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A unified method for constructing developable surface pencils interpolating characteristic curves

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Developable surface is a simple and common surface in surface modeling. Geodesic, line of curvature, asymptotic curve, and D-type curve are important characteristic curves on the surfaces. This study gives a unified method for constructing developable surface pencils interpolating these four kinds of characteristic curves. Given a regular space curve $\boldsymbol{R}(r)$, we derive a new condition that a surface pencil $\boldsymbol{P}(r, t)$ interpolating $\boldsymbol{R}(r)$ is developable. The result shows that the condition completely depends on a univalent function λ and an angle θ. By choosing different λ and θ, we can not only control the shape of $\boldsymbol{P}(r, t)$, but also make $\boldsymbol{R}(r)$ become any kind of characteristic curve on $\boldsymbol{P}(r, t)$. Furthermore, we take natural and conjugate curve pairs as those characteristic curves to construct developable surface pairs. Finally, an example of a slant helix shows that the proposed unified method is more general than other methods, and has good interactivity and convenience.

KEYWORDS

geodesic, line of curvature, asymptotic curve, D-type curve, natural pair, conjugate pair, developable surface pencils, simple and common surfaces

1 Introduction

A developable surface can unfold into a flat without scaling, folding, or tearing, which is widely used in the manufacture of many products, such as the design of leather, paper, and metal plates [1–6]. With the in-depth research of many scholars on the developable surface, its construction theory has gradually become a system. At present, there are three main construction methods: the point geometry method [7, 8], the dual method [9, 10], and the construction method with certain geometric constraints [11–13]. In addition, there are also scholars who focused on its related research, such as [14–19].

Geodesic, line of curvature, and asymptotic curve are crucial characteristic curves on the surfaces, which determine the properties and shapes of the surfaces. The third method mentioned earlier, namely, the surface construction method with certain geometric constraints, is the method of constructing a surface pencil interpolating a given geodesic, line of curvature, or asymptotic curve. This method was first proposed by Wang and Tang [20], which constructed a surface pencil interpolating a given geodesic, then Zhao and Wang [11] extended the method to the developable surface. Li and Wang

[21] proved the constraint conditions that need to be satisfied when a given curve is a common line of curvature on a surface pencil. Later, they [12] also proposed a brand new method of constructing a developable surface pencil interpolating a given line of curvature. Bayram and Güler [22] explored the construction of a surface pencil with a given asymptotic curve, and Liu and Wang [13] extended the results of [22] to the developable surface. In recent years, it has become popular to use special curve pairs to construct surface pairs. Atalay [23, 24], constructed surface pencil pairs interpolating Mannheim curve pairs as common asymptotic curves and geodesics, and extended the surface pairs to the ruled surface pairs. Wang and Jiang [25] took natural and conjugate curve pairs as common asymptotic curves to construct developable surface pencil pairs. Jiang [26] took the Bertrand curve pair as asymptotic curves to construct surface pencil pair in Galilean space, and the result is extended to the ruled surface pair.

Kaya and Önder [27] introduced a new characteristic curve called D-type curve, which satisfies $W_0 \cdot n = constant$, where W_0 is the unit Darboux vector of a surface curve, n is the unit surface normal vector along the surface curve, and $W_0 \cdot n$ denotes the dot product of W_0 and n. The conditions required for a surface pencil interpolating a common D-type curve are proposed in [27]. The D-type curve is more general because it contains the geodesic and the asymptotic curve. However, it does not contain the line of curvature, and the study on it is not extended to the developable surface. This study discusses whether a unified method can be proposed to construct developable surfaces interpolating geodesic, line of curvature, asymptotic curve, and D-type curve.

In addition, Wang and Jiang [25] only proposed developable surface pencil pairs interpolating natural and conjugate curve pairs as asymptotic curves. To complete the results, we put forward developable surface pencil pairs interpolating these two kinds of curve pairs as common geodesics, lines of curvature, and D-type curves, respectively.

In Section 2, we introduce the judgment methods of four kinds of characteristic curves on the surfaces. In Section 3, the unified method for constructing developable surface pencils interpolating four kinds of characteristic curves is given. In Section 4, we give the specific expressions for developable surface pencil pairs interpolating natural and conjugate curve pairs as common geodesics, lines of curvature, and D-type curves, respectively. In Section 5, an example of a slant helix is given to verify the efficiency and convenience of this unified method.

