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Quaternionic quantum Turing machines

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Quaternionic quantum theory is an extension of the standard complex quantum theory. Inspired by this, we study the quaternionic quantum computation using quaternions. We first develop a theory of quaternionic quantum Turing machines as a model of quaternionic quantum computation. Quaternionic quantum Turing machines can also be seen as a generalization of the complex quantum Turing machine. Then, we introduce the weighted sum of quaternionic quantum Turing machines and establish some of their basic properties.

KEYWORDS

quantum Turing machine, quantum computation, quaternionic quantum Turing machine, quaternionic quantum computation, quaternionic quantum theory

1 Introduction

In recent years, quantum computation, which integrates computer science with quantum physics, has attracted extensive attention [1]. In 1980, Benioff [2] proved that quantum computing devices are at least as powerful as classical computers. Then, in 1982, Feynman [3] suggested the quantum computer for simulating a quantum mechanical system. Afterward, in 1985, Deutsch [4] defined the quantum Turing machine as a formal model of quantum computation. In 1993, Bernstein and Vazirani [5] introduced the quantum complexity theory. In the same year, Yao [6] introduced the quantum circuit model for simulation of quantum computation. As another theoretical model of quantum computation, the quantum automata theory has been well-studied [7–9]. In 1994, Shor [10] developed the quantum polynomial-time algorithms for factorization and discrete logarithm problems. Shor's algorithm is also applied to solve other types of discrete logarithm problems [11, 12]. In 1996, Grover [13] developed a quantum searching algorithm in a database including n items in time $O(\sqrt{n})$. In 2009, Harrow, Hassidim, and Lloyd [14] proposed a quantum algorithm for solving linear systems of equations.

Due to its wide application potential in many fields, quantum computation has been an important research area. Indeed, the aforementioned quantum computation models and quantum algorithms are based on the standard complex quantum mechanics. It is important and interesting to further study quantum computation based on other versions of quantum mechanics. Quaternionic quantum mechanics, as an extension of the standard complex quantum mechanics, has been considered. In 1936, Birkhoff and Von Neumann [15] suggested the quaternionic quantum theory. They showed that the mathematical model of orthogonal vector subspaces of Hilbert spaces over the quaternions also has properties of the propositional calculus suggested by quantum mechanics. Yang [16] also pointed out the interest of the possibility of using quaternion algebra as the language of quantum mechanics. Kaneno [17] first attempted to introduce the quaternions into quantum mechanics, called quaternionic quantum mechanics (QQM). Reference [18] studied the QQM from a purely logical point of view. They also [19] gave some general features of QQM. Davies and McKellar [20] considered the observability of QQM. Adler [21] proposed a comprehensive treatment of the rules of QQM. Recently, QQM has interested many researchers. For instance,Reference [22] studied the Ramsauer–Townsend effect in QQM. Graydon [23] proposed a quaternionic quantum formalism for the description of quantum dynamics. Giardino [24] proposed the non-anti-Hermitian QQM. He [25] also studied the virial theorem and quantum quaternionic Lorentz force in QQM.

As we know, QQM has existed for a long time. Recently, the computation model based on QQM has aroused the concern of some scholars. References [26, 27] developed the quaternion quantum neural network (QQNN) in the quaternion algebra framework. Bayro–Corrochano [28] also studied quantum computing using geometric algebra, specifically quaternion algebra and rotor algebra. Altamirano–Escobedo and Bayro–Corrochano [29] proposed a quaternionic quantum neural network for classification. Konno [30] extended the QW to a walk determined by a unitary matrix, the component of which is quaternion, and called this model quaternionic quantum walk. Afterward, Konno, Mitsuhashi, and Sato [31] studied the discrete-time quaternionic quantum walk on a graph. Dai [32] extended complex quantum automata to quaternionic quantum automata. When we consider the computation model based on QQM, the Turing machine was an inevitable model of computation. Although the quantum Turing machine has been studied for many years [33–35], it might not be suitable for the case of QQM. The purpose of this paper is to establish a theoretical model of quaternionic quantum computation, called quaternionic quantum Turing machine (QQTM). Actually, to the best of our knowledge, this paper is the first attempt on the study of the QQTM. We hope that the results obtained in the QQTM may offer new insights into quantum computation.

The paper is organized as follows: Section 2 presents some preliminaries that help understand our analysis. Section 3 presents the concept of a QQTM and a multitape QQTM. Section 4 describes the study of the weighted sum of QQTM. Section 5 concludes our research studies.

2 Preliminaries

2.1 Quaternions

The quaternion was first proposed by Hamilton [36]. For more details, the reader is referred to [37].

