Analytical analysis of fractional nonlinear Jaulent-Miodek system with energy-dependent Schrödinger potential

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In this work, a novel technique is considered for analyzing the fractional-order Jaulent-Miodek system. The suggested approach is based on the use of the residual power series technique in conjunction with the Laplace transform and Caputo operator to solve the system of equations. The Caputo derivative is applied to express the fractional operator, which is more suitable for modeling real-world phenomena with memory effects. As a real example, the proposed technique is implemented for analyzing the Jaulent-Miodek equation under suitable initial conditions. Additionally, the proposed technique's validity (accuracy and effectiveness) is examined by studying some numerical examples. The obtained solutions show that the suggested technique can provide a reliable solution for the fractional-order Jaulent-Miodek system, making it a helpful tool for researchers in different areas, including engineering, physics, and mathematics. We also analyze the absolute error between the derived approximations and the analytical solutions to check the validation and accuracy of the obtained approximations. Many researchers can benefit from both the obtained approximations and the suggested method in analyzing many complicated nonlinear systems in plasma physics and nonlinear optics, and many others.

1 Introduction

Fractional differential equations (DEs) are types of DEs that involve fractional derivatives (FDs). Unlike ordinary DEs, where the order of the derivative is a positive integer, fractional DEs (FDEs) involve operators of non-integer orders. These equations are applied to model various physical and biological phenomena, such as anomalous diffusion, viscoelasticity, and fractal growth [1–3]. FDEs have several unique properties that differentiate them from ordinary DEs, such as non-locality and memory effects. Solving the FDEs requires specialized techniques, such as fractional calculus and numerical methods [4–8]. These equations have become an active area of research in recent decades due to their potential application in various fields of science [9–12].

Fractional nonlinear systems of partial DEs (PDEs) are mathematics model that describes the behavior of complex models in different areas, such as chemistry, biology, physics, engineering, and finance [13–15]. These systems are characterized by the presence of FDs, which are generalizations of the classical integer-order derivatives. FDs describe the system's memory and long-range interactions and allow for modeling anomalous diffusion, power-law behavior, and other non-local effects that classical models cannot capture. Nonlinearities are another essential feature of fractional systems, as they can lead to the emergence of rich and diverse phenomena, such as chaos, bifurcations, solitons, and patterns. Nonlinear systems are ubiquitous in nature and technology, and understanding their dynamics is crucial for predicting and controlling their behavior [16–23].

Fractional nonlinear systems of PDEs are challenging to study due to their non-locality, nonlinearity, and complexity. They require developing new analytical and numerical tools, such as fractional calculus (FC), dynamical systems theory, and computer simulations [24–26]. Despite the difficulties, fractional nonlinear systems of PDEs have attracted increasing attention in recent years due to their relevance in many applications. They provide a powerful framework for modeling and understanding complex phenomena and offer new opportunities for scientific and technological advances [27, 28].

It has been found that FDEs describe real-world problems more precisely than integral order DEs. The study of coupled systems of FDEs is also quite interesting. Because mathematical models of many phenomena in bio-mathematics, physics, psychology, and other fields are coupled systems of DEs [29, 30]. Among such coupled systems of fractional PDEs (FPDEs), we have the coupled Jaulent-Miodek models with Schrodinger energy-dependent potential. This type of equation system is widely applied as a model for the solution of several real worlds problems in the areas of applied sciences [31, 32]. Extensive analysis of nonlinear coupled fractional-order Jaulent-Miodek models a key role in many areas fields of science, such as plasma physics [33], condensed matter physics [34, 35]. There are a variety of techniques applied in achieving analytic and numeric results to linear and nonlinear FPDE, such as the homotopy perturbation technique (HPT) [36–39], the variational iteration technique [40], the q-homotopy analysis transformation technique [41], the fractional natural decomposition technique [42], the fractional multi-step differential transform technique [43], the new iterative technique [44, 45] and the homotopy analysis technique [46, 47], Residual power series technique [48].

The suggested method is called Laplace residual power series method (LRPSM) which was introduced recently to address nonlinear DEs (NLDEs) with fractional orders [49, 50]. This method is a combination between Laplace transform (LT) and RPSM which provides a more accurate solution, requiring less time and simpler calculations than other analytical methods. Unlike other methods, LRPSM does not involve differentiation or linearization and only utilizes the LT, followed by taking the limit at infinity. Thus, the current work aims to apply an innovative analytical technique (LRPSM) to obtain highly accurate estimated fractional solutions to the Jaulent-Miodek equation in the Caputo sense subject to suitable initial conditions. Also, the outcomes of the

LRPSM will be compared with the precise answer by creating graphs and tables for the numerical problem. The suggested technique has been used to make exact results for emerging realistic models of physical phenomena by using fast convergent power series. This method succeeded because it is straightforward and handles different kinds of initial conditions directly. Also, it doesn't need linearization or restrictive assumptions, doesn't need a lot of computing power, takes less time, and is more accurate.

The framework of this study is detailed as follows: Section 2 reviews certain essential concepts, properties, and theorems related to FC, LT, and Laplace fractional expansion. The general methodology of LRPSM for the proposed model is presented in Section 3. Fractional solution Jaulent-Miodek equations are provided applying the LRPSM in Section 4. Section 5 contains the result and discussion. Finally, the conclusion is given in Section 6.

2 Basic definitions

Definition 1. *For at least n time differentiable function, the fractional Caputo derivative of order ρ reads [51]*

$$
{}^{C}D_{\tau}^{\rho}\psi(\gamma,\tau) = \begin{cases} \int_{0}^{\tau} \dfrac{(\tau-w)^{n-\rho-1}\psi(\eta,w)}{\Gamma(n-\rho)}dw, & n-1<\rho\le n, \\ \dfrac{\partial^{n}\psi(\eta,\tau)}{\partial\tau^{n}}, & n=\rho, \end{cases} \tag{1}
$$

where n ∈ N and the fractional Riemann–Liouville (RL) of Ω(γ, τ) of order κ becomes

$$
J_{\tau}^{\rho}\psi(\eta,\tau) = \frac{1}{\Gamma(\rho)}\int_{0}^{\tau}(\tau-w)^{\rho-1}\psi(\eta,w)dw, \tag{2}
$$

assuming that the given integral exists.

