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# *ISI* spectral radii and *ISI* energies of graph operations

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Graph energy is defined to be the $p$-norm of adjacency matrix associated to the graph for $p = 1$ elaborated as the sum of the absolute eigenvalues of adjacency matrix. The graph's spectral radius represents the adjacency matrix's largest absolute eigenvalue. Applications for graph energies and spectral radii can be found in both molecular computing and computer science. On similar lines, Inverse Sum Indeg, (*ISI*) energies, and (*ISI*) spectral radii can be constructed. This article's main focus is the *ISI* energies, and *ISI* spectral radii of the generalized splitting and shadow graphs constructed on any regular graph. These graphs can be representation of many physical models like networks, molecules and macromolecules, chains or channels. We actually compute the relations about the *ISI* energies and *ISI* spectral radii of the newly created graphs to those of the original graph.

KEYWORDS

**ISI spectral radius, splitting graph, ISI energy, shadow graph, eigenvalues**

## 1 Introduction

Since the proof of Kirchhoff's renowned matrix-tree theorem in 1876, the relationship between graph eigenvalues and graph structure has been well established. It has been a quest whether the structure of a graph can be described by the eigenvalues of some matrix associated to the graph. In network sciences, many real-world problems can be described in terms of a graph or a network. In most of these problems, eigenvalues play a significant role. One of the best-known applications is the epidemic control model, where nodes indicate either healthy peers who are vulnerable to infection or diseased peers [1]. The detection of such nodes can be done using the spectrum of the graph. It has been proven that there exists a logarithmic growth relationship between the average distance and the overall number of nodes, [2]. The connection between Laplacian energy and network coherence was studied by Liu et al. [3]. Fractional derivative of the Gabor—Morlet wavelet is computed by Guariglia et al. in [4]. X. Zheng et al. proposed a new framework of adaptive multiscale graph wavelet decomposition for signals in [5]. Guariglia et al. in [6] analyzed Chebyshev wavelets properties by computing their Fourier transform. Some properties relating to operators which approximate a signal at a given resolution has been given in [7]. Some aspects of fractional calculus of zeta functions along with application of Shannon entropy has been discussed in [8]. The dimer problem and Huckel's theory are two examples of usage of graph spectra in statistical physics and chemistry, respectively [9]. Applications in physics and chemistry provided inspiration for the theory of graph spectra to be developed. In physics, treating the membrane vibration problem by approximatively solving the related partial differential equation results in examination of the eigenvalues of a graph that is a discrete model of the membrane [10]. The topic of membrane vibration served as the inspiration for the first mathematical publication on graph spectra [11]. The Huckel molecular orbital theory, which describes unsaturated conjugated hydrocarbons, uses graph spectra as one of its primary tools in chemistry. Several statistical physics issues contain the spectra of specific matrices that are strongly related to adjacency matrices [12–14]. The process of counting 1-factors takes into account the eigenvalues and walks in the associated graphs [10]. The problem of counting the number of 1-factors in a graph becomes the dimer problem in physics. The enumeration of 1-factors can be used to solve a variety of issues in physics, not only the dimer problem. The famous Ising problem that emerged from the idea of ferromagnetism is the most well-known [12, 13]. Physicists are interested in the graph-walk problem for reasons other than the 1-factor enumeration problem. The random-walk and self-avoiding-walk problems are two such examples [12, 13]. The eigenvalues can also be used to calculate the independence number, chromatic number, partitioning, ranking, and epidemic spreading in networks and clustering [15]. The second largest eigenvalue of a regular graph can be used in coding theory to represent the minimum Hamming distance of a linear code [16, 17]. According to Shannon information theory, the eigenvalues of the channel graph can be used to represent the channel capacity, which is the maximum amount of information that can be communicated over a channel or stored in a storage medium [17, 18]. For a given code, an encoder or decoder is constructed based on the spectral radius of the channel graph. A graph is used in quantum chemistry to represent the skeleton of an unsaturated hydrocarbon. In such molecules, the eigenvalues of molecular graph correspond to electron energy levels. A close relationship exists between the spectrum of the graph and the stability of the molecules [19]. The idea of using the spectral radius of the graph $G$ as a gauge of branching was first put forth by Cvetkovic and Gutman in 1977 [20]. After this, spectral radii have been discussed extensively for different purposes [20–23].

In this article, we only restrict to a regular graph $G$ without isolated vertices referred as base graph. *ISI* matrix was established by Zangi et al. having entries $\frac{d_i d_j}{d_i + d_j}$ when $i \sim j$, all entries are 0 elsewhere, [24]. It has been established that *ISI* index would be a good indicator of the total surface area of the octane isomers. A number of topological indices are used to define the energy of a graph, many of which are useful in chemistry. This article relates the *ISI* energies and *ISI* spectral radii of larger graphs with *ISI* energies and *ISI* spectral radii of the base graph. Gutman et al. first proposed the idea of $\varepsilon(G)$ in 1978 [25]. This idea is currently gaining a lot of attention due to its usage and applications in different areas. The symbol $A(G)$ represents the adjacency matrix of the graph $G$, whose all entries are 1 when vertices are adjacent and 0 when vertices are not adjacent. There are several uses for the greatest eigenvalue of the $A(G)$ matrix in algebraic graph theory, which is also known as the graph's spectral radius and is indicated by the symbol $\wp(G)$. Billal et al. established closed relations among different versions of energies and spectral radii of splitting and shadow graphs with energies and spectral radii of the base graph, [26–28]. The following [29–31] provides information and sources related to spectral radii. It is possible to obtain *ISI* Spectral radius and *ISI* energy using Nordhaus-Gaddum-type results, [32]. Some lower bounds for the adjacency spectral radius and the Laplacian spectral radius in terms of the degrees and the 2-degrees of vertices are presented by Yu et al. in [33]. Zhou et al. in [34] provided lower and upper bounds for the distance energy and spectral radius of bipartite graphs. Matrix analysis has been studied in relation to graph energies by Gatmacher [35]. Meenakshi et al. discussed several energies connected to a graph and the bounds of various matrices energies connected to a graph in a survey [36].

