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A Note on Resistance Distances of Graphs

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Let G be a connected graph with vertex set $V(G)$. The resistance distance between any two vertices u, $v \in V(G)$ is the net effective resistance between them in the electric network constructed from G by replacing each edge with a unit resistor. Let $S \subset V(G)$ be a set of vertices such that all the vertices in S have the same neighborhood in $G - S$, and let $G[S]$ be the subgraph induced by S. In this note, by the {1}-inverse of the Laplacian matrix of G, formula for resistance distances between vertices in S is obtained. It turns out that resistance distances between vertices in S could be given in terms of elements in the inverse matrix of an auxiliary matrix of the Laplacian matrix of $G[S]$, which derives the reduction principle obtained in [J. Phys. A: Math. Theor. 41 (2008) 445203] by algebraic method.

Keywords: resistance distance, Laplacian matrix, {1}-inverse, moore-penrose inverse, reduction principle

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1 INTRODUCTION

The novel concept of resistance distance was introduced by Klein and Randić [8] in 1993. For a connected graph G with vertex set $V(G) = \{1, 2, \ldots, n\}$, the *resistance distance* between $u, v \in V(G)$, denoted by $\Omega_G(u, v)$, is defined to be the effective resistance between u and v in the corresponding electric network obtained from G by replacing each edge with a unit resistor. Since resistance distance is an intrinsic graph metric and an important component of circuit theory, with potential applications in chemistry, it has been extensively studied in mathematics, physics, and chemistry. For more information, we refer the readers to recent papers [2, 4, 6, 7, 10, 11, 15] and references therein.

Let G be a connected graph of order n. For any set of vertices $U \subset V(G)$, we use $G[U]$ to denote the subgraph induced by U, and $G - U$ to denote the subgraph obtained from G by removing all the vertices in U as well as all the edges incident to vertices of U. The *adjacency matrix* A_G of G is an $n \times n$ matrix such that the (i, j)-th element of A_G is equal to 1 if vertices i and j are adjacent and 0 otherwise. The *Laplacian matrix* of G is $L_G = D_G - A_G$, where D_G is the diagonal matrix of vertex degrees of G. Clearly, L_G is real symmetric and singular.

Let M be an $n \times m$ real matrix. An $m \times n$ real matrix X is called a $\{1\}$-inverse of M and denoted by $M^{(1)}$, if X satisfies the following equation:

$$MXM = M.$$

If M is singular, then it has infinite $\{1\}$-inverses. It is well known that resistance distances in a connected graph G can be obtained from any $\{1\}$-inverse of L_G (see [1]). So far, there are many well-established results on this inverse. For example, in 2014, Bu *et al* [4] obtained the $\{1\}$-inverse of the Laplacian matrix for a class of connected graphs, and investigated resistance distances in subdivision-vertex join and subdivision-edge join of graphs. Then in 2015, an exact expression for the $\{1\}$-inverse of the Laplacian matrix of connected graphs was obtained by Sun *et al.* [13]. After that, Liu *et al.* [9] obtained the $\{1\}$-inverses for the Laplacian matrix of subdivision-vertex and subdivision-edge coronae networks. Recently, Cao *et al.* [5] also characterised the {1}-inverses for the Laplacian of corona and neighborhood corona networks. Sardar *et al.* [12] determined resistance distances of some classes of rooted product graphs via the Laplacian {1}-inverses method.

In this paper, some results on the {1}-inverses for Laplacian matrices of graphs with given special properties are established. As an application, for any given vertex set $S \subset V(G)$ such that all the vertices in S have the same neighborhood N in $G - S$, explicit formula for resistance distances between vertices in S is obtained. It turns out that resistance distances between vertices in S could be given in terms of elements in the inverse matrix of an auxiliary matrix of the Laplacian matrix of $G[S]$, which derives the reduction principle obtained in [J. Phys. A: Math. Theor. 41 (2008) 445203] by algebraic method.

2 PRELIMINARY RESULTS

In this section, we present some preliminary results. We first introduce the concept of group inverse and Moore-Penrose inverse of a matrix.

Definition 2.1. For a square matrix X, the *group inverse* of X, denoted by $X^{\#}$, is the unique matrix H that satisfies matrix equations:

$$XHX = X, \quad HXH = H, \quad XH = HX.$$

Definition 2.2. Let M be an $n \times m$ matrix. An $m \times n$ matrix X is called the *Moore-Penrose inverse* of M, if X satisfies the following conditions:

$$MXM = M, \quad XMX = X, \quad (MX)^H = MX, \quad (XM)^H = XM.$$

where X^H represents the conjugate transpose of the matrix X.