Lemma 1. *For n − 1 < ρ ≤ n, q > − 1, τ ≥ 0 and λ ∈ R, we have [52]:*

1. $D_{\tau}^{\rho}\tau^{q} = \frac{\Gamma(q+1)}{\Gamma(q-\rho+1)}\tau^{q-\rho}$,
2. $D_{\tau}^{\rho}\lambda = 0$,
3. $D_{\tau}^{\rho}I_{\tau}^{\rho}\psi(\eta,\tau) = \psi(\eta,\tau)$,
4. $I_{\tau}^{\rho}D_{\tau}^{\rho}\psi(\eta,t) = \psi(\eta,\tau) - \sum_{j=0}^{n-1}\partial^{j}\psi(\eta,0)\frac{\tau^{j}}{j!}$.

Definition 2. *The function of LT ψ(η, τ) is given as [52]*

$$
\psi(\eta,s) = \mathcal{L}_{\tau}[\psi(\eta,\tau)] = \int_{0}^{\infty}e^{-s\tau}\psi(\eta,\tau)d\tau, \quad s>\mu. \tag{3}
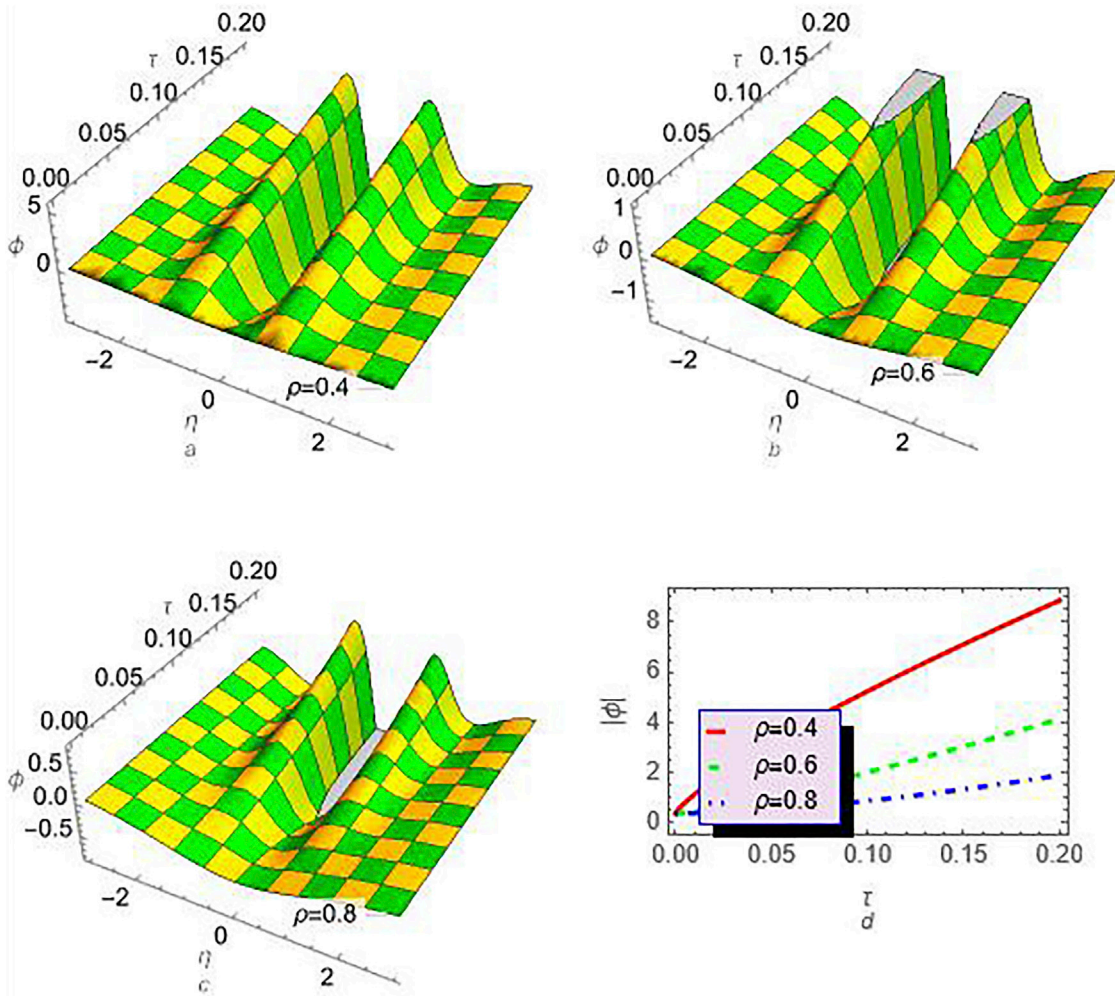$$

The expression for the inverse of LT reads

$$
\psi(\eta,\tau) = \mathcal{L}_{\tau}^{-1}[\psi(\eta,s)] = \int_{l-i\infty}^{l+i\infty}e^{s\tau}\psi(\eta,s)ds, \quad l=Re(s)>l_{0}, \tag{4}
$$

where l_0 is in the right half-plane of absolute convergence the Laplace integral's.

Lemma 2. *Assuming that ψ(η, τ) is a piecewise continuous function with exponential-order δ, we can obtain ψ(η, s) = $\mathcal{L}_{\tau}$[ψ(η, τ)] by taking the LT of ψ(η, τ).*

1. $\mathcal{L}_{\tau}[J_{\tau}^{\rho}\psi(\eta,\tau)] = \frac{\psi(\eta,s)}{s^{\rho}}, \quad \kappa>0$.