2 Preliminaries

Let $\boldsymbol{R}(r)$, $L_1 \le r \le L_2$ be a regular curve in $\mathbb{E}^3$, where L_1, L_2 are two real numbers, and r is an arbitrary parameter. Let $\kappa(r)$, $\tau(r)$, $\boldsymbol{T}(r)$, $\boldsymbol{N}(r)$ and $\boldsymbol{B}(r)$ represent the curvature, torsion, unit tangent vector, principal normal vector, and binormal vector of $\boldsymbol{R}(r)$, respectively, which follow [28].

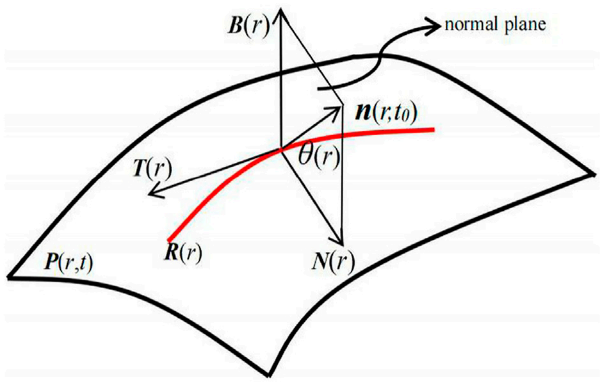

FIGURE 1
Angle between $\boldsymbol{n}$ (r, t_0) and $\boldsymbol{N}(r)$.

$$\kappa = \frac{|\boldsymbol{R}' \times \boldsymbol{R}''|}{|\boldsymbol{R}'|^3}, \tau = \frac{(\boldsymbol{R}', \boldsymbol{R}'', \boldsymbol{R}''')}{(\boldsymbol{R}' \times \boldsymbol{R}'')^2}, \boldsymbol{T} = \frac{\boldsymbol{R}'}{|\boldsymbol{R}'|}, \boldsymbol{B} = \frac{\boldsymbol{R}' \times \boldsymbol{R}''}{|\boldsymbol{R}' \times \boldsymbol{R}''|}, \boldsymbol{N} = \boldsymbol{B} \times \boldsymbol{T},$$

where κ, τ, $\boldsymbol{R}$, $\boldsymbol{T}$, $\boldsymbol{N}$, and $\boldsymbol{B}$ denote the shorthand of $\kappa(r)$, $\tau(r)$, $\boldsymbol{R}(r)$, $\boldsymbol{T}(r)$, $\boldsymbol{N}(r)$, and $\boldsymbol{B}(r)$, respectively, "×"denotes the cross product, and $(\cdot, \cdot, \cdot)$ denotes the mixed product. $\boldsymbol{T}$, $\boldsymbol{N}$, and $\boldsymbol{B}$ form the Frenet frame of $\boldsymbol{R}$, which follows [28].

$$\begin{pmatrix} \boldsymbol{T}' \\ \boldsymbol{N}' \\ \boldsymbol{B}' \end{pmatrix} = |\boldsymbol{R}'| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B}. \end{pmatrix} \tag{1}$$

Suppose [20].

$$\boldsymbol{P}(r,t) = \boldsymbol{R}(r) + (\alpha(r,t), \beta(r,t), \gamma(r,t)) \times \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix}, L_1 \le r \le L_2, T_1 \le t \le T_2 \tag{2}$$

is a surface pencil interpolating $\boldsymbol{R}(r)$, and $\boldsymbol{P}$ $(r, t_0) = \boldsymbol{R}(r)$, where $\alpha(r, t)$, $\beta(r, t)$, and $\gamma(r, t)$ are called the marching-scale functions, T_1, T_2 are two real numbers, and $t_0 \in [T_1, T_2]$.

Let the angle between the principal normal vector $\boldsymbol{N}(r)$ of $\boldsymbol{R}(r)$ and the surface normal vector $\boldsymbol{n}$ (r, t_0) along $\boldsymbol{R}(r)$ be $\theta(r)$ (Figure 1). Because the surface normal vector $\boldsymbol{n}$ (r, t_0) is in the normal plane, and the normal plane is made up of the principal normal vector $\boldsymbol{N}(r)$ and the binormal vector $\boldsymbol{B}(r)$, the surface normal vector $\boldsymbol{n}$ (r, t_0) can be represented as a linear combination of $\boldsymbol{N}(r)$ and $\boldsymbol{B}(r)$, that is.

$$\boldsymbol{n}(r, t_0) = \lambda(r)[\cos\theta(r)\boldsymbol{N} + \sin\theta(r)\boldsymbol{B}], \tag{3}$$

where $\lambda(r)$ is a univalent function.

