 1,\quad \frac{m(2-\lambda)\sqrt{B}}{m\sqrt{B}+\sqrt{A}}<\eta\leq 1,\quad 0\leq\delta<m^{-1}\eta\sqrt{A}-(2-\lambda-\eta)\sqrt{B}$$

satisfied

$$\varepsilon=\sum_{j=1}^{m}\left[(1-\lambda+1-\eta)\sqrt{B}+\delta\right]\eta^{-1}=\left[m(2-\lambda-\eta)\sqrt{B}+m\delta\right]\eta^{-1}<\sqrt{A}.$$

By Corollary 3.10, the family $\{\{\lambda\eta^{-1}f_{ij}-\delta\eta^{-1}e_i\}_{i=1}^{\infty}|j\in[m]\}$ is woven for $H$ with the universal lower and upper bounds

$$A\left[1-\frac{m(2-\lambda-\eta)\sqrt{B}+m\delta}{\eta\sqrt{A}}\right]^2,\quad B\left[1+\frac{m(2-\lambda-\eta)\sqrt{B}+m\delta}{\eta\sqrt{B}}\right]^2.$$

We complete the proof.

From Theorem 3.7 or Corollary 3.10, we can obtain Theorem 3.2, Theorem 3.3, Proposition 3.4, and Corollary 3.5 in [7], and obtain Theorem 6.1 in [1]. The following corollary is obvious from Corollary 3.10.

**Corollary 3.12** Suppose that the frames family $\{\{f_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ is woven for $H$ with the universal lower and upper bounds $A$, $B$ and $\{\{g_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ is a family of sequences for $H$. If there exist non-negative numbers $\alpha_j$, $\mu_j$ satisfied $\varepsilon=\sum_{j=1}^{m}(\alpha_j\sqrt{B}+\mu_j)<\sqrt{A}$ such that for any $c_1,c_2,\ldots,c_n\,(n\in\mathbb{N})$ and $j\in[m]$, we have

$$\|\sum_{i=1}^{n}c_if_{ij}-\sum_{i=1}^{n}c_ig_{ij}\|\leq\alpha_j\|\sum_{i=1}^{n}c_if_{ij}\|+\mu_j\left(\sum_{i=1}^{n}|c_i|^2\right)^{\frac{1}{2}},$$

and then, $\{\{g_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ is woven for $H$ with the universal lower and upper bounds $(\sqrt{A}-\varepsilon)^2$ and $(\sqrt{B}+\varepsilon)^2$.

**Corollary 3.13** Suppose that the frames family $\{\{f_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ is woven for $H$ with universal lower and upper bounds $A$, $B$ and $\{\{g_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ is a family of Bessel sequences for $H$. If there exist non-negative numbers $\alpha_j$, $\mu_j$ satisfied $\varepsilon=\sum_{j=1}^{m}(\alpha_j\sqrt{B}+\mu_j)<\sqrt{A}$ such that for any $j\in[m]$, we have

$$\left(\sum_{i=1}^{\infty}|\langle f,f_{ij}-g_{ij}\rangle|^2\right)^{\frac{1}{2}}\leq\alpha_j\left(\sum_{i=1}^{\infty}|\langle f,f_{ij}\rangle|^2\right)^{\frac{1}{2}}+\mu_j\|f\|,\quad f\in H,$$

and then, $\{\{g_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ is woven for $H$ with the universal lower and upper bounds $(\sqrt{A}-\varepsilon)^2$ and $(\sqrt{B}+\varepsilon)^2$.

**Proof**. We compute

$$\left(\sum_{i=1}^{\infty}|\langle f,f_{ij}-g_{ij}\rangle|^2\right)^{\frac{1}{2}}\leq\alpha_j\left(\sum_{i=1}^{\infty}|\langle f,f_{ij}\rangle|^2\right)^{\frac{1}{2}}+\mu_j\|f\|\leq\left(\alpha_j\sqrt{B}+\mu_j\right)\|f\|,\quad f\in H,$$

i.e., $\|T_j^{(k)*}-T_j^*\|\leq(\alpha_j\sqrt{B}+\mu_j)$, where

$$T_j^*(f)=\left\{\langle f,f_{ij}\rangle\right\}_{i=1}^{\infty},\quad T_j^{(k)*}(f)=\left\{\langle f,g_{ij}\rangle\right\}_{i=1}^{\infty},\quad f\in H.$$

Hence $\sum_{j=1}^{m}\|T_j^{(k)^*}-T_j^*\|\leq\sum_{j=1}^{m}(\alpha_j\sqrt{B}+\mu_j)=\varepsilon<\sqrt{A}$. From Theorem 3.7, we have that $\{\{g_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ is woven for $H$ with the universal lower and upper bounds $(\sqrt{A}-\varepsilon)^2$ and $(\sqrt{B}+\varepsilon)^2$.