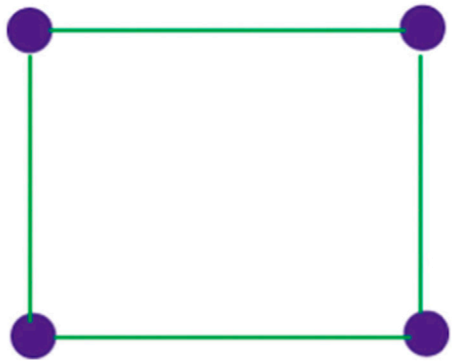

FIGURE 1
A base graph $C_4$.

One-splitting and two-shadow graphs of simple connected graph were constructed by Samir et al. [37], and it was demonstrated that the adjacency energies of these newly created graphs are constant multiples of the energies of the original graph. Later, Samir et al. [38] developed these ideas and came up results relating to adjacency energies. Liu et al. investigated distance and adjacency energies of multilevel wheel networks in [39]. In [40] Chu et al. established the signless Laplacian and Laplacian energies as well as their spectra using multilevel wheels. Gutman et al. discussed graph energy and its applications that provided details on more than a hundred different varieties of graph energies and their applications in diverse areas, [41]. We refer to [42] for additional information and fundamental concepts on graph energies. Various applications of graph energies can be traced down in [43–45]. There are crystallographic uses for various graph energies [46, 47], the theory of macromolecules [48, 49], protein sequencing [50–52], biology [53], challenges related to air travel [54], and construction of spacecraft [55].

Present article focuses on the *ISI* spectral radii and energy of the generalized splitting and shadow graphs. To be more precise, we link the spectral radii and energies of the new graph to those of the base graphs. The *ISI* spectral radius and energy of the $p$-splitting graph are determined in Section 2. We obtain further results relating to shadow graphs Section 3.

## 2 Preliminaries

We outline the main ideas and background data related to our main findings in this section. For additional details and sources relevant to this section, see [24]. According to [24], the inverse sum indeg matrix $[ISI(G)]$ for the graph $G$ has entries $k_{ij}$.

$$k_{ij} = \begin{cases} \dfrac{d_i d_j}{d_i + d_j}, & when \quad v_i \sim v_j, \\ 0, & otherwise. \end{cases}$$

The degrees of the vertices $v_i$ and $v_i$ are $d_i$ and $d_j$, respectively. The distinct eigenvalues of the Inverse Sum Indeg *ISI* matrix of the

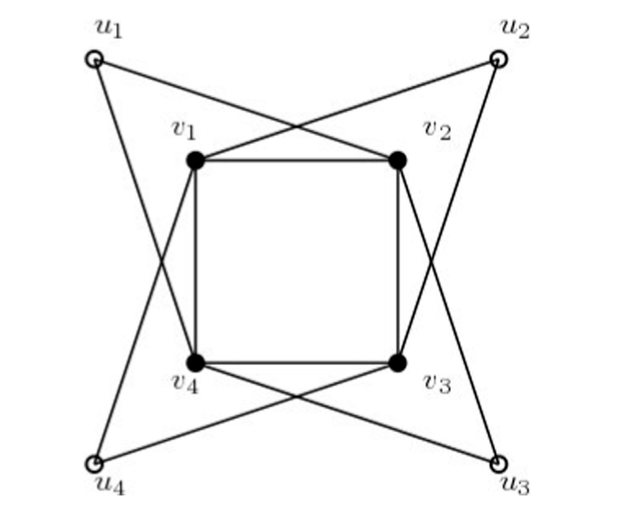


FIGURE 2
$Spl_1(C_4)$, the 1-splitting graph of $C_4$.

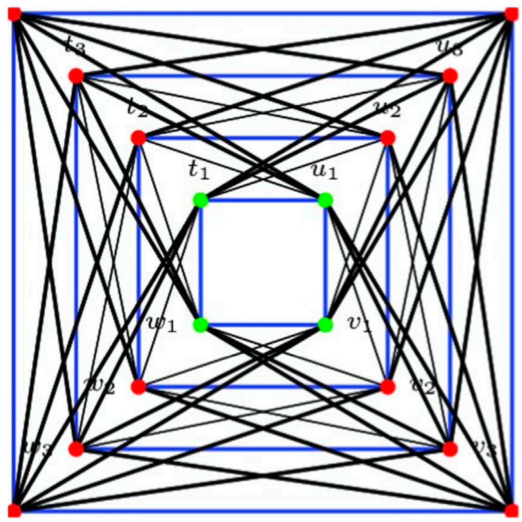

FIGURE 3
$Sh_4(C_4)$, the 4-shadow graph of $C_4$.

graph $G$ are $\zeta_1, \zeta_2, \cdots \zeta_n$. If different *ISI* eigenvalues of the graph $G$ have multiplicities of $m_1, m_2, \cdots m_n$, respectively, then

$$specISI = \begin{pmatrix} \zeta_1 & \zeta_2 \cdots & \zeta_n \\ m_1 & m_2 \cdots & m_n \end{pmatrix}. \tag{2.1}$$

Then *ISI* energy is defined as

$$ISI\varepsilon(G) = \sum_{i=1}^{n} |\zeta_i|.$$

The spectral radius for *ISI* matrix is

$$\wp ISI(G) = \max_{i=1}^{n} |\zeta_i|,$$

where the eigenvalues of the *ISI* matrix are $\zeta_1, \zeta_2, \ldots, \zeta_n$. It is worth mentioning that these two invariants are quite different as the *ISI* energy is the sum whereas the *ISI* spetral radius is the largest value. Our findings are fundamentally dependent on the following definitions. In order to generate the $p$-splitting graph $Spl_p(G)$ of the graph $G$, new $p$ vertices are added to each vertex $v$ of the graph $G$, ensuring that each of the new vertices is also connected to each vertices that is adjacent to $v$ in $G$, [56]. Base graph $C_4$ is given in Figure 1 and 1-splitting graph of $C_4$ is given in Figure 2.