If $\theta(r)$ satisfies a specific condition, then the curve $\boldsymbol{R}(r)$ will become the corresponding characteristic curve of $\boldsymbol{P}$ (r, t). That is

Lemma 2.1 [29]. $\mathbf{R}(r)$ *is a geodesic of* $\boldsymbol{P}$ (r, t) *as* $\theta = 0$ *or* π.

Lemma 2.2 [13]. *$\boldsymbol{R}(r)$ is an asymptotic curve of $\boldsymbol{P}$ (r, t) as* $\theta = \frac{\pi}{2}$.

Lemma 2.3 [21]. *$\boldsymbol{R}(r)$ is a line of curvature of $\boldsymbol{P}$ (r, t) as* $\theta' = -\tau|\boldsymbol{R}'|$.

Lemma 2.4. *$\boldsymbol{R}(r)$ is a D-type curve of $\boldsymbol{P}$ (r, t) as* $\sin\theta = \frac{C\sqrt{\kappa^2+\tau^2}}{\kappa}$, *where C is a real constant satisfying* $0 \le C \le \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}$.

In Lemmas 2.1–2.4, θ denotes the shorthand of $\theta(r)$.

Proof. According to Eq. 3, suppose the unit surface normal vector along $\boldsymbol{R}(r)$ is

$$\bar{\boldsymbol{n}}(r, t_0) = \cos\theta \boldsymbol{N} + \sin\theta \boldsymbol{B}, \tag{4}$$

and the unit Darboux vector of $\boldsymbol{R}(r)$ is [27].

$$\boldsymbol{W}_0 = \frac{\tau\boldsymbol{T} + \kappa\boldsymbol{B}}{\sqrt{\kappa^2 + \tau^2}}, \tag{5}$$

if $\boldsymbol{R}(r)$ is a D-type curve of $\boldsymbol{P}(r, t)$, then [27].

$$\boldsymbol{W}_0 \cdot \bar{\boldsymbol{n}}(r, t_0) = C. \tag{6}$$

According to Eqs. 4–6,

$$\begin{aligned}\boldsymbol{W}_0 \cdot \bar{\boldsymbol{n}}(r, t_0) &= \frac{(\tau\boldsymbol{T} + \kappa\boldsymbol{B}) \cdot (\cos\theta\boldsymbol{N} + \sin\theta\boldsymbol{B})}{\sqrt{\kappa^2+\tau^2}} \\ &= \frac{\kappa\sin\theta}{\sqrt{\kappa^2+\tau^2}} = C,\end{aligned}$$

then we get

$$\sin\theta = \frac{C\sqrt{\kappa^2+\tau^2}}{\kappa}\left(0 \le C \le \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right).$$

3 Unified necessary and sufficient conditions for developable surface pencils interpolating characteristic curves

The results of developable surface pencils interpolating a common geodesic, a common line of curvature, and a common asymptotic curve can be found in [11–13], respectively, but there is no unified method for constructing developable surface pencils interpolating these characteristic curves. In this section, we make up for this shortcoming. First, we recall a known lemma.

Lemma 3.1 [13]. $\boldsymbol{P}(r, t)$ (Eq. (2)) is developable if and only if

$$(\alpha(r,t), \beta(r,t), \gamma(r,t)) = (t - t_0)(\alpha(r), \beta(r), \gamma(r)), \tag{7}$$

and

$$\begin{aligned}&[\beta(r)\gamma'(r) - \beta'(r)\gamma(r)] - \kappa|\boldsymbol{R}'|\alpha(r)\gamma(r) + \tau|\boldsymbol{R}'|\left[\beta^2(r) + \gamma^2(r)\right] \\ &\quad = 0,\end{aligned} \tag{8}$$

where $\alpha(r)$, $\beta(r)$, and $\gamma(r)$ are marching-scale functions.

By Eq. 3, Lemma 3.1 can be rewritten as Theorem 3.1.

Theorem 3.1. $\boldsymbol{P}(r, t)$ (Eq. (2)) is developable if and only if the functions α, β, γ, λ, and θ satisfy Eq. (7) and the following conditions

$$\begin{cases}\lambda\left[\lambda\left(\theta' + \tau|\boldsymbol{R}'|\right) + \kappa\alpha\cos\theta|\boldsymbol{R}'|^2\right] = 0, \\ \beta = \dfrac{\lambda\sin\theta}{|\boldsymbol{R}'|}, \gamma = -\dfrac{\lambda\cos\theta}{|\boldsymbol{R}'|},\end{cases} \tag{9}$$

where α, β, γ, λ, and θ represent the shorthand of $\alpha(r)$, $\beta(r)$, $\gamma(r)$, $\lambda(r)$, and $\theta(r)$, respectively.