The quaternion is an extension of real and complex numbers. Let $\mathbb{H}$ be the set of quaternions. Any quaternion $h \in \mathbb{H}$ can be written in the form

$$h = h_0 + h_1 i + h_2 j + h_3 k, \tag{1}$$

where h_s ($s = 0, 1, 2, 3$) are real numbers and i, j, and k are three different imaginary roots of -1, i.e.,

$$i^2 = j^2 = k^2 = ijk = -1, \tag{2}$$

Moreover, they obey

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \tag{3}$$

The real and quaternionic imaginary parts of h are denoted by $Re(h) = h_0$ and $Qim(h) = h_1 i + h_2 j + h_3 k$, respectively.

Given a quaternion $h \in \mathbb{H}$, its "quaternion conjugate" $\bar{h}$ is defined as

$$\bar{h} = h_0 - h_1 i - h_2 j - h_3 k. \tag{4}$$

Its modulus

$$|h| = \sqrt{h\bar{h}} = \sqrt{\bar{h}h} = \sqrt{h_0^2 + h_1^2 + h_2^2 + h_3^2}. \tag{5}$$

For any two quaternions $h, h' \in \mathbb{H}$, we have

$$\overline{hh'} = \overline{h'}\,\bar{h} \tag{6}$$

Quaternion addition is defined as

$$h + h' = (h_0 + h_0') + (h_1 + h_1')i + (h_2 + h_2')j + (h_3 + h_3')k. \tag{7}$$

Quaternion multiplication is defined as

$$\begin{aligned} hh' = &(h_0h_0' - h_1h_1' - h_2h_2' - h_3h_3') \\ &+(h_0h_1' + h_1h_0' + h_2h_3' - h_3h_2')i \\ &+(h_0h_2' + h_2h_0' - h_1h_3' + h_3h_1')j \\ &+(h_0h_3' + h_3h_0' + h_1h_2' - h_2h_1')k. \end{aligned} \tag{8}$$

Quaternion multiplication is non-commutative, i.e.,

$$hh' \neq h'h, \quad \exists h, h', \in \mathbb{H}. \tag{9}$$

Quaternion addition and multiplication are distributive, i.e., $\forall h, h', h'' \in \mathbb{H}$,

$$h(h' + h'') = hh' + hh'', \tag{10}$$

$$(h'' + h')h = h''h + h'h. \tag{11}$$

For any two vectors $h = (v_1, v_2, \ldots, v_n)$, $h' = (u_1, u_2, \ldots, u_n) \in \mathbb{H}^n$, their direct sum is $h \oplus h' = (v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n)$. Their inner product is $h \cdot h' = \Sigma_{r=1}^n v_r u_r$. Their pointwise addition and multiplication are $h + h' = (v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n)$ and $h \cdot h' = (v_1u_1, v_2u_2, \ldots, v_nu_n)$, respectively.

Let $\mathbb{H}^{n\times m}$ be the set of all $n \times m$ quaternionic matrices. For any $U \in \mathbb{H}^{n\times m}$, its adjoint of U is defined as U^*, where $(U^*)_{r,s} = \overline{(U)_{s,r}}$.

2.2 Quaternionic quantum formalism

We give a brief introduction to QQM [17, 21, 22].

The state of a quaternionic quantum system is described by a unit vector of quaternions. The dimension of a quaternionic quantum system is the number of quaternions in the vector. A column vector is written $|h\rangle$, and its quaternion conjugate $|h\rangle^\dagger$ is the row vector $\langle h|$. Similar to quantum information in an ordinary complex field, a quaterbit in quaternion Hilbert space has the general form [38].

$$|\varphi\rangle = h_0|0\rangle + h_1|1\rangle \tag{12}$$

where h_0 and h_0 are two quaternion numbers with $|h_0| + |h_1| = 1$.

As usual, a quaternionic matrix $U \in \mathbb{H}^{n\times n}$ is said to be unitary if $UU^* = I$, Hermitian if $U^* = U$, and positive semi-definite if $\langle h|Uh\rangle \geq, \forall h \in \mathbb{H}^n$. A linear operator from $\mathbb{H}^n$ to $\mathbb{H}^m$ corresponds to a quaternionic matrix $U \in \mathbb{H}^{n\times m}$.

The trace of a quaternionic matrix $U \in \mathbb{H}^{n\times n}$ with respect to a basis $\Theta = \{e_1, e_2, \ldots, e_n\}$ for $\mathbb{H}^n$ is defined by

$$tr(U) = \mathrm{Re}\left(\sum_{r=1}^{n} \langle e_r | U e_r \rangle\right). \tag{13}$$

The norm of U is defined by $|U| = \sqrt{tr(U^2)}$.

2.3 Complex quantum Turing machine

Complex quantum Turing machines (CQTMs) play an important role in the theory of complex quantum computing. We, here, present a formal definition for the CQTM given by Bernstein and Vazirani [5] as follows.

