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# On Non-Convexity of the Nonclassicality Measure *via* Operator Ordering Sensitivity

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In quantum optics, nonclassicality in the sense of Glauber-Sudarshan is a valuable resource related to the quantum aspect of photons. A desirable and intuitive requirement for a consistent measure of nonclassicality is convexity: Classical mixing should not increase nonclassicality. We show that the recently introduced nonclassicality measure [Phys. Rev. Lett. **122**, 080402 (2019)] is not convex. This nonclassicality measure is defined via operator ordering sensitivity, which is an interesting and significant probe (witness) of nonclassicality without convexity but can be intrinsically connected to the convex Wigner-Yanase skew information [Proc. Nat. Acad. Sci. United States **49**, 910 (1963)] *via* the square root operation on quantum states. Motivated by the Wigner-Yanase skew information, we also propose a faithful measure of nonclassicality, although it cannot be readily computed, it is convex.

**Keywords: coherent states, nonclassicality, operator ordering sensitivity, convexity, Wigner-Yanase skew information**



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## 1 INTRODUCTION

In the conventional scheme of Glauber-Sudarshan, nonclassicality of light refers to quantum optical states that cannot be expressed as classical (probabilistic) mixtures of Glauber coherent states [1–7]. Its detection and quantification are of both theoretical and experimental importance in quantum optics [8–17]. Recently, a remarkable and interesting nonclassicality measure is introduced in Ref. [18]. This measure is well motivated and has operational significance stemmed from operator ordering sensitivity [18], which is also known as squared quadrature coherence scale in measuring quadrature coherence [19], and proved to be closely related to the entanglement [20]. Here we demonstrate that this nonclassicality measure, as well as the operator ordering sensitivity, are not convex. This means that classical (probabilistic) mixing of states can increase nonclassicality, as quantified by this nonclassicality measure via the operator ordering sensitivity. Our result complements the key contribution in Ref. [18].

By the way, we show that the operator ordering sensitivity, though not convex, can be connected to a convex quantity via the very simple and straightforward operation of square root. The modified quantity has both physical and information-theoretic significance, and is actually rooted in an amazing quantity of Wigner and Yanase, introduced in 1963 [21]. Motivated by the Wigner-Yanase skew information, we also propose a faithful measure of nonclassicality which is convex.

To be precise, let us first recall the basic idea and the key quantities in Ref. [18]. Consider a single-mode bosonic field with annihilation operator $a$ and creation operator $a^\dagger$ satisfying the commutation relation

$$[a, a^\dagger] = \mathbf{1}.$$

Let $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$ be the Weyl displacement operators with amplitudes $\alpha \in \mathbb{C}$, then $|\alpha\rangle = D(\alpha)|0\rangle$ are the coherent states [1–3]. For a bosonic field state $\rho$, consider the parameterized phase space distributions [18]

$$W_s(z) = \frac{1}{\pi^2}\int_{\mathbb{C}} e^{s|z|^2/2 + \alpha z^* - \alpha^* z}\mathrm{tr}(\rho D(\alpha))d^2\alpha$$

on the phase space $\mathbb{C}$, where $s \in [-1, 1]$, $d^2\alpha = dxdy$ with $\alpha = x + iy, x, y \in \mathbb{R}$, and tr denotes operator trace. In particular, for $s = 1, 0, -1$, the corresponding phase space distributions are the Glauber-Sudarshan $P$ functions, the Wigner functions, and the Husimi functions, respectively.