Proof. According to Eqs 2, 7, $\boldsymbol{P}(r, t)$ can be represented as

$$\boldsymbol{P}(r,t) = \boldsymbol{R}(r) + (t - t_0)(\alpha\boldsymbol{T} + \beta\boldsymbol{N} + \gamma\boldsymbol{B}). \tag{10}$$

By taking the partial derivative of Eq. 10 and using Eq. 1, we obtain that

$$\boldsymbol{P}_r(r, t_0) = |\boldsymbol{R}'|\boldsymbol{T}, \boldsymbol{P}_t(r, t_0) = \alpha\boldsymbol{T} + \beta\boldsymbol{N} + \gamma\boldsymbol{B}, \tag{11}$$

then, according to Eq. 11, the surface normal vector is

$$\boldsymbol{n}(r, t_0) = \boldsymbol{P}_r(r, t_0) \times \boldsymbol{P}_t(r, t_0) = -|\boldsymbol{R}'|(\gamma\boldsymbol{N} - \beta\boldsymbol{B}). \tag{12}$$

By Eqs 3, 12, we get

$$\beta = \frac{\lambda\sin\theta}{|\boldsymbol{R}'|}, \gamma = -\frac{\lambda\cos\theta}{|\boldsymbol{R}'|}, \tag{13}$$

substituting Eq. 13 into Eq.8, we obtain

$$\lambda\left[\lambda\left(\theta' + \tau|\boldsymbol{R}'|\right) + \kappa\alpha\cos\theta|\boldsymbol{R}'|^2\right] = 0.$$

Hence, the theorem is proved.

Obviously, if $\lambda = 0$, $\boldsymbol{P}(r, t)$ is the tangent surface pencil of $\boldsymbol{R}(r)$, that is, it is always developable, so next, we consider only the case $\lambda \neq 0$.

By Lemmas 2.1–2.4, if θ further satisfies a particular condition, the curve $\boldsymbol{R}(r)$ can become the corresponding characteristic curve on the developable surface, so we get Theorem 3.2.

Theorem 3.2. $\boldsymbol{P}(r, t)$ (Eq. (2)) is a developable surface pencil interpolating $\boldsymbol{R}(r)$ as a common characteristic curve if and only if the functions α, β, γ, λ, and θ satisfy Eqs. (7), (9), and the following conditions:

1. If this characteristic curve is a geodesic, then θ = 0 or π.
2. If this characteristic curve is an asymptotic curve, then $\theta = \frac{\pi}{2}$.
3. If this characteristic curve is a line of curvature, then $\theta' = -\tau|\mathbf{R}'|$.
4. If this characteristic curve is a D-type curve, then $\sin\theta = \frac{C\sqrt{\kappa^2+\tau^2}}{\kappa}$ $\left(0 \le C \le \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\right)$.

Theorem 3.2 unifies the conditions for interpolating four kinds of characteristic curves to construct developable surface pencils, it not only contains the results of [11–13], but also puts forward interpolating a D-type curve to construct developable surface pencil, simultaneously, which is not involved in all current studies.

According to Theorems 3.1 and 3.2, the functions α, β, and γ of the developable surface are only related to a function λ and an angle θ, we can change the shape of the developable surface by choosing different λ, and make the given curve become any kind of characteristic curve on the developable surface by choosing different θ. Therefore, the method in this study has better interactivity than other methods. Now we continue to use this method to complete the study of developable surface pairs interpolating natural and conjugate curve pairs.

4 Developable surface pencil pairs interpolating natural and conjugate curve pairs as common characteristic curves

Developable surface pencil pairs interpolating natural and conjugate curve pairs as asymptotic curves have been proposed in [25]. To make this result complete, we continue to give developable surface pencil pairs interpolating these two kinds of curve pairs as the other three kinds of characteristic curves.

Definition 4.1 [25]. Given a unit speed curve $\boldsymbol{R}_0(s)$, $L_1 \le s \le L_2$. $\boldsymbol{R}_1(s) = \int \boldsymbol{N}_0(s)ds$ and $\boldsymbol{R}_2(s) = \int \boldsymbol{B}_0(s)ds$ are called the natural mate curve and the conjugate mate curve of $\boldsymbol{R}_0(s)$, respectively, where $\boldsymbol{N}_0(s)$ and $\boldsymbol{B}_0(s)$ are the principal normal vector and binormal vector of $\boldsymbol{R}_0(s)$, respectively. $\{\boldsymbol{R}_0(s), \boldsymbol{R}_1(s)\}$ and $\{\boldsymbol{R}_0(s), \boldsymbol{R}_2(s)\}$ are called the natural curve pair and the conjugate curve pair, respectively.