A fresh $p$ copy of the graph $G$ is first considered when creating the $p$-shadow graph $Sh_p(G)$ of the graph. The neighbors of the corresponding vertex $V$ in $G_j$ are then connected to each vertex $U$ in $G_i$. The 4-shadow graph of $C_4$ is given in Figure 3.

Let $A \epsilon R^{m\times n}$, $B \epsilon R^{p\times q}$. Then $A \otimes B$ [38], is given by

$$A \otimes B = \begin{pmatrix} a_{11}B & . & . & .a_{1n}B \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{m1}B & . & . & .a_{mn}B \end{pmatrix}.$$

The following proposition will be frequently used to prove the main results. In fact, it relates the eigenvalues of the tensor product of two matrices with the eigenvalues of these matrices.

**Proposition 1.1.** [57] Assuming that $\alpha$ is an eigenvalue of $A$ and $\beta$ is an eigenvalue of $B$. Then an eigenvalue of $A \otimes B$ is $\alpha\beta$.

## 3 *ISI* energies and *ISI* spectral radii of *p* splitting graph of *G*

The *ISI* energies and *ISI* spectral radii of the $p$-splitting graph of $G$ and the *ISI* energies and *ISI* spectral radii of the base graph are compared in this section. We wish to reiterate that $G$ is any regular graph.

**Theorem 1.** *The relation between the ISI energy of the base graph G and the ISI energy of the p-splitting graph of G is*

$$ISI\varepsilon\big(Spl_p(G)\big) = \frac{p+1\sqrt{p^2+20p+4}}{p+2} ISI\varepsilon(G).$$

Proof

$$ISI\left(Spl_p(G)\right) = \left[\begin{array}{cccc|cccc|c|cccc}
0 & k_{11} & \dots & k_{1n} & 0 & t_{11} & \dots & t_{1n} & \dots & 0 & t_{12} & \dots & t_{1n} \\
k_{21} & 0 & \dots & k_{2n} & t_{21} & 0 & \dots & t_{2n} & \dots & t_{21} & 0 & \dots & t_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
k_{n1} & k_{n2} & \dots & 0 & t_{n1} & t_{n2} & \dots & 0 & \dots & t_{n1} & t_{n2} & \dots & 0 \\
\hline
0 & t_{12} & \dots & t_{1n} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
t_{21} & 0 & \dots & t_{2n} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
t_{n1} & t_{n2} & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & t_{q2} & \dots & t_{qn} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
t_{q1} & 0 & \dots & t_{qn} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\
t_{q1} & t_{q2} & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0
\end{array}\right]$$

$ISI(Spl_p(G))$ matrix may be written as follows

$$ISI\left(Spl_p(G)\right) = \begin{bmatrix} \aleph_1 & \aleph_2 & \dots & \aleph_2 \\ \aleph_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \aleph_2 & 0 & \cdots & 0 \end{bmatrix}.$$

Matrix $\aleph_1$ is given by $\aleph_1 = (p+1)ISI(G)$. Matrix $\aleph_2$ is given by $\aleph_2 = \left(\frac{2(p+1)}{p+2}\right)ISI(G)$.

$$ISI\left(Spl_p(G)\right) = \begin{bmatrix} (p+1)ISI(G) & \frac{2(p+1)}{p+2}ISI(G) & \dots & \frac{2(p+1)}{p+2}ISI(G) \\ \frac{2(p+1)}{p+2}ISI(G) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2(p+1)}{p+2}ISI(G) & 0 & \cdots & 0 \end{bmatrix}_{p+1}$$
$$= ISI(G) \otimes \begin{bmatrix} (p+1) & \frac{2(p+1)}{p+2} & \dots & \frac{2(p+1)}{p+2} \\ \frac{2(p+1)}{p+2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2(p+1)}{p+2} & 0 & \cdots & 0 \end{bmatrix}_{p+1}$$

Let $[A] = c_{ij}$ having entries

$$c_{ij} = \begin{bmatrix} (p+1) & \frac{2(p+1)}{p+2} & \dots & \frac{2(p+1)}{p+2} \\ \frac{2(p+1)}{p+2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2(p+1)}{p+2} & 0 & \cdots & 0 \end{bmatrix}_{p+1}$$

We are looking for $ISI\varepsilon(Spl_p(G))$, therefore obtaining all eigenvalues of $[A]$ is necessary. The eigenvalues of $[A]$ are being calculated right now. Due to its rank, $[A]$ has only two non-zero eigenvalues. The eigenvalues of $[A]$ are represented by the symbols $\alpha_1$ and $\alpha_1$. Clearly, then, we have

$$\alpha_1 + \alpha_2 = tr(A) = p + 1. \tag{3.1}$$

Trace of the matrix $A$ is denoted by $tr(A)$. Consider $[A^2] = d_{ij}$ having entries

$$d_{ij} = \begin{bmatrix} (p+1)^2 + p\frac{4(p+1)^2}{(p+2)^2} & \frac{2(p+1)^2}{p+2} & \dots & \frac{2(p+1)^2}{p+2} \\ \frac{2(p+1)^2}{p+2} & \frac{4(p+1)^2}{(p+2)^2} & \dots & \frac{4(p+1)^2}{(p+2)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2(p+1)^2}{p+2} & \frac{4(p+1)^2}{(p+2)^2} & \dots & \frac{4(p+1)^2}{(p+2)^2} \end{bmatrix}_{p+1}$$

