



A New Modified Analytical Approach for the Solution of Time-Fractional Convection–Diffusion Equations With Variable Coefficients

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In this article, a new modification of the Adomian decomposition method is performed for the solution fractional order convection–diffusion equation with variable coefficient and initial–boundary conditions. The solutions of the suggested problems are calculated for both fractional and integer orders of the problems. The series of solutions of the problems with variable coefficients have been provided for the first time. To verify and illustrate our new technique, four numerical examples are presented and solved by using the proposed technique. The derived results are plotted, and the dynamics are shown for both fractional and integer orders of the problems. An excellent variation among the solutions at various fractional orders is observed. It is analyzed that the new technique based on the Adomian decomposition method is accurate and effective. The present method fits both the initial and boundary conditions with double approximations simultaneously, which increases the accuracy of the present method. For the first time, the present technique is used for the solutions of the problems with variable coefficients along with initial and boundary conditions. It is therefore suggested to apply the present procedure for the solutions of other problems with variable order and coefficients along with initial and boundary conditions.

Keywords: Adomian decomposition method, initial–boundary value problems, Caputo derivative, fractional convection–diffusion equations, analytical method

1 INTRODUCTION

Fractional Calculus (FC) is the branch of mathematics that focuses on the study of fractional-order operators and their applications in mathematical theory, which is the expansion of the basic classical concept in calculus and has attracted the attention of many researchers in the last few decades. FC has useful applications in numerous fields of physical and numerical sciences, such as relaxation processes, diffusion, visco-elasticity, hydrology, biomedical engineering, electro-chemistry, relaxation vibration, finance, probability, seismology, and so on [1–5]. Other FC applications

include earthquake's oscillations modeling [6], statistical and mechanic continuum [7], control theory [8], engineering and physics [9, 10], entropy [11], image processing [10], field theories, optimal control, classical and quantum mechanics [12, 13], chaos theory [14], fractional diabetes model [15], and human diseases [16]. The main contribution to the development of (FC) includes Podlubny et al. [17]; Miller and Ross [18]; Oldham and Spanier [19]; and Kilbas and Trujillo [20].

A well-known fact about fractional-order partial differential equations (FPDESs) is that they are involved in many natural and physical phenomena. Many researchers have attracted towards them due to the wonderful applications in various fields especially anomalous diffusion, dielectric polarization, relaxation vibration, scattering, system identification, fluid flow, mathematical physics [21], psychology, and acoustics [22].

Over the last 10 years, attempts were made to find robust analytical and numerical techniques to solve (FPDES). Some of them are the homotopy analysis method [23], the wavelet operational method [24, 25], the Legendre-based method [26], the fractional recti sub-equation method [27], the generalized fractional Taylor series method [28], the Laplace Adomian decomposition method [29], the fractional wavelet method [30], the Adomian decomposition method [31], the discrete Adomian decomposition method [32], the Mohand decomposition method [33], the Wlash function method [34], the Chebyshev spectral approximation [35], the wavelet optimization method [36], the solitary ansatz method [37], the multiple exp-function method [38], the Hirota's direct method [39], the Laplace transform decomposition method [40], the natural transform decomposition method [41], and so on. The time-fractional convection–diffusion equations (CDEs) have been conventional used in the mathematical models of computations and simulations, such as the transport of energy and mass, oil reservoir simulations, flow of heat, particles, dispersion of chemicals in reactors, and global weather prediction [42, 43]. Time-fractional CDEs are operated in the abnormal diffusion processes related to time.

Several research work have been done on the Adomian decomposition method (ADM), such as Wazwaz A.M. [44] used it to solve linear and non-linear operators, initial-value problems solved by Abdelrazec and Pelinovsky [45] and Ray and Bera [46] applied it to the differential equations. Hashim [47] applied it to the boundary-value problems, fractional differential equations solved by Hu and Luo [45], Duan J.S. applied to the boundary-value problems [48], Song and Wang [49] solved fractional differential equations, initial value problems solved by Wu [50] and Almazmumy et al. [51]. Patel et al. applied the ADM sumudu transform method in [52, 53] and the fractional reduced differential transform method in [54, 55]. For the convergence of ADM, see [56, 57]. Fatoorehchi et al. solved explicit Frost–Kalkwarf type equations by ADM [58]. The one thing about all the above appreciable work is that ADM is used either with initial or only boundary conditions. Furthermore, a new modification is generated in ADM by Ali in [31, 59] for initial–boundary problems. He used the same new technique in [60] with the variation iteration method, and the results are good. We

extend his work by applying it to fractional order problems. This procedure gives accuracy because we construct a new initial approximate for each iteration. Abdelrazec et al. have provided a comprehensive study on the convergence of ADM for initial value problems in [61]. Similarly, Aminataei et al. have done the stability of ADM with other methods in [62].

In the present work, time-fractional CDEs with variable coefficients are considered as follows:

$$\frac{\partial^\gamma u(\zeta, t)}{\partial t^\gamma} + c(\zeta) \frac{\partial u(\zeta, t)}{\partial \zeta} + d(\zeta) \frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} = \kappa(\zeta), \quad 0 < \zeta < 1, \quad 0 < t \leq 1,$$

with the initial conditions

$$u(\zeta, 0) = f(\zeta), \quad 0 < \zeta < 1,$$

and the boundary conditions

$$u(0, t) = g_0(t), \quad u(1, t) = g_1(t) \quad 0 < t \leq 1,$$

where $c(\zeta)$, $d(\zeta) \neq 0$ are continuous functions and $0 < \gamma < 1$. The fractional derivative is in the Caputo sense.

Many techniques have been used to obtain the solutions of time-fractional CDEs such as the Gegenbauer spectral method [63], the Sinc–Legendre collocation method [64], the Bernstein polynomials method [65], the flatlet oblique multi wavelets [66], and the improved differential transform method [67]. Marjan et al. applied the radial basis interpolation method to compute the solution of this equation with constant coefficients [68]. Luo et al. computed the equation by the quadratic spline collocation method with constant coefficient and without convection [69]. In [70], Pirkhedri and Javadi computed the equations with variable coefficient by the collocation approach.

In this article, we have investigated the analytical solutions of various time-fractional CDEs by using a new technique of the Adomian decomposition method (ADM). In the year 1980, Adomian had developed an effective technique, called ADM to solve differential equations involved in physical phenomena [71]. Later on, some short coming related to ADM, the researchers have modified ADM and construct some effective tools for the solutions of various fractional partial differential equations and their systems. Generally, many techniques are not efficient for the solutions of fractional initial and boundary conditions together. However, in this work, the solutions of initial and boundary value problems related to time-fractional CDEs are obtained by using a new technique of ADM. The obtained solutions are represented through graphs and tables. The 2D and 3D graphs have confirmed a closed contact between the exact and new version of ADM results. The fractional-order solutions represent the valuable dynamics of the suggested problems. The solution convergence of the fractional solutions towards integer-order solutions confirmed the validity of the suggested method. The present method is simple and required fewer calculations and therefore can be extended for the solutions of other higher non-linear fractional PDEs and their systems.

2 PRELIMINARIES

In this section, a few definitions related to our work are taken into consideration.

2.1 Definition

The expression for the Caputo derivative of fractional order γ is as follows:

$$(D^\gamma h)(\zeta) = \frac{\partial^\gamma h(\zeta)}{\partial t^\gamma} = \begin{cases} I^{p-\gamma} \left[\frac{\partial^\gamma h(\zeta)}{\partial t^\gamma} \right], & p-1 < \gamma \leq p, \\ \frac{\partial^\gamma h(\zeta)}{\partial t^\gamma}, & \end{cases}$$

where $p \in \mathbb{N}$, $\zeta > 0$, $g \in \mathbb{C}_t$, and $t \geq 1$.

2.2 Definition

The Reimann–Liouville integral operator having order γ is given by [72]

$$(I_\zeta^\gamma h)(\zeta) = \begin{cases} \frac{1}{\Gamma(\gamma)} \int_0^\zeta (\zeta - \nu)^{\gamma-1} d\nu, & \gamma > 0, \\ h(\zeta), & \gamma = 0, \end{cases}$$

where Γ denotes the gamma function and can be written as

$$\Gamma(\omega) = \int_0^\infty e^{-\zeta} \zeta^{\omega-1} d\zeta, \quad \omega \in \mathbb{C}.$$

2.3 Lemma

For $p-1 < \gamma \leq p$ with $p \in \mathbb{N}$ and $h \in \mathbb{C}_t$ with $t \geq -1$, then [73].

$$\begin{cases} I^\gamma I^b = I^{\gamma+b} h(\zeta), & b, \gamma \geq 0, \\ I^\gamma \zeta^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\gamma+\lambda+1)} \zeta^{\gamma+\lambda}, & \gamma > 0, \lambda > -1, \\ I^\gamma D^\gamma h(\zeta) = h(\zeta) - \sum_{k=0}^{p-1} h^k(0^+) \frac{\zeta^k}{k!}, \end{cases}$$

where $\zeta > 0$ and $p-1 < \gamma \leq p$.

2.4 Definition

The Mittag-Leffler function $E_\gamma(\rho)$ for $\gamma > 0$ is [74]

$$E_\gamma(\rho) = \sum_{n=0}^\infty \left(\frac{\rho^n}{n\gamma+1} \right), \quad \gamma > 0, \quad \rho \in \mathbb{C}.$$

3 ADOMIAN DECOMPOSITION METHOD

This method was discovered by Adomian for the solution of linear and nonlinear differential and integro-differential equations. To understand about the method, consider the following equation [31]:

$$F(u(\zeta)) = g(\zeta), \tag{1}$$

where g is the known function and $F(u(x))$ is the differential operator, which can be decomposed as follows:

$$F(u(\zeta)) = Lu + Ru + Nu,$$

where L is the invertible operator of highest order derivative, R represents the linear operator, and N is the non-linear term. Then, Eq. (1) has the following representation:

$$Lu + Ru + Nu = g. \tag{2}$$

Apply the inverse operator L^{-1} on both sides of Eq. (2),

$$u = \varphi + L^{-1}(g) - L^{-1}(Ru) - L^{-1}(Nu),$$

where φ is the constant of integration and $L\varphi = 0$.

The ADM solution is represented as an infinite series,

$$u = \sum_{n=0}^\infty u_n.$$

The non-linear term Nu is denoted by A_n and is defined as follows:

$$Nu = \sum_{n=0}^\infty A_n.$$

With the help of the following formula, we can calculate A_n as

$$A_n = \frac{1}{n!} \frac{d^n}{d\psi^n} N \left(\sum_{k=0}^\infty (\psi^k u_k) \right), \quad n = 0, 1, \dots$$

The other method to calculate Adomian polynomials can be seen in [75]. The series has the following relation to represent the solution of Eq. (1):

$$\begin{cases} u_0 = \varphi + L^{-1}(g), & n = 0, \\ u_{n+1} = L^{-1}(Ru_n) - L^{-1}(A_n), & n \geq 0. \end{cases}$$

4 MODIFICATION OF ADOMIAN DECOMPOSITION METHOD

To understand the main idea of the proposed technique, consider Eq. (1), Eq.(1) and Eq.(1). The new initial solution (u_n^*) is calculated by new iteration for Eq. (1) with the help of the proposed technique [31],

$$u_n^* = u_n(\zeta, t) + (1 - \zeta)[g_0(t) - u_n(0, t)] + \zeta[g_1(t) - u_n(1, t)].$$

Using ADM, the operator form of Eq. (1) is

$$Lu = \left(-c(\zeta) \frac{\partial u(\zeta, t)}{\partial \zeta} - d(\zeta) \frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} + \kappa(\zeta, t) \right), \tag{3}$$

where the differential operator L is defined as

$$L = \frac{\partial^\gamma}{\partial t^\gamma}.$$

Hence, L^{-1} is defined as

$$L^{-1}(\cdot) = I^\gamma(\cdot)dt.$$

Applying L^{-1} to Eq. (3),

$$u(\zeta, t) = u(\zeta, 0) + L^{-1} \left(-a(\zeta) \frac{\partial u(\zeta, t)}{\partial \zeta} - b(\zeta) \frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} + \kappa(\zeta, t) \right),$$

and using the new technique, the initial approximation becomes

$$u_0(\zeta, t) = u(\zeta, 0) + L^{-1}(\kappa(\zeta, t)),$$

and the iteration formula becomes

$$u_{n+1}(\zeta, t) = L^{-1} \left(-a(\zeta) \frac{\partial u_n^*(\zeta, t)}{\partial \zeta} - b(\zeta) \frac{\partial^2 u_n^*(\zeta, t)}{\partial \zeta^2} \right),$$

where $n = 0, 1, \dots$. It is obvious that initial solutions u_n^* of Eq. (1) satisfied both the initial and boundary conditions as follows:

$$\begin{aligned} \text{at } t = 0, & \quad u_n^*(\zeta, 0) = u_n(\zeta, 0), \\ \zeta = 0, & \quad u_n^*(0, t) = g_0(t), \\ \zeta = 1, & \quad u_n^*(1, t) = g_1(t). \end{aligned}$$

The proposed technique works effectively for the two-dimensional problems.