FIGURE 1
The profile of the approximation $\phi(\eta, \tau)$ (24) is plotted at different values of ρ and with $\lambda = 1$: **(A)** $\rho = 0.4$, **(B)** $\rho = 0.6$, **(C)** $\rho = 0.8$., and **(D)** the comparison between different values of ρ at $x = 0$.

2. $\mathcal{L}_\tau \left[D_\tau^\rho \psi(\eta, \tau) \right] = s^\rho \psi(\eta, s) - \sum_{k=0}^{m-1} s^{\rho-k-1} \psi^k(\eta, 0), \quad m-1 < \rho \le m.$

3. $\mathcal{L}_\tau \left[D_\tau^{n\rho} \psi(\eta, \tau) \right] = s^{n\rho} \psi(\eta, s) - \sum_{k=0}^{n-1} s^{(n-k)\rho-1} D_\tau^{k\rho} \psi(\eta, 0),$
$0 < \rho \le 1.$

Theorem 1. *Consider a function $\Omega(\gamma, \tau)$ that is continuous and piecewise-defined over the interval $I \times [0, \infty)$ and has an exponential order of ζ. Let us define the term $\Omega(\gamma, s)$ as the Laplace transform of $\Omega(\gamma, \tau)$ with respect to τ. It is worth noting that $\Omega(\gamma, s)$ has a fractional expansion.*

$$\Omega(\gamma, s) = \sum_{n=0}^{\infty} \frac{f_n(\gamma)}{s^{1+n\mu}}, 0 < \mu \le 1, \gamma \in I, s > \zeta. \tag{5}$$

Then, $f_n(\gamma) = D_\tau^{n\mu} \Omega(\gamma, 0)$.

3 General implementation laplace residual power series method

Consider the general FPDE

$$D_\tau^\rho \psi(\eta, \tau) + N \left[\psi(\eta, \tau) \right] + A \left[\psi(\eta, \tau) \right] = 0, \quad \text{where} \quad 0 < \rho \le 1, \tag{6}$$

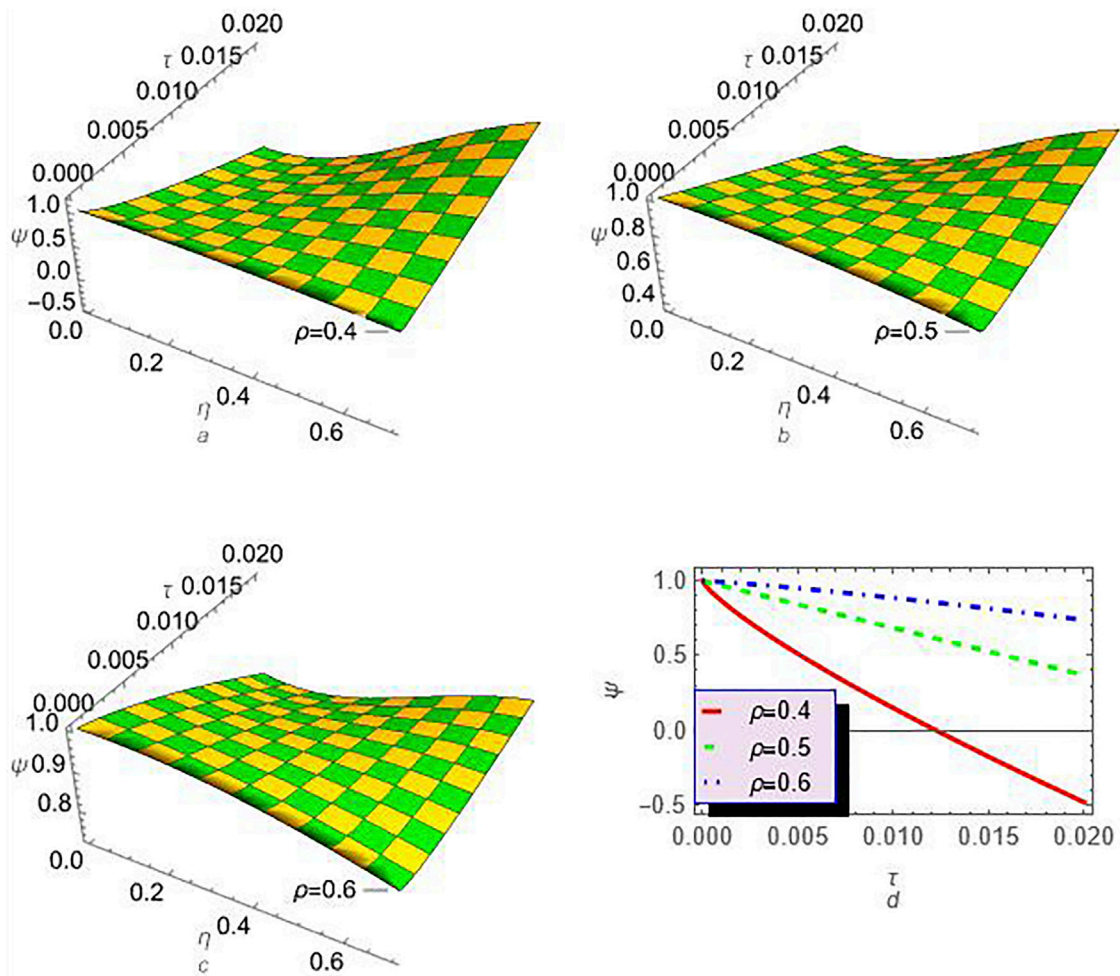subject to initial condition:

$$\psi(\eta, \tau) = f_0(\eta). \tag{7}$$

The function $d\psi(\eta, \tau)$ is unknown and depends on the independent variables η and τ, where the operator A is linear and N is nonlinear. Applying LT to Eq. 6 and making use of Eq. 7 we get

$$\psi(\eta, s) - \frac{f_0(\eta, s)}{s} + \frac{1}{s^\rho} \mathcal{L}_\tau \left[N \left[\mathcal{L}_\tau^{-1} \left[\psi(\eta, s) \right] \right] + A \left[\psi(\eta, \tau) \right] \right] = 0. \tag{8}$$