Then

$$\alpha_1^2 + \alpha_2^2 = tr(A^2) = (p+1)^2 + (2p)\frac{4(p+1)^2}{(p+2)^2}. \tag{3.2}$$

Eqs 3.1, 3.2 when solved yield the following results

$$\alpha_1 = \frac{p+1}{2(p+2)}\left(p + 2 + \sqrt{p^2 + 20p + 4}\right), \tag{3.3}$$

and

$$\alpha_2 = \frac{p+1}{2(p+2)}\left(p + 2 - \sqrt{p^2 + 20p + 4}\right). \tag{3.4}$$

The notation $Ch(A)$ stands for the characteristic equation of $[A]$. Finally, we may find $Ch(A)$, which is denoted by $Ch(A) = \alpha^{p-1}(\alpha - \frac{p+1}{2(p+2)}(p+2+\sqrt{p^2+20p+4}))(\alpha - \frac{p+1}{2(p+2)}(p+2-\sqrt{p^2+20p+4})) = 0$.

Consequently, we reach at the following spectrum,

$$specA = \begin{pmatrix} 0 & \frac{p+1}{2(p+2)}\left(p+2+\sqrt{p^2+20p+4}\right) & \frac{p+1}{2(p+2)}\left(p+2-\sqrt{p^2+20p+4}\right) \\ p-1 & 1 & 1 \end{pmatrix}. \tag{3.5}$$

In light of the fact that $ISI(Spl_p(G)) = ISI(G) \otimes A$. By applying Proposition 1.1, we get

$$\begin{aligned} &ISI\varepsilon\left(Spl_p(G)\right) \\ &= \sum_{i=1}^{n} \left|\frac{p+1\left(p+2 \pm \sqrt{p^2+20p+4}\right)}{2(p+2)}\zeta_i\right| \\ &= \sum_{i=1}^{n} |\zeta_i| \left[\frac{p+1\left(p+2+\sqrt{p^2+20p+4}\right)}{2(p+2)} + \frac{p+1\left(\sqrt{p^2+20p+4}-(p+2)\right)}{2(p+2)}\right] \\ &= \frac{p+1\sqrt{p^2+20p+4}}{p+2} ISI\varepsilon(G). \end{aligned}$$

**Proposition 3.1.** *ISI energy of p-splitting graph of $C_s$ is*

$$ISI\varepsilon\left(Spl_p(C_s)\right) = \begin{Bmatrix} 4\cot\frac{\pi}{s}, & if \quad s \equiv 0\,(mod 4), \\ 4\csc\frac{\pi}{s}, & if \quad s \equiv 2\,(mod 4), \\ 2\csc\frac{\pi}{2s}, & if \quad s \equiv 1\,(mod 2). \end{Bmatrix} \frac{p+1\sqrt{p^2+20p+4}}{p+2}.$$

Proof

$$ISI(C_s) = \begin{bmatrix} 0 & \frac{d_1 d_2}{d_1+d_2} & 0 & 0 & \cdots & 0 & \frac{d_1 d_s}{d_1+d_s} \\ \frac{d_2 d_1}{d_2+d_1} & 0 & \frac{d_2 d_3}{d_2+d_3} & 0 & \cdots & 0 & 0 \\ 0 & \frac{d_3 d_2}{d_3+d_2} & 0 & \frac{d_3 d_4}{d_3+d_4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{d_s d_1}{d_s+d_1} & 0 & 0 & 0 & \cdots & \frac{d_s d_{s-1}}{d_s+d_{s-1}} & 0 \end{bmatrix}.$$

Since each vertex in $C_s$ has degree 2 because $C_s$ is a 2 regular graph, we get

$$ISI(C_s) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

*ISI* eigenvalues of $C_s$ are easily observed as follows

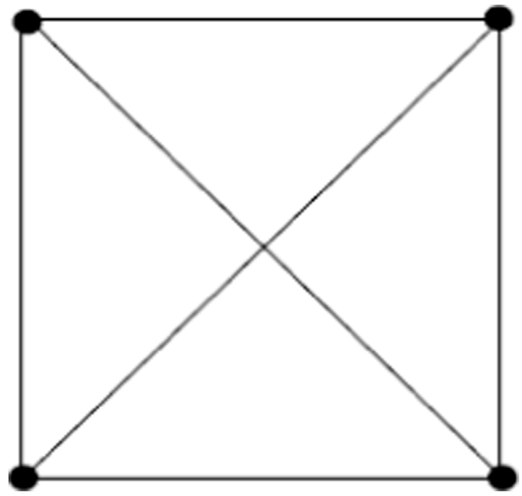

FIGURE 4
A base graph $K_4$.

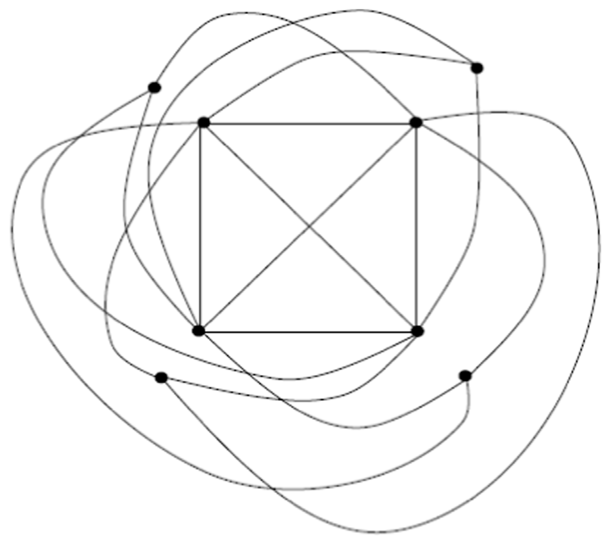

FIGURE 5
$Spl_1(K_4)$, the 1-splitting graph of $K_4$.

$$s_z = 2\cos\frac{2\pi z}{s}, \quad z = 0, 1, 2, \ldots, s-1. \tag{3.6}$$

Then utilizing Theorem 6 of [58], we have

$$ISI\varepsilon(C_s) = \left\{\begin{array}{l} 4\cot\frac{\pi}{s},\ if\ \ s \equiv 0\,(mod4), \\ 4\csc\frac{\pi}{s},\ if\ \ s \equiv 2\,(mod4), \\ 2\csc\frac{\pi}{2s},\ if\ \ s \equiv 1\,(mod2). \end{array}\right\}. \tag{3.7}$$

Theorem 1 can be used to get the desired result since cycle graph is a regular graph.