5 NUMERICAL RESULTS

In this section, we will present the solution of some illustrative examples by using the new technique based on ADM.

5.1 Example 1

Consider TFCDE of the form [44]

$$\begin{aligned} \frac{\partial^\gamma u(\zeta, t)}{\partial t^\gamma} + \zeta \frac{\partial u(\zeta, t)}{\partial t} + \frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} &= 2t^\gamma + 2\zeta^2 \\ &+ 2, \quad 0 < \zeta < 1, 0 < \gamma < 1, \end{aligned} \tag{4}$$

with the following initial and boundary conditions:

$$\begin{aligned} u(\zeta, 0) &= \zeta^2, \quad u(0, t) = \frac{2\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma}, \\ u(1, t) &= 1 + \frac{2\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma}. \end{aligned}$$

The problem has the exact solution at $\gamma = 1$,

$$u(\zeta, t) = \zeta^2 + \frac{\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma}.$$

Apply the new technique based on ADM to Eq. (4), we get the following result:

$$u_n^*(\zeta, t) = u_n(\zeta, t) + (1 - \zeta)[0 - u_n(0, t)] + \zeta[0 - u_n(1, t)], \tag{5}$$

where $n = 0, 1, \dots$

Applying L to Eq. (4), we have

$$Lu = 2t^\gamma + 2\zeta^2 + 2 - \zeta \frac{\partial u(\zeta, t)}{\partial \zeta} - \frac{\partial^2 u(\zeta, t)}{\partial \zeta^2}, \tag{6}$$

where $L = \frac{\partial^\gamma}{\partial t^\gamma}$ and L^{-1} is

$$L^{-1}(\cdot) = I^\gamma(\cdot)dt.$$

Operating Eq. (6) by L^{-1} , we have

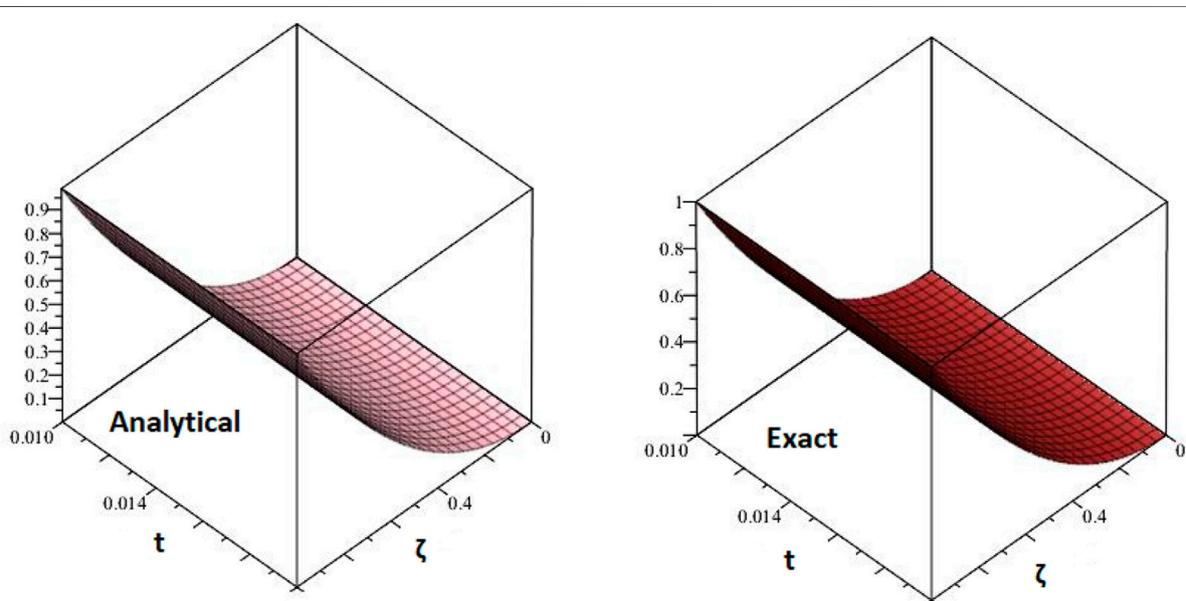


FIGURE 1 | 3D plots for the exact and analytical solution at $\gamma = 1$ for Example 1.

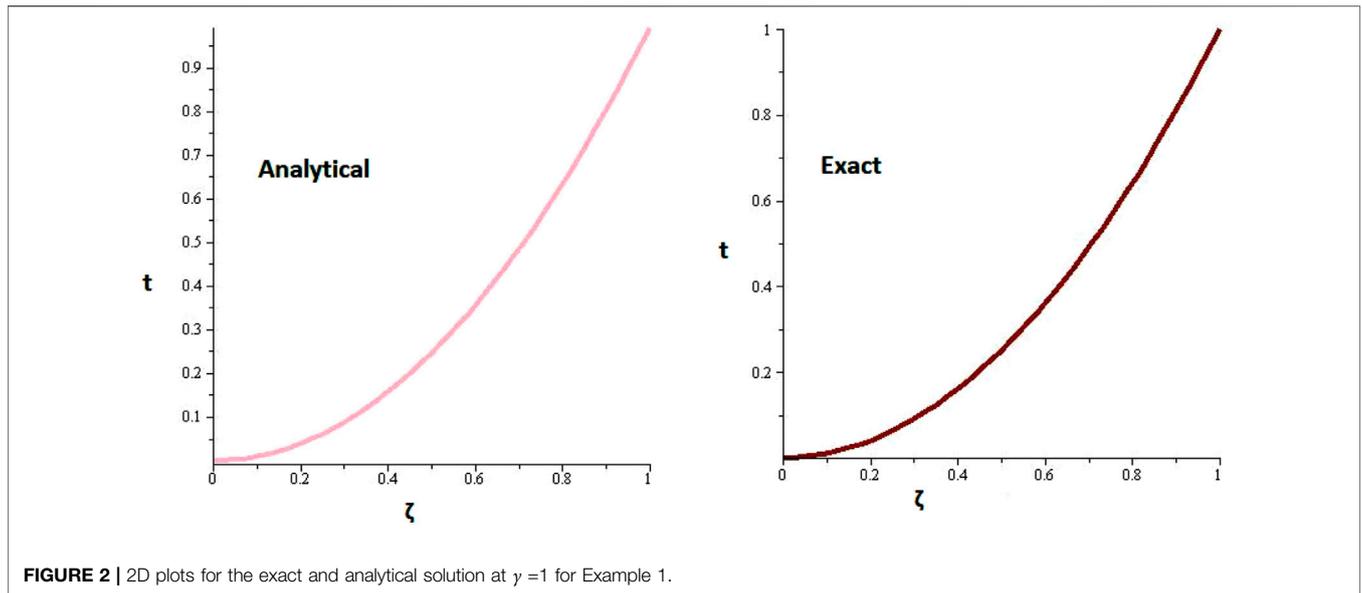


FIGURE 2 | 2D plots for the exact and analytical solution at $\gamma = 1$ for Example 1.

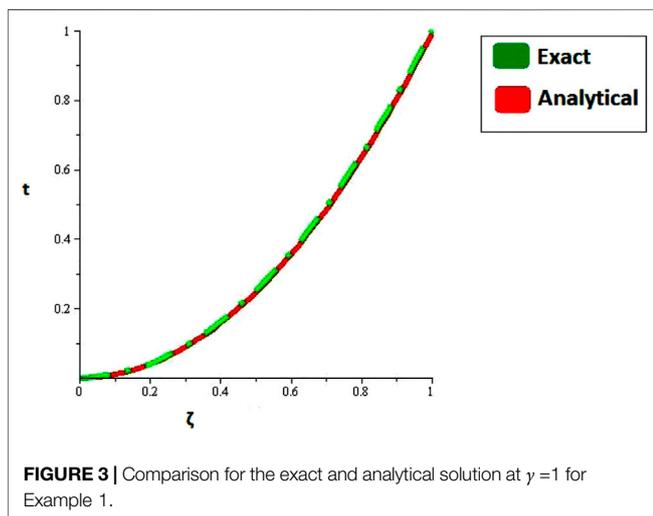


FIGURE 3 | Comparison for the exact and analytical solution at $\gamma = 1$ for Example 1.

$$u(\zeta, t) = u(\zeta, 0) + L^{-1} \left[2t^\gamma + 2\zeta^2 + 2 - 2 - \zeta \frac{\partial u(\zeta, t)}{\partial \zeta} - \frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} \right].$$

Using the ADM solution, the initial approximation becomes

$$u_0(\zeta, t) = u(\zeta, 0) + L^{-1} (2t^\gamma + 2\zeta^2 + 2),$$

$$= \zeta^2 + \frac{2\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma} + \frac{2\zeta^2 t^\gamma}{\Gamma(\gamma + 1)} + \frac{2t^\gamma}{\Gamma(\gamma + 1)},$$

and using the new technique of initial approximation u_n^* , the iteration formula becomes

$$u_{n+1}(\zeta, t) = L^{-1} \left(-\zeta \frac{\partial u_n^*(\zeta, t)}{\partial \zeta} - \frac{\partial^2 u_n^*(\zeta, t)}{\partial \zeta^2} \right). \quad (7)$$

By putting initial and boundary conditions in Eq.(5), for $n = 0$,

$$u_0^*(\zeta, t) = u_0(\zeta, t) + (1 - \zeta) \left[\frac{2\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma} - u_0(0, t) \right]$$

$$+ \zeta \left[1 + \frac{2\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma} - u_0(1, t) \right],$$

$$= \zeta^2 + \frac{2\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma} + \frac{2\zeta^2 t^\gamma}{\Gamma(\gamma + 1)} + \frac{2t^\gamma}{\Gamma(\gamma + 1)}$$

$$+ (1 - \zeta) \left[\frac{2\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma} - \frac{2\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma} - \frac{2t^\gamma}{\Gamma(\gamma + 1)} \right]$$

$$+ \zeta \left[1 + \frac{2\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma} - 1 - \frac{2\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma} - \frac{2t^\gamma}{\Gamma(\gamma + 1)} - \frac{2t^\gamma}{\Gamma(\gamma + 1)} \right],$$

$$= \zeta^2 + \frac{2\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma} + \frac{2\zeta^2 t^\gamma}{\Gamma(\gamma + 1)} + \frac{2t^\gamma}{\Gamma(\gamma + 1)}$$

$$+ (1 - \zeta) \left[-\frac{2t^\gamma}{\Gamma(\gamma + 1)} \right] + \zeta \left[-\frac{4t^\gamma}{\Gamma(\gamma + 1)} \right], = \zeta^2 + \frac{2\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma}$$

$$+ \frac{2\zeta^2 t^\gamma}{\Gamma(\gamma + 1)} + \frac{2t^\gamma}{\Gamma(\gamma + 1)} - \frac{2t^\gamma}{\Gamma(\gamma + 1)} + \frac{2\zeta t^\gamma}{\Gamma(\gamma + 1)} - \frac{4\zeta t^\gamma}{\Gamma(\gamma + 1)}, u_0^*$$

$$(\zeta, t) = \zeta^2 + \frac{2\Gamma(\gamma + 1)}{\Gamma(2\gamma + 1)} t^{2\gamma} + \frac{2\zeta^2 t^\gamma}{\Gamma(\gamma + 1)} - \frac{2\zeta t^\gamma}{\Gamma(\gamma + 1)}.$$