Motivated by operator ordering due to noncommutativity and in terms of the Hilbert-Schmidt norm, the quantity

$$S_o(\rho) = -\frac{d}{ds}\ln(\pi\|W_s\|^2)|_{s=0}$$

is introduced as a probe of nonclassicality of $\rho$ in Ref. [18], and is called operator ordering sensitivity. Here

$$\|W_s\|^2 = \int_{\mathbb{C}} |W_s(z)|^2 d^2 z.$$

It turns out that.

$$S_o(\rho) = -\frac{1}{2\mathrm{tr}(\rho^2)}\left(\mathrm{tr}\left([\rho, Q]^2\right) + \mathrm{tr}\left([\rho, P]^2\right)\right), \tag{1}$$

where $[X, Y] = XY - YX$ denotes operator commutator, and

$$Q = \frac{a + a^\dagger}{\sqrt{2}}, \qquad P = \frac{a - a^\dagger}{\sqrt{2}i}$$

are the conjugate quadrature operators. Simple manipulation shows that

$$S_o(\rho) = \frac{1}{\mathrm{tr}(\rho^2)}\mathrm{tr}\left([\rho, a][\rho, a]^\dagger\right). \tag{2}$$

Moreover, the following nonclassicality measure

$$\mathcal{N}(\rho) = \inf_{\sigma\in\mathcal{C}} |||\tilde{\rho} - \tilde{\sigma}||| \tag{3}$$

is introduced as a key result [18]. Here $\mathcal{C}$ is the set of classical states (i.e., mixtures of coherent states), $\tilde{\rho} = \rho/\sqrt{\mathrm{tr}(\rho^2)}$, $\tilde{\sigma} = \sigma/\sqrt{\mathrm{tr}(\sigma^2)}$, and the norm $|||\cdot|||$ is defined as

$$|||X|||^2 = \frac{1}{2}\mathrm{tr}\left([X^\dagger, Q][Q, X] + [X^\dagger, P][P, X]\right).$$

In particular,

$$|||\tilde{\rho}|||^2 = S_o(\rho)$$

is precisely the operator ordering sensitivity.

The purpose of this work is to demonstrate that the nonclassicality measure $\mathcal{N}(\cdot)$ defined by **Eq. 3** is not convex. Consequently, this quantity cannot be a consistent measure of nonclassicality if one imposes the fundamental rationale that classical mixing of quantum states should not increase nonclassicality, which resembles the idea that classical mixing of quantum states should not increase entanglement. By the way, we also demonstrate that the operator ordering sensitivity $S_o(\cdot)$ defined by **Eq. 2** is not convex either.

The structure of the remainder of the paper is as follows. In **Section 2**, we demonstrate that the nonclassicality measure $\mathcal{N}(\cdot)$ is not convex through counterexamples. In **Section 3**, we show that although the operator ordering sensitivity $S_o(\cdot)$ is not convex, it can be directly connected to a convex quantity related to the celebrated Wigner-Yanase skew information. By the way, we also present a simple proof of the fact that $S_o(\rho) \le 1$ for any classical state. In **Section 4**, we bring up a convex measure of nonclassicality based on the Wigner-Yanase skew information. Finally, a summary is presented in **Section 5**.

## 2 NON-CONVEXITY OF THE NONCLASSICALITY MEASURE $\mathcal{N}(\rho)$

In this section, we show that $\mathcal{N}(\rho)$ defined by **Eq. 3**, the nonclassicality measure introduced in Ref. [18], is not convex. First recall that by the triangle inequality for norm and the fact that the set $\tilde{\mathcal{C}}$, the image of $\mathcal{C}$ under the map $\rho \to \tilde{\rho} = \rho/\sqrt{\mathrm{tr}(\rho^2)}$, is contained inside the unit ball, it is shown that [18]

$$|||\tilde{\rho}||| - 1 \le \mathcal{N}(\rho) \le |||\tilde{\rho}||| \tag{4}$$

with $|||\tilde{\rho}||| = \sqrt{S_o(\rho)}$.