The result of Eq. 8 is given as

$$\psi(\xi, s) = \sum_{n=0}^{\infty} \frac{f_n(\xi, s)}{s^{n\rho+1}}, \tag{9}$$

the kth-truncate terms series are

FIGURE 2
The profile of the approximation $\psi(\eta, \tau)$ (25) is plotted at different values of ρ and with $\lambda = 1$: **(A)** $\rho = 0.4$, **(B)** $\rho = 0.5$, **(C)** $\rho = 0.6$., and **(D)** the comparison between different values of ρ at $x = 0$.

$$\psi(\eta, s) = \frac{f_0(x, s)}{s} + \sum_{n=1}^{k} \frac{f_n(\eta, s)}{s^{n\rho+1}}, \qquad (10)$$
$$n = 1, 2, 3, 4 \cdots$$

The residual Laplace function reads [53].

$$\mathcal{L}_\tau Res(\eta, s) = \psi(\eta, s) - \frac{f_0(\eta, s)}{s} + \frac{1}{s^\rho}\mathcal{L}_\tau\Big[N[\mathcal{L}_\tau^{-1}[\psi(\eta, s)]] \\ + A[\psi(\eta, \tau)]\Big]. \qquad (11)$$

And the kth-LRFs as:

$$\mathcal{L}_\tau Res_k(\eta, s) = \psi_k(\eta, s) - \frac{f_0(\eta, s)}{s} + \frac{1}{s^\rho}\mathcal{L}_\tau\Big[N[\mathcal{L}_\tau^{-1}[\psi_k(\eta, s)]] \\ + A[\psi_k(\eta, \tau)]\Big]. \qquad (12)$$

The few properties of the LRPSM [53], is expressed as.

- $\mathcal{L}_\tau Res(\eta, s) = 0$ and $\lim_{j\to\infty}\mathcal{L}_\tau Res_k(\eta, s) = \mathcal{L}_\tau Res_\psi(\eta, s)$ for each $s > 0$,

- $\lim_{s\to\infty} s\mathcal{L}_\tau Res_\psi(\eta, s) = 0 \Rightarrow \lim_{s\to\infty} s\mathcal{L}_\tau Res_{\psi,k}(\eta, s) = 0$,
- $\lim_{s\to\infty} s^{k\rho+1}\mathcal{L}_\tau Res_{\psi,k}(\eta, s) = \lim_{s\to\infty} s^{k\rho+1}\mathcal{L}_\tau Res_{\psi,k}(\eta, s) = 0$, $0 < \rho \leq 1$, $k = 1, 2, 3, \cdots$.

To investigate the coefficient $f_n(\eta, s)$ and $g_n(\eta, s)$, we find the solution of the following system

$$\lim_{s\to\infty} s^{k\rho+1}\mathcal{L}_\tau Res_{\psi,k}(\eta, s) = 0, \quad \lim_{s\to\infty} s^{k\rho+1}\mathcal{L}_\tau Res_{\phi,k}(\eta, s) = 0, \quad k = 1, 2, \cdots.$$
$$(13)$$

Finally, we apply the inverse of the LT to Eq. 9, to get the kth analytical solution of $\psi_k(\eta, \tau)$ and $\phi_k(\eta, \tau)$.

4 Numerical problem

Consider the coupled fractional-order Jaulent-Miodek equations:

TABLE 1 The approximate solution $\phi(\eta, \tau)$ (24) is considered at different values ρ and with ($\tau = 0.0099$, $\lambda = 0.2$). Also, the absolute error of $\phi(\eta, \tau)$ (24) at $\rho = 1$ as compared to the exact solution (26) is estimated.

η	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.8$	LRPSM ($\rho = 1$)	exact	Abs.error
0.1	−0.01	−0.0150	−0.0150	−0.0150	−0.0150	8.67132×10^{-8}
0.2	−0.01	−0.01499	−0.01499	−0.01499	−0.01499	1.69551×10^{-7}
0.3	−0.01	−0.01498	−0.01498	−0.01498	−0.01498	2.51868×10^{-7}
0.4	−0.01	−0.01497	−0.01497	−0.01497	−0.01497	3.33415×10^{-7}
0.5	−0.01	−0.01495	−0.01495	−0.01495	−0.01495	4.1395×10^{-7}
0.6	−0.01	−0.01493	−0.01493	−0.01493	−0.01493	4.93232×10^{-7}
0.7	−0.01	−0.0149	−0.0149	−0.0149	−0.0149	5.71027×10^{-7}
0.8	−0.01	−0.01488	−0.01487	−0.01487	−0.01487	6.47109×10^{-7}
0.9	−0.01	−0.01484	−0.01484	−0.01484	−0.01484	7.21256×10^{-7}
1	−0.01	−0.0148	−0.0148	−0.0148	−0.0148	7.93258×10^{-7}
1.1	−0.01	−0.01476	−0.01476	−0.01476	−0.01476	8.62914×10^{-7}
1.2	−0.01	−0.01472	−0.01472	−0.01472	−0.01471	9.30033×10^{-7}
1.3	−0.01	−0.01467	−0.01467	−0.01467	−0.01467	9.94433×10^{-7}
1.4	−0.01	−0.01462	−0.01462	−0.01461	−0.01461	1.05595×10^{-6}
1.5	−0.01	−0.01456	−0.01456	−0.01456	−0.01456	1.11442×10^{-6}
1.6	−0.01	−0.0145	−0.0145	−0.0145	−0.0145	1.16971×10^{-6}
1.7	−0.01	−0.01444	−0.01444	−0.01443	−0.01443	1.22168×10^{-6}
1.8	−0.01	−0.01437	−0.01437	−0.01437	−0.01437	1.27022×10^{-6}
1.9	−0.01	−0.0143	−0.0143	−0.0143	−0.01429	1.31522×10^{-6}
2	−0.01	−0.01423	−0.01422	−0.01422	−0.01422	1.35661×10^{-6}