Base graph $K_4$ is given in Figure 4 and 1-splitting graph of $K_4$ is given in Figure 5.

**Proposition 3.2.** *ISI energy of p-splitting graph of $K_s$ is* $ISI\varepsilon(Spl_p(K_s)) = ((s-1)^2)\frac{p+1\sqrt{p^2+20p+4}}{p+2}$.

Proof

$$ISI(K_s) = \begin{bmatrix} 0 & \frac{d_1d_2}{d_1+d_2} & \frac{d_1d_3}{d_1+d_3} & \cdots & \frac{d_1d_s}{d_1+d_s} \\ \frac{d_2d_1}{d_2+d_1} & 0 & \frac{d_2d_3}{d_2+d_3} & \cdots & \frac{d_2d_s}{d_2+d_s} \\ \frac{d_3d_1}{d_3+d_1} & \frac{d_3d_2}{d_3+d_2} & 0 & \cdots & \frac{d_3d_s}{d_3+d_s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{d_sd_1}{d_s+d_1} & \frac{d_sd_2}{d_s+d_2} & \frac{d_sd_3}{d_s+d_3} & \cdots & 0 \end{bmatrix}.$$

Since each vertex in $K_s$ has a degree of $s-1$ because $K_s$ is a $s-1$ regular graph, we get

$$ISI(K_s) = \begin{bmatrix} 0 & \frac{s-1}{2} & \frac{s-1}{2} & \cdots & \frac{s-1}{2} \\ \frac{s-1}{2} & 0 & \frac{s-1}{2} & \cdots & \frac{s-1}{2} \\ \frac{s-1}{2} & \frac{s-1}{2} & 0 & \cdots & \frac{s-1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{s-1}{2} & \frac{s-1}{2} & \frac{s-1}{2} & \cdots & 0 \end{bmatrix}.$$

$$ISIspec(K_s) = \begin{pmatrix} \frac{(s-1)^2}{2} & \frac{-(s-1)}{2} \\ 1 & s-1 \end{pmatrix}. \tag{3.8}$$

$ISI\varepsilon(K_s) = |\frac{(s-1)^2}{2}| + |\frac{-(s-1)^2}{2}|.ISI\varepsilon(K_s) = \frac{(s-1)^2}{2} + \frac{(s-1)^2}{2}$.

Finally, we have

$$ISI\varepsilon(K_s) = (s-1)^2. \tag{3.9}$$

Theorem 1 can be used to get the desired result since complete graph is a regular graph.

Base graph $K_{3,3}$ is given in Figure 6 and 1-splitting graph of $K_{3,3}$ is given in Figure 7.

**Proposition 3.3.** *ISI energy of p-splitting graph of $K_{s,s}$ is* $ISI\varepsilon(Spl_p(K_{s,s})) = \frac{p+1\sqrt{p^2+20p+4}}{p+2}(s^2)$.

*Proof.* $$ISI(K_{s,s}) = \left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & \frac{d_1d_{1'}}{d_1+d_{1'}} & \frac{d_1d_{2'}}{d_1+d_{2'}} & \cdots & \frac{d_1d_{s'}}{d_1+d_{s'}} \\ 0 & 0 & \cdots & 0 & \frac{d_2d_{1'}}{d_2+d_{1'}} & \frac{d_2d_{2'}}{d_2+d_{2'}} & \cdots & \frac{d_2d_{s'}}{d_2+d_{s'}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{d_sd_{1'}}{d_s+d_{1'}} & \frac{d_sd_{2'}}{d_s+d_{2'}} & \cdots & \frac{d_sd_{s'}}{d_s+d_{s'}} \\ \hline \frac{d_{1'}d_1}{d_{1'}+d_1} & \frac{d_{1'}d_2}{d_{1'}+d_2} & \cdots & \frac{d_{1'}d_s}{d_{1'}+d_s} & 0 & 0 & \cdots & 0 \\ \frac{d_{2'}d_1}{d_{2'}+d_1} & \frac{d_{2'}d_2}{d_{2'}+d_2} & \cdots & \frac{d_{2'}d_s}{d_{2'}+d_s} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{d_{s'}d_1}{d_{s'}+d_1} & \frac{d_{s'}d_2}{d_{s'}+d_2} & \cdots & \frac{d_{s'}d_s}{d_{s'}+d_s} & 0 & 0 & \cdots & 0 \end{array}\right].$$

Since each vertex in $K_{s,s}$ has a degree of $s$ because $K_{s,s}$ is a $s$-regular graph, we get

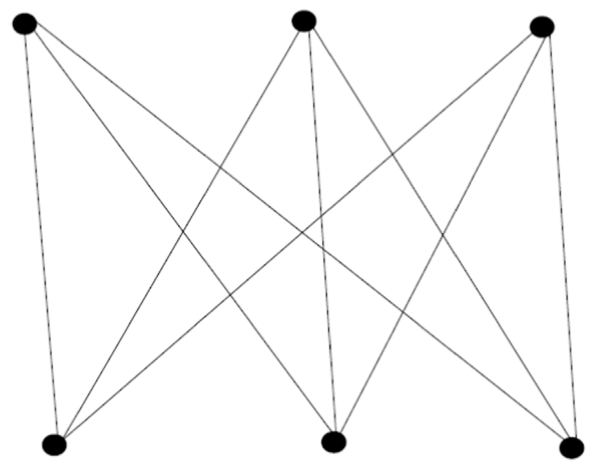

FIGURE 6
A base graph $K_{3,3}$.