If M is real symmetric, then there exists a unique $M^{\#}$ and $M^{\#}$ is the symmetric {1}-inverse of M. In particular, $M^{\#}$ is equal to the Moore-Penrose inverse of M because M is symmetric [3].

Let $(M)_{ij}$ denote the (i, j)-entry of M. It is well known that resistance distances in a connected graph G can be obtained from any {1}-inverse of L_G according to the following lemma.

Lemma 2.3. *[3] Let G be a connected graph. Then for vertices i and j,*

$$\begin{aligned}\Omega_G(i,j) &= \left(L_G^{(1)}\right)_{ii} + \left(L_G^{(1)}\right)_{jj} - \left(L_G^{(1)}\right)_{ij} - \left(L_G^{(1)}\right)_{ji} \\ &= \left(L_G^{\#}\right)_{ii} + \left(L_G^{\#}\right)_{jj} - 2\left(L_G^{\#}\right)_{ij}.\end{aligned}$$

Let $\mathbf{0}$ and $\mathbf{e}$ be all-zero and all-one column vectors, respectively. Let $J_{n\times m}$ be the $n \times m$ all-one matrix. The following result is due to Sun et al. [13] which characterizes the {1}-inverse of the Laplacian matrix.

Lemma 2.4. *[13] Let* $L_G = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix}$ *be the Laplacian matrix of a connected graph. If* L_1 *is nonsingular, then* $X = \begin{bmatrix} L_1^{-1} + L_1^{-1}L_2S^{\#}L_2^TL_1^{-1} & -L_1^{-1}L_2S^{\#} \\ -S^{\#}L_2^TL_1^{-1} & S^{\#} \end{bmatrix}$ *is a symmetric {1}-inverse of* L_G*, where* $S = L_3 - L_2^TL_1^{-1}L_2$.

In particular, if each column vector of L_2^T is $-\mathbf{e}$ or $\mathbf{0}$, then X can be further simplified. For convenience, in the rest of this section (see Lemmas 2.5, 2.6, 2.7), we always assume that $L_G = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix}$, with the property that L_1 is nonsingular, and each column vector of L_2^T is $-\mathbf{e}$ or $\mathbf{0}$.

Lemma 2.5. *[4] Let* L_G *be defined as above. Then* $X = \begin{bmatrix} L_1^{-1} & 0 \\ 0 & S^{\#} \end{bmatrix}$ *is a symmetric {1}-inverse of* L_G*, where* $S = L_3 - L_2^TL_1^{-1}L_2$.

According to Lemma 2.4, we could get the following results.

Lemma 2.6. *Let* L_G *be defined as above. If each row of* L_1 *sums to k, then each column vector of* $-L_1^{-1}L_2S^{\#}$ *is proportional to the all-one vector, where* $S = L_3 - L_2^TL_1^{-1}L_2$.

Proof. Suppose that the number of columns of L_2 is n_2 and let $L_2 = [\,\mathbf{r}_1 \;\; \mathbf{r}_2 \;\; \ldots \;\; \mathbf{r}_{n_2}\,]$ with $\mathbf{r}_i$ being its i-th column vector, $i = 1, 2, \ldots, n_2$. First we show that for any $\mathbf{r}_i$, all the elements of $L_1^{-1}\mathbf{r}_i$ are the same. If $\mathbf{r}_i = \mathbf{0}$, then the assertion holds since $L_1\mathbf{r}_i = \mathbf{0}$. Otherwise, $\mathbf{r}_i = -\mathbf{e}$. Since L_1 is nonsingular with each row sum being k, it follows that each row of L_1^{-1} sums to $\frac{1}{k}$. Thus $L_1^{-1}\mathbf{r}_i = L_1^{-1}(-\mathbf{e}) = -\frac{1}{k}(\mathbf{e})$, which also implies that all the elements of $L_1^{-1}\mathbf{r}_i$ are the same. Hence, each column of $-L_1^{-1}L_2$ is proportional to the all-one vector, that is, all the row vectors of $-L_1^{-1}L_2$ are the same. It thus follows that each column of $-L_1^{-1}L_2S^{\#}$ is proportional to the all-one vector, i.e. all the elements in any given column of $-L_1^{-1}L_2S^{\#}$ are the same. □

According to Lemma 2.6, we have the following result.