$$D_\tau^\rho \phi(\eta,\tau) + \frac{\partial^3 \phi(\eta,\tau)}{\partial \eta^3} + \frac{3}{2}\psi(\eta,\tau)\frac{\partial^3 \psi(\eta,\tau)}{\partial \eta^3} + \frac{9}{2}\frac{\partial \psi(\eta,\tau)}{\partial \eta}\frac{\partial^2 \psi(\eta,\tau)}{\partial \eta^2}$$
$$- 6\phi(\eta,\tau)\frac{\partial \phi(\eta,\tau)}{\partial \eta} - 6\phi(\eta,\tau)\psi(\eta,\tau)\frac{\partial \psi(\eta,\tau)}{\partial \eta}$$
$$- \frac{3}{2}\psi^2(\eta,\tau)\frac{\partial \phi(\eta,\tau)}{\partial \eta} = 0,$$
$$D_\tau^\rho \psi(\eta,\tau) + \frac{\partial^3 \psi(\eta,\tau)}{\partial \eta^3} - 6\frac{\partial \phi(\eta,\tau)}{\partial \eta}\psi(\eta,\tau) - 6\phi(\eta,\tau)\frac{\partial \psi(\eta,\tau)}{\partial \eta}$$
$$- \frac{15}{2}\frac{\partial \psi(\eta,\tau)}{\partial \eta}\psi^2(\eta,\tau) = 0,$$

where $\quad 0 < \rho \le 1$, $\qquad\qquad\qquad$ (14)

with the initial conditions (ICs)

$$\begin{cases} \phi(\eta,0) = \frac{\lambda^2}{8}\left(1 - Sech^2\left(\frac{\lambda\eta}{2}\right)\right), \\ \\ \psi(\eta,0) = \lambda Sech^2\left(\frac{\lambda\eta}{2}\right). \end{cases} \qquad (15)$$

By utilizing Eq. 14 and taking advantage of Eq. 15, we get

$$\phi(\eta,s) - \frac{\phi(\eta,0)}{s} + \frac{1}{s^\rho}\frac{\partial^3 \phi(\eta,s)}{\partial \eta^3} + \frac{3}{2s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\psi(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial^3 \psi(\eta,s)}{\partial \eta^3}\right]$$
$$+ \frac{9}{2s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\frac{\partial \psi(\eta,s)}{\partial \eta}\mathcal{L}_\tau^{-1}\frac{\partial^2 \psi(\eta,s)}{\partial \eta^2}\right]$$
$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\phi(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial \phi(\eta,s)}{\partial \eta}\right]$$
$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\phi(\eta,s)\mathcal{L}_\tau^{-1}\psi(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial \psi(\eta,s)}{\partial \eta}\right]$$
$$- \frac{3}{2s^\rho}\mathcal{L}_\tau\left[\left(\mathcal{L}_\tau^{-1}\psi(\eta,s)\right)^2\mathcal{L}_\tau^{-1}\frac{\partial \phi(\eta,s)}{\partial \eta}\right] = 0,$$
$$\psi(\eta,s) - \frac{\phi(\eta,0)}{s} + \frac{1}{s^\rho}\frac{\partial^3 \psi(\eta,s)}{\partial \eta^3} - \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\frac{\partial \phi(\eta,s)}{\partial \eta}\mathcal{L}_\tau^{-1}\psi(\eta,s)\right]$$
$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\phi(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial \psi(\eta,s)}{\partial \eta}\right]$$
$$- \frac{15}{2s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\frac{\partial \psi(\eta,s)}{\partial \eta}\left(\mathcal{L}_\tau^{-1}\psi(\eta,s)\right)^2\right] = 0.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (16)

By applying the ICs, we get

$$\phi(\eta,s) - \frac{\frac{\lambda^2}{8}\left(1 - Sech^2\left(\frac{\lambda\eta}{2}\right)\right)}{s} + \frac{1}{s^\rho}\frac{\partial^3\phi(\eta,s)}{\partial\eta^3}$$

$$+ \frac{3}{2s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\psi(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial^3\psi(\eta,s)}{\partial\eta^3}\right]$$

$$+ \frac{9}{2s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\frac{\partial\psi(\eta,s)}{\partial\eta}\mathcal{L}_\tau^{-1}\frac{\partial^2\psi(\eta,s)}{\partial\eta^2}\right]$$

$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\phi(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial\phi(\eta,s)}{\partial\eta}\right]$$

$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\phi(\eta,s)\mathcal{L}_\tau^{-1}\psi(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial\psi(\eta,s)}{\partial\eta}\right]$$

$$- \frac{3}{2s^\rho}\mathcal{L}_\tau\left[\left(\mathcal{L}_\tau^{-1}\psi(\eta,s)\right)^2\mathcal{L}_\tau^{-1}\frac{\partial\phi(\eta,s)}{\partial\eta}\right] = 0, \qquad (17)$$

$$\psi(\eta,s) - \frac{\lambda Sech^2\left(\frac{\lambda\eta}{2}\right)}{s} + \frac{1}{s^\rho}\frac{\partial^3\psi(\eta,s)}{\partial\eta^3}$$

$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\frac{\partial\phi(\eta,s)}{\partial\eta}\mathcal{L}_\tau^{-1}\psi(\eta,s)\right]$$

$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\phi(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial\psi(\eta,s)}{\partial\eta}\right]$$

$$- \frac{15}{2s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\frac{\partial\psi(\eta,s)}{\partial\eta}\left(\mathcal{L}_\tau^{-1}\psi(\eta,s)\right)^2\right] = 0.$$