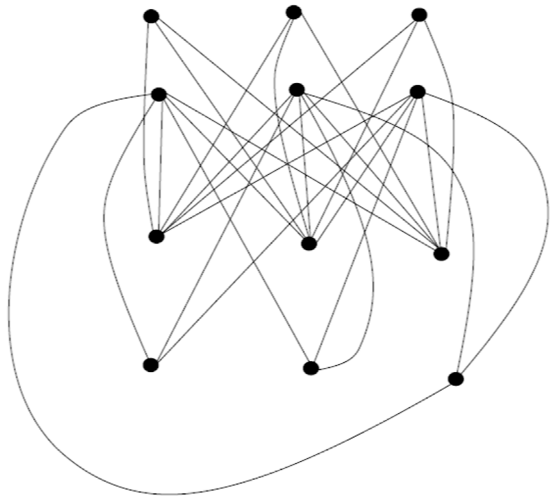

FIGURE 7
$Spl_1(K_{3,3})$, the 1-splitting graph of $K_{3,3}$.

$$ISI(K_{s,s}) = \left[\begin{array}{cccc|cccc} 0 & 0 & \dots & 0 & \frac{s}{2} & \frac{s}{2} & \dots & \frac{s}{2} \\ 0 & 0 & \dots & 0 & \frac{s}{2} & \frac{s}{2} & \dots & \frac{s}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \frac{s}{2} & \frac{s}{2} & \dots & \frac{s}{2} \\ \hline \frac{s}{2} & \frac{s}{2} & \dots & \frac{s}{2} & 0 & 0 & \dots & 0 \\ \frac{s}{2} & \frac{s}{2} & \dots & \frac{s}{2} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{s}{2} & \frac{s}{2} & \dots & \frac{s}{2} & 0 & 0 & \dots & 0 \end{array}\right].$$

$$ISIspec(K_{s,s}) = \begin{pmatrix} \frac{(s)^3}{2s} & 0 & \frac{-(s)^3}{2s} \\ 1 & 2s-2 & 1 \end{pmatrix}. \tag{3.10}$$

$ISI\varepsilon(K_s) = |\frac{(s)^3}{2s}| + |\frac{-(s)^3}{2s}|$.
$ISI\varepsilon(K_s) = \frac{(s)^3}{2s} + \frac{(s)^3}{2s}$.
Finally, we have

$$ISI\varepsilon(K_{s,s}) = (s)^2. \tag{3.11}$$

Theorem 1 can be used to get the desired result since complete graph is a regular graph.

**Theorem 2.** *The relation between the ISI spectral radius of the base graph G and the ISI spectral radius of the p-splitting graph of G is* $\wp ISI(Spl_p(G)) = \wp ISI(G)(\frac{p+1(p+2+\sqrt{p^2+20p+4})}{2(p+2)})$.
Proof Using the same justifications as Formula 3.5 in Theorem 1,

$$specA = \begin{pmatrix} 0 & \frac{p+1}{2(p+2)}\left(p+2+\sqrt{p^2+20p+4}\right) & \frac{p+1}{2(p+2)}\left(p+2-\sqrt{p^2+20p+4}\right) \\ p-1 & 1 & 1 \end{pmatrix}.$$

In light of the fact that $ISI(Spl_p(G)) = ISI(G) \otimes A$. By applying Proposition 1.1, we get

$$\begin{aligned} \wp ISI\left(Spl_p(G)\right) &= max_{i=1}^{n} \left|(specA))\zeta_i\right| \\ &= max_{i=1}^{n} |\zeta_i| \left[\frac{p+1\left(p+2 \pm \sqrt{p^2+20p+4}\right)}{2(p+2)}\right] \\ &= \wp ISI(G)\left(\frac{p+1\left(p+2+\sqrt{p^2+20p+4}\right)}{2(p+2)}\right). \end{aligned}$$

**Proposition 3.4.** *ISI spectral radius of p-splitting graph of $C_s$ is* $\wp ISI(Spl_p(C_s)) = \frac{p+1(p+2+\sqrt{p^2+20p+4})}{2(p+2)}(2)$.
Proof Using Eq. 3.6, all eigenvalues of $C_s$ are

$$s_z = 2\cos\frac{2\pi z}{s}, \quad z = 0, 1, 2, \dots, s-1.$$

The largest absolute eigenvalue of $C_s$ is 2. So, we arrive at

$$\wp ISI(C_s) = 2. \tag{3.12}$$

Theorem 2 can be used to get the desired result since cycle graph is a regular graph.

The following result give the *ISI* spectral radius of *p*-splitting graph of $K_s$.

**Proposition 3.5.** *ISI spectral radius of p-splitting graph of $K_s$ is* $\wp ISI(Spl_p(K_s)) = \frac{p+1(p+2+\sqrt{p^2+20p+4})}{2(p+2)}\left(\frac{(s-1)^2}{2}\right)$.
Proof Using the same justifications as Formula 3.8 in Proposition 3.2,

$$ISIspec(K_s) = \begin{pmatrix} \frac{(s-1)^2}{2} & \frac{-(s-1)}{2} \\ 1 & s-1 \end{pmatrix}.$$

The largest absolute eigenvalue of $K_s$ is $\frac{(s-1)^2}{2}$. So, we arrive at

$$\wp ISI(K_s) = \frac{(s-1)^2}{2}. \tag{3.13}$$

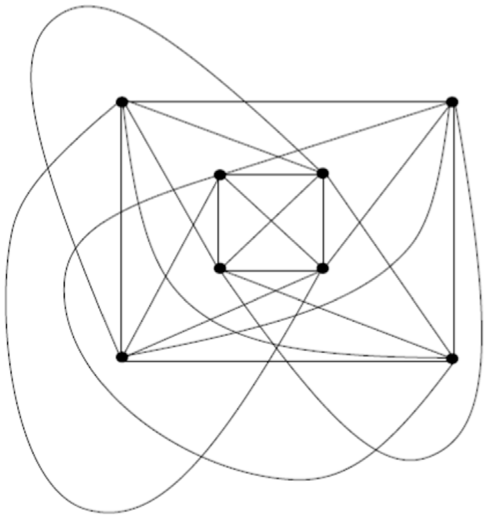

FIGURE 8
$Sh_2(K_4)$, the 2-shadow graph of $K_4$.