A CQTM is a 7-tuple $QM =< Q, \Gamma, \Sigma, q_0, \delta, B, q_f >$ where

(i) Q is a finite set of control states.
(ii) Γ is a finite set of allowable tape symbols.
(iii) $\Sigma \subseteq \Gamma - \{B\}$ is a finite input alphabet, where $B \in \Gamma$ is the blank.
(iv) $q_0 \in Q$ is an identified initial state.
(v) $q_f \in Q$ is an identified accepting states.
(vi) δ: $Q \times \Gamma \times Q \times \Gamma \times \{R, L\} \to \mathbb{C}$ is a complex transition function satisfying the well-formedness conditions that make the evolution unitary.

3 Quaternionic quantum Turing machine

In this section, we shall introduce the concepts of QQTMs.

Definition 1. A QQTM is a 7-tuple $\mathcal{M} = < Q, \Gamma, \Sigma, S, \delta, B, F >$, where

(i) Q is a finite set of control states.
(ii) Γ is a finite set of allowable tape symbols.
(iii) $\Sigma \subseteq \Gamma - \{B\}$ is a finite input alphabet, where $B \in \Gamma$ is the blank.
(iv) $S = \{(s_i, h_i): s_i \in Q, h_i \in \mathbb{H}, \quad i = 1, 2, \ldots, k\}$ with $\sum_{i=1}^{k} |h_i| = 1$ is called the set of initial symbols.
(v) $F \subseteq Q$ is the set of accepting states.
(vi) δ: $Q \times \Gamma \times Q \times \Gamma \times \{R, L\} \to \mathbb{H}$ is a quaternionic transition function satisfying the following:

(a) For any $p \in Q$ and $\gamma \in \Gamma$,

$$\sum_{d\in\{R,L\}, q\in Q, \tau\in\Gamma} |\delta(p, \gamma, q, \tau, d)| = 1 \tag{14}$$

(b) For any (p, γ), $(p_1, \gamma_2) \in (Q, \Gamma)$ with $(p, \gamma) \neq (p_1, \gamma_2)$,

$$\sum_{d\in\{R,L\}, q\in Q, \tau\in\Gamma} \delta(p, \gamma, q, \tau, d)\overline{\delta(p_1, \gamma_1, q, \tau, d)} = 0 \tag{15}$$

(c) For any p, $p_1 \in Q$ and γ, γ_1, τ, $\tau_1 \in \Gamma$,

$$\sum_{q\in Q} \delta(p, \gamma, q, \tau, R)\overline{\delta(p_1, \gamma_1, q, \tau_1, L)} = 0 \tag{16}$$

S can be viewed as a quaternionic unit length vector denoting an initial distribution of quaternionic amplitudes over the control states.

To each $(p, \gamma, q, \tau, d) \in Q \times\Gamma \times Q \times\Gamma \times\{R, L\}$, the transition function assigns a quaternionic amplitude $\delta(p, \gamma, q, \tau, d)$ with which the current state p turns to state q, the tape symbol τ being scanned replaces symbol γ, and the head moves left (when $d = L$) or right (when $d = R$).

We, here, construct an example of QQTM that is not a CQTM.

Example 1. Let $\mathcal{M} = < Q, \Gamma, \Sigma, S, \delta, B, F >$, where $Q = \{q_0, q_1\}$, $\Gamma = \{B\}$ $F = \{q_1\}$, $S = \{q_0\}$ is the initial state, and the transition function δ is defined as follows:

$$\begin{aligned}
&\delta(q_0, B, q_0, B, L) = \frac{i}{\sqrt{2}}, \quad \delta(q_0, B, q_0, B, R) = 0,\\
&\delta(q_0, B, q_1, B, L) = 0, \quad \delta(q_0, B, q_1, B, R) = \frac{1}{\sqrt{2}},\\
&\delta(q_1, B, q_0, B, L) = \frac{j}{\sqrt{2}}, \quad \delta(q_1, B, q_0, B, R) = 0,\\
&\delta(q_1, B, q_1, B, L) = 0, \quad \delta(q_1, B, q_1, B, R) = \frac{-k}{\sqrt{2}}.
\end{aligned}$$

We can check that δ meets (vi) (a–c) in Definition 1.δ meets (vi) (a) since

$$\begin{aligned}
&|\delta(q_0, B, q_0, B, L)| + |\delta(q_0, B, q_1, B, L)| + |\delta(q_0, B, q_0, B, R)| + |\delta(q_0, B, q_1, B, R)| = 1,\\
&|\delta(q_1, B, q_0, B, L)| + |\delta(q_1, B, q_1, B, L)| + |\delta(q_1, B, q_0, B, R)| + |\delta(q_1, B, q_1, B, R)| = 1.
\end{aligned}$$