The kth-truncated term series reads

$$\phi(\eta,s) = \frac{\frac{\lambda^2}{8}\left(1 - Sech^2\left(\frac{\lambda\eta}{2}\right)\right)}{s} + \sum_{n=1}^{k}\frac{f_n(\eta,s)}{s^{n\rho+1}},$$

$$\psi(\eta,s) = \frac{\lambda Sech^2\left(\frac{\lambda\eta}{2}\right)}{s} + \sum_{n=1}^{k}\frac{f_n(\eta,s)}{s^{n\rho+1}}, \qquad (18)$$

$$n = 1,2,3,4\cdots.$$

The Laplace residual functions (LRFs) reads

$$\mathcal{L}_\tau Res(\eta,s) = \phi(\eta,s) - \frac{\frac{\lambda^2}{8}\left(1 - Sech^2\left(\frac{\lambda\eta}{2}\right)\right)}{s} + \frac{1}{s^\rho}\frac{\partial^3\phi(\eta,s)}{\partial\eta^3}$$

$$+ \frac{3}{2s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\psi(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial^3\psi(\eta,s)}{\partial\eta^3}\right]$$

$$+ \frac{9}{2s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\frac{\partial\psi(\eta,s)}{\partial\eta}\mathcal{L}_\tau^{-1}\frac{\partial^2\psi(\eta,s)}{\partial\eta^2}\right]$$

$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\phi(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial\phi(\eta,s)}{\partial\eta}\right]$$

$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\phi(\eta,s)\mathcal{L}_\tau^{-1}\psi(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial\psi(\eta,s)}{\partial\eta}\right]$$

$$- \frac{3}{2s^\rho}\mathcal{L}_\tau\left[\left(\mathcal{L}_\tau^{-1}\psi(\eta,s)\right)^2\mathcal{L}_\tau^{-1}\frac{\partial\phi(\eta,s)}{\partial\eta}\right],$$

$$\mathcal{L}_\tau Res(\eta,s) = \psi(\eta,s) - \frac{\lambda Sech^2\left(\frac{\lambda\eta}{2}\right)}{s} + \frac{1}{s^\rho}\frac{\partial^3\psi(\eta,s)}{\partial\eta^3}$$

$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\frac{\partial\phi(\eta,s)}{\partial\eta}\mathcal{L}_\tau^{-1}\psi(\eta,s)\right]$$

$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\phi(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial\psi(\eta,s)}{\partial\eta}\right]$$

$$- \frac{15}{2s^\rho}\mathcal{L}_\ell\left[\mathcal{L}_\tau^{-1}\frac{\partial\psi(\eta,s)}{\partial\eta}\left(\mathcal{L}_\tau^{-1}\psi(\eta,s)\right)^2\right]. \qquad (19)$$

The kth-LRFs are given by

$$\mathcal{L}_\tau Res_k(\eta,s) = \phi_k(\eta,s) - \frac{\frac{\lambda^2}{8}\left(1 - Sech^2\left(\frac{\lambda\eta}{2}\right)\right)}{s} + \frac{1}{s^\rho}\frac{\partial^3\phi_k(\eta,s)}{\partial\eta^3}$$

$$+ \frac{3}{2s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\psi_k(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial^3\psi_k(\eta,s)}{\partial\eta^3}\right]$$

$$+ \frac{9}{2s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\frac{\partial\psi_k(\eta,s)}{\partial\eta}\mathcal{L}_\tau^{-1}\frac{\partial^2\psi_k(\eta,s)}{\partial\eta^2}\right]$$

$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\phi_k(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial\phi_k(\eta,s)}{\partial\eta}\right]$$

$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\phi_k(\eta,s)\mathcal{L}_\tau^{-1}\psi_k(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial\psi_k(\eta,s)}{\partial\eta}\right]$$

$$- \frac{3}{2s^\rho}\mathcal{L}_\tau\left[\left(\mathcal{L}_\tau^{-1}\psi_k(\eta,s)\right)^2\mathcal{L}_\tau^{-1}\frac{\partial\phi_k(\eta,s)}{\partial\eta}\right],$$

$$\mathcal{L}_\tau Res_k(\eta,s) = \psi_k(\eta,s) - \frac{\lambda Sech^2\left(\frac{\lambda\eta}{2}\right)}{s} + \frac{1}{s^\rho}\frac{\partial^3\psi_k(\eta,s)}{\partial\eta^3}$$

$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\frac{\partial\phi_k(\eta,s)}{\partial\eta}\mathcal{L}_\tau^{-1}\psi_k(\eta,s)\right]$$

$$- \frac{6}{s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\phi_k(\eta,s)\mathcal{L}_\tau^{-1}\frac{\partial\psi_k(\eta,s)}{\partial\eta}\right]$$

$$- \frac{15}{2s^\rho}\mathcal{L}_\tau\left[\mathcal{L}_\tau^{-1}\frac{\partial\psi_k(\eta,s)}{\partial\eta}\left(\mathcal{L}_\tau^{-1}\psi_k(\eta,s)\right)^2\right]. \qquad (20)$$

To compute $f_k(\eta,s)$ and $g_k(\eta,s)$ for $k = 1,2,3,\ldots$, we can use Eq. 18 which gives the nth-truncated series, and substitute it into Eq. 20, which gives the nth-Laplace residual term. Then, we can multiply the solution of the equation by $s^{n\rho+1}$ and solve the relation recursively for $\lim_{s\to\infty}(s^{n\rho+1}\mathcal{L}\tau Res\phi, n(\eta,s)) = 0$ and $\lim_{s\to\infty}(s^{n\rho+1}\mathcal{L}\tau Res\psi, n(\eta,s)) = 0$ for $n = 1, 2, 3, \cdots$. The following are the first few terms:

$$f_0 = \frac{\lambda^2}{8}\left(1 - Sech^2\left(\frac{\lambda\eta}{2}\right)\right), g_0 = \lambda Sech^2\left(\frac{\lambda\eta}{2}\right),$$

$$f_1 = -\frac{1}{128}\lambda^5 Sech^7\left(\frac{\lambda\eta}{2}\right)\left(794Sinh\left(\frac{\lambda\eta}{2}\right) - 165Sinh\left(\frac{3\lambda\eta}{2}\right) + \sinh\left(\frac{5\lambda\eta}{2}\right)\right)$$