Lemma 2.7. *Let* L_G *be defined as above. If each row of* L_1 *sums to k, then there exists a real number* ξ *such that* $L_1^{-1}L_2S^{\#}L_2^TL_1^{-1} = \xi J_{n_1\times n_1}$*, where* $S = L_3 - L_2^TL_1^{-1}L_2$.

Proof. Let $M_1 = L_1^{-1}L_2S^{\#}$. According to the argument in the proof of Lemma 2.6, all the row vectors in M_1 are the same. On the other hand, since L_1 is real symmetric, it follows that

$$L_2^TL_1^{-1} = L_2^T\left(L_1^{-1}\right)^T = \left(L_1^{-1}L_2\right)^T.$$

Let $M_2 = L_2^TL_1^{-1}$. Then all the column vectors in M_2 are the same since all the row vectors in $-L_1^{-1}L_2$ are the same. Thus, we conclude that there exists a real number ξ such that

$$L_1^{-1}L_2S^{\#}L_2^TL_1^{-1} = M_1M_2 = \xi J_{n_1\times n_1}.$$

This completes the proof. □

3 MAIN RESULTS

In this section, we consider resistance distances between vertices in a specific subset S of $V(G)$. Let $S \subset V(G)$ such that all the vertices in S have the same neighborhood N in $G - S$. In the

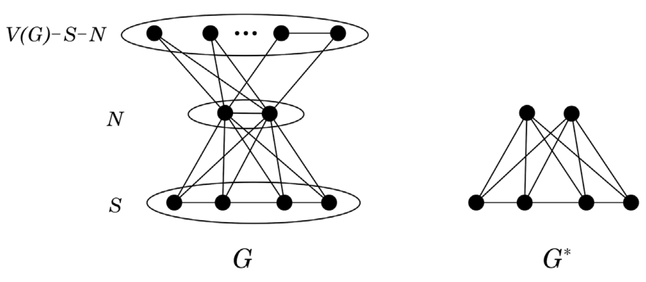

FIGURE 1 | Illustration of graphs G and its subgraph G^*.

following, we give explicit formula for resistance distances between vertices in S. For simplicity, we use L_S to denote the Laplacian matrix of the subgraph induced by S. Suppose that the cardinalities of S and N are n_1 and k, respectively. Then the Laplacian matrix of G can be written as follows.

$$L_G = \begin{bmatrix} L_S + kI_{n_1} & L_2 \\ L_2^T & L_3 \end{bmatrix},$$

where I_{n_1} is the identity matrix of order n_1.

Now we are ready to give formula for resistance distances between vertices in S.

Theorem 3.1. *Let $S \subset V(G)$ such that all the vertices in S have the same neighborhood N in $G - S$. Then for $i, j \in S$, we have*

$$\Omega_G(i,j) = (L_1^{-1})_{ii} + (L_1^{-1})_{jj} - 2(L_1^{-1})_{ij}.$$

where $L_1 = L_S + kI_{n_1}$.

Proof. Let $L_1 = L_S + kI_{n_1}$. Clearly, L_1 is nonsingular, and each row of L_1 sums to k and each column vector of L_2 is $-\mathbf{e}$ or $\mathbf{0}$. Then by Lemma 2.7, there exists a real number ξ such that $L_1^{-1}L_2S^{\#}L_2^TL_1^{-1} = \xi J_{n_1\times n_1}$, where $S = L_3 - L_2^TL_1^{-1}L_2$. Then by Lemma 2.5, we can obtain the {1}-inverse of L_G as follows.

$$X = \begin{bmatrix} L_1^{-1} + \xi J_{n_1\times n_1} & -L_1^{-1}L_2S^{\#} \\ -S^{\#}L_2^TL_1^{-1} & S^{\#} \end{bmatrix}.$$

Thus, for vertices $i, j \in S$, by Lemma 2.3, we have

$$\begin{aligned}\Omega_G(i,j) &= (X)_{ii} + (X)_{jj} - (X)_{ij} - (X)_{ji} \\ &= (L_1^{-1} + \xi J_{n_1\times n_1})_{ii} + (L_1^{-1} + \xi J_{n_1\times n_1})_{jj} - (L_1^{-1} + \xi J_{n_1\times n_1})_{ij} - (L_1^{-1} + \xi J_{n_1\times n_1})_{ji} \\ &= (L_1^{-1})_{ii} + \xi + (L_1^{-1})_{jj} + \xi - (L_1^{-1})_{ij} - \xi - (L_1^{-1})_{ji} - \xi \\ &= (L_1^{-1})_{ii} + (L_1^{-1})_{jj} - 2(L_1^{-1})_{ij}.\end{aligned}$$