Definition 4.2 [25]. Let the developable surface pencils $\boldsymbol{P}_i(s, t)$ interpolate $\boldsymbol{R}_i(s)$, $i = 0, 1, 2$, $\{\boldsymbol{P}_0(s, t), \boldsymbol{P}_1(s, t)\}$ and $\{\boldsymbol{P}_0(s, t), \boldsymbol{P}_2(s, t)\}$ are called the natural developable surface pencil pair and the conjugate developable surface pencil pair, respectively.

Let $\{\boldsymbol{T}_i, \boldsymbol{N}_i, \boldsymbol{B}_i\}$, κ_i, and τ_i denote the Frenet frame, curvature, and torsion of $\boldsymbol{R}_i(s)$, $i = 0, 1, 2$, respectively.

According to Theorem 3.1, if $t_0 = 0$ and $\theta_i \neq \frac{\pi}{2}$, then

$$\boldsymbol{P}_i(s,t) = \boldsymbol{R}_i(s) - t\lambda_i\left(\frac{\theta_i' + \tau_i}{\kappa_i \cos\theta_i}\boldsymbol{T}_i - \sin\theta_i \boldsymbol{N}_i + \cos\theta_i \boldsymbol{B}_i\right), \quad L_1 \le s \le L_2, T_1 \le t \le T_2, \quad i = 0,1,2, \tag{14}$$

where θ_i are the angles between the surface normal vectors $\boldsymbol{n}_i(s, t_0)$ along $\boldsymbol{R}_i(s)$ and the principal normal vectors $\boldsymbol{N}_i$ of $\boldsymbol{R}_i(s)$, λ_i are functions that control the shapes of $\boldsymbol{P}_i(s, t)$.

According to Theorem 3.2, substituting $\theta_i = 0$ or π, $\theta_i' = -\tau_i$, and $\sin\theta_i = \frac{C\sqrt{\kappa_i^2+\tau_i^2}}{\kappa_i^2}$ into Eq. 14, respectively, the following three theorems can be obtained.

Theorem 4.1. $\boldsymbol{R}_i(s)$ are common geodesics of $\boldsymbol{P}_i(s, t)$ (see Eq. (14)) if and only if

$$\boldsymbol{P}_i(s,t) = \boldsymbol{R}_i(s) \mp t\frac{\lambda_i}{\kappa_i}(\tau_i \boldsymbol{T}_i + \kappa_i \boldsymbol{B}_i), \quad i = 0,1,2. \tag{15}$$

Theorem 4.2. $\boldsymbol{R}_i(s)$ are common lines of curvature of $\boldsymbol{P}_i(s, t)$ (Eq. (14)) if and only if

$$\boldsymbol{P}_i(s,t) = \boldsymbol{R}_i(s) + t\lambda_i(\sin\theta_i \boldsymbol{N}_i - \cos\theta_i \boldsymbol{B}_i), \quad i = 0,1,2, \tag{16}$$

where $\theta_i' = -\tau_i$

Theorem 4.3. $\boldsymbol{R}_i(s)$ are common D-type curves of $\boldsymbol{P}_i(s, t)$ (see Eq. (14)) if and only if

$$\boldsymbol{P}_i(s,t) = \boldsymbol{R}_i(s) - t\lambda_i(g_i \boldsymbol{T}_i - \sin\theta_i \boldsymbol{N}_i + \cos\theta_i \boldsymbol{B}_i), \quad i = 0,1,2, \tag{17}$$

where $g_i = \frac{(\sin\theta_i)' + \tau_i\cos\theta_i}{\kappa_i\cos^2\theta_i}$, $\sin\theta_i = \frac{C\sqrt{\kappa_i^2+\tau_i^2}}{\kappa_i}$, $(0 \le C \le \frac{\kappa_i}{\sqrt{\kappa_i^2+\tau_i^2}})$.

