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Response to: Commentary: Multidimensional discrete chaotic maps

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A Commentary on
Response to commentary: Multidimensional discrete chaotic maps

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The paper [1] gives the concept of multidimensional map as

$$\mathrm{X}(k+1) = \mathrm{f}(\mathrm{X}(k)) \tag{1}$$

with $\mathrm{X} \in \mathbb{R}^{n \times n}$ and f is generated from the gene scalar function

$$x(k+1) = f(x(k)) \tag{2}$$

with $x \in \mathbb{R}$.

The multidimensional logistic map is therefore given by

$$\mathrm{X}(k+1) = a\mathrm{X}(k)(\mathrm{I} - \mathrm{X}(k)) \tag{3}$$

as defined in [2].

In [1] the discussion is oriented to the invariance of the bifurcation diagrams of the multidimensional maps with respect to that of the corresponding gene scalar map, not limiting to the logistic map case. We remark that in [1] the interest is towards ensuring as a first condition the non-explosivity of the multidimensional map, therefore the aim of Theorems 1 and 2 in [1] is to determine the conditions under which non-explosivity is guaranteed. Theorems 1 and 2, in fact, guarantee that the multidimensional maps are not explosive if the eigenvalues of the diagonalizable initial condition matrix are inside the basin of attraction of the corresponding scalar map.

In the Commentary [3], an example on the multidimensional logistic map is reported with the aim of invalidating the theory presented in [1]. It assumed as initial condition the matrix $\mathrm{X}(0) = \begin{bmatrix} -0.1 & 0.4 \\ -0.1 & 0.3 \end{bmatrix}$ with eigenvalues $\lambda_1 = \lambda_2 = 0.1$. The authors show the explositivity of the map by using a numerical simulation.

Fact. The example does not match the assumptions of Theorem 1. In fact, the assumed matrix X(0) is not diagonalizable. In the proof of Theorem 1 [1], Eq. 9 is immediately followed by the sentence: "with $\mathrm{X}(0) = \mathrm{T}\mathrm{X}_D(0)\mathrm{T}^{-1}$ being $\mathrm{X}_D(0)$ a diagonal matrix containing the eigenvalues of X(0) and $\mathrm{T} \in \mathbb{R}^{N \times N}$ the matrix of eigenvectors." The proof is performed under this hypothesis. Therefore, even if the eigenvalues of the considered X(0) are in basin of attraction of the scalar logistic map, it is verified that it is not diagonalizable being X(0) a 2×2 not diagonal matrix with identical eigenvalues. For this matrix a Jordan form can be derived as

$$X(0) = TJT^{-1} = T\begin{bmatrix} 0.1 & 1 \\ 0 & 0.1 \end{bmatrix}T^{-1} \tag{4}$$

with $T = \begin{bmatrix} -0.2 & 1 \\ -0.1 & 0 \end{bmatrix}$ and being J a not diagonal matrix and thus not respecting the assumptions of Theorem 1.

Moreover, in the case of $X(0) \in \mathbb{R}^{n \times n}$ with $n > 2$ admitting eigenvalues with multiplicity greater than 1, the theory does hold if X(0) is diagonalizable. As an example, let us consider as initial condition the matrix $X(0) = \begin{bmatrix} 0.15 & 0 & -0.05 \\ 0 & 0.1 & 0 \\ -0.05 & 0 & 0.15 \end{bmatrix}$. This matrix admits two identical eigenvalues $\lambda_1 = \lambda_2 = 0.1$ and a third eigenvalue $\lambda_3 = 0.2$. Matrix X(0) is symmetrical and hence diagonalizable, and all eigenvalues are in the basin of attraction of the scalar logistic map. The multidimensional logistic map is, in fact, not explosive, as guaranteed by Theorems 1 and 2.

The authors of [3] chose a not correct example to put in crisis the validity of paper [1].

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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