δ meets (vi) (b) since

$$\begin{aligned}
&\delta(q_0, B, q_0, B, L)\overline{\delta(q_1, B, q_0, B, L)}\\
&\quad + \delta(q_0, B, q_1, B, L)\overline{\delta(q_1, B, q_1, B, L)}\\
&\quad + \delta(q_0, B, q_0, B, R)\overline{\delta(q_1, B, q_0, B, R)}\\
&\quad + \delta(q_0, B, q_1, B, R)\overline{\delta(q_1, B, q_1, B, R)}\\
&\quad + \delta(q_1, B, q_0, B, L)\overline{\delta(q_0, B, q_0, B, L)}\\
&\quad + \delta(q_1, B, q_1, B, L)\overline{\delta(q_0, B, q_1, B, L)}\\
&\quad + \delta(q_1, B, q_0, B, R)\overline{\delta(q_0, B, q_0, B, R)}\\
&\quad + \delta(q_1, B, q_1, B, R)\overline{\delta(q_0, B, q_1, B, R)}\\
&= \frac{i}{\sqrt{2}}\frac{-j}{\sqrt{2}} + 0 + 0 + \frac{1}{\sqrt{2}}\frac{k}{\sqrt{2}} + \frac{j}{\sqrt{2}}\frac{-i}{\sqrt{2}} + 0 + 0 + 0 + \frac{-k}{\sqrt{2}}\frac{1}{\sqrt{2}} = 0.
\end{aligned}$$

δ meets (vi) (c) since

$$\begin{aligned}
&\delta(q_0, B, q_0, B, R)\overline{\delta(q_0, B, q_0, B, L)}\\
&\quad + \delta(q_0, B, q_1, B, R)\overline{\delta(q_0, B, q_1, B, L)} = 0,\\
&\delta(q_0, B, q_0, B, R)\overline{\delta(q_1, B, q_0, B, L)}\\
&\quad + \delta(q_0, B, q_1, B, R)\overline{\delta(q_1, B, q_1, B, L)} = 0,\\
&\delta(q_1, B, q_0, B, R)\overline{\delta(q_1, B, q_0, B, L)}\\
&\quad + \delta(q_1, B, q_1, B, R)\overline{\delta(q_1, B, q_1, B, L)} = 0,\\
&\delta(q_1, B, q_0, B, R)\overline{\delta(q_0, B, q_0, B, L)}\\
&\quad + \delta(q_1, B, q_1, B, R)\overline{\delta(q_0, B, q_1, B, L)} = 0.
\end{aligned}$$

So the aforementioned definition $\mathcal{M}$ is a QQTM.

Then, we give the definition of a multitape QQTM.

Definition 2. Suppose that $k \geq 1$ is an integer. A k-tape QQTM is a 7-tuple $\mathcal{M} = < Q, \Gamma, \Sigma, S, \delta, B, F >$. where Q, Γ, Σ, S, B, F are the same in Definition 1, and δ: $Q \times \Gamma^k \times Q \times (\Gamma \times \{R, L\})^k \to \mathbb{H}$ is a quaternionic transition function satisfying the following:

(a) For any $p \in Q$ and $\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma$,

$$\sum_{d\in\{R,L\},q\in Q,\tau_1,\tau_2,\ldots,\tau_k\in\Gamma} \left|\delta(p,\gamma_1,\gamma_2,\ldots,\gamma_k,q,\tau_1,\tau_2,\ldots,\tau_k,d)\right| = 1 \quad (17)$$

(b) For any $(p, \gamma_{11}, \gamma_{12}, \ldots, \gamma_{1k}), (p_1, \gamma_{21}, \gamma_{22}, \ldots, \gamma_{2k}) \in (Q, \Gamma^k)$ with $(p, \gamma_{11}, \gamma_{12}, \ldots, \gamma_{1k}) \neq (p_1, \gamma_{21}, \gamma_{22}, \ldots, \gamma_{2k})$,

$$\sum_{d\in\{R,L\},q\in Q,\tau_1,\tau_2,\ldots,\tau_k\in\Gamma} \delta(p,\boldsymbol{\gamma}_1,q,\tau_1,\tau_2,\ldots,\tau_k,d)\overline{\delta(p_1,\boldsymbol{\gamma}_2,q,\tau_1,\tau_2,\ldots,\tau_k,d)} = 0 \quad (18)$$

where $\boldsymbol{\gamma}_1 = (\gamma_{11}, \gamma_{12}, \ldots, \gamma_{1k})$ and $\boldsymbol{\gamma}_2 = (\gamma_{21}, \gamma_{22}, \ldots, \gamma_{2k})$

(c) For any $p, p_1 \in Q$, $\boldsymbol{\gamma}_1 = (\gamma_{11}, \gamma_{12}, \ldots, \gamma_{1k}) \in \Gamma^k$, $\boldsymbol{\gamma}_2 = (\gamma_{21}, \gamma_{22}, \ldots, \gamma_{2k}) \in \Gamma^k$, $\boldsymbol{\tau}_1 = (\tau_{11}, \tau_{12}, \ldots, \tau_{1k}) \in \Gamma^k$ and $\boldsymbol{\tau}_2 = (\tau_{21}, \tau_{22}, \ldots, \tau_{2k}) \in \Gamma^k$

$$\sum_{q\in Q} \delta(p,\boldsymbol{\gamma}_1,q,\boldsymbol{\tau}_1,R)\overline{\delta(p_1,\boldsymbol{\gamma}_2,q,\boldsymbol{\tau}_2,L)} = 0 \quad (19)$$

amplitude with which thIntuitively, $\delta(p, \gamma_1, \gamma_2, \ldots, \gamma_k, q, \tau_1, \tau_2, \ldots, \tau_k, d)$ is a quaternionice current state p turns to state q, each tape symbol $\tau_1, \tau_2, \ldots, \tau_k$ being scanned replaces symbol $\gamma_1, \gamma_2, \ldots, \gamma_k$, and each head moves left (when $d = L$) or right (when $d = R$) respectively.