Now we give a family of counterexamples to show that $\mathcal{N}(\rho)$ is not convex with respect to $\rho$. Considering the mixture

$$\rho = \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2$$

of the vacuum state $\rho_1 = |0\rangle\langle 0|$ (which is classical) and the Fock state $\rho_2 = |n\rangle\langle n|$ with $n > 1$, then by direct calculation, we have

$$S_o(\rho_1) = 1, \qquad S_o(\rho_2) = 1 + 2n.$$

To evaluate $S_o(\rho)$, noting that

$$S_o(\rho) = 1 + \frac{2}{\mathrm{tr}(\rho^2)}\left(\mathrm{tr}(a\rho^2 a^\dagger) - \mathrm{tr}(\rho a^\dagger \rho a)\right),$$

we have, by direct calculation, that

$$\mathrm{tr}(\rho^2) = \frac{1}{2}, \qquad \mathrm{tr}(a\rho^2 a^\dagger) = \frac{n}{4}, \qquad \mathrm{tr}(\rho a^\dagger \rho a) = 0,$$

from which we obtain

$$S_o(\rho) = 1 + n.$$

It follows from the inequality chain (4) that

$$\mathcal{N}(\rho_1) \le \sqrt{S_o(\rho_1)} = 1,$$
$$\mathcal{N}(\rho_2) \le \sqrt{S_o(\rho_2)} = \sqrt{1 + 2n},$$

while

$$\mathcal{N}(\rho) \ge \sqrt{S_o(\rho)} - 1 = \sqrt{1 + n} - 1.$$

Consequently,

$$\frac{1}{2}\mathcal{N}(\rho_1) + \frac{1}{2}\mathcal{N}(\rho_2) \le \frac{1+\sqrt{1+2n}}{2}.$$

Since when $n > 24$, the following inequality holds

$$\sqrt{1+n} - 1 > \frac{1+\sqrt{1+2n}}{2},$$

it follows that

$$\mathcal{N}(\rho) \ge \sqrt{1+n} - 1 > \frac{1+\sqrt{1+2n}}{2} \ge \frac{1}{2}\mathcal{N}(\rho_1) + \frac{1}{2}\mathcal{N}(\rho_2).$$

This implies that $\mathcal{N}(\cdot)$ is not convex. In this sense, $\mathcal{N}(\cdot)$ cannot be a consistent measure of nonclassicality because classical mixing should not increase nonclassicality. Of course, $\mathcal{N}(\cdot)$ still captures certain features of nonclassicality and can be used as a probe of nonclassicality.

## 3 RELATING THE OPERATOR ORDERING SENSITIVITY $S_O(\rho)$ TO THE WIGNER-YANASE SKEW INFORMATION

As a side issue, in this section, we show that although the operator ordering sensitivity $S_o(\rho)$ is not convex either with respect to $\rho$, it can be intrinsically related to the celebrated Wigner-Yanase skew information, which is convex.

First, we illustrate non-convexity of $S_o(\rho)$ through the following counterexamples. Take

$$\rho_1 = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|), \qquad \rho_2 = |2\rangle\langle 2|, \qquad p_1 = \frac{1}{4}, \qquad p_2 = \frac{3}{4},$$

where $|n\rangle$ are the Fock (number) states with

$$a|0\rangle = 0, \qquad a|n\rangle = \sqrt{n}|n-1\rangle, \qquad n = 1, 2, \ldots,$$

and

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \qquad n = 0, 1, \cdots.$$

Now direct evaluation yields

$$[\rho_1, a] = \frac{1}{\sqrt{2}}|1\rangle\langle 2|, \qquad [\rho_2, a] = \sqrt{3}|2\rangle\langle 3| - \sqrt{2}|1\rangle\langle 2|,$$

and

$$[p_1\rho_1 + p_2\rho_2, a] = -\frac{5\sqrt{2}}{8}|1\rangle\langle 2| + \frac{3\sqrt{3}}{4}|2\rangle\langle 3|.$$

Substituting the above into **Eq. 2**, we obtain

$$S_o(\rho_1) = 1, \qquad S_o(\rho_2) = 5,$$

and

$$S_o(p_1\rho_1 + p_2\rho_2) = \frac{79}{19} > p_1 S_o(\rho_1) + p_2 S_o(\rho_2) = 4.$$

This implies that $S_o(\rho)$ is not convex.