From Eq.(7), we have

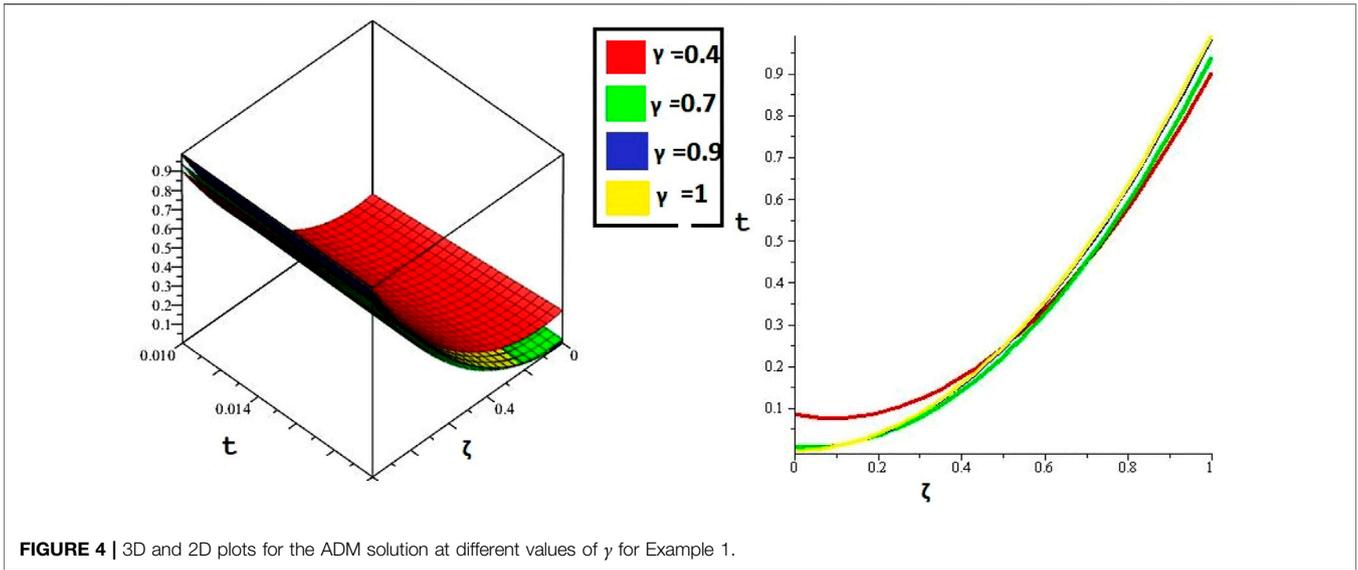
$$u_1(\zeta, t) = L^{-1} \left(-\zeta \frac{\partial u_0^*(\zeta, t)}{\partial \zeta} - \frac{\partial^2 u_0^*(\zeta, t)}{\partial \zeta^2} \right),$$

$$= L^{-1} \left[-\zeta \left(2\zeta + \frac{4\zeta t^\gamma}{\Gamma(\gamma + 1)} - \frac{2t^\gamma}{\Gamma(\gamma + 1)} \right) - \left(2 + \frac{4t^\gamma}{\Gamma(\gamma + 1)} \right) \right],$$

$$\times L^{-1} \left(-2\zeta^2 - \frac{4\zeta^2 t^\gamma}{\Gamma(\gamma + 1)} + \frac{2\zeta t^\gamma}{\Gamma(\gamma + 1)} - 2 - \frac{4t^\gamma}{\Gamma(\gamma + 1)} \right),$$

$$u_1(\zeta, t) = \frac{2\zeta^2 t^\gamma}{\Gamma(\gamma + 1)} - \frac{4\zeta^2 t^{2\gamma}}{\Gamma(2\gamma + 1)} + \frac{2\zeta t^{2\gamma}}{\Gamma(2\gamma + 1)} - \frac{2t^\gamma}{\Gamma(\gamma + 1)} - \frac{4t^{2\gamma}}{\Gamma(2\gamma + 1)}.$$

For $n = 1$, Eq.(5) becomes



$$\begin{aligned}
 u(\zeta, t) &= u_0(\zeta, t) + u_1(\zeta, t) + u_2(\zeta, t) + u_3(\zeta, t) + \dots \\
 &= \zeta^2 + \frac{2\Gamma(\gamma+1)}{\Gamma(2\gamma+1)}t^{2\gamma} + \frac{2\zeta^2 t^\gamma}{\Gamma(\gamma+1)} + \frac{2t^\gamma}{\Gamma(\gamma+1)} \\
 &\quad - \frac{2\zeta^2 t^\gamma}{\Gamma(\gamma+1)} - \frac{4\zeta^2 t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{2\zeta^2 t^{2\gamma}}{\Gamma(\gamma+1)} - \frac{2t^\gamma}{\Gamma(\gamma+1)} - \frac{4t^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{4\zeta^2 t^{2\gamma}}{\Gamma(2\gamma+1)} \\
 &\quad + \frac{8\zeta^2 t^{3\gamma}}{8t^{3\gamma}} - \frac{2\zeta^2 t^{3\gamma}}{2\zeta^2 t^{3\gamma}} - \frac{4\zeta^2 t^{2\gamma}}{4\zeta^2 t^{2\gamma}} - \frac{\zeta t^\gamma}{\zeta t^\gamma} + \frac{4t^{2\gamma}}{4t^{2\gamma}} \\
 &\quad + \frac{\Gamma(3\gamma+1)}{8t^{3\gamma}} - \frac{\Gamma(3\gamma+1)}{-8\zeta^2 t^{3\gamma}} - \frac{\Gamma(2\gamma+1)}{16\zeta^2 t^{4\gamma}} - \frac{\Gamma(\gamma+1)}{8\zeta^2 t^{4\gamma}} + \frac{\Gamma(2\gamma+1)}{4\zeta^2 t^{3\gamma}} \\
 &\quad + \frac{\Gamma(3\gamma+1)}{8t^{3\gamma}} - \frac{\Gamma(3\gamma+1)}{16t^{4\gamma}} - \frac{\Gamma(4\gamma+1)}{\Gamma(4\gamma+1)} + \frac{\Gamma(4\gamma+1)}{\Gamma(4\gamma+1)} + \frac{\Gamma(3\gamma+1)}{\Gamma(3\gamma+1)} - \frac{\zeta t^\gamma}{\Gamma(\gamma+1)} \\
 &\quad - \frac{\Gamma(3\gamma+1)}{\Gamma(3\gamma+1)} - \frac{16t^{4\gamma}}{\Gamma(4\gamma+1)} + \dots u(\zeta, t) = \zeta^2 + \frac{2\Gamma(\gamma+1)}{\Gamma(2\gamma+1)}t^{2\gamma}.
 \end{aligned}$$

$$\frac{\partial^\gamma u(\zeta, t)}{\partial t^\gamma} + \zeta \frac{\partial u(\zeta, t)}{\partial \zeta} - \frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} = g(\zeta, t), \quad 0 < \zeta < 1, 0 < \gamma < 1. \tag{8}$$

With the following initial and boundary conditions,

$$\begin{aligned}
 u(\zeta, 0) &= \zeta - \zeta^3, \\
 u(0, t) &= u(1, t) = 0,
 \end{aligned}$$

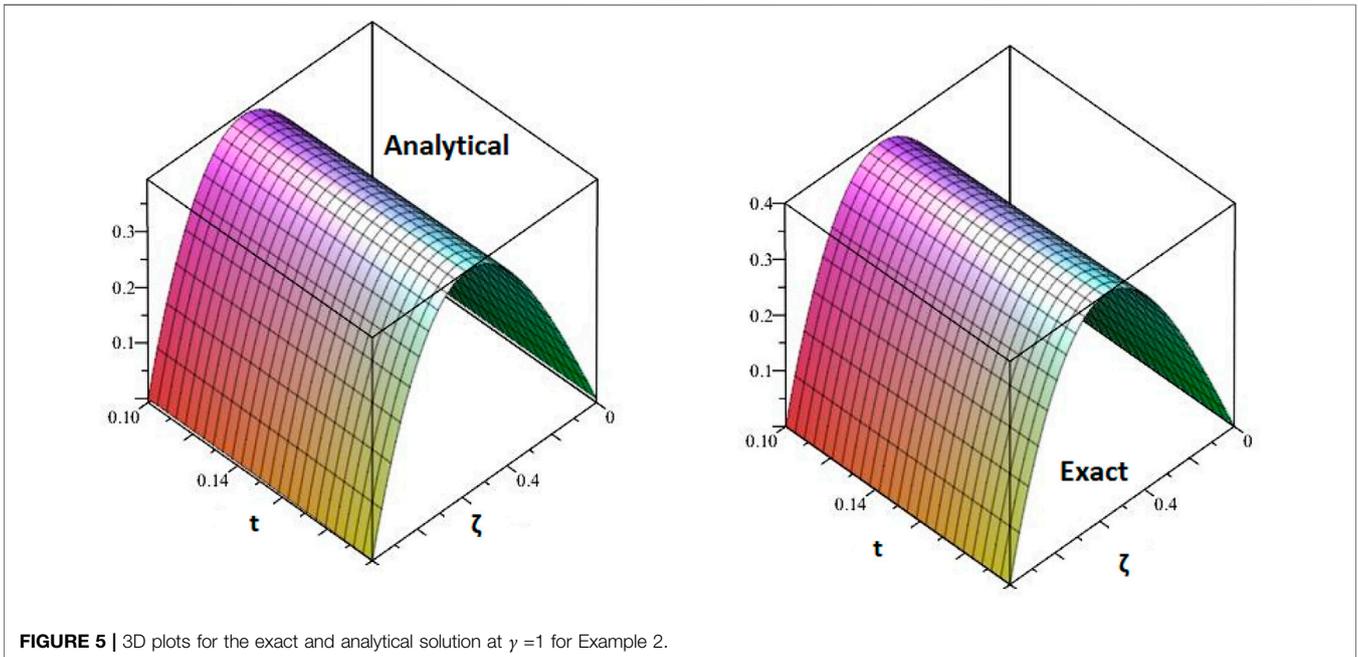
where

$$g(\zeta, t) = \frac{\Gamma(1+2\gamma)}{\Gamma(1+\gamma)}t^\gamma(\zeta - \zeta^3) + (1+t^{2\gamma})(7\zeta - 3\zeta^3).$$

The problem has the exact solution at $\gamma = 1$ is,

5.2 Example 2

Consider the TFCDE of the following form [76]:



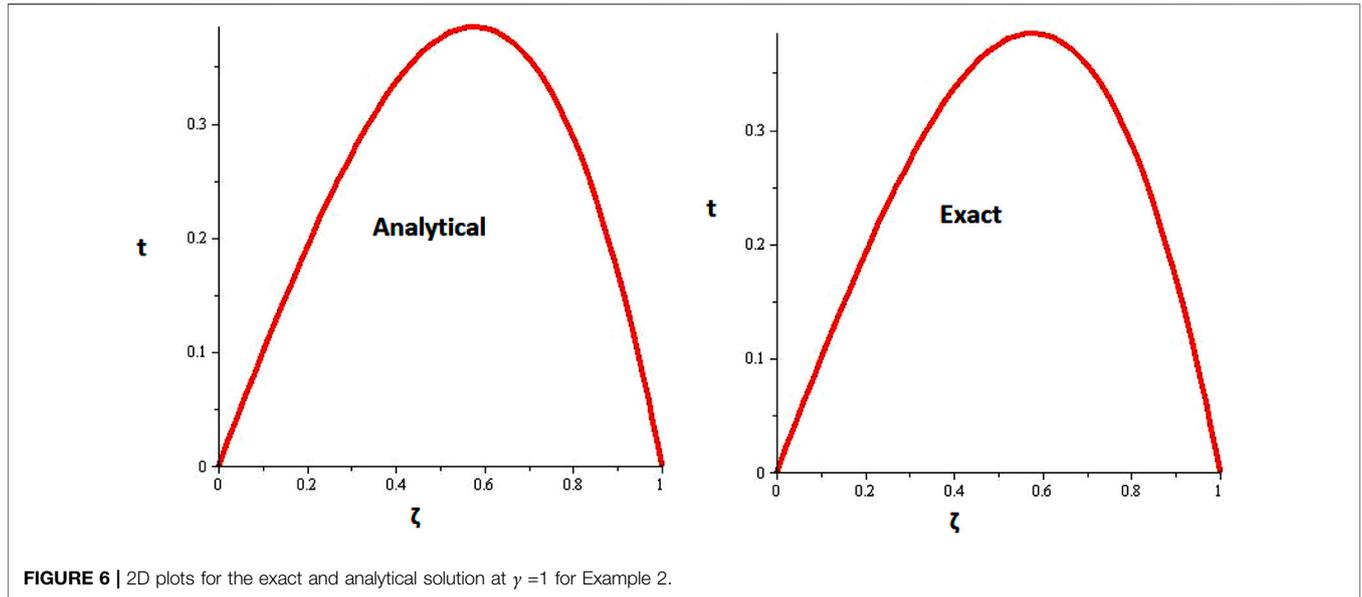


FIGURE 6 | 2D plots for the exact and analytical solution at $\gamma = 1$ for Example 2.