**Corollary 3.14** Suppose that the frames family $\{\{f_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ is woven for $H$ with the universal lower and upper bounds $A$, $B$ and $\{\{g_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ is a family of Bessel sequences for $H$. If there exist non-negative numbers $\alpha_j$, $\mu_j$ satisfied $\varepsilon=\sum_{j=1}^{m}\sqrt{\alpha_jB+\mu_j}<\sqrt{A}$ such that for any $j\in[m]$, we have

$$\sum_{i=1}^{\infty}|\langle f,f_{ij}-g_{ij}\rangle|^2\leq\alpha_j\sum_{i=1}^{\infty}|\langle f,f_{ij}\rangle|^2+\mu_j\|f\|^2,\quad f\in H,$$

and then, $\{\{g_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ is woven for $H$ with the universal lower and upper bounds $(\sqrt{A}-\varepsilon)^2$ and $(\sqrt{B}+\varepsilon)^2$.

**Proof**. We compute

$$\sum_{i=1}^{\infty}|\langle f,f_{ij}-g_{ij}\rangle|^2\leq\alpha_j\sum_{i=1}^{\infty}|\langle f,f_{ij}\rangle|^2+\mu_j\|f\|^2\leq\left(\alpha_jB+\mu_j\right)\|f\|^2,\quad f\in H,$$

i.e., $\|T_j^{(k)*}-T_j^*\|\leq\sqrt{\alpha_jB+\mu_j}$, where

$$T_j^*(f)=\left\{\langle f,f_{ij}\rangle\right\}_{i=1}^{\infty},\quad T_j^{(k)*}(f)=\left\{\langle f,g_{ij}\rangle\right\}_{i=1}^{\infty},\quad f\in H.$$

Hence, $\sum_{j=1}^{m}\|T_j^{(k)^*}-T_j^*\|\leq\sum_{j=1}^{m}\sqrt{\alpha_jB+\mu_j}=\varepsilon<\sqrt{A}$. From Theorem 3.7, we have that $\{\{g_{ij}\}_{i=1}^{\infty}|j\in[m]\}$ is woven for $H$ with the universal lower and upper bounds $(\sqrt{A}-\varepsilon)^2$ and $(\sqrt{B}+\varepsilon)^2$.

**Example 3.15** Suppose that $\{f_i\}_{i=1}^{\infty}$ is a Parseval frame for $H$ and $f_{ij} = f_i$ for all $i \in \mathbb{N}$, $j \in [m]$, then $\{\{f_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ is woven for $H$ with the universal lower and upper bounds 1. Take $g_{ij} = -f_i + \frac{1}{m^2} f_i$ for all $i \in \mathbb{N}$, $j \in [m]$, we have $\{\{g_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ is woven for $H$.

**Proof**. Computing

$$\|\sum_{i\in\sigma_j} \langle f, f_{ij}\rangle f_{ij} - \sum_{i\in\sigma_j} \langle f, g_{ij}\rangle g_{ij}\| = \frac{2m^2-1}{m^4} \|\sum_{i\in\sigma_j} \langle f, f_j\rangle f_j\| \le \frac{2m^2-1}{m^4}\|f\|$$

and

$$\|\sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle f, f_{ij}\rangle f_{ij} - \sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle f, g_{ij}\rangle g_{ij}\| = \frac{2m^2-1}{m^4} \|\sum_{i=1}^{\infty} \langle f, f_j\rangle f_j\| = \frac{2m^2-1}{m^4}\|f\|$$

for $f \in H$, $j \in [m]$ and $\sigma \in \Omega$, from Theorem 3.1, Corollary 3.2, or Theorem 3.4, we have $\{\{g_{ij}\}_{i=1}^{\infty} | j \in [m]\}$ that is woven for $H$. Furthermore, we can obtain that the universal lower and upper bounds $(1 - \frac{1}{m^2})^2$ and $1 + \frac{2m^2-1}{m^4}$.

Note that Example 3.15 can be proved by Theorem 3.1, Corollary 3.2, or Theorem 3.4, but it cannot be proved from Theorem 3.7.