$$g_1 = -\frac{1}{32}\lambda^4\left(-189 + 52Cosh(\lambda\eta) + Cosh(2\lambda\eta)\right)Sech^6\left(\frac{\lambda\eta}{2}\right)Tanh\left(\frac{\lambda\eta}{2}\right),$$

$$f_2 = \frac{\lambda^8}{16384}\left(10003020 - 11000862Cosh(\lambda\eta)\right.$$

$$+ 1410960Cosh(2\eta\lambda))Sech^{12}\left(\frac{\lambda\eta}{2}\right)$$

$$+ \frac{\lambda^8}{16384}\left(61341Cosh(3\lambda\eta)\right.$$

$$- 4700Cosh(4\lambda\eta) + Cosh(5\lambda\eta))Sech^{12}\left(\frac{\lambda\eta}{2}\right),$$

$$g_2 = \frac{\lambda^7}{8192}\left(-1713684 + 1217538Cosh(\lambda\eta)315984Cosh(2\lambda\eta)\right)Sech^{12}\left(\frac{\lambda\eta}{2}\right)$$

$$+ \frac{\lambda^7}{8192}\left(-79491Cosh(3\lambda\eta) + 1348Cosh(4\lambda\eta)\right.$$

$$+ Cosh(5\lambda\eta))Sech^{12}\left(\frac{\lambda\eta}{2}\right),$$

$$\qquad (21)$$

Now, by using the values of $f_k(\eta)$, $k = 1, 2, 3, \ldots$, we get

TABLE 2 The approximate solution $\psi(\eta, \tau)$ (25) is considered at different values ρ and with ($\tau = 0.0095$, $\lambda = 0.02$). Also, the absolute error of $\psi(\eta, \tau)$ (25) at $\rho = 1$ as compared to the exact solution (276) is estimated.

η	$\rho = 0.56$	$\rho = 0.78$	$\rho = 0.85$	LRPSM ($\rho = 1$)	exact	Abs.error
0.1	0.0200	0.0200	0.0200	0.0200	0.0200	1.00061×10^{-8}
0.2	0.0200	0.0200	0.0200	0.0200	0.0200	4.0012×10^{-8}
0.3	0.0200	0.0200	0.0200	0.0200	0.0200	9.00175×10^{-8}
0.4	0.0200	0.0200	0.0200	0.0200	0.0200	1.60022×10^{-7}
0.5	0.020	0.0200	0.0200	0.0200	0.0200	2.50025×10^{-7}
0.6	0.019999	0.019999	0.019999	0.019999	0.02000	3.60025×10^{-7}
0.7	0.019999	0.019999	0.019999	0.019999	0.02000	4.9002110×10^{-7}
0.8	0.019999	0.019999	0.019999	0.019999	0.019999	6.40011×10^{-7}
0.9	0.019998	0.019998	0.019998	0.019998	0.019999	8.09995×10^{-7}
1	0.019998	0.019998	0.019998	0.019998	0.019999	9.99969×10^{-7}
1.1	0.019998	0.019998	0.019998	0.019998	0.019999	1.20993×10^{-6}
1.2	0.019997	0.019997	0.019997	0.019997	0.019999	1.43988×10^{-6}
1.3	0.019997	0.019997	0.019997	0.019997	0.019998	1.68982×10^{-6}
1.4	0.019996	0.019996	0.019996	0.019996	0.019998	1.95973×10^{-6}
1.5	0.019996	0.019996	0.019996	0.019996	0.019998	2.24963×10^{-6}
1.6	0.019995	0.019995	0.019995	0.019995	0.019997	2.5595×10^{-6}
1.7	0.019994	0.019994	0.019994	0.019994	0.019997	2.88934×10^{-6}
1.8	0.019994	0.019994	0.019994	0.019994	0.019997	3.23915×10^{-6}
1.9	0.019993	0.019993	0.019993	0.019993	0.019996	3.60892×10^{-6}
2	0.019992	0.019992	0.019992	0.019992	0.019996	3.99866×10^{-6}

$$\phi(\eta,s) = \frac{\frac{\lambda^2}{8}\left(1 - Sech^2\left(\frac{\lambda\eta}{2}\right)\right)}{s}$$
$$- \frac{\frac{1}{128}\lambda^5 Sech^7\left(\frac{\lambda\eta}{2}\right)\left(794 Sinh\left(\frac{\lambda\eta}{2}\right) - 165 Sinh\left(\frac{3\lambda\eta}{2}\right) + sinh\left(\frac{5\lambda\eta}{2}\right)\right)}{s^{\kappa+1}}$$
$$+ \frac{\frac{\lambda^8}{16384}\left(10003020 - 11000862 Cosh(\lambda\eta) + 1410960 Cosh(2\eta\lambda)\right) Sech^{12}\left(\frac{\lambda\eta}{2}\right)}{s^{2\kappa+1}}$$
$$+ \frac{\frac{\lambda^8}{16384}\left(61341 Cosh(3\eta) - 4700 Cosh(4\eta) + Cosh(5\lambda\eta)\right) Sech^{12}\left(\frac{\lambda\eta}{2}\right)}{s^{2\kappa+1}} + \cdots,$$

$$(22)$$

$$\psi(\eta,s) = \frac{\lambda Sech^2\left(\frac{\lambda\eta}{2}\right)}{s}$$
$$- \frac{\frac{1}{32}\lambda^4\left(-189 + 52 Cosh(\lambda\eta) + Cosh(2\lambda\eta)\right) Sech^6\left(\frac{\lambda\eta}{2}\right) Tanh\left(\frac{\lambda\eta}{2}\right)}{s^{\kappa+1}}$$
$$+ \frac{\frac{\lambda^7}{8192}\left(-1713684 + 1217538 Cosh(\lambda\eta) 315984 Cosh(2\lambda\eta)\right) Sech^{12}\left(\frac{\lambda\eta}{2}\right)}{s^{2\kappa+1}}$$
$$+ \frac{\frac{\lambda^7}{8192}\left(-79491 Cosh(3\eta) + 1348 Cosh(4\eta) + Cosh(5\lambda\eta)\right) Sech^{12}\left(\frac{\lambda\eta}{2}\right)}{s^{2\kappa+1}} + \cdots.$$