$$u(\zeta, t) = (1 + t^{2\gamma})(\zeta - \zeta^3).$$

Applying the new technique based on ADM to Eq.(15), we get the following result:

$$u_n^*(\zeta, t) = u_n(\zeta, t) + (1 - \zeta)[\zeta - \zeta^3 - u_n(0, t)] + \zeta[0 - u_n(1, t)], \tag{9}$$

where $n = 0, 1, \dots$

Applying L to Eq.(15), we have

$$Lu = \frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} - \zeta \frac{\partial u(\zeta, t)}{\partial \zeta} + g(\zeta, t), \tag{10}$$

where $L = \frac{\partial^\gamma}{\partial t^\gamma}$ and L^{-1} is

$$L^{-1}(\cdot) = I^\gamma(\cdot)dt.$$

Operating by L^{-1} , we have

$$u(\zeta, t) = u(\zeta, 0) + L^{-1}\left(\frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} - \zeta \frac{\partial u(\zeta, t)}{\partial \zeta} + g(\zeta, t)\right).$$

Using the ADM solution, the initial approximation becomes

$$\begin{aligned} u_0(\zeta, t) &= u(\zeta, 0) + L^{-1}(g(\zeta, t)), \\ &= (\zeta - \zeta^3) + L^{-1}\left(\frac{\Gamma(1+2\gamma)}{\Gamma(1+\gamma)} t^\gamma (\zeta - \zeta^3) + (1+t^{2\gamma})(7\zeta - 3\zeta^3)\right), \\ &= (\zeta - \zeta^3) + t^{2\gamma}(\zeta - \zeta^3) + (7\zeta - 3\zeta^3) \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)}\right), u_0(\zeta, t) \\ &= (\zeta - \zeta^3)(1+t^{2\gamma}) + (7\zeta - 3\zeta^3) \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)}\right). \end{aligned}$$

Using the new technique of initial approximation u_n^* , the iteration formula becomes

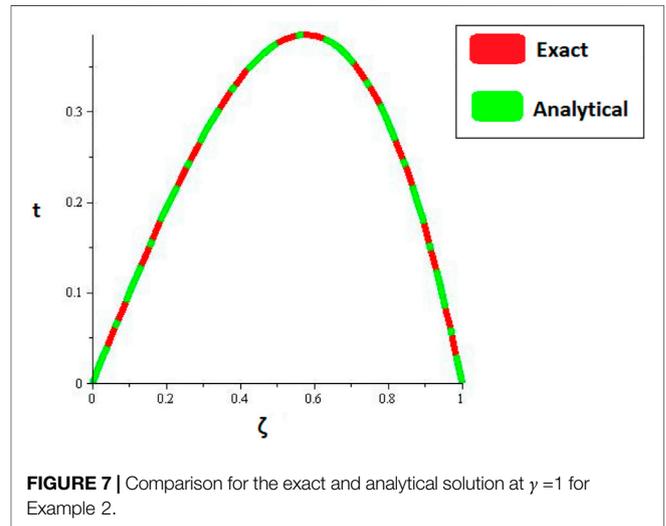


FIGURE 7 | Comparison for the exact and analytical solution at $\gamma = 1$ for Example 2.

$$u_{n+1}(\zeta, t) = L^{-1}\left(\frac{\partial^2 u_n^*}{\partial \zeta^2} - \zeta \frac{\partial u_n^*}{\partial \zeta}\right). \tag{11}$$

By putting the initial and boundary condition in Eq.(16), for $n = 0$,

$$\begin{aligned} u_0^*(\zeta, t) &= u_0(\zeta, t) + (1 - \zeta)[0 - u_0(0, t)] + \zeta[0 - u_0(1, t)], \\ &= (\zeta - \zeta^3)(1+t^{2\gamma}) + (7\zeta - 3\zeta^3) \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)}\right) \\ &\quad + (1 - \zeta)[0 - 0] + \zeta \left[0 - 4 \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)}\right)\right], \\ &= (\zeta - \zeta^3)(1+t^{2\gamma}) + (7\zeta - 3\zeta^3) \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)}\right) \\ &\quad - 4\zeta \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)}\right), \\ u_0^*(\zeta, t) &= (\zeta - \zeta^3)(1+t^{2\gamma}) + (7\zeta - 3\zeta^3) \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)}\right). \end{aligned}$$

From Eq. (9), we have

$$\begin{aligned}
 u_1(\zeta, t) &= L^{-1} \left(\frac{\partial^2 u_0^*}{\partial \zeta^2} - \zeta \frac{\partial u_0^*}{\partial \zeta} \right), \\
 &= L^{-1} \left[-6\zeta(1+t^{2\gamma}) - 18\zeta \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \right. \\
 &\quad \left. - \zeta \left((1-3\zeta^2)(1+t^{2\gamma}) + (3-9\zeta^2) \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \right) \right], \\
 &= L^{-1} \left[(1+t^{2\gamma})(-6\zeta - \zeta + 3\zeta^3) + \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) (-18\zeta - 3\zeta + 9\zeta^3) \right], \\
 &= L^{-1} \left[(1+t^{2\gamma})(-7\zeta + 3\zeta^3) + \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) (-21\zeta + 9\zeta^3) \right], \\
 u_1(\zeta, t) &= \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) (-7\zeta + 3\zeta^3) \\
 &\quad + \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) (-21\zeta + 9\zeta^3).
 \end{aligned}$$

For $n = 1$, Eq.(9), we have

$$\begin{aligned}
 u_1^*(\zeta, t) &= u_1(\zeta, t) + (1-\zeta)[0 - u_1(0, t)] + \zeta[0 - u_1(1, t)], \\
 &= \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) (-7\zeta + 3\zeta^3) \\
 &\quad + \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) (-21\zeta + 9\zeta^3) + (1-\zeta)[0 - 0] \\
 &\quad + \zeta \left[0 + 4 \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) + 12 \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \right], \\
 &= (-7\zeta + 4\zeta + 3\zeta^3) \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \\
 &\quad + \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) (-21\zeta + 9\zeta^3 + 12\zeta), \\
 u_1'(\zeta, t) &= (-3\zeta + 3\zeta^3) \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \\
 &\quad + \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) (-9\zeta + 9\zeta^3),
 \end{aligned}$$

From Eq. (9), we have

$$\begin{aligned}
 u_2(\zeta, t) &= L^{-1} \left(\frac{\partial^2 u_1^*}{\partial \zeta^2} - \zeta \frac{\partial u_1^*}{\partial \zeta} \right), \\
 &= L^{-1} \left[18\zeta \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) + 54\zeta \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \right. \\
 &\quad \left. - \zeta \left[(-3 + 9\zeta^2) \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \right. \right. \\
 &\quad \left. \left. + (-9 + 27\zeta^2) \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \right] \right], \\
 &= L^{-1} \left[(18\zeta + 3\zeta - 9\zeta^3) \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \right. \\
 &\quad \left. + (54\zeta + 9\zeta - 27\zeta^3) \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \right], \\
 &= L^{-1} \left[(21\zeta - 9\zeta^3) \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \right. \\
 &\quad \left. + (63\zeta - 27\zeta^3) \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \right], \\
 u_2(\zeta, t) &= (21\zeta - 9\zeta^3) \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \\
 &\quad + (63\zeta - 27\zeta^3) \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right).
 \end{aligned}$$

For $n = 2$, Eq.(11) becomes

$$\begin{aligned}
 u_2^*(\zeta, t) &= u_2(\zeta, t) + (1-\zeta)[0 - u_2(0, t)] + \zeta[0 - u_2(1, t)], \\
 &= (21\zeta - 9\zeta^3) \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \\
 &\quad + (63\zeta - 27\zeta^3) \left(\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \frac{t^{5\gamma}}{\Gamma(5\gamma+1)} \right) \\
 &\quad + (1-\zeta)[0 - 0] + \zeta \left[0 - 12 \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \right. \\
 &\quad \left. - 36 \left(\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \frac{t^{5\gamma}}{\Gamma(5\gamma+1)} \right) \right], \\
 &= (21\zeta - 12\zeta - 9\zeta^3) \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \\
 &\quad + (63\zeta - 36\zeta - 27\zeta^3) \left(\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \frac{t^{5\gamma}}{\Gamma(5\gamma+1)} \right), \\
 u_2^*(\zeta, t) &= (9\zeta - 9\zeta^3) \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \\
 &\quad + (27\zeta - 27\zeta^3) \left(\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \frac{t^{5\gamma}}{\Gamma(5\gamma+1)} \right).
 \end{aligned}$$

From Eq. (9), we have

$$\begin{aligned}
 u_3(\zeta, t) &= L^{-1} \left(\frac{\partial^2 u_2^*}{\partial \zeta^2} - \zeta \frac{\partial u_2^*}{\partial \zeta} \right), \\
 &= L^{-1} \left[-54\zeta \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) - 162\zeta \left(\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \frac{t^{5\gamma}}{\Gamma(5\gamma+1)} \right) \right. \\
 &\quad \left. - \zeta \left((9 - 27\zeta^2) \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \right. \right. \\
 &\quad \left. \left. + (27 - 81\zeta^2) \left(\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \frac{t^{5\gamma}}{\Gamma(5\gamma+1)} \right) \right) \right], \\
 &= L^{-1} \left[(-54\zeta - 9\zeta + 27\zeta^3) \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \right. \\
 &\quad \left. + (-27\zeta - 162\zeta + 81\zeta^3) \left(\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \frac{t^{5\gamma}}{\Gamma(5\gamma+1)} \right) \right], \\
 &= L^{-1} \left[(-63\zeta + 27\zeta^3) \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \right. \\
 &\quad \left. + (-189\zeta + 81\zeta^3) \left(\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \frac{t^{5\gamma}}{\Gamma(5\gamma+1)} \right) \right], \\
 u_3(\zeta, t) &= (-63\zeta + 27\zeta^3) \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \\
 &\quad + (-189\zeta + 81\zeta^3) \left(\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \frac{t^{5\gamma}}{\Gamma(5\gamma+1)} \right).
 \end{aligned}$$

⋮

Thus, the ADM solution in a series form is

$$\begin{aligned}
 u(\zeta, t) &= u_0(\zeta, t) + u_1(\zeta, t) + u_2(\zeta, t) + u_3(\zeta, t) + \dots \\
 &= (\zeta - \zeta^3)(1+t^{2\gamma}) + (7\zeta - 3\zeta^3) \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \\
 &\quad + \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) (-7\zeta + 3\zeta^3) \\
 &\quad + \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) (-21\zeta + 9\zeta^3) \\
 &\quad + (21\zeta - 9\zeta^3) \left(\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) \\
 &\quad + (63\zeta - 27\zeta^3) \left(\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \frac{t^{5\gamma}}{\Gamma(5\gamma+1)} \right) \\
 &\quad + (-63\zeta + 27\zeta^3) \left(\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \frac{t^{5\gamma}}{\Gamma(5\gamma+1)} \right) \\
 &\quad + (-189\zeta + 81\zeta^3) \left(\frac{t^{4\gamma}}{\Gamma(4\gamma+1)} + \frac{t^{6\gamma}}{\Gamma(6\gamma+1)} \right) + \dots \\
 u(\zeta, t) &= (1+t^{2\gamma})(\zeta - \zeta^3).
 \end{aligned}$$