In the above counterexamples showing non-convexity of $S_o(\rho)$, both the constituent states $\rho_1$ and $\rho_2$ are nonclassical in the sense that they cannot be represented as probabilistic mixtures of coherent states [1–3]. The following counterexamples illustrates that even the mixture of a classical thermal state and a nonclassical state can demonstrate non-convexity. Considering the thermal state

$$\tau_1 = (1-\lambda)\sum_{n=0}^{\infty}\lambda^n|n\rangle\langle n|, \qquad \lambda \in (0,1), \tag{5}$$

which is classical and the Fock state $\tau_2 = |1\rangle\langle 1|$, and their mixture

$$\tau = \frac{1}{2}(\tau_1 + \tau_2),$$

then by direct calculation, we have

$$S_o(\tau_1) = \frac{1-\lambda}{1+\lambda}, \qquad S_o(\tau_2) = 3.$$

To evaluate $S_o(\tau)$, noting that from **Eq. 2**, we have

$$S_o(\tau) = 1 + \frac{2}{\mathrm{tr}(\tau^2)}\left(\mathrm{tr}(a\tau^2 a^\dagger) - \mathrm{tr}(a\tau a^\dagger \tau)\right).$$

Now direct calculation leads to

$$\begin{aligned}
\mathrm{tr}(\tau^2) &= \frac{1+\lambda(1-\lambda^2)}{2(1+\lambda)},\\
\mathrm{tr}(a\tau^2 a^\dagger) &= \frac{1+4\lambda+4\lambda^2-2\lambda^3-2\lambda^4}{4(1+\lambda)^2},\\
\mathrm{tr}(a\tau a^\dagger\tau) &= \frac{\lambda+(1-\lambda)(1+2\lambda^2)(1+\lambda)^2}{4(1+\lambda)^2},
\end{aligned}$$

from which we obtain

$$S_o(\tau) = 1 + \frac{\lambda(2+3\lambda-3\lambda^2+2\lambda^4)}{(1+\lambda-\lambda^3)(1+\lambda)}, \qquad \lambda \in (0,1).$$

Clearly

$$\lim_{\lambda\to 1} S_o(\tau) = 3 > \frac{1}{2}\lim_{\lambda\to 1} S_o(\tau_1) + \frac{1}{2}\lim_{\lambda\to 1} S_o(\tau_2) = \frac{3}{2}.$$

By continuity, this implies that $S_o(\cdot)$ is not convex for $\lambda$ close to 1. More explicitly, for $\lambda = 0.9$, we have

$$S_o(\tau) \approx 2.45 > \frac{1}{2}(S_o(\tau_1) + S_o(\tau_2)) \approx 1.53,$$

which explicitly shows that $S_o(\cdot)$ is not convex.

The non-convex quantity $S_o(\rho)$ can be modified to a convex one if we formally replace $\rho$ by the square root $\sqrt{\rho}$ in **Eq. 1** and define

$$\hat{S}_o(\rho) = -\frac{1}{2}\left(\mathrm{tr}\left(\left[\sqrt{\rho}, Q\right]^2\right) + \mathrm{tr}\left(\left[\sqrt{\rho}, P\right]^2\right)\right), \tag{6}$$

which is precisely the sum of the Wigner-Yanase skew information [21]

$$I(\rho, Q) = -\frac{1}{2}\mathrm{tr}\left(\left[\sqrt{\rho}, Q\right]^2\right), \qquad I(\rho, P) = -\frac{1}{2}\mathrm{tr}\left(\left[\sqrt{\rho}, P\right]^2\right).$$