$$(23)$$

Applying the inverse of LT, we get

$$\phi(\eta,\rho) = \frac{\lambda^2}{8}\left(1 - Sech^2\left(\frac{\lambda\eta}{2}\right)\right)$$
$$- \frac{\frac{1}{128}\lambda^5 Sech^7\left(\frac{\lambda\eta}{2}\right)\left(794 Sinh\left(\frac{\lambda\eta}{2}\right) - 165 Sinh\left(\frac{3\lambda\eta}{2}\right) + sinh\left(\frac{5\lambda\eta}{2}\right)\right)}{\Gamma(\rho+1)}\tau^\rho$$
$$+ \frac{\frac{\lambda^8}{16384}\left(10003020 - 11000862 Cosh(\lambda\eta) + 1410960 Cosh(2\eta\lambda)\right) Sech^{12}\left(\frac{\lambda\eta}{2}\right)}{\Gamma(2\rho+1)}\tau^{2\rho}$$
$$+ \frac{\frac{\lambda^8}{16384}\left(61341 Cosh(3\eta) - 4700 Cosh(4\eta) + Cosh(5\lambda\eta)\right) Sech^{12}\left(\frac{\lambda\eta}{2}\right)}{\Gamma(2\rho+1)}\tau^{2\rho} + \cdots,$$

$$(24)$$

$$\psi(\eta,\rho) = \lambda Sech^2\left(\frac{\lambda\eta}{2}\right)$$
$$- \frac{\frac{1}{32}\lambda^4\left(-189 + 52 Cosh(\lambda\eta) + Cosh(2\lambda\eta)\right) Sech^6\left(\frac{\lambda\eta}{2}\right) Tanh\left(\frac{\lambda\eta}{2}\right)}{\Gamma(\rho+1)}\tau^\rho$$
$$+ \frac{\frac{\lambda^7}{8192}\left(-1713684 + 1217538 Cosh(\lambda\eta) 315984 Cosh(2\lambda\eta)\right) Sech^{12}\left(\frac{\lambda\eta}{2}\right)}{\Gamma(2\rho+1)}\tau^{2\rho}$$
$$+ \frac{\frac{\lambda^7}{8192}\left(-79491 Cosh(3\eta) + 1348 Cosh(4\eta) + Cosh(5\lambda\eta)\right) Sech^{12}\left(\frac{\lambda\eta}{2}\right)}{\Gamma(2\rho+1)}\tau^{2\rho} + \cdots.$$

$$(25)$$

The exact solutions of $\phi(\eta, \tau)$ and $\psi(\eta, \tau)$ are, respectively, given by

$$\phi(\eta, \tau) = \frac{\lambda^2}{8}\left(1 - Sech^2\left(\frac{\lambda}{2}\left(\eta + \frac{\lambda^2\tau}{2}\right)\right)\right), \qquad (26)$$

$$\psi(\eta, \tau) = \lambda Sech\left(\frac{\lambda}{2}\left(\eta + \frac{\lambda^2\tau}{2}\right)\right). \qquad (27)$$

In the results section, we will discuss the profile of the obtained solutions as well as will make a comparison with the exact solutions and the other literature approximations.

5 Results and discussion

Figures 1, 2 represent both two- and three-dimensional graphs for the approximations (24) and (25), respectively, using the LRPSM at different values of fractional order derivative ρ. It is clear that the absolute amplitude of the approximation $\phi(\eta, \tau)$ decreases with the enhancement of ρ while the amplitude of the approximation $\psi(\eta, \tau)$ has an opposite behavior, i.e., its amplitude increases with increasing ρ. Moreover, the two approximations are presented in Tables 1, 2 at different values of ρ and discrete values for η and τ. Furthermore, the absolute error for the approximations $\phi(\eta, \tau)$ and $\phi(\eta, \tau)$ at $\rho = 1$ as compared to the exact solutions is estimated in Tables 1, 2, respectively. After analyzing the obtained results, we can conclude that the obtained approximations using the proposed technique are closely aligned with the exact solutions, indicating a strong level of agreement. We have applied the proposed method to the fractional-order Jaulent-Miodek system to analyze it, and this is not the only example, and it can be applied to a wide range of complicated systems related to nonlinear mediums such as plasma physics and optical fibers.

6 Conclusion

In conclusion, the fractional-order Jaulent-Miodek system has been solved using a novel technique, called the residual power series method with the help of the Laplace transform (LT) in the sense of the Caputo operator. The LT, in conjunction with the Caputo operator, has been used to transform the fractional differential equation into an algebraic equation, which can then be solved using the residual power series method. The suggested method (Laplace residual power series method (LRPSM)) involves using a truncated power series to approximate the solution, and the residual error is minimized by adjusting the power series coefficients. The study of fractional-order systems has gained significant attention in recent years due to their ability to model complex phenomena in various fields of science and engineering. For instance, the Jaulent-Miodek system is a well-known example of such a fractional system, and its solution has been a topic of interest for many researchers. Accordingly, the LRPSM has been applied for derived high accurate approximations for the fractional-order Jaulent-Miodek system. The derived approximations have been compared with the analytical solutions. It was found great harmony and agreement with a very small absolute error between both the approximate

solutions and the analytical solutions. This method has proven to be an effective tool for solving fractional-order systems, as it can provide accurate and efficient solutions. The findings of this study may be helpful in different areas of applied sciences where fractional-order systems are commonly encountered.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

MH: conceptualization (equal), formal analysis (equal), investigation (equal), and methodology (equal). AA: conceptualization (equal), formal analysis (equal), investigation (equal), and methodology (equal). RS: conceptualization (equal), formal analysis (equal), investigation (equal), and methodology (equal). SI: validation (equal), formal analysis (equal), investigation (equal), data curation (equal). SA-T: methodology (equal), writing—review and editing (equal), visualization (equal), supervision (equal). All authors contributed to the article and approved the submitted version.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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