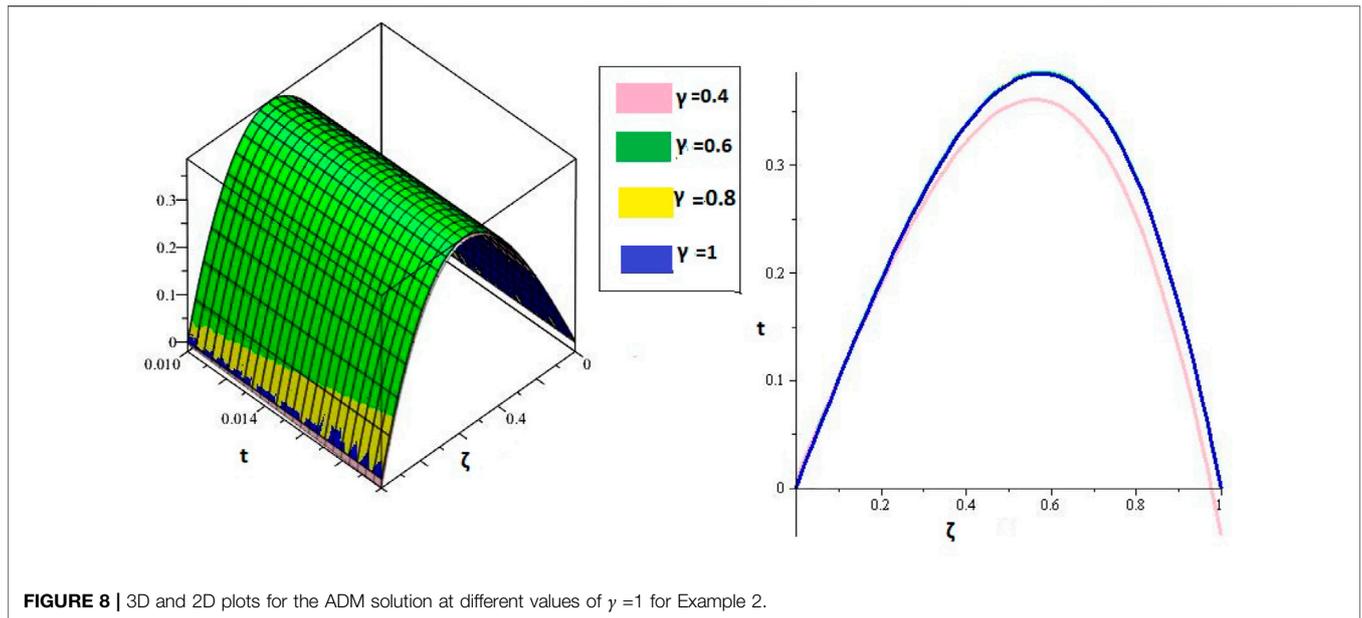


FIGURE 8 | 3D and 2D plots for the ADM solution at different values of $\gamma=1$ for Example 2.

5.3 Example 3

Consider TFCDE of the form [77]

$$g(\zeta, t) = \frac{2t^{2-\gamma}(\zeta^2 - \zeta^3)}{\Gamma(3-\gamma)} + (t^2 + 1)(2\zeta^2 - 3\zeta^3 + 6\zeta - 2).$$

$$\frac{\partial^\gamma u(\zeta, t)}{\partial t^\gamma} + \zeta \frac{\partial u(\zeta, t)}{\partial \zeta} - \frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} = g(\zeta, t), \quad 0 < \zeta < 1, \quad 0 < \gamma \leq 1. \tag{12}$$

The problem has the exact solution at $\gamma = 1$ is,

$$u(\zeta, t) = (t^2 + 1)(\zeta^2 - \zeta^3).$$

With the following initial and boundary conditions,

$$\begin{aligned} u(\zeta, 0) &= \zeta - \zeta^3, \\ u(0, t) = u(1, t) &= 0, \end{aligned}$$

Applying the new technique based on ADM to Eq. (12), we get the following result:

where

$$u_n^*(\zeta, t) = u_n(\zeta, t) + (1 - \zeta)[0 - u_n(0, t)] + \zeta[0 - u_n(1, t)], \tag{13}$$

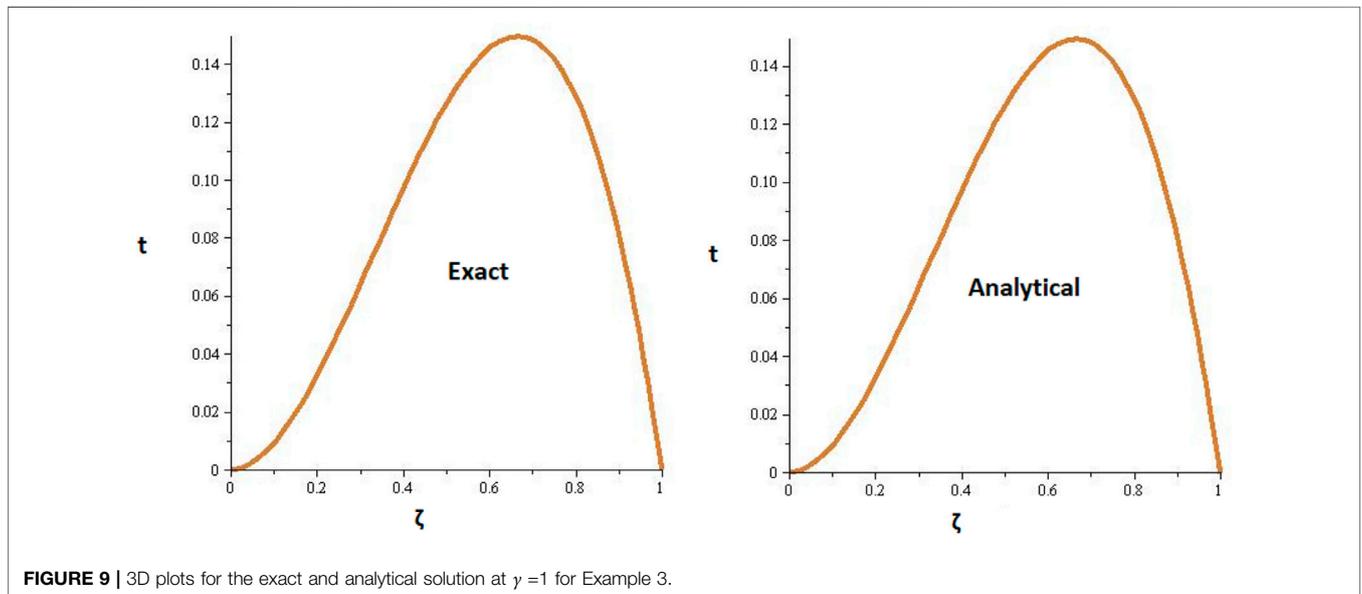


FIGURE 9 | 3D plots for the exact and analytical solution at $\gamma=1$ for Example 3.

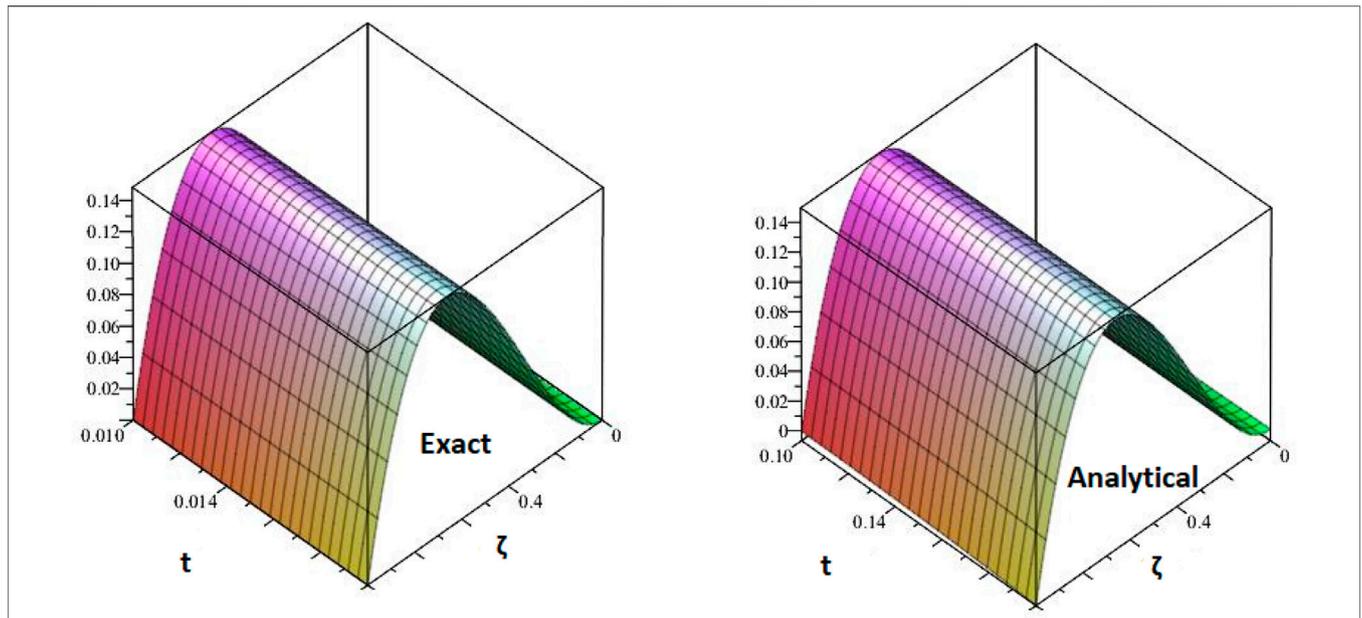


FIGURE 10 | 2D plots for the exact and analytical solution at $\gamma = 1$ for Example 3.

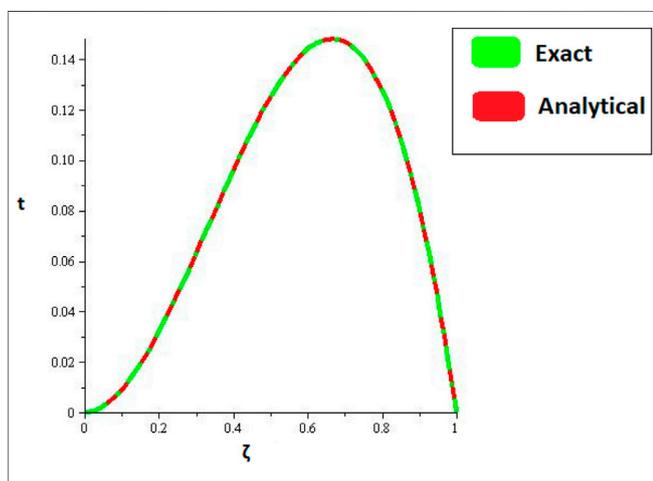


FIGURE 11 | Comparison for the exact and analytical solution at $\gamma = 1$ for Example 3.

where $n = 0, 1, \dots$. Applying L to Eq. (12), we have

$$Lu = \frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} - \zeta \frac{\partial u(\zeta, t)}{\partial \zeta} + g(\zeta, t), \tag{14}$$

where $L = \frac{\partial^\gamma}{\partial t^\gamma}$ and L^{-1} is

$$L^{-1}(\cdot) = I^\gamma(\cdot)dt.$$

Operating Eq. (12) by L^{-1} , we have

$$u(\zeta, t) = u(\zeta, 0) + L^{-1}\left(\frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} - \zeta \frac{\partial u(\zeta, t)}{\partial \zeta} + g(\zeta, t)\right).$$