**Example 3.16** Let $\{\psi_j\}_{j\in[m]}, \{\varphi_j\}_{j\in[m]} \subset L^2(\mathbb{R})$ and $a > 1$, $b > 0$ be given, and assume that the wavelet frames family $\left\{\{a^{n/2}\psi_j(a^n x - kb)\}_{n,k\in\mathbb{Z}} \,|\, j \in [m]\right\}$ is woven for $L^2(\mathbb{R})$ with the universal lower and upper bounds $A$, $B$. If

$$R := \frac{1}{b}\max_{j\in[m]}\sup_{|\gamma|\in[1,a]} \sum_{n,k\in\mathbb{Z}} \left|\left(\hat{\psi}_j - \hat{\varphi}_j\right)(a^j\gamma)\left(\hat{\psi}_j - \hat{\varphi}_j\right)(a^j\gamma + k/b)\right| < \frac{A}{m^2},$$

then $\left\{\{a^{n/2}\varphi_j(a^n x - kb)\}_{n,k\in\mathbb{Z}} \,|\, j \in [m]\right\}$ is woven for $L^2(\mathbb{R})$ with the universal lower and upper bounds

$$A\left(1 - m\sqrt{\frac{R}{A}}\right)^2, \quad B\left(1 + m\sqrt{\frac{R}{B}}\right)^2.$$

**Proof**. From $\{\psi_j\}_{j\in[m]}, \{\varphi_j\}_{j\in[m]} \subset L^2(\mathbb{R})$, we have $\{\psi_j - \varphi_j\}_{j\in[m]} \subset L^2(\mathbb{R})$. Since

$$R = \frac{1}{b}\max_{j\in[m]}\sup_{|\gamma|\in[1,a]} \sum_{n,k\in\mathbb{Z}} \left|\left(\hat{\psi}_j - \hat{\varphi}_j\right)(a^j\gamma)\left(\hat{\psi}_j - \hat{\varphi}_j\right)(a^j\gamma + k/b)\right| < \frac{A}{m^2},$$

i.e.,

$$\frac{1}{b}\sup_{|\gamma|\in[1,a]} \sum_{n,k\in\mathbb{Z}} \left|\left(\hat{\psi}_j - \hat{\varphi}_j\right)(a^j\gamma)\left(\hat{\psi}_j - \hat{\varphi}_j\right)(a^j\gamma + k/b)\right| \le R < A$$

for all $j \in [m]$, by Theorem 15.2.3 and Theorem 22.5.1 in [3], $\{a^{n/2}(\psi_j - \varphi_j)(a^n x - kb)\}_{n,k\in\mathbb{Z}}$ is a Bessel sequence for $L^2(\mathbb{R})$ with bound $R$ and $\{a^{n/2}\varphi_j(a^n x - kb)\}_{n,k\in\mathbb{Z}}$ is a wavelet frame for $L^2(\mathbb{R})$ for all $j \in [m]$. Let $T_j$ and $T_j'$ be the synthesis operators of $\{a^{n/2}\psi_j(a^n x - kb)\}_{n,k\in\mathbb{Z}}$ and $\{a^{n/2}\varphi_j(a^n x - kb)\}_{n,k\in\mathbb{Z}}$ respectively. Then, $T_j - T_j'$ is the synthesis operators of $\{a^{n/2}(\psi_j - \varphi_j)(a^n x - kb)\}_{n,k\in\mathbb{Z}}$ and

$$\varepsilon = \sum_{j\in[m]} \|T_j - T_j'\| \le m\sqrt{R} < m\sqrt{\frac{A}{m^2}} = \sqrt{A}.$$

It is known that there is a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}^2$, by Theorem 3.7 or Corollary 3.14, we have that $\left\{\{a^{n/2}\varphi_j(a^n x - kb)\}_{n,k\in\mathbb{Z}} \,|\, j \in [m]\right\}$ is woven for $L^2(\mathbb{R})$ with the universal lower and upper bounds $(\sqrt{A} - \varepsilon)^2$ and $(\sqrt{B} + \varepsilon)^2$. We compute

$$A\left(1 - m\sqrt{\frac{R}{A}}\right)^2 = \left(\sqrt{A} - m\sqrt{R}\right)^2 \le \left(\sqrt{A} - \varepsilon\right)^2$$

and

$$B\left(1 + m\sqrt{\frac{R}{B}}\right)^2 = \left(\sqrt{B} + m\sqrt{R}\right)^2 \ge \left(\sqrt{B} + \varepsilon\right)^2,$$

and this implies that $\left\{\{a^{n/2}\varphi_j(a^n x - kb)\}_{n,k\in\mathbb{Z}} \,|\, j \in [m]\right\}$ has the universal lower and upper bounds

$$A\left(1 - m\sqrt{\frac{R}{A}}\right)^2, \quad B\left(1 + m\sqrt{\frac{R}{B}}\right)^2.$$

We complete the proof.