Remarkably, $\hat{S}_o(\rho)$ defined by **Eq. 6** can be more succinctly expressed as

$$\hat{S}_o(\rho) = \mathrm{tr}\left(\left[\sqrt{\rho}, a\right]\left[\sqrt{\rho}, a\right]^\dagger\right), \tag{7}$$

which is essentially (up to a constant factor 1/2) an extension of the Wigner-Yanase skew information, as can be readily seen if we recast the original Wigner-Yanase skew information [21]

$$I(\rho, K) = -\frac{1}{2}\mathrm{tr}\left(\left[\sqrt{\rho}, K\right]^2\right)$$

of the quantum state $\rho$ with respect to (skew to) the observable (Hermitian operator) $K$ as

$$I(\rho, K) = \frac{1}{2}\mathrm{tr}\left(\left[\sqrt{\rho}, K\right]\left[\sqrt{\rho}, K\right]^\dagger\right),$$

and formally replace the Hermitian operator $K$ by the non-Hermitian annihilation operator $a$. An apparent interpretation of $\hat{S}_o(\rho)$ is the quantum uncertainty of the conjugate pair $(Q, P)$ in the state $\rho$ [22–24].

Due to the convexity of the Wigner-Yanase skew information [21], $\hat{S}_o(\rho)$ is convex with respect to $\rho$, in sharp contrast to $S_o(\rho)$. Moreover, $\hat{S}_o(\rho)$ has many nice features as guaranteed by the fundamental properties of the Wigner-Yanase skew information and its various physical and information-theoretic interpretations [24].

It is amusing to note the analogy between the passing from classical probability distributions to quantum mechanical amplitudes and that from $S_o(\rho)$ to $\hat{S}_o(\rho)$: Both involve the square root of states.

By the way, we present an alternative and simple proof of the interesting fact that [18]

$$S_o(\rho) \le 1$$

for any classical state $\rho$, which implies that $S_o(\cdot)$ is convex when the component states are restricted to coherent states (noting that $S_o(|\alpha\rangle\langle\alpha|) = 1$ for any coherent state $|\alpha\rangle$), though it is not convex in the whole state space. To this end, let the Glauber-Sudarhsan $P$ representation of $\rho$ be

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha,$$

then

$$\mathrm{tr}\left(a\rho^2 a^\dagger\right) = \int P(\alpha)P(\beta)\alpha{*}\beta e^{-|\alpha-\beta|^2} d^2\alpha d^2\beta = \int P(\alpha)P(\beta)\beta{*}\alpha e^{-|\alpha-\beta|^2} d^2\alpha d^2\beta,$$
$$\mathrm{tr}\left(a\rho a^\dagger\rho\right) = \int P(\alpha)P(\beta)|\alpha|^2 e^{-|\alpha-\beta|^2} d^2\alpha d^2\beta = \int P(\alpha)P(\beta)|\beta|^2 e^{-|\alpha-\beta|^2} d^2\alpha d^2\beta,$$

from which we obtain

$$\begin{aligned} S_o(\rho) &= 1 + \frac{2}{\mathrm{tr}(\rho^2)}\left(\mathrm{tr}\left(a\rho^2 a^\dagger\right) - \mathrm{tr}\left(a\rho a^\dagger\rho\right)\right) \\ &= 1 - \frac{2}{\mathrm{tr}(\rho^2)}\int P(\alpha)P(\beta)|\alpha-\beta|^2 e^{-|\alpha-\beta|^2} d^2\alpha d^2\beta. \end{aligned}$$

In particular, if $\rho$ is a classical state, then $P(\alpha) \ge 0$, and this implies that $S_o(\rho) \le 1$ for any classical state $\rho$. In contrast, the fact that

$$\hat{S}_o(\rho) \le 1 \tag{8}$$

for any classical state follows readily from the convexity of $\hat{S}_o(\rho)$ and $\hat{S}_o(|\alpha\rangle\langle\alpha|) = 1$ for any coherent state $|\alpha\rangle$.