The configuration of a Turing machine is described by a string $\alpha_1 q \alpha_2$ for $q \in Q$ and $\alpha_1, \alpha_2 \in \Gamma^*$, where Γ^* denotes all the finite strings over Γ including the empty string ε, and the tape head scans the leftmost symbol of α_2 or the blank B in case $\alpha_2 = \varepsilon$.

Let $\mathbb{D} = \Gamma^* \times Q \times \Gamma^*$ be the set of configurations. A move from $D_1 \in \mathbb{D}$ to another $D_2 \in \mathbb{D}$, denoted by $D_1 \vdash D_2$, is defined as follows: for any $\alpha_1, \alpha_2 \in \Gamma^*$, $x, y, z \in \Gamma$, and $p, q \in Q$,

$$D_1 \vdash D_2 = \begin{cases} \delta(q,x,p,y,R), & \text{if } D_1 = \alpha_1 q x \alpha_2,\ D_2 = \alpha_1 y p \alpha_2, \\ \delta(q,x,p,y,L), & \text{if } D_1 = \alpha_1 z q x \alpha_2,\ D_2 = \alpha_1 p z y \alpha_2, \\ \delta(q,B,p,y,R), & \text{if } D_1 = \alpha_1 q,\ D_2 = \alpha_1 y p, \\ \delta(q,B,p,y,R), & \text{if } D_1 = \alpha_1 z q,\ D_2 = \alpha_1 p z y, \\ 0, & \text{otherwise.} \end{cases}$$

In the quaternionic quantum case, the quaternionic transition function δ is a quaternion. A chain of derivatives from $s_i\omega$ to $\alpha_n q_n \beta_n$ is expressed as $s_i\omega \vdash \alpha_1 q_1 \beta_1 \vdash \cdots \vdash \alpha_{n-1} q_{n-1} \beta_{n-1} \vdash \alpha_n q_n \beta_n$ with the probability $|(s_i\omega \vdash \alpha_1 q_1 \beta_1)(\alpha_1 q_1 \beta_1 \vdash \alpha_2 q_2 \beta_2)\cdots(\alpha_{n-1} q_{n-1} \beta_{n-1} \vdash \alpha_n q_n \beta_n)|$.

A QQTM $\mathcal{M}$ defined previously induces a function $f_{\mathcal{M}}$: $\Sigma^* \to [0,1]$ as follows: for any $\omega \in \Sigma^*$,

$$f_{\mathcal{M}}(\omega) = \sum_{q_{n_i}\in F} \left| \sum_{(s_i,h_i)\in S} h_i \sum_{q_1,\ldots,q_{n_i-1}\in Q,\alpha_1,\ldots,\alpha_{n_i},\beta_1,\ldots,\beta_{n_i}\in\Gamma^*} (s_i\omega \vdash \alpha_1 q_1 \beta_1) (\alpha_1 q_1 \beta_1 \vdash \alpha_2 q_2 \beta_2)\cdots\left(\alpha_{n_i-1} q_{n_i-1} \beta_{n_i-1} \vdash \alpha_{n_i} q_{n_i} \beta_{n_i}\right) \right|, \quad (20)$$

which represents the probability that $\mathcal{M}$ accepts ω. Particularly, if $F = \emptyset$, then $f_{\mathcal{M}}(\omega) = 0$. If $S = \{q_0\}$, then

$$f_{\mathcal{M}}(\omega) = \sum_{q_n\in F} \left| \sum_{q_1,\ldots,q_{n-1}\in Q,\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n\in\Gamma^*} (q_0\omega \vdash \alpha_1 q_1 \beta_1) (\alpha_1 q_1 \beta_1 \vdash \alpha_2 q_2 \beta_2)\cdots(\alpha_{n-1} q_{n-1} \beta_{n-1} \vdash \alpha_n q_n \beta_n) \right|. \quad (21)$$

4 Weighted sum of QQTM

How to construct a desired machine is an important issue. In [5], the dovetailing lemma and the branching lemma are given and used to construct the universal QTM. The weighted sum of complex quantum automata, a theoretical model of quantum computation, has been well-studied [39, 40].

In this section, we study the weighted sum of QQTM.