Using the ADM solution, the initial approximation becomes

$$\begin{aligned} u_0(\zeta, t) &= u(\zeta, 0) + L^{-1}(g(\zeta, t)), \\ &= \zeta^2 - \zeta^3 + L^{-1}\left(\frac{2t^{2-\gamma}(\zeta^2 - \zeta^3)}{\Gamma(3-\gamma)} + (t^2 + 1)(2\zeta^2 - 3\zeta^3 + 6\zeta - 2)\right), \\ &= \zeta^2 - \zeta^3 + t^2(\zeta^2 - \zeta^3) + \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)}\right)(2\zeta^2 - 3\zeta^3 + 6\zeta - 2), \\ u_0(\zeta, t) &= (1 + t^2)(\zeta^2 - \zeta^3) + \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)}\right)(2\zeta^2 - 3\zeta^3 + 6\zeta - 2), \end{aligned}$$

and using the new technique of initial approximation u_n^* , the iteration formula becomes

$$u_{n+1}(\zeta, t) = L^{-1}\left(\frac{\partial^2 u_n^*}{\partial \zeta^2} - \zeta \frac{\partial u_n^*}{\partial \zeta}\right). \tag{15}$$

By putting the initial and boundary conditions in Eq. (13), for $n = 0$,

$$\begin{aligned} u_0^*(\zeta, t) &= u_0(\zeta, t) + (1 - \zeta)[0 - u_0(0, t)] + \zeta[0 - u_0(1, t)], \\ &= (1 + t^2)(\zeta^2 - \zeta^3) + \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)}\right)(2\zeta^2 - 3\zeta^3 + 6\zeta - 2) \\ &\quad + (1 - \zeta)\left[0, +, 2\left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)}\right)\right] + \zeta\left[0, -, 3\left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)}\right)\right], \\ &= (1 + t^2)(\zeta^2 - \zeta^3) + \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)}\right)(2\zeta^2 - 3\zeta^3 + 6\zeta - 2) \\ &\quad + 2\left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)}\right) - 2\zeta\left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)}\right) \\ &\quad - 3\zeta\left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)}\right) = (1 + t^2)(\zeta^2 - \zeta^3) \\ &\quad + \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)}\right)(2\zeta^2 - 3\zeta^3 + 6\zeta - 2 - 5\zeta + 2), \\ u_0^*(\zeta, t) &= (1 + t^2)(\zeta^2 - \zeta^3) + \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)}\right)(2\zeta^2 - 3\zeta^3 + \zeta). \end{aligned} \tag{16}$$

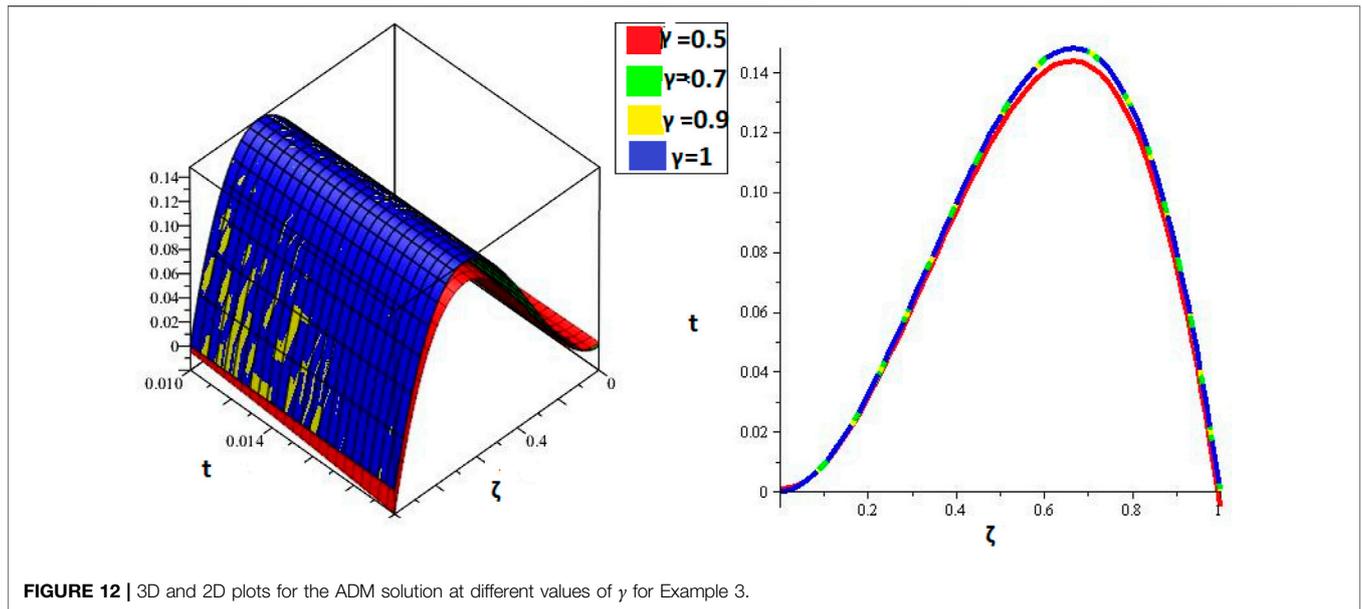


FIGURE 12 | 3D and 2D plots for the ADM solution at different values of γ for Example 3.

From Eq. (15), we have

$$\begin{aligned}
 u_1(\zeta, t) &= L^{-1} \left(\frac{\partial^2 u_0^*}{\partial \zeta^2} - \zeta \frac{\partial u_0^*}{\partial \zeta} \right), \\
 &= L^{-1} \left[(1+t^2)(2-6\zeta) + (4-18\zeta) \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)} \right) \right. \\
 &\quad \left. - \zeta \left((1+t^2)(2\zeta-3\zeta^2) + (4\zeta-9\zeta^2+1) \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)} \right) \right) \right], L^{-1} \\
 &\quad \times \left[(1+t^2)(2-6\zeta-2\zeta^2+3\zeta^3) + (9\zeta^3-4\zeta^2-19\zeta+4) \right. \\
 &\quad \left. \times \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)} \right) \right], \\
 u_1(\zeta, t) &= \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)} \right) (2-6\zeta-2\zeta^2+3\zeta^3) \\
 &\quad + (9\zeta^3-4\zeta^2-19\zeta+4) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right).
 \end{aligned}$$

For $n = 1$, Eq. (13), we have

$$\begin{aligned}
 u_1^*(\zeta, t) &= u_1(\zeta, t) + (1-\zeta)[0-u_1(0, t)] + \zeta[0-u_1(1, t)], \\
 &= \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)} \right) (2-6\zeta-2\zeta^2+3\zeta^3) \\
 &\quad + (9\zeta^3-4\zeta^2-19\zeta+4) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \\
 &\quad + (1-\zeta) \left[0 - 2 \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} \right) - 4 \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{2t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \right] \\
 &\quad + \zeta \left[0 + 3 \left(\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} \right) + 10 \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \right], \\
 &= \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)} \right) (2-6\zeta-2\zeta^2+3\zeta^3-2+2\zeta+3\zeta) \\
 &\quad + \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{2t^{2\gamma}}{\Gamma(2\gamma+1)} \right) (9\zeta^3-4\zeta^2-19\zeta+4-4+4\zeta+10\zeta), \\
 u_1^*(\zeta, t) &= \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)} \right) (3\zeta^3-2\zeta^2-\zeta) \\
 &\quad + \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{2t^{2\gamma}}{\Gamma(2\gamma+1)} \right) (9\zeta^3-4\zeta^2-5\zeta).
 \end{aligned}$$

From Eq. (15), we have

$$\begin{aligned}
 u_2(\zeta, t) &= L^{-1} \left(\frac{\partial^2 u_1^*}{\partial \zeta^2} - \zeta \frac{\partial u_1^*}{\partial \zeta} \right), \\
 &= L^{-1} \left[(18\zeta-4) \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)} \right) + (54\zeta-8) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{2t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \right. \\
 &\quad \left. - \zeta \left((9\zeta^2-4\zeta-1) \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)} \right) \right) \right. \\
 &\quad \left. - (27\zeta^2-8\zeta-5) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{2t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \right], \\
 &= L^{-1} \left[(18\zeta-4-9\zeta^3+4\zeta^2+\zeta) \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)} \right) \right. \\
 &\quad \left. + (54\zeta-8-27\zeta^3+8\zeta^2+5\zeta) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{2t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \right], \\
 &= L^{-1} \left[(19\zeta-9\zeta^3-4+4\zeta^2) \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)} \right) \right. \\
 &\quad \left. + (59\zeta-27\zeta^3+8\zeta^2-3) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{2t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \right], \\
 u_2(\zeta, t) &= (19\zeta-9\zeta^3-4+4\zeta^2) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \\
 &\quad + (59\zeta-27\zeta^3+8\zeta^2-8) \left(\frac{2t^{3\gamma+2}}{\Gamma(3\gamma+3)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right).
 \end{aligned}$$

For $n = 2$, Eq. (13) becomes

$$\begin{aligned}
 u_2^*(\zeta, t) &= u_2(\zeta, t) + (1-\zeta)[0-u_2(0, t)] + \zeta[0-u_2(1, t)], \\
 &= (19\zeta-9\zeta^3-4+4\zeta^2) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \\
 &\quad + (59\zeta-27\zeta^3+8\zeta^2-8) \left(\frac{2t^{3\gamma+2}}{\Gamma(3\gamma+3)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \\
 &\quad + (1-\zeta) \left[0 + 4 \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) + 8 \left(\frac{2t^{3\gamma+2}}{\Gamma(3\gamma+3)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \right] \\
 &\quad + \zeta \left[0 - 10 \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) - 32 \left(\frac{2t^{3\gamma+2}}{\Gamma(3\gamma+3)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \right], \\
 &= (19\zeta-9\zeta^3-4+4\zeta^2+4-4\zeta-10\zeta) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \\
 &\quad + (59\zeta-27\zeta^3+8\zeta^2-8+8-8\zeta-32\zeta) \left(\frac{2t^{3\gamma+2}}{\Gamma(3\gamma+3)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right), \\
 u_2^*(\zeta, t) &= (5\zeta-9\zeta^3+4\zeta^2) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \\
 &\quad + (19\zeta-27\zeta^3+8\zeta^2) \left(\frac{2t^{3\gamma+2}}{\Gamma(3\gamma+3)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right).
 \end{aligned}$$

From Eq. (15), we have

$$\begin{aligned}
 u_3(\zeta, t) &= L^{-1} \left(\frac{\partial^2 u_2^*}{\partial \zeta^2} - \zeta \frac{\partial u_2^*}{\partial \zeta} \right), \\
 &= L^{-1} \left[(-54\zeta + 8) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \right. \\
 &\quad + (-162\zeta + 16) \left(\frac{2t^{3\gamma+2}}{\Gamma(3\gamma+3)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \\
 &\quad - \zeta \left((5 - 27\zeta^2 + 8\zeta) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \right. \\
 &\quad \left. \left. + (19 - 81\zeta^2 + 16\zeta) \left(\frac{2t^{3\gamma+2}}{\Gamma(3\gamma+3)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \right) \right], \\
 &= L^{-1} \left[(-54\zeta + 8 - 5\zeta + 27\zeta^3 - 8\zeta^2) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \right. \\
 &\quad \left. + (-162\zeta + 16 - 19\zeta + 81\zeta^3 - 8\zeta^2) \left(\frac{2t^{3\gamma+2}}{\Gamma(3\gamma+3)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \right], \\
 &= L^{-1} \left[(-59\zeta + 27\zeta^3 - 8\zeta^2 + 8) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \right. \\
 &\quad \left. + (-181\zeta + 81\zeta^3 - 8\zeta^2 + 16) \left(\frac{2t^{3\gamma+2}}{\Gamma(3\gamma+3)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \right], \\
 u_3(\zeta, t) &= (-59\zeta + 27\zeta^3 - 8\zeta^2 + 8) \left(\frac{2t^{2\gamma+2}}{\Gamma(3\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(3\gamma+1)} \right) \\
 &\quad + (-181\zeta + 81\zeta^3 - 8\zeta^2 + 16) \left(\frac{2t^{4\gamma+2}}{\Gamma(4\gamma+3)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right).
 \end{aligned}$$