## 4 A CONVEX MEASURE OF NONCLASSICALITY

Motivated by the Wigner-Yanase skew information, we propose a measure of nonclassicality defined as

$$\begin{aligned} \hat{\mathcal{N}}(\rho) &= \inf_{\sigma\in\mathcal{C}} \left|\left|\left| \sqrt{|\rho-\sigma|} \right|\right|\right|^2 \\ &= \inf_{\sigma\in\mathcal{C}} \mathrm{tr}\left(\left[\sqrt{|\rho-\sigma|}, a\right]\left[\sqrt{|\rho-\sigma|}, a\right]^\dagger\right). \end{aligned}$$

Here $|A| = \sqrt{A^\dagger A}$ is the square root of $A^\dagger A$, and $\mathcal{C}$ is the set of classical states.

It is clear that $\hat{\mathcal{N}}(\rho)$ is a faithful measure of nonclassicality, $\hat{\mathcal{N}}(\rho) > 0$ for all nonclassical states and $\hat{\mathcal{N}}(\rho) = 0$ for all classical states. Compared with the nonclassicality measure $\mathcal{N}(\rho)$ which is not convex, we prove below that $\hat{\mathcal{N}}(\rho)$ is convex.

Considering the convex combination of quantum states $\rho_1$ and $\rho_2$ with probabilities $p_1 = p$ and $p_2 = 1 - p$ respectively, the mixed state is denoted by

$$\rho = p_1\rho_1 + p_2\rho_2.$$

Supposing that

$$\begin{aligned} \hat{\mathcal{N}}(\rho_1) &= \inf_{\sigma\in\mathcal{C}} \left|\left|\left| \sqrt{|\rho_1-\sigma|} \right|\right|\right|^2 = \left|\left|\left| \sqrt{|\rho_1-\sigma_1|} \right|\right|\right|^2, \\ \hat{\mathcal{N}}(\rho_2) &= \inf_{\sigma\in\mathcal{C}} \left|\left|\left| \sqrt{|\rho_2-\sigma|} \right|\right|\right|^2 = \left|\left|\left| \sqrt{|\rho_2-\sigma_2|} \right|\right|\right|^2, \end{aligned}$$

due to the fact that the convex combination of classical states is also a classical state, we have $\sigma_c = p_1\sigma_1 + p_2\sigma_2 \in \mathcal{C}$, therefore

$$\begin{aligned} \hat{\mathcal{N}}(\rho) &= \inf_{\sigma\in\mathcal{C}} \left|\left|\left| \sqrt{|\rho-\sigma|} \right|\right|\right|^2 \\ &\le \left|\left|\left| \sqrt{|p_1\rho_1 + p_2\rho_2 - \sigma_c|} \right|\right|\right|^2 = \left|\left|\left| \sqrt{|p_1(\rho_1-\sigma_1) + p_2(\rho_2-\sigma_2)|} \right|\right|\right|^2 \\ &\le \left|\left|\left| \sqrt{p_1|\rho_1-\sigma| + p_2|\rho_2-\sigma|} \right|\right|\right|^2 \\ &\le p_1\left|\left|\left| \sqrt{|\rho_1-\sigma_1|} \right|\right|\right|^2 + p_2\left|\left|\left| \sqrt{|\rho_2-\sigma_2|} \right|\right|\right|^2 = p_1\hat{\mathcal{N}}(\rho_1) + p_2\hat{\mathcal{N}}(\rho_2). \end{aligned}$$

Here the second inequality holds due to

$$|A + B| \le |A| + |B|, \tag{10}$$

which can be obtained from the fact that $|A + \lambda B|^2 \ge 0$ for all real $\lambda$. While the third inequality follows from the convexity of the celebrated Wigner-Yanase skew information, the convexity of the measure $\hat{\mathcal{N}}(\rho)$ is easily proved. We point out here that similar to other measures involving optimization, this nonclassicality measure $\hat{\mathcal{N}}(\rho)$ can not be readily computed. It would be desirable if tight bounds of this quantity can be given.