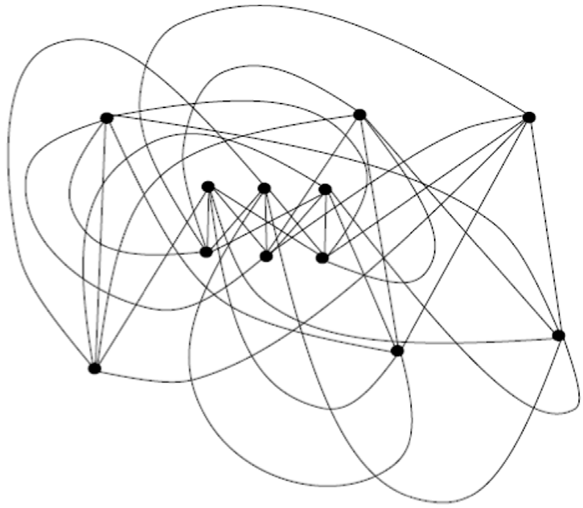

FIGURE 9
$Sh_2(K_{3,3})$, the 2-shadow graph of $K_{3,3}$.

Theorem 2 can be used to get the desired result since complete graph is a regular graph.

**Proposition 3.6.** *ISI spectral radius of p-splitting graph of $K_{s,s}$ is* $\wp ISI(Spl_p(K_{s,s})) = \frac{p+1(p+2+\sqrt{p^2+20p+4})}{2(p+2)}\left(\frac{s^2}{2}\right)$.

Proof Using the same justifications as Formula 3.10 in Proposition 3.3,

$$ISIspec(K_{s,s}) = \begin{pmatrix} \frac{(s)^3}{2s} & 0 & \frac{-(s)^3}{2s} \\ 1 & 2s-2 & 1 \end{pmatrix}.$$

The largest absolute eigenvalue of $K_{s,s}$ is $\frac{(s)^3}{2s}$. So, we arrive at

$$\wp ISI(K_{s,s}) = \frac{s^2}{2}. \tag{3.14}$$

Theorem 2 can be used to get the desired result since complete bipartite graph is a regular graph.

## 4 *ISI* energies and *ISI* spectral radii of *p*-shadow graph of *G*

The *ISI* energies and *ISI* spectral radii of the *p*-shadow graph of *G* and the *ISI* energies and *ISI* spectral radii of the base graph are compared in this section. We wish to reiterate that *G* is any regular graph.

**Theorem 3.** *The relation between the ISI energy of the base graph G and the ISI energy of the p-shadow graph of G is*

$$ISI\varepsilon\big(Sh_p(G)\big) = p^2 ISI\varepsilon(G).$$

Proof You may write $ISI(Sh_p(G))$ matrix as follows

$$ISI\big(Sh_p(G)\big) = \begin{bmatrix} \aleph_3 & \aleph_3 & \dots & \aleph_3 \\ \aleph_3 & \aleph_3 & \dots & \aleph_3 \\ \vdots & \vdots & \ddots & \vdots \\ \aleph_3 & \aleph_3 & \cdots & \aleph_3 \end{bmatrix}.$$

Matrix $\aleph_3$ is given by $\aleph_3 = (p)ISI(G)$.

$$ISI\big(Sh_p(G)\big) = \begin{bmatrix} (p)ISI(G) & (p)ISI(G) & \dots & (p)ISI(G) \\ (p)ISI(G) & (p)ISI(G) & \dots & (p)ISI(G) \\ \vdots & \vdots & \ddots & \vdots \\ (p)ISI(G) & (p)ISI(G) & \cdots & (p)ISI(G) \end{bmatrix}_p$$
$$= ISI(G) \otimes \begin{bmatrix} p & p & \dots & p \\ p & p & \dots & p \\ \vdots & \vdots & \ddots & \vdots \\ p & p & \cdots & p \end{bmatrix}_p$$

Let $[M] = m_{ij}$ having entries

$$m_{ij} = \begin{bmatrix} p & p & \dots & p \\ p & p & \dots & p \\ \vdots & \vdots & \ddots & \vdots \\ p & p & \cdots & p \end{bmatrix}_p$$

We are looking for $ISI\varepsilon(Sh_p(G))$, therefore obtaining all eigenvalues of $M$ is necessary. The eigenvalues of $[M]$ are being calculated right now. Due to its rank, $[M]$ has only one non-zero eigenvalue. The notation $Ch(M)$ stands for characteristic equation of $[M]$. Finally, we may find $Ch(M)$, which is denoted by$Ch(M) = \alpha^{p-1}(\alpha - p^2) = 0$.