5 Example

Given a unit speed slant helix [25].

$$\boldsymbol{R}_0(s) = \left(\frac{\cos 3s + 9\cos s}{12}, \frac{\sin 3s + 9\sin s}{12}, \frac{-\sqrt{3}\cos s}{2}\right).$$

The Frenet frame, curvature, and torsion of $\boldsymbol{R}_0(s)$ are

$$\begin{cases} \boldsymbol{T}_0 = \left(-\frac{3}{4}\sin s - \frac{1}{4}\sin 3s, \frac{3}{4}\cos s + \frac{1}{4}\cos 3s, \frac{\sqrt{3}}{2}\sin s\right), \\ \boldsymbol{N}_0 = \left(-\frac{\sqrt{3}}{2}\cos 2s, -\frac{\sqrt{3}}{2}\sin 2s, \frac{1}{2}\right), \\ \boldsymbol{B}_0 = \left(\frac{3}{4}\cos s - \frac{1}{4}\cos 3s, \frac{3}{4}\sin s - \frac{1}{4}\sin 3s, \frac{\sqrt{3}}{2}\cos s\right), \end{cases}$$

$$\kappa_0 = \sqrt{3}|\cos s|, \tau_0 = \sqrt{3}\sin s.$$

According to Definition 4.1, the natural mate curve is

$$\boldsymbol{R}_1(s) = \left(-\frac{\sqrt{3}}{4}\sin 2s, \frac{\sqrt{3}}{4}\cos 2s, \frac{s}{2}\right),$$

and the conjugate mate curve is

$$\boldsymbol{R}_2(s) = \left(\frac{3}{4}\sin s - \frac{\sin 3s}{12}, -\frac{3}{4}\cos s + \frac{\cos 3s}{12}, \frac{\sqrt{3}}{2}\sin s\right).$$

By [30], the Frenet frame, curvature, and torsion of $\boldsymbol{R}_1(s)$ are

$$\begin{cases} \boldsymbol{T}_1 = -\frac{1}{2}(\sqrt{3}\cos 2s, \sqrt{3}\sin 2s, -1), \\ \boldsymbol{N}_1 = (\sin 2s, -\cos 2s, 0), \qquad \kappa_1 = \sqrt{3}, \tau_1 = 1. \\ \boldsymbol{B}_1 = \frac{1}{2}(\cos 2s, \sin 2s, \sqrt{3}), \end{cases}$$

Also, the Frenet frame, curvature, and torsion of $\boldsymbol{R}_2(s)$ are

$$\begin{cases} \boldsymbol{T}_2 = \left(\frac{3\cos s - \cos 3s}{4}, \frac{3\sin s - \sin 3s}{4}, \frac{\sqrt{3}}{2}\cos s\right), \\ \boldsymbol{N}_2 = \frac{\varepsilon}{2}(\sqrt{3}\cos 2s, \sqrt{3}\sin 2s, -1), \\ \boldsymbol{B}_2 = \varepsilon\left(-\frac{3\sin s + \sin 3s}{4}, \frac{3\cos s + \cos 3s}{4}, \frac{\sqrt{3}}{2}\sin s\right), \end{cases}$$

$$\kappa_2 = \sqrt{3}|\sin s|, \tau_2 = -\sqrt{3}\varepsilon\cos s,$$

where $\varepsilon = sgn\,\tau_0$.

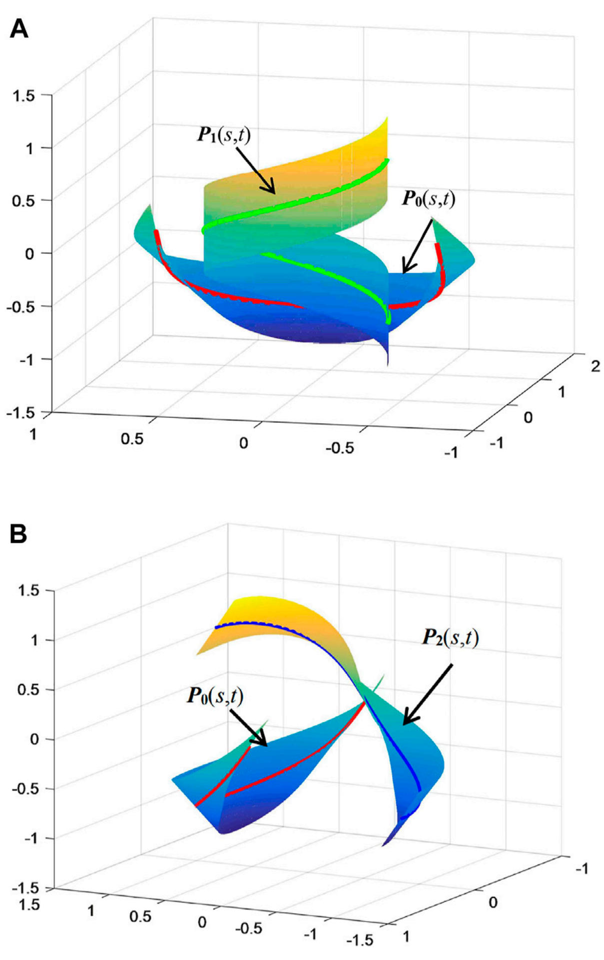

FIGURE 2
$\{P_0(s, t), P_i(s, t)\}$ interpolating $\{R_0(s), R_i(s)\}$, $(i = 1, 2)$ as common geodesics.