Similarly from inequality (10) and the convexity of the Wigner-Yanase skew information, we have

$$\begin{aligned} \hat{\mathcal{N}}(\rho) &= \inf_{\sigma\in\mathcal{C}} \left|\left|\left| \sqrt{|\rho-\sigma|} \right|\right|\right|^2 \le \inf_{\tau_1\in\mathcal{T}} \left|\left|\left| \sqrt{|\rho-\tau_1|} \right|\right|\right|^2 \\ &\le \inf_{\tau_1\in\mathcal{T}} \left|\left|\left| \sqrt{|\rho| + |\tau_1|} \right|\right|\right|^2 = 2\inf_{\tau_1\in\mathcal{T}} \left|\left|\left| \sqrt{\frac{1}{2}|\rho| + \frac{1}{2}|\tau_1|} \right|\right|\right|^2 \\ &\le \left|\left|\left| \sqrt{|\rho|} \right|\right|\right|^2 + \inf_{\tau_1\in\mathcal{T}} \left|\left|\left| \sqrt{|\tau_1|} \right|\right|\right|^2 = \left|\left|\left| \sqrt{|\rho|} \right|\right|\right|^2 = \hat{S}_o(\rho), \end{aligned}$$

where $\mathcal{T}$ is the set of thermal states as defined in **Eq. 5**, the first inequality follows from the fact that thermal states are classical states (that is, $\mathcal{T} \subseteq \mathcal{C}$), and the last inequality holds since $\inf_{\tau_1} |||\sqrt{|\tau_1|}|||^2 = \inf_{\tau_1\in\mathcal{T}}\hat{S}_o(\tau_1) = \inf_{\lambda\in(0,1)}(1-\sqrt{\lambda})/(2+2\sqrt{\lambda}) = 0$, as shown in Ref. [24]. Analogously, we notice that

$$
\begin{aligned}
\hat{S}_o(\rho) &= |||\sqrt{|\rho|}\,|||^2 \\
&\le |||\sqrt{|\rho-\sigma|+|\sigma|}\,|||^2 = 2|||\sqrt{\frac{1}{2}|\rho-\sigma|+\frac{1}{2}|\sigma|}\,|||^2 \\
&\le |||\sqrt{|\rho-\sigma|}\,|||^2 + |||\sqrt{|\sigma|}\,|||^2 \\
&\le |||\sqrt{|\rho-\sigma|}\,|||^2 + 1,
\end{aligned}
$$

where $\sigma$ is a classical state, and the last inequality can be directly obtained from inequality (8). So we have

$$\hat{S}_o(\rho) - 1 \le \hat{\mathcal{N}}(\rho) \le \hat{S}_o(\rho).$$

In other words, $\hat{\mathcal{N}}(\rho)$ may be well estimated by the convex nonclassicality quantifier $\hat{S}_o(\rho)$ for highly nonclassical states.

## 5. CONCLUSION

We have demonstrated that $\mathcal{N}(\cdot)$, a recently introduced significant nonclassicality measure based on the operator ordering sensitivity, is not convex, and thus cannot be a consistent measure of the conventional nonclassicality of light in the sense of Glauber-Sudarshan. This non-convexity should be borne in mind whenever one wants to employ $\mathcal{N}(\cdot)$ to quantify nonclassicality in quantum optics in the customary fashion. We have proposed a faithful measure of nonclassicality $\hat{\mathcal{N}}(\cdot)$ which is convex. One obstacle of applying this measure is that it can not be readily computed due to the optimization over the set of classical states.

By the way, we have also demonstrated that although the important operator ordering sensitivity $S_o(\cdot)$ is not convex either, it can be simply connected to the convex Wigner-Yanase skew information via the square root operation on quantum states, which is reminiscent of the passing from probabilities to amplitudes via square roots, so fundamental in going from classical to quantum.

Due to the remarkable properties and information-theoretic significance of the Wigner-Yanase skew information, it is desirable to employ this quantity to study nonclassicality of light in particular, and nonclassicality of arbitrary quantum states in general.

## DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material further inquiries can be directed to the corresponding author.

## AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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