The proof is complete. □

Theorem 3.1indicates that, if $S \subset V(G)$ satisfies that all the vertices in S have the same neighborhood N in $G - S$, then resistance distances between vertices in S depends only on the subgraph $G[S]$ and the cardinality of N. In other words, if we use G^* to denote the subgraph obtained from $G[S \cup N]$ by deleting all the edges between vertices in N (see **Figure 1**), then resistance distances between vertices in S depends only on G^*. In fact, for $i, j \in S$, $\Omega_G(i,j) = \Omega_{G^*}(i,j)$, as shown in the following.

Theorem 3.2. *Let $S \subset V(G)$ such that all the vertices in S have the same neighborhood N in $G - S$. Let G^* the graph obtained from $G[S \cup N]$ by deleting all the edges between vertices in N. Then for $i, j \in S$, we have*

$$\Omega_{G^*}(i,j) = (L_1^{-1})_{ii} + (L_1^{-1})_{jj} - 2(L_1^{-1})_{ij}.$$

where $L_1 = L_S + kI_{n_1}$.

Proof. According to the definition of G^*, it is readily to see that the Laplacian matrix of G^* is

$$L_{G^*} = \begin{bmatrix} L_1 & -J_{n_1\times k} \\ -J_{k\times n_1} & kI_k \end{bmatrix}.$$

Since each column vector of $-J_{k\times n_1}$ is $-\mathbf{e}$, by Lemma 2.5, we can obtain the symmetric {1}-inverse of L_{G^*} as follows:

$$Y = \begin{bmatrix} L_1^{-1} & 0 \\ 0 & S^{\#} \end{bmatrix},$$

where $S = kI_k - J_{k\times n_1}L_1^{-1}J_{n_1\times k}$. Hence by Lemma 2.3, we have

$$\Omega_{G^*}(i,j) = (L_1^{-1})_{ii} + (L_1^{-1})_{jj} - 2(L_1^{-1})_{ij},$$

as required. □

Remark 1. Combining Theorems 3.1 and 3.2, we could conclude that if $S \subset V(G)$ satisfies that all the vertices in S have the same neighborhood N in $G - S$, then resistance distances between vertices in S can be computed as in the subgraph obtained from $G[S \cup N]$ by deleting all the edges between vertices in N. It should be mentioned that this fact, known as the reduction principle, was established in [14]. We confirm this result by algebraic method, rather than electric network method as used in [14]. Furthermore, we also give an exact formula for resistance distances between vertices in S. By Theorem 3.1, we are able to establish some interesting properties.

Theorem 3.3. *Let $S \subset V(G)$ such that all the vertices in S have the same neighborhood N in $G - S$. Then for $i, j \in S$ and $u \in G - S$, we have*

$$\Omega_G(i,u) - \Omega_G(j,u) = (L_1^{-1})_{ii} - (L_1^{-1})_{jj}.$$

where $L_1 = L_S + kI_{n_1}$.

Proof. As given in the proof of Theorem 3.1, we know that the {1}-inverse of L_G is

$$X = \begin{bmatrix} L_1^{-1} + \xi J_{n_1\times n_1} & -L_1^{-1}L_2S^{\#} \\ -S^{\#}L_2^TL_1^{-1} & S^{\#} \end{bmatrix}.$$

where ξ be a real number and $S = L_3 - L_2^TL_1^{-1}L_2$. By Lemma 2.3, we have

$$\begin{aligned}\Omega_G(i,u) - \Omega_G(j,u) = {} & (X)_{ii} + (X)_{uu} - (X)_{iu} - (X)_{ui} \\ & - \left[(X)_{jj} + (X)_{uu} - (X)_{ju} - (X)_{uj}\right].\end{aligned}$$

Note that L_1 is nonsingular and every row sums to k and each column vector of L_2 is $-\mathbf{e}$ or a zero vector. So by Lemma 2.6, we know that each column of $-L_1^{-1}L_2S^{\#}$ is proportional to all-one vector, which implies that $(X)_{iu} = (X)_{ju}$. Since X is real symmetric, we also have $(X)_{ui} = (X)_{uj}$. It follows that

$$\begin{aligned}\Omega_G(i,u) - \Omega_G(j,u) &= (X)_{ii} + (X)_{uu} - (X)_{jj} - (X)_{uu}\\ &= (L_1^{-1})_{ii} + \xi + (L_1^{-1})_{uu} + \xi - (L_1^{-1})_{jj} - \xi - (L_1^{-1})_{uu} - \xi\\ &= (L_1^{-1})_{ii} - (L_1^{-1})_{jj}.\end{aligned}$$