Thus, the ADM solution as a series is

$$\begin{aligned}
 u(\zeta, t) &= u_0(\zeta, t) + u_1(\zeta, t) + u_2(\zeta, t) + u_3(\zeta, t) + \dots, \\
 &= (1 + t^2)(\zeta^2 - \zeta^3) + \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)} \right) (2\zeta^2 - 3\zeta^3 + 6\zeta - 2) \\
 &\quad + \left(\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^\gamma}{\Gamma(\gamma+1)} \right) (2 - 6\zeta - 2\zeta^2 + 3\zeta^3) \\
 &\quad + (9\zeta^3 - 4\zeta^2 - 19\zeta + 4) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \\
 &\quad + (19\zeta - 9\zeta^3 - 4 + 4\zeta^2) \left(\frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} \right) \\
 &\quad + (59\zeta - 27\zeta^3 + 8\zeta^2 - 8) \left(\frac{2t^{3\gamma+2}}{\Gamma(3\gamma+3)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \\
 &\quad + (-59\zeta + 27\zeta^3 - 8\zeta^2 + 8) \left(\frac{2t^{3\gamma+2}}{\Gamma(3\gamma+3)} + \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} \right) \\
 &\quad + (-181\zeta + 81\zeta^3 - 8\zeta^2 + 16) \left(\frac{2t^{4\gamma+2}}{\Gamma(4\gamma+3)} + \frac{t^{4\gamma}}{\Gamma(4\gamma+1)} \right) + \dots, \\
 u(\zeta, t) &= (1 + t^2)(\zeta^2 - \zeta^3).
 \end{aligned}$$

5.4 Example 4

Consider the TFCDE of the form [77]

$$\frac{\partial^\gamma u(\zeta, t)}{\partial t^\gamma} + \frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} = g(\zeta, t), \quad 0 < \zeta < 1, 0 < \gamma < 1. \tag{17}$$

With the following initial and boundary conditions,

$$\begin{aligned}
 u(\zeta, 0) &= 0, \\
 u(0, t) &= u(1, t) = 0.
 \end{aligned}$$

where

$$g(\zeta, t) = \frac{2}{\Gamma(3-\gamma)} t^{2-\gamma} \sin(2\pi\zeta) + 4\pi^2 t^2 \sin(2\pi\zeta).$$

The problem has the exact solution at $\gamma = 1$ is,

$$u(\zeta, t) = t^2 \sin(2\pi\zeta).$$

Applying the new technique based on ADM to Eq. (17), we get the following result:

$$u_n^*(\zeta, t) = u_n(\zeta, t) + (1 - \zeta)[0 - u_n(0, t)] + \zeta[0 - u_n(1, t)], \tag{18}$$

where $n = 0, 1, \dots$

Applying L to Eq.(17), we have

$$Lu = \frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} + g(\zeta, t), \tag{19}$$

where $L = \frac{\partial^\gamma}{\partial t^\gamma}$ and L^{-1} is

$$L^{-1}(\cdot) = I^\gamma(\cdot)dt.$$

Operating Eq. (19) by L^{-1} , we have

$$u(\zeta, t) = u(\zeta, 0) + L^{-1} \left(\frac{\partial^2 u(\zeta, t)}{\partial \zeta^2} + g(\zeta, t) \right).$$

Using the ADM solution, the initial approximation becomes

$$\begin{aligned}
 u_0(\zeta, t) &= u(\zeta, 0) + L^{-1}(g(\zeta, t)), \\
 &= 0 + L^{-1} \left(\frac{2}{\Gamma(3-\gamma)} t^{2-\gamma} \sin(2\pi\zeta) + 4\pi^2 t^2 \sin(2\pi\zeta) \right), \\
 u_0(\zeta, t) &= t^2 \sin(2\pi\zeta) + \frac{8\pi^2 \sin(2\pi\zeta) t^{2+\gamma}}{\Gamma(3+\gamma)},
 \end{aligned}$$

and using the new technique of initial approximation u_n^* , the iteration formula becomes

$$u_{n+1}(\zeta, t) = L^{-1} \left(\frac{\partial^2 u_n^*}{\partial \zeta^2} \right). \tag{20}$$

By putting the initial and boundary condition in Eq.(18), for $n = 0$,

$$\begin{aligned}
 u_0^*(\zeta, t) &= u_0(\zeta, t) + (1 - \zeta)[0 - u_0(0, t)] + \zeta[0 - u_0(1, t)], \\
 &= t^2 \sin(2\pi\zeta) + \frac{8\pi^2 \sin(2\pi\zeta) t^{2+\gamma}}{\Gamma(3+\gamma)} + (1 - \zeta) \left[0 - t^2 \sin(0) - \frac{8\pi^2 \sin(0)}{\Gamma(3+\gamma)} \right] \\
 &\quad + \zeta \left[0 - t^2 \sin(2\pi) t^{2+\gamma} - \frac{8\pi^2 \sin(2\pi) t^{2+\gamma}}{\Gamma(3+\gamma)} \right], \\
 &= t^2 \sin(2\pi\zeta) + \frac{8\pi^2 \sin(2\pi\zeta) t^{2+\gamma}}{\Gamma(3+\gamma)} + 0 + 0, \\
 u_0^*(\zeta, t) &= t^2 \sin(2\pi\zeta) + \frac{8\pi^2 \sin(2\pi\zeta) t^{2+\gamma}}{\Gamma(3+\gamma)}.
 \end{aligned}$$

From Eq. (20), we have

$$\begin{aligned}
 u_1(\zeta, t) &= L^{-1} \left(\frac{\partial^2 u_0^*}{\partial \zeta^2} \right), \\
 &= L^{-1} \left(-4\pi^2 t^2 \sin(2\pi\zeta) - \frac{32\pi^4 \sin(2\pi\zeta) t^{2+\gamma}}{\Gamma(3+\gamma)} \right), \\
 u_1(\zeta, t) &= \frac{-4\pi^2 t^{2+\gamma} \sin(2\pi\zeta)}{\Gamma(\gamma+3)} - \frac{32\pi^4 \sin(2\pi\zeta) t^{2+2\gamma}}{\Gamma(2\gamma+3)}.
 \end{aligned}$$

For $n = 1$ Eq. (18), we have

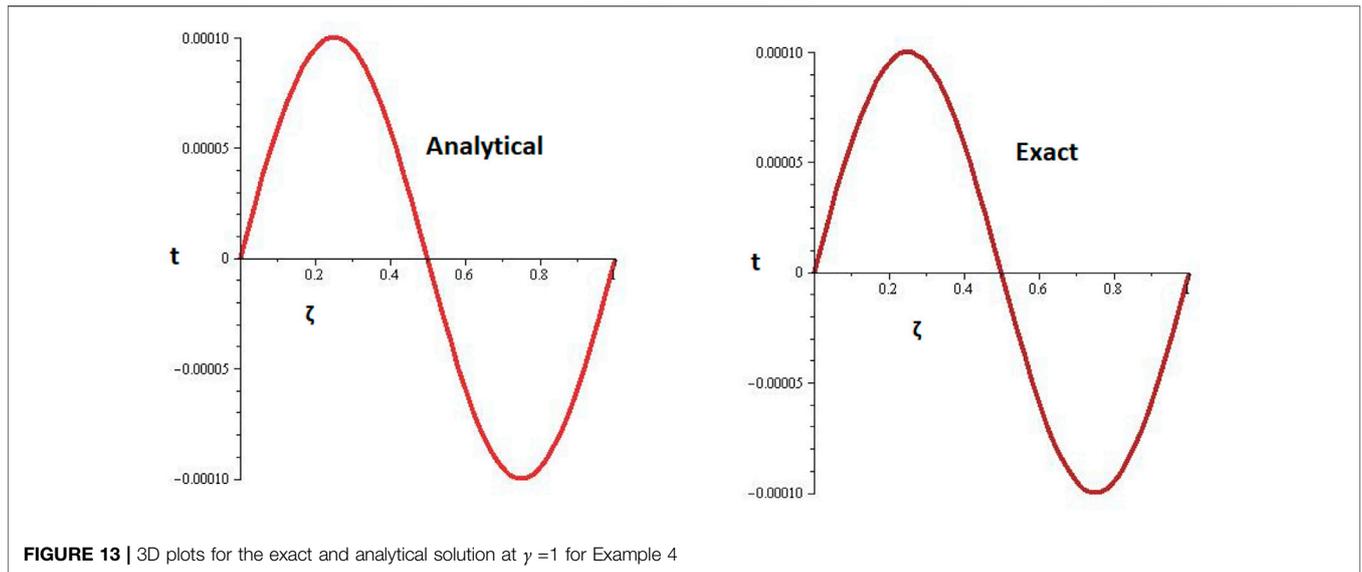


FIGURE 13 | 2D plots for the exact and analytical solution at $\gamma = 1$ for Example 4

$$\begin{aligned}
 u_1^*(\zeta, t) &= u_1(\zeta, t) + (1 - \zeta)[0 - u_1(0, t)] + \zeta[0 - u_1(1, t)], \\
 &= \frac{-4\pi^2 t^{2+\gamma} \sin(2\pi\zeta)}{\Gamma(\gamma+3)} - \frac{32\pi^4 \sin(2\pi\zeta) t^{2+2\gamma}}{\Gamma(3+2\gamma)} \\
 &\quad + (1 - \zeta) \left[0 - \left(\frac{-4\pi^2 t^{2+\gamma} \sin(0)}{\Gamma(\gamma+3)} - \frac{32\pi^4 \sin(0) t^{2+2\gamma}}{\Gamma(3+2\gamma)} \right) \right] \\
 &\quad + \zeta \left[0 - \left(\frac{-4\pi^2 t^{2+\gamma} \sin(2\pi)}{\Gamma(\gamma+3)} - \frac{32\pi^4 \sin(2\pi) t^{2+2\gamma}}{\Gamma(3+2\gamma)} \right) \right], \\
 &= \frac{-4\pi^2 t^{2+\gamma} \sin(2\pi\zeta)}{\Gamma(\gamma+3)} - \frac{32\pi^4 \sin(2\pi\zeta) t^{2+2\gamma}}{\Gamma(3+2\gamma)} + (1 - \zeta)[0 + 0 + 0] + \zeta[0 + 0 + 0], \\
 u_1^*(\zeta, t) &= \frac{-4\pi^2 t^{2+\gamma} \sin(2\pi\zeta)}{\Gamma(\gamma+3)} - \frac{32\pi^4 \sin(2\pi\zeta) t^{2+2\gamma}}{\Gamma(3+2\gamma)}.
 \end{aligned}$$

From Eq. (20), we have

$$\begin{aligned}
 u_2(\zeta, t) &= L^{-1} \left(\frac{\partial^2 u_1^*}{\partial \zeta^2} \right), = L^{-1} \left(\frac{32\pi^4 t^{2+\gamma} \sin(2\pi\zeta)}{\Gamma(\gamma+3)} + \frac{128\pi^6 \sin(2\pi\zeta) t^{2+2\gamma}}{\Gamma(3+2\gamma)} \right), \\
 u_2(\zeta, t) &= \frac{32\pi^4 t^{2+2\gamma} \sin(2\pi\zeta)}{\Gamma(3+2\gamma)} + \frac{128\pi^6 \sin(2\pi\zeta) t^{2+3\gamma}}{\Gamma(3+3\gamma)}.
 \end{aligned}$$