Consequently, we reach at the following spectrum,

$$specM = \begin{pmatrix} 0 & p^2 \\ p-1 & 1 \end{pmatrix}. \tag{4.1}$$

In light of the fact that $ISI(Sh_p(G)) = ISI(G)\otimes M$. By applying Proposition 1.1, we get

$$\begin{aligned} ISI\varepsilon\left(Sh_p(G)\right) &= \sum_{i=1}^{n} |p^2\zeta_i| \\ &= \sum_{i=1}^{n} |\zeta_i|\left[p^2\right] \\ &= p^2 ISI\varepsilon(G). \end{aligned}$$

**Proposition 4.1.** *ISI energy of p-shadow graph of $C_s$ is* $ISI\varepsilon(Sh_p(C_s))$

$$= p^2 \left\{ \begin{array}{ll} 4\cot\frac{\pi}{s}, & if \quad s \equiv 0\,(mod 4), \\ 4\csc\frac{\pi}{s}, & if \quad s \equiv 2\,(mod 4), \\ 2\csc\frac{\pi}{2s}, & if \quad s \equiv 1\,(mod 2). \end{array} \right\}.$$

Proof Using the same justifications as Formula 3.7 in Proposition 3.1, we have $ISI\varepsilon(C_s) = \left\{ \begin{array}{ll} 4\cot\frac{\pi}{s}, & if \quad s \equiv 0\,(mod 4), \\ 4\csc\frac{\pi}{s}, & if \quad s \equiv 2\,(mod 4), \\ 2\csc\frac{\pi}{2s}, & if \quad s \equiv 1\,(mod 2). \end{array} \right\}$. Theorem 3 can be used to get the desired result since cycle graph is a regular graph.

2-shadow graph of $K_4$ is given in Figure 8.

**Proposition 4.2.** *ISI energy of p-shadow graph of $K_s$ is*

$$ISI\varepsilon\left(Sh_p(K_s)\right) = p^2 s^2 + p^2 - 2p^2 s.$$

**Proof.** Using the same justifications as Formula 3.9 in Proposition 3.2, we have

$$ISI\varepsilon(K_s) = s^2 + 1 - 2s.$$

Theorem 3 can be used to get the desired result since complete graph is a regular graph.

2-shadow graph of $K_{3,3}$ is given in Figure 9.

**Proposition 4.3.** *ISI energy of p-shadow graph of $K_{s,s}$ is*

$$ISI\varepsilon\left(Sh_p(K_{s,s})\right) = p^2 s^2.$$

Proof Using the same justifications as Formula 3.11 in Proposition 3.3, we have $ISI\varepsilon(K_{s,s}) = s^2$.

Theorem 3 can be used to get the desired result since complete bipartite graph is a regular graph.

**Theorem 4.** *The relation between the ISI spectral radius of the base graph G and the ISI spectral radius of the p-shadow graph of G is*

$$\wp ISI\left(Sh_p(G)\right) = \wp ISI(G)\left(p^2\right).$$

Proof Using the same justifications as Formula 4.1 in Theorem 3,

$$specM = \begin{pmatrix} 0 & p^2 \\ p-1 & 1 \end{pmatrix}.$$

In light of the fact that $ISI(Sh_p(G)) = ISI(G) \otimes M$. By applying Proposition 1.1, we get

$$\begin{aligned} \wp ISI\left(Sh_p(G)\right) &= max_{i=1}^{n}\left|(specA)\right)\zeta_i| \\ &= max_{i=1}^{n}|\zeta_i|\left[p^2\right] \\ &= \wp ISI(G)\left(p^2\right). \end{aligned}$$

**Proposition 4.4.** *ISI spectral radius of p-shadow graph of $C_s$ is*

$$\wp ISI\left(Sh_p(C_s)\right) = 2p^2.$$

Proof Utilizing Eq. 3.12 of Proposition 3.4, we have

$$ISI\varepsilon(c_s) = 2.$$

Theorem 4 can be used to get the desired result since cycle graph is a regular graph.

**Proposition 4.5.** *ISI spectral radius of p-shadow graph of $K_s$ is*

$$\wp ISI\left(Sh_p(K_s)\right) = \frac{p^2 s^2 + p^2 - 2p^2 s}{2}.$$

Proof Utilizing Eq. 3.13 of Proposition 3.5, we have

$$\wp ISI(K_s) = \frac{s^2 + 1 - 2s}{2}.$$

Theorem 4 can be used to get the desired result since complete graph is a regular graph.

**Proposition 4.6.** *ISI spectral radius of p-shadow graph of $K_{s,s}$ is*

$$\wp ISI\left(Sh_p(K_{s,s})\right) = \frac{p^2 s^2}{2}.$$

Proof Utilizing Eq. 3.14 of Proposition 3.6, we have

$$\wp ISI(K_{s,s}) = \frac{s^2}{2}.$$

Theorem 4 can be used to get the desired result since complete bipartite graph is a regular graph.

## 5 Conclusion and applications

Most well known theories in spectral graph theory are graph energy and the spectral radius. These thoughts establish a connection between mathematics and chemistry. The literature contains a huge amount of writing on these ideas. Exploring the spectral radii and energies of bigger graphs is a task that we must rise to. By focusing on splitting and shadow graphs, we arrived at the conclusions that the spectral radii and energies of the newly developed graphs are multiples of the spectral radii and energies of the original graphs. As propositions, we derived these particular relations for the basic families of graphs such as cycle, complete and complete bipartite graphs.

Recently it has been observed that physical and chemical properties of anticancer drugs were well correlated with *ISI* energies and spectral radius. Moreover, this work implied that these anticancer drugs may be utilized for further study by pharmacists and chemists in designing new drugs, using the concept of these topological indices. The more correlated drugs may have a better impact on the treatment of cancer. For a better treatment of cancer, a future study may be carried out by interdisciplinary researchers as a joint venture, [59]. *ISI* energy and its variants have diverse, amazing and, to some extent, unanticipated utilizations in crystallography and total surface are of octane isomers [60]. *ISI* energy has some connection with protein sequences [49, 52]. *ISI* energies and

*ISI* spectrum has applications in network analysis and resilience [54, 61, 62]. Similarly other key invariants of graphs like chromatic number can be estimated using *ISI* energy.

## Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

## Author contributions

Article was conceived by MMM, computations have been done by AB and MIQ and drafted by MA.

## Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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