**Example 3.17** Let $\{\psi_j\}_{j\in[m]}, \{\varphi_j\}_{j\in[m]} \subset L^2(\mathbb{R})$ and $a$, $b > 0$ be given, and assume that the Gabor frames family $\left\{E_{mb}T_{na}\psi_j\right\}_{m,n\in\mathbb{Z}} \,|\, j \in [m]\}$ is woven for $L^2(\mathbb{R})$ with the universal lower and upper bounds $A$, $B$. If

$$R := \frac{1}{b}\max_{j\in[m]}\sup_{x\in[0,a]} \sum_{k\in\mathbb{Z}}\left|\sum_{n\in\mathbb{Z}}\left(\psi_j - \varphi_j\right)(x - na)\overline{\left(\psi_j - \varphi_j\right)(x - na - k/b)}\right| < \frac{A}{m^2},$$

then the family $\left\{E_{mb}T_{na}\varphi_j\right\}_{m,n\in\mathbb{Z}} \,|\, j \in [m]\}$ is woven for $L^2(\mathbb{R})$ with the universal lower and upper bounds

$$A\left(1 - m\sqrt{\frac{R}{A}}\right)^2, \quad B\left(1 + m\sqrt{\frac{R}{B}}\right)^2.$$

**Proof**. From $\{\psi_j\}_{j\in[m]}, \{\varphi_j\}_{j\in[m]} \subset L^2(\mathbb{R})$, we have $\{\psi_j - \varphi_j\}_{j\in[m]} \subset L^2(\mathbb{R})$. Since

$$R = \frac{1}{b}\max_{j\in[m]}\sup_{x\in[0,a]} \sum_{k\in\mathbb{Z}}\left|\sum_{n\in\mathbb{Z}}\left(\psi_j - \varphi_j\right)(x - na)\overline{\left(\psi_j - \varphi_j\right)(x - na - k/b)}\right| < \frac{A}{m^2},$$

i.e.,

$$\frac{1}{b}\sup_{x\in[0,a]} \sum_{k\in\mathbb{Z}}\left|\sum_{n\in\mathbb{Z}}\left(\psi_j - \varphi_j\right)(x - na)\overline{\left(\psi_j - \varphi_j\right)(x - na - k/b)}\right| \le R < A$$

for all $j \in [m]$, by Theorem 11.4.2 and Theorem 22.4.1 in [3], $\{E_{mb}T_{na}(\psi_j - \varphi_j)\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for $L^2(\mathbb{R})$ with bound $R$ and $\{E_{mb}T_{na}\varphi_j\}_{m,n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ for all $j \in [m]$. Let $T_j$ and $T'_j$ be the synthesis operators of $\{E_{mb}T_{na}\psi_j\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}\varphi_j\}_{m,n\in\mathbb{Z}}$ respectively. Then, $T_j - T'_j$ is the synthesis operators of $\{E_{mb}T_{na}(\psi_j - \varphi_j)\}_{m,n\in\mathbb{Z}}$ and

$$\varepsilon = \sum_{j\in[m]} \|T_j - T'_j\| \le m\sqrt{R} < m\sqrt{\frac{A}{m^2}} = \sqrt{A}.$$

It is known that there is a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}^2$, by Theorem 3.7 or Corollary 3.14, we have that $\left\{E_{mb}T_{na}\varphi_j\right\}_{m,n\in\mathbb{Z}} \mid j \in [m]\}$ is woven for $L^2(\mathbb{R})$ with the universal lower and upper bounds $(\sqrt{A} - \varepsilon)^2$ and $(\sqrt{B} + \varepsilon)^2$. Computing

$$A\left(1 - m\sqrt{\frac{R}{A}}\right)^2 = \left(\sqrt{A} - m\sqrt{R}\right)^2 \le \left(\sqrt{A} - \varepsilon\right)^2$$

and

$$B\left(1 + m\sqrt{\frac{R}{B}}\right)^2 = \left(\sqrt{B} + m\sqrt{R}\right)^2 \ge \left(\sqrt{B} + \varepsilon\right)^2$$

implies that $\left\{E_{mb}T_{na}\varphi_j\right\}_{m,n\in\mathbb{Z}} \mid j \in [m]\}$ has the universal lower and upper bounds

$$A\left(1 - m\sqrt{\frac{R}{A}}\right)^2, \quad B\left(1 + m\sqrt{\frac{R}{B}}\right)^2.$$

We complete the proof.