Let $\mathcal{A} = \langle Q_A, \Gamma, \Sigma, S_A, \delta_A, B, F_A \rangle$ and $\mathcal{B} = \langle Q_B, \Gamma, \Sigma, S_B, \delta_B, B, F_B \rangle$ be two QQTMs over Σ, where $S_A = \{(s_{A_i}, h_{A_i})\colon s_{A_i} \in Q, h_{A_i} \in \mathbb{H},\ i = 1,2,\ldots,k\}$ with $\sum_{i=1}^k |h_{A_i}| = 1$ and $S_B = \{(s_{B_i}, h_{B_i})\colon s_{B_i} \in Q, h_{B_i} \in \mathbb{H},\ i = 1,2,\ldots,l\}$ with $\sum_{i=1}^l |h_{B_i}| = 1$. We assume that $Q_A \cap Q_B = \emptyset$. Let $\alpha, \beta \in \mathbb{H}$ and $|\alpha| + |\beta| = 1$. Then, their weighted sum $\mathcal{C} = \mathcal{A} +_{\alpha,\beta} \mathcal{B} = \langle Q_C, \Sigma, \psi_C, \delta_C, F_C \rangle$ is defined as follows:

(i) $Q_C = Q_A \cup Q_B$.
(ii) $S_C = \{(s_{A_i}, \alpha h_{A_i})\} \cup \{(s_{B_i}, \beta h_{B_i})\}$
(iii) $F_C = F_A \cup F_B \subseteq Q_C$.
(iv) δ: $Q_C \times \Gamma \times Q_C \times \Gamma \times \{R, L\} \to \mathbb{H}$ is defined as follows:

$$\delta_C(p,\gamma,q,\tau,d) = \begin{cases} \delta_A(p,\gamma,q,\tau,d), & \text{if } q, p \in Q_A, \\ \delta_B(p,\gamma,q,\tau,d), & \text{if } q, p \in Q_B, \\ 0, & \text{otherwise,} \end{cases} \quad (22)$$

where $d \in \{R, L\}$.

Theorem 1. Let $\alpha, \beta \in \mathbb{H}$ and $|\alpha| + |\beta| = 1$. If $\mathcal{A} = \langle Q_A, \Gamma, \Sigma, q_A, \delta_A, B, F_A \rangle$ and $\mathcal{B} = \langle Q_B, \Gamma, \Sigma, q_B, \delta_B, B, F_B \rangle$ be two QQTMs over Σ, then their weighted sum $\mathcal{A} +_{\alpha,\beta} \mathcal{B}$ is a QQTM over Σ.

Proof. Let $\mathcal{C} = \mathcal{A} +_{\alpha,\beta} \mathcal{B}$. First, S_C satisfies $\sum_{i=1}^k |\alpha h_{A_i}| + \sum_{i=1}^l |\beta h_{B_i}| = |\alpha| + |\beta| = 1$. Then, we check that δ_C meets (iv) (a–c) in Definition 1.

(a) For any $p \in Q_C$ and $\gamma \in \Gamma$, if $p \in Q_A$, since $\delta(p, \gamma, q, \tau, d) = 0$ for any $q \in Q_B$, then

$$\sum_{d\in\{R,L\},q\in Q_C,\tau\in\Gamma} \left|\delta(p,\gamma,q,\tau,d)\right| = \sum_{d\in\{R,L\},q\in Q_A,\tau\in\Gamma} \left|\delta(p,\gamma,q,\tau,d)\right| = 1. \quad (23)$$

If $p \in Q_B$, since $\delta(p, \gamma, q, \tau, d) = 0$ for any $q \in Q_A$, then

$$\sum_{d\in\{R,L\},q\in Q_C,\tau\in\Gamma} \left|\delta(p,\gamma,q,\tau,d)\right| = \sum_{d\in\{R,L\},q\in Q_B,\tau\in\Gamma} \left|\delta(p,\gamma,q,\tau,d)\right| = 1, \quad (24)$$

(b) For any $(p, \gamma), (p_1, \gamma_2) \in (Q_C, \Gamma)$ with $(p, \gamma) \neq (p_1, \gamma_2)$, if $q \in Q_A$, since $\delta(p, \gamma, q, \tau, d) = 0$ for any $p \in Q_B$, then

$$\sum_{d\in\{R,L\},q\in Q_C,\tau\in\Gamma} \delta(p,\gamma,q,\tau,d)\overline{\delta(p_1,\gamma_1,q,\tau,d)} = \sum_{d\in\{R,L\},q\in Q_A,\tau\in\Gamma} \delta(p,\gamma,q,\tau,d)\overline{\delta(p_1,\gamma_1,q,\tau,d)} = 0 \quad (25)$$

If $q \in Q_B$, since $\delta(p, \gamma, q, \tau, d) = 0$ for any $p \in Q_A$, then

$$\sum_{d\in\{R,L\},q\in Q_C,\tau\in\Gamma}\delta(p,\gamma,q,\tau,d)\overline{\delta(p_1,\gamma_1,q,\tau,d)} = \sum_{d\in\{R,L\},q\in Q_B,\tau\in\Gamma}\delta(p,\gamma,q,\tau,d)\overline{\delta(p_1,\gamma_1,q,\tau,d)} = 0 \tag{26}$$