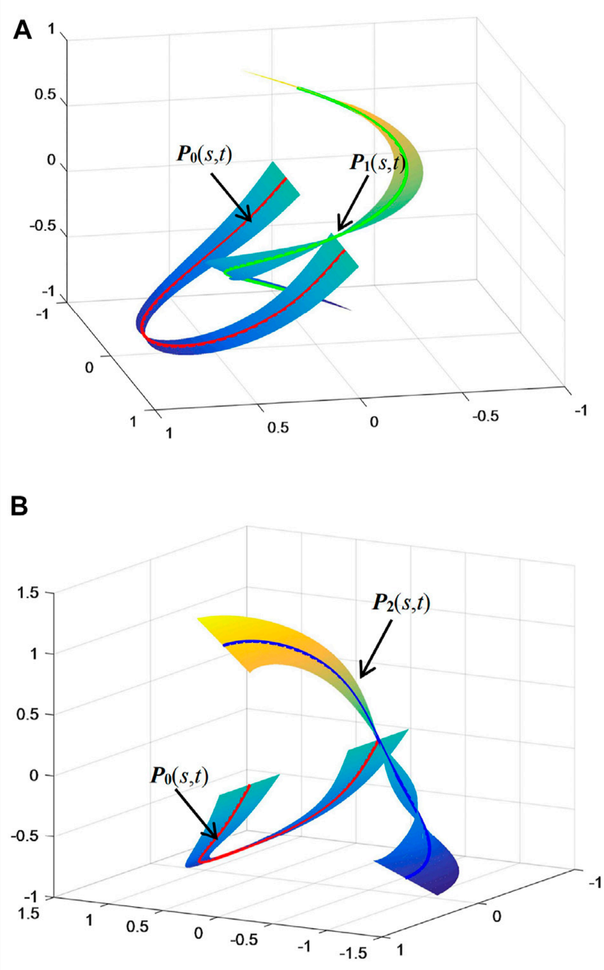

FIGURE 3
$\{P_0(s, t), P_i(s, t)\}$ interpolating $\{R_0(s), R_i(s)\}$, $(i = 1, 2)$ as common lines of curvature.

Case 1. By taking $\lambda_i = -\kappa_i$ in Eq. 15, $\{\boldsymbol{P}_0(s, t), \boldsymbol{P}_i(s, t)\}$ interpolating $\{\boldsymbol{R}_0(s), \boldsymbol{R}_i(s)\}$, $(i = 1, 2)$ as common geodesics are shown in Figure 2, and

$$\boldsymbol{P}_i(s,t) = \boldsymbol{R}_i(s) + t(\tau_i \boldsymbol{T}_i + \kappa_i \boldsymbol{B}_i),$$
$$i = 0, 1, 2, \left(-\frac{\pi}{2} \le s \le \frac{\pi}{2}, -0.2 \le t \le 0.2\right).$$

Case 2. By taking $\theta_0 = \sqrt{3}\cos s, \theta_1 = -s, \theta_2 = -\varepsilon\sqrt{3}\sin s$, and $\lambda_i = s$ in Eq.16, $\{\boldsymbol{P}_0(s, t), \boldsymbol{P}_i(s, t)\}$ interpolating $\{\boldsymbol{R}_0(s), \boldsymbol{R}_i(s)\}$, $(i = 1, 2)$ as common lines of curvature are shown in Figure 3, and

$$\boldsymbol{P}_i(s,t) = \boldsymbol{R}_i(s) + ts(\sin\theta_i \boldsymbol{N}_i - \cos\theta_i \boldsymbol{B}_i), \quad i = 0, 1, 2,$$
$$\left(-\frac{\pi}{2} \le s \le \frac{\pi}{2}, -0.2 \le t \le 0.2\right).$$