Considering the Wiener space

$$W := \left\{ g\colon \mathbb{R} \to \mathbb{C} \;\middle|\; g \text{ measurable and } \sum_{k\in\mathbb{Z}} \|g\chi_{[ka,(k+1)a]}\|_\infty < \infty \right\}$$

which is a Banach space with respect to the norm

$$\|g\|_{W,a} = \sum_{k\in\mathbb{Z}} \|g\chi_{[ka,(k+1)a]}\|_\infty,$$

we can obtain the following example.

**Example 3.18** Let $\{\psi_j\}_{j\in[m]}, \{\varphi_j\}_{j\in[m]} \subset L^2(\mathbb{R})$ and $a$, $b > 0$ be given, and assume that the Gabor frames family $\left\{E_{mb}T_{na}\psi_j\right\}_{m,n\in\mathbb{Z}} \mid j \in [m]\}$ is woven for $L^2(\mathbb{R})$ with the universal lower and upper bounds $A$, $B$. If $ab \le 1$ and

$$R: = \sqrt{\frac{2}{b}} \max_{j\in[m]} \|\psi_j - \varphi_j\|_{W,a} < \sqrt{\frac{A}{m^2}},$$

then the family $\left\{E_{mb}T_{na}\varphi_j\right\}_{m,n\in\mathbb{Z}} \mid j \in [m]\}$ is woven for $L^2(\mathbb{R})$ with the universal lower and upper bounds $(\sqrt{A} - mR)^2$ and $(\sqrt{B} + mR)^2$.

**Proof**. From $\{\psi_j\}_{j\in[m]}, \{\varphi_j\}_{j\in[m]} \subset L^2(\mathbb{R})$, we have $\{\psi_j - \varphi_j\}_{j\in[m]} \subset L^2(\mathbb{R})$. Since

$$R = \sqrt{\frac{2}{b}} \max_{j\in[m]} \|\psi_j - \varphi_j\|_{W,a} < \sqrt{\frac{A}{m^2}},$$

i.e.,

$$\sqrt{\frac{2}{b}} \|\psi_j - \varphi_j\|_{W,a} \le R < \sqrt{A}, \quad j \in [m]$$

by Proposition 11.5.2 and Theorem 22.4.1 in [3], $\{E_{mb}T_{na}(\psi_j - \varphi_j)\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence for $L^2(\mathbb{R})$ with bound $R^2$ and $\{E_{mb}T_{na}\varphi_j\}_{m,n\in\mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ for all $j \in [m]$. Let $T_j$ and $T'_j$ be the synthesis operators of $\{E_{mb}T_{na}\psi_j\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}\varphi_j\}_{m,n\in\mathbb{Z}}$ respectively. Then, $T_j - T'_j$ is the synthesis operators of $\{E_{mb}T_{na}(\psi_j - \varphi_j)\}_{m,n\in\mathbb{Z}}$ and

$$\varepsilon = \sum_{j\in[m]} \|T_j - T'_j\| \le mR < m\sqrt{\frac{A}{m^2}} = \sqrt{A}.$$

It is known that there is a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}^2$, by Theorem 3.7 or Corollary 3.14, we have that $\left\{E_{mb}T_{na}\varphi_j\right\}_{m,n\in\mathbb{Z}} \mid j \in [m]\}$ is woven for $L^2(\mathbb{R})$ with the universal lower and upper bounds $(\sqrt{A} - \varepsilon)^2$ and $(\sqrt{B} + \varepsilon)^2$. Computing

$$\left(\sqrt{A} - mR\right)^2 \le \left(\sqrt{A} - \varepsilon\right)^2, \qquad \left(\sqrt{B} + mR\right)^2 \ge \left(\sqrt{B} + \varepsilon\right)^2$$

implies that $\left\{E_{mb}T_{na}\varphi_j\right\}_{m,n\in\mathbb{Z}} \mid j \in [m]\}$ has the universal lower and upper bounds $(\sqrt{A} - mR)^2$ and $(\sqrt{B} + mR)^2$.

## DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/**Supplementary Material**. Further inquiries can be directed to the corresponding author.

## AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

## SUPPLEMENTARY MATERIAL

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fphy.2022.873955/full#supplementary-material

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