(c) For any p, $p_1 \in Q_C$ and γ, γ_1, τ, $\tau_1 \in \Gamma$, if p, $p_1 \in Q_A$, then

$$\sum_{q\in Q_C}\delta(p,\gamma,q,\tau,R)\overline{\delta(p_1,\gamma_1,q,\tau_1,L)} = \sum_{q\in Q_A}\delta(p,\gamma,q,\tau,R)\overline{\delta(p_1,\gamma_1,q,\tau_1,L)} = 0 \tag{27}$$

If p, $p_1 \in Q_B$, then

$$\sum_{q\in Q_C}\delta(p,\gamma,q,\tau,R)\overline{\delta(p_1,\gamma_1,q,\tau_1,L)} = \sum_{q\in Q_B}\delta(p,\gamma,q,\tau,R)\overline{\delta(p_1,\gamma_1,q,\tau_1,L)} = 0 \tag{28}$$

If $p \in Q_A$, $p_1 \in Q_B$, then

$$\begin{aligned}\sum_{q\in Q_C}\delta(p,\gamma,q,\tau,R)\overline{\delta(p_1,\gamma_1,q,\tau_1,L)} &= \sum_{q\in Q_A}\delta(p,\gamma,q,\tau,R)\overline{\delta(p_1,\gamma_1,q,\tau_1,L)} + \sum_{q\in Q_B}\delta(p,\gamma,q,\tau,R)\overline{\delta(p_1,\gamma_1,q,\tau_1,L)}\\ &= \sum_{q\in Q_A}\delta(p,\gamma,q,\tau,R)0 + \sum_{q\in Q_B}0\overline{\delta(p_1,\gamma_1,q,\tau_1,L)} = 0\end{aligned} \tag{29}$$

If $p \in Q_B$, $p_1 \in Q_A$, then

$$\begin{aligned}\sum_{q\in Q_C}\delta(p,\gamma,q,\tau,R)\overline{\delta(p_1,\gamma_1,q,\tau_1,L)} &= \sum_{q\in Q_A}\delta(p,\gamma,q,\tau,R)\overline{\delta(p_1,\gamma_1,q,\tau_1,L)} + \sum_{q\in Q_B}\delta(p,\gamma,q,\tau,R)\overline{\delta(p_1,\gamma_1,q,\tau_1,L)}\\ &= \sum_{q\in Q_A}\delta(p,\gamma,q,\tau,R)0\overline{\delta(p_1,\gamma_1,q,\tau_1,L)} + \sum_{q\in Q_B}\delta(p,\gamma,q,\tau,R)0 = 0\end{aligned} \tag{30}$$

So, $\mathcal{C}$ is a QQTM.

Theorem 2. Let $\alpha,\beta\in\mathbb{H}$ with $|\alpha| + |\beta| = 1$, $\mathcal{A} = <Q_A,\Gamma,\Sigma,q_A,\delta_A,B,F_A>$ and $\mathcal{B} = <Q_B,\Gamma,\Sigma,q_B,\delta_B,B,F_B>$ be two QQTMs over Σ. If $f_{\mathcal{A}}$ is the function induced by $\mathcal{A}$, and $f_{\mathcal{B}}$ is the function induced by $\mathcal{B}$, then $f_{\mathcal{A}+_{\alpha,\beta}\mathcal{B}} = |\alpha| f_{\mathcal{A}} + |\beta| f_{\mathcal{B}}$.

Proof. Let $\mathcal{C} = \mathcal{A}+_{\alpha,\beta}\mathcal{B}$.

$$\begin{aligned}f_{\mathcal{C}}(\omega) = \sum_{q_{n_i}\in F_C}\Bigg|&\sum_{(s_{A_i},h_{A_i})\in S}\alpha h_{A_i}\sum_{q_1,\ldots,q_{n_i-1}\in Q,\alpha_1,\ldots,\alpha_{n_i},\beta_1,\ldots,\beta_{n_i}\in\Gamma^*}(s_i\omega\vdash\alpha_1 q_1\beta_1)\\ &\times(\alpha_1 q_1\beta_1\vdash\alpha_2 q_2\beta_2)\cdots(\alpha_{n_i-1}q_{n_i-1}\beta_{n_i-1}\vdash\alpha_{n_i}q_{n_i}\beta_{n_i})\\ &+\sum_{(s_{A_i},h_{A_i})\in S}\beta h_{A_i}\sum_{q_1,\ldots,q_{n_i-1}\in Q,\alpha_1,\ldots,\alpha_{n_i},\beta_1,\ldots,\beta_{n_i}\in\Gamma^*}(s_i\omega\vdash\alpha_1 q_1\beta_1)\\ &\times(\alpha_1 q_1\beta_1\vdash\alpha_2 q_2\beta_2)\cdots(\alpha_{n_i-1}q_{n_i-1}\beta_{n_i-1}\vdash\alpha_{n_i}q_{n_i}\beta_{n_i})\Bigg|.\end{aligned}$$