For $n = 2$, Eq. (18) becomes

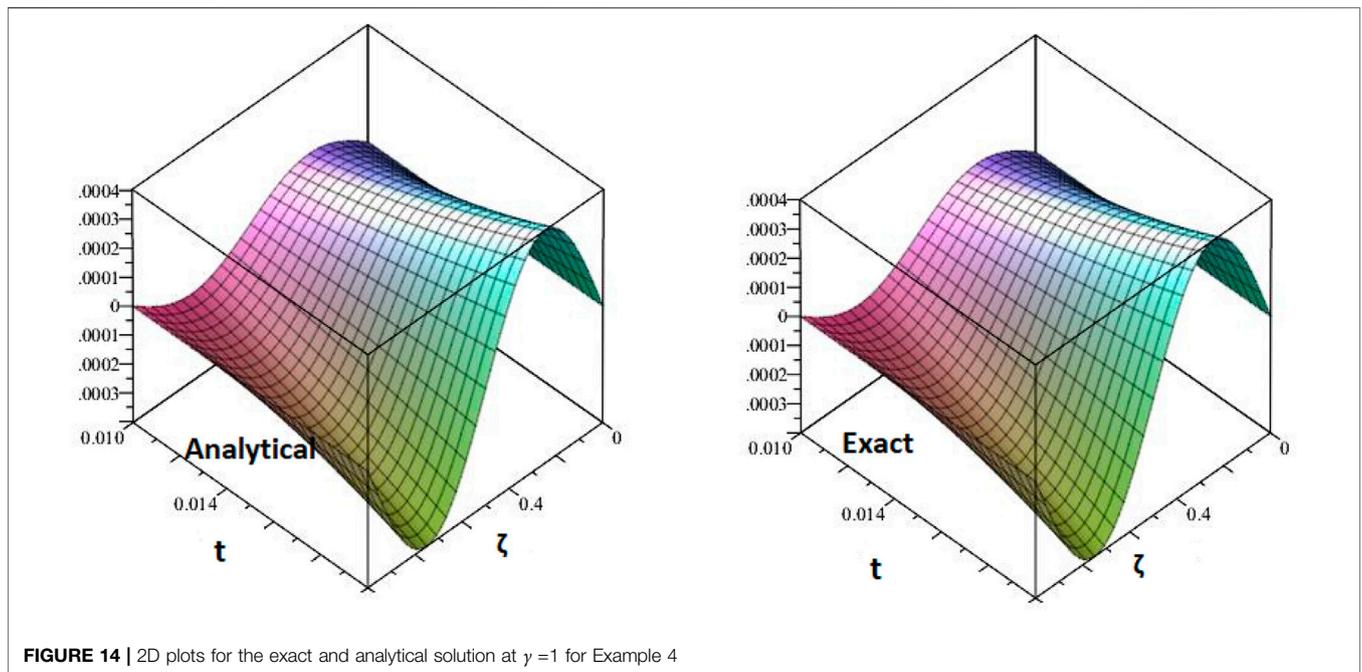


FIGURE 14 | 3D plots for the exact and analytical solution at $\gamma = 1$ for Example 4

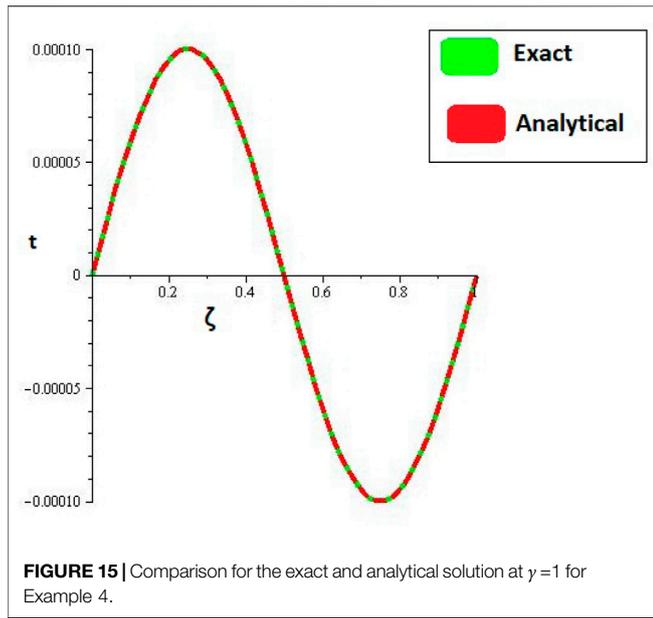


FIGURE 15 | Comparison for the exact and analytical solution at $\gamma = 1$ for Example 4.

$$\begin{aligned}
 u_2^*(\zeta, t) &= u_2(\zeta, t) + (1 - \zeta)[0 - u_2(0, t)] + \zeta[0 - u_2(1, t)], \\
 &= \frac{32\pi^4 t^{2+2\gamma} \sin(2\pi\zeta)}{\Gamma(3 + 2\gamma)} + \frac{128\pi^6 \sin(2\pi\zeta) t^{2+3\gamma}}{\Gamma(3 + 3\gamma)} \\
 &\quad + (1 - \zeta) \left[0 - \frac{32\pi^4 t^{2+2\gamma} \sin(0)}{\Gamma(3 + 2\gamma)} - \frac{128\pi^6 \sin(0) t^{2+3\gamma}}{\Gamma(3 + 3\gamma)} \right] \\
 &\quad + \zeta \left[0 - \frac{32\pi^4 t^{2+2\gamma} \sin(2\pi)}{\Gamma(3 + 2\gamma)} - \frac{128\pi^6 \sin(2\pi) t^{2+3\gamma}}{\Gamma(3 + 3\gamma)} \right], \\
 &= \frac{32\pi^4 t^{2+2\gamma} \sin(2\pi\zeta)}{\Gamma(3 + 2\gamma)} + \frac{128\pi^6 \sin(2\pi\zeta) t^{2+3\gamma}}{\Gamma(3 + 3\gamma)} \\
 &\quad + (1 - \zeta)[0 + 0 + 0] + \zeta[0 + 0 + 0], \\
 u_2^*(\zeta, t) &= \frac{32\pi^4 t^{2+2\gamma} \sin(2\pi\zeta)}{\Gamma(3 + 2\gamma)} + \frac{128\pi^6 \sin(2\pi\zeta) t^{2+3\gamma}}{\Gamma(3 + 3\gamma)}.
 \end{aligned}$$

From Eq.(20), we have

$$\begin{aligned}
 u_3(\zeta, t) &= L^{-1} \left(\frac{\partial^2 u_2^*}{\partial \zeta^2} \right), \\
 &= L^{-1} \left(\frac{-128\pi^6 t^{2+2\gamma} \sin(2\pi\zeta)}{\Gamma(3 + 2\gamma)} - \frac{512\pi^8 \sin(2\pi\zeta) t^{2+3\gamma}}{\Gamma(3 + 3\gamma)} \right), \\
 u_3(\zeta, t) &= \frac{-128\pi^6 t^{2+3\gamma} \sin(2\pi\zeta)}{\Gamma(3 + 3\gamma)} - \frac{512\pi^8 \sin(2\pi\zeta) t^{2+4\gamma}}{\Gamma(3 + 4\gamma)}.
 \end{aligned}$$

⋮

Thus, the ADM solution as a series is

$$\begin{aligned}
 u(\zeta, t) &= u_0(\zeta, t) + u_1(\zeta, t) + u_2(\zeta, t) + u_3(\zeta, t) + \dots, \\
 &= t^2 \sin(2\pi\zeta) + \frac{8\pi^2 \sin(2\pi\zeta) t^{2+\gamma}}{\Gamma(3 + \gamma)} \\
 &\quad + \frac{-4\pi^2 t^{2+\gamma} \sin(2\pi\zeta)}{\Gamma(\gamma + 3)} - \frac{32\pi^4 \sin(2\pi\zeta) t^{2+2\gamma}}{\Gamma(2\gamma + 3)} \\
 &\quad + \frac{32\pi^4 t^{2+2\gamma} \sin(2\pi\zeta)}{\Gamma(3 + 2\gamma)} + \frac{128\pi^6 \sin(2\pi\zeta) t^{2+3\gamma}}{\Gamma(3 + 3\gamma)} \\
 &\quad \times \frac{-128\pi^6 t^{2+3\gamma} \sin(2\pi\zeta)}{\Gamma(3 + 3\gamma)} - \frac{512\pi^8 \sin(2\pi\zeta) t^{2+4\gamma}}{\Gamma(3 + 4\gamma)} + \dots, \\
 u(\zeta, t) &= t^2 \sin(2\pi\zeta).
 \end{aligned}$$

6 RESULTS AND DISCUSSION

Figures 1, 2 are the 2D and 3D representations of the Exact and approximate solutions of Example 1 while Figure 3 is the comparison plot of exact and approximate solutions. Figure 4 is the 2D and 3D plots for different fractional orders of Example 1. Figures 5, 6 represent 2D and 3D plots of exact and approximate solutions for Example 2 while Figure 7 is the comparison plot of the exact and approximate solutions. In Figure 8, the 2D and 3D plots at different fractional orders of

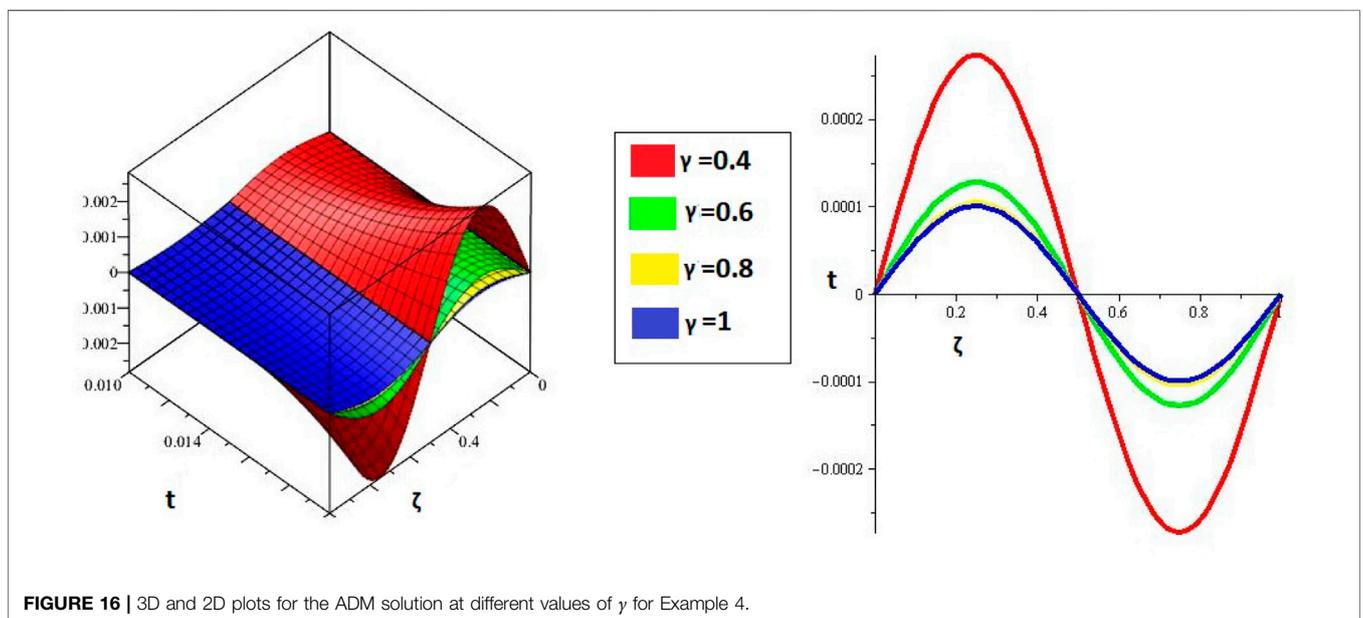


FIGURE 16 | 3D and 2D plots for the ADM solution at different values of γ for Example 4.

the derivatives are discussed for Example 2. **Figures 9, 10** have shown the 2D and 3D plots of exact and approximate solutions of Example 3 are presented while **Figure 11** has displayed the comparison plot of exact and approximate solutions. **Figure 12** is the 2D and 3D plots for different fractional orders of the derivative of example 3. **Figures 13, 14** represent the 2D and 3D plots of exact and approximate solutions of Example 4 while **Figure 15** is the comparison plot of the exact and approximate solutions. In **Figure 16**, the 2D and 3D plots are displayed for different fractional orders of the derivative of Example 4.

7 CONCLUSION

In many situations, the analytical solutions of fractional partial differential equations with boundary conditions are very difficult to investigate. Here, a very useful and successful attempt is made to determine the solution of fractional convection–diffusion equations with variable coefficients along with the initial–boundary conditions. The simulations for the given problems are derived and applied it for the solutions of some numerical examples. The fractional-order solutions are obtained in a very sophisticated manner. The solution graphs are plotted with higher accuracy for both fractional and integer orders of the problems. It is shown that the solution sequence of fractional order problems is convergent towards the integer-order solutions. Moreover, it is investigated that the present method has a unique capability to solve fractional partial differential equations with variable coefficients with initial

and boundary conditions. In the light of the above varieties and features, the proposed technique can be extended to solve other problems with higher accuracy.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material, and further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

HK supervision, PK funding, Hajira methodology, QK Investigation, FT Project Administrator, KS Funding, ID draft writing.

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