This completes the proof. □

It is interesting to note from Theorem 3.2 that the difference between $\Omega_G(i, u)$ and $\Omega_G(j, u)$ depends only on the subgraph $G[S]$ and the cardinality of N, no matter the chosen of u. Then we have the following result.

Corollary 3.4. *Let $S \subset V(G)$ such that all the vertices in S have the same neighborhood N in $G - S$. Then for $i, j \in S$ and $u, v \in G - S$, we have*

$$\Omega_G(i,u) - \Omega_G(j,u) = \Omega_G(i,v) - \Omega_G(j,v).$$

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

YY contributed to conception and design of the study. WS performed the theoretical analysis and wrote the first draft of the manuscript. YY revised the manuscript. Both authors read, and approved the submitted version.

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REFERENCES

1. Bapat RB. Resistance Distance in Graphs. *Math Stud* (1999) 68(1-4):87–98.
2. Bapat RB, Gupta S. Resistance Distance in Wheels and Fans. *Indian J Pure Appl Math* (2010) 41(1):1–13. doi:10.1007/s13226-010-0004-2
3. Bu C, Sun L, Zhou J, Wei Y. A Note on Block Representations of the Group Inverse of Laplacian Matrices. *Electron J Linear Algebra* (2012) 23:866–76. doi:10.13001/1081-3810.1562
4. Bu C, Yan B, Zhou X, Zhou J. Resistance Distance in Subdivision-Vertex Join and Subdivision-Edge Join of Graphs. *Linear Algebra its Appl* (2014) 458: 454–62. doi:10.1016/j.laa.2014.06.018
5. Cao J, Liu J, Wang S. Resistance Distances in corona and Neighborhood corona Networks Based on Laplacian Generalized Inverse Approach. *J Algebra Appl* (2019) 18(3):1950053. doi:10.1142/s0219498819500531
6. Chen H, Zhang F. Resistance Distance and the Normalized Laplacian Spectrum. *Discrete Appl Maths* (2007) 155:654–61. doi:10.1016/j.dam.2006.09.008
7. Fowler PW. Resistance Distances in Fullerene Graphs. *Croat Chem Acta* (2002) 75(2):401–8.
8. Klein DJ, Randić M. Resistance Distance. *J Math Chem* (1993) 12:81–95. doi:10.1007/bf01164627
9. Liu JB, Pan XF, Hu FT. The {1}-inverse of the Laplacian of Subdivision-Vertex and Subdivision-Edge Coronae with Applications. *Linear and Multilinear Algebra* (2017) 65(1):178–91. doi:10.1080/03081087.2016.1179249
10. Liu JB, Wang WR, Zhang YM, Pan XF. On Degree Resistance Distance of Cacti. *Discrete Appl Maths* (2016) 203:217–25. doi:10.1016/j.dam.2015.09.006
11. Palacios JL. Resistance Distance in Graphs and Random Walks. *Int J Quant Chem* (2001) 81(1):29–33. doi:10.1002/1097-461x(2001)81:1<29::aid-qua6>3.0.co;2-y
12. Sardar MS, Alaeiyan M, Farahani MR, Cancan M, Ediz S. Resistance Distance in Some Classes of Rooted Product Graphs Obtained by Laplacian Generalized Inverse Method. *J Inf Optimization Sci* (2021) 42(7):1447–67. doi:10.1080/02522667.2021.1899210
13. Sun L, Wang W, Zhou J, Bu C. Some Results on Resistance Distances and Resistance Matrices. *Linear and Multilinear Algebra* (2015) 63(3):523–33. doi:10.1080/03081087.2013.877011
14. Yang Y, Zhang H. Some Rules on Resistance Distance with Applications. *J Phys A: Math Theor* (2008) 41(44):445203. doi:10.1088/1751-8113/41/44/445203
15. Zhang H, Yang Y. Resistance Distance and Kirchhoff index in Circulant Graphs. *Int J Quan Chem.* (2007) 107(2):330–9. doi:10.1002/qua.21068

Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.