Because $F_A \subseteq Q_A$, $F_B \subseteq Q_B$, and $Q_A \cap Q_B = \varnothing$, we have

$$\begin{aligned}f_{\mathcal{C}}(\omega) &= \sum_{q_{n_i}\in F_A}\Bigg|\sum_{(s_{A_i},h_{A_i})\in S}\alpha h_{A_i}\sum_{q_1,\ldots,q_{n_i-1}\in Q,\alpha_1,\ldots,\alpha_{n_i},\beta_1,\ldots,\beta_{n_i}\in\Gamma^*}(s_i\omega\vdash\alpha_1 q_1\beta_1)\\ &\quad(\alpha_1 q_1\beta_1\vdash\alpha_2 q_2\beta_2)\cdots(\alpha_{n_i-1}q_{n_i-1}\beta_{n_i-1}\vdash\alpha_{n_i}q_{n_i}\beta_{n_i})\Bigg|\\ &+\sum_{q_{n_i}\in F_B}\Bigg|\sum_{(s_{A_i},h_{A_i})\in S}\beta h_{A_i}\sum_{q_1,\ldots,q_{n_i-1}\in Q,\alpha_1,\ldots,\alpha_{n_i},\beta_1,\ldots,\beta_{n_i}\in\Gamma^*}(s_i\omega\vdash\alpha_1 q_1\beta_1)\\ &\quad(\alpha_1 q_1\beta_1\vdash\alpha_2 q_2\beta_2)\cdots(\alpha_{n_i-1}q_{n_i-1}\beta_{n_i-1}\vdash\alpha_{n_i}q_{n_i}\beta_{n_i})\Bigg|\\ &= |\alpha|\sum_{q_{n_i}\in F_A}\Bigg|\sum_{(s_{A_i},h_{A_i})\in S}h_{A_i}\sum_{q_1,\ldots,q_{n_i-1}\in Q,\alpha_1,\ldots,\alpha_{n_i},\beta_1,\ldots,\beta_{n_i}\in\Gamma^*}(s_i\omega\vdash\alpha_1 q_1\beta_1)\\ &\quad(\alpha_1 q_1\beta_1\vdash\alpha_2 q_2\beta_2)\cdots(\alpha_{n_i-1}q_{n_i-1}\beta_{n_i-1}\vdash\alpha_{n_i}q_{n_i}\beta_{n_i})\Bigg|\\ &+|\beta|\sum_{q_{n_i}\in F_B}\Bigg|\sum_{(s_{A_i},h_{A_i})\in S}h_{A_i}\sum_{q_1,\ldots,q_{n_i-1}\in Q,\alpha_1,\ldots,\alpha_{n_i},\beta_1,\ldots,\beta_{n_i}\in\Gamma^*}(s_i\omega\vdash\alpha_1 q_1\beta_1)\\ &\quad(\alpha_1 q_1\beta_1\vdash\alpha_2 q_2\beta_2)\cdots(\alpha_{n_i-1}q_{n_i-1}\beta_{n_i-1}\vdash\alpha_{n_i}q_{n_i}\beta_{n_i})\Bigg|\\ &= |\alpha| f_{\mathcal{A}}(\omega) + |\beta| f_{\mathcal{B}}(\omega).\end{aligned}$$

So, $|\alpha| f_{\mathcal{A}}(\omega) + |\beta| f_{\mathcal{B}}(\omega)$ is the function induced by $\mathcal{C}$.

5 Conclusion

The main purpose of this paper is to understand the quaternionic quantum computation. In this paper, we have defined quaternionic quantum versions of the Turing machine and multitape Turing machine. The QQTM is based on quaternionic quantum mechanics, which is a generalization of the standard complex quantum mechanics. The QQTM provides a new perception of quantum computation which is different from the traditional complex quantum computation.

In our view, it is a natural mathematical progression from the real to the complex to the quaternionic numbers. Then, there is a corresponding natural progression also in computer science that uses these numbers. This paper considers the computation model in this direction, i.e., from the complex quantum Turing machine to the QQTM. To conclude this paper, we would like to mention some research questions for further studies.

1) We focus on the Turing machine model based on quaternionic quantum mechanics. There are various models of quantum computation. As future work, we can consider other models of quaternionic quantum computation.
2) It is also interesting to consider the quantum information from the complex quantum case to quaternionic quantum case. This will help us understand the quantum information theory.
3) Whether it is necessary to study quaternionic quantum computation. From a practical viewpoint, one of the most

important problems is to examine the applicability of quaternionic quantum computation.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

Investigations and writing: SD.

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Conflict of interest

The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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