Case 3. By taking

$$C = \frac{1}{4}, \sin\theta_0 = \frac{1}{4}\sec s, \cos\theta_0 = -\sqrt{1 - \frac{1}{16}\sec^2 s}, \sin\theta_1 = \frac{1}{2\sqrt{3}},$$
$$\cos\theta_1 = -\sqrt{\frac{11}{12}}, \sin\theta_2 = \frac{1}{4}|\csc s|, \cos\theta_2 = \sqrt{1 - \frac{1}{16}\csc^2 s}, \lambda_i = s.$$

In Eq. 17, $\{\boldsymbol{P}_0(s, t), \boldsymbol{P}_i(s, t)\}$ interpolating $\{\boldsymbol{R}_0(s), \boldsymbol{R}_i(s)\}$, $(i = 1, 2)$, as common D-type curves are shown in Figure 4, and

$$\boldsymbol{P}_i(s,t) = \boldsymbol{R}_i(s) - ts\left(\frac{(\sin\theta_i)' + \tau_i\cos\theta_i}{\kappa_i\cos^2\theta_i}\boldsymbol{T}_i - \sin\theta_i\boldsymbol{N}_i + \cos\theta_i\boldsymbol{B}_i\right), i = 0,1,2,$$

where $-0.2 \le t \le 0.2$, and if $i = 0, 1$, then $-\frac{\pi}{3} \le s \le \frac{\pi}{3}$; if $i = 2$, then $-\frac{\pi}{3} \le s \le -\frac{\pi}{8}$ and $\frac{\pi}{8} \le s \le \frac{\pi}{3}$.

In Figures 2–4, the red curve, the green curve, and the blue curve represent $\boldsymbol{R}_0(s)$, $\boldsymbol{R}_1(s)$, and $\boldsymbol{R}_2(s)$, respectively.

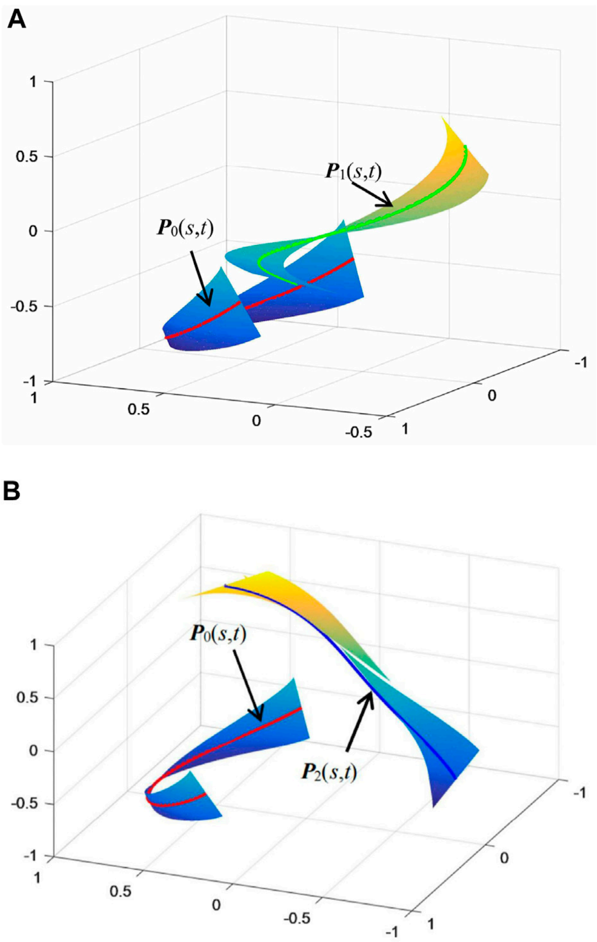

FIGURE 4
{$P_0(s, t)$, $P_i(s, t)$} interpolating {$R_0(s)$, $R_i(s)$}, (i = 1, 2) as common D-type curves.

6 Conclusion

We give a unified method for constructing developable surface pencils interpolating four kinds of characteristic curves. This method not only contains the results of [11–13], but also includes the method for constructing a developable surface pencil interpolating a common D-type curve. We find that the marching-scale functions are completely determined by a univalent function λ and an angle θ, where λ controls the shape of the developable surface and θ determines which kind of characteristic curve the given curve is. Therefore, the method in this study is more general and interactive than other methods. Furthermore, we obtain specific expressions of developable surface pencil pairs interpolating natural and conjugate curve pairs as common geodesics, lines of curvature, and D-type curves, respectively, which complete the results of [25].

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

JW, MC, and DW contributed to the conception and design of the study. DW collected methods for constructing developable surfaces. JW and MC provided the theoretical derivation of the method and implemented the method through examples. JW wrote the first draft of the manuscript. All authors contributed to manuscript revision, read, and approved the submitted version.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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