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New symmetry reduction method for (1+1)-dimensional differential-difference equations

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We propose a new symmetry reduction method for (1+1)-dimensional differential-difference equations (DDEs), namely, the λ-symmetry reduction method of solving ordinary differential equations is generalized to DDEs. Order-reduction processes are a consequence of the invariance of the given DDE under vector fields of the new class. These vector fields satisfy a new prolongation formula. A simple example of order-reduction is provided to illustrate the application.

KEYWORDS

λ-symmetry, differential-difference equation, order-reduction, vector field, reduction method

1 Introduction

Symmetry is closely related to the integrability of the nonlinear evolution equations (NLEEs) in various specific meanings. For example, the existence of infinite Lie-Bäcklund symmetry is a criterion for the integrability of NLEEs, so the study of symmetry of NLEEs is particularly important. The symmetry of the NLEEs is studied systematically by Lie point symmetry theory [1–3]. Although the Lie point symmetry method has relatively mature theories, it also has great limitations [1–10]. When a given NLEE does not allow enough non-trivial Lie point symmetries, this method cannot be applied. Therefore, it is necessary to extend the classical Lie point symmetry concept from various angles [11–20]. For example, if the infinitesimal also depends on the higher derivative, the corresponding Lie-Bäcklund symmetry is obtained [21, 22].

The concept of λ-symmetry proposed by Muriel and Romero [23], aims to show that many of the known order-reduction processes can be explained by the invariance of the equation under some special vector fields that are neither Lie symmetries nor Lie-Bäcklund symmetries. The λ-symmetry reduction method for ordinary differential equations (ODEs) has attracted the attention of more and more scientists [24]. For example, Levi and Rodriguez successfully extended this method to the case of difference equations [25]. Again, the μ-symmetry reduction method is used to deal with partial differential equations (PDEs) [26–30].

For the sake of readability, we will briefly introduce the λ-symmetry reduction method for ODEs in Section 2. Then we extend the λ-symmetry reduction method to the case of (1+1)-dimensional differential-difference equations (DDEs) in Section 3. The last section is devoted to conclusions and discussions.

2 The λ-symmetry reduction method of ODEs

In this section we briefly review the λ-symmetry reduction method of ODEs. For a given mth-order ODE

$$\Delta_1 \equiv \Delta\left(x, u^{(0)}, u^{(1)}, \ldots, u^{(m)}\right) = 0, \tag{1}$$

we can set a vector field

$$v = X(x,u)\frac{\partial}{\partial x} + U(x,u)\frac{\partial}{\partial u}, \tag{2}$$

where $u^{(i)} = \frac{\mathrm{d}^i u(x)}{\mathrm{d}x^i}$, $(i = 0, 1, \ldots, m)$ means the ith-order derivative with respect to the independent variable x. Thus we can construct high-order infinitesimal prolongation vector field

$$v^{[(m)]} = v + \sum_{i=1}^{m} U^{[(i)]}\frac{\partial}{\partial u^{(i)}}, \tag{3}$$

where

$$\begin{aligned} U^{[(0)]} &= U, \\ U^{[(i+1)]} &= D_x U^{[(i)]} - u^{(i+1)} D_x X, \qquad i = 0, 1, \ldots, m. \end{aligned} \tag{4}$$

Here D_x means the total derivative with respect to x. So the invariance of Eq. 1 needs

$$v^{[(m)]}(\Delta_1)\big|_{\Delta_1=0} = 0. \tag{5}$$

Solving this equation, the expressions for X and U can be derived. For complex high-order ODEs or systems, we need to use symbolic computing software to calculate X and U. Theoretically, all of the similarity variables be derived by solving the following characteristic equation

$$\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}u}{U}, \tag{6}$$

and then we can reduce and solve Eq. 1.

The above method is the Lie point symmetry method, also known as the classical symmetry reduction method. In Ref. [23], authors have introduced a new class of symmetries, that strictly includes Lie point symmetries, for which there exists an algorithm that lets us reduce the order of a given ODE. This method is now called the λ-symmetry reduction method. The key step of this generalized method is that the infinitesimal prolongation is modified to the following form

$$\begin{aligned} U^{[\lambda,(i)]}\left(x,u^{(i)}\right) = D_x\left(U^{[\lambda,(i-1)]}\left(x,u^{(i-1)}\right)\right) - D_x\left(X(x,u)\right)u^{(i)} \\ +\lambda\left(U^{[\lambda,(i-1)]}\left(x,u^{(i-1)}\right) - X(x,u)u^{(i)}\right), \end{aligned} \tag{7}$$

where λ is a smooth function that is determined simultaneously with the coefficients of the infinitesimal generators X and U. Thus the infinitesimal prolongation vector field is modified to

$$v^{[\lambda,(m)]} = X(x,u)\frac{\partial}{\partial x} + \sum_{i=0}^{m} U^{[\lambda,(i)]}\left(x,u^{(i)}\right)\frac{\partial}{\partial u^{(i)}}. \tag{8}$$

The following theorem that is important for the λ-symmetry reduction method, which is first obtained by Muriel and Romero [23].

Theorem 1. (Muriel, Romero [23]). *Let us suppose that, for some smooth functions λ, the vector field v is a λ-symmetry of the following ODE*

$$u^{(m)} = F\left(x, u^{(0)}, u^{(1)}, \ldots, u^{(m-1)}\right). \tag{9}$$

Then

$$\left[v^{[\lambda,(m-1)]}, A\right] = \lambda \cdot v^{[\lambda,(m-1)]} + \mu \cdot A, \tag{10}$$

for some smooth functions μ. Here A is the vector field of Eq. 9,

$$A = \frac{\partial}{\partial x} + u^{(1)}\frac{\partial}{\partial u} + \cdots + F\left(x, u^{(0)}, u^{(1)}, \ldots, u^{(m-1)}\right)\frac{\partial}{\partial u^{(m-1)}}. \tag{11}$$

Conversely, if

$$K = X(x,u)\frac{\partial}{\partial x} + U^{(0)}(x,u)\frac{\partial}{\partial u} + \sum_{i=1}^{m-1} U^{(i)}\left(x,u^{(i)}\right)\frac{\partial}{\partial u_i}, \tag{12}$$

is a vector field such that

$$[K, A] = \lambda \cdot K + \mu \cdot A, \tag{13}$$

for some smooth functions λ, μ, then the vector field

$$v = X(x,u)\frac{\partial}{\partial x} + U^{(0)}(x,u)\frac{\partial}{\partial u}, \tag{14}$$

is a λ-symmetry of Eq. 9 *and $K = v^{[\lambda,(m-1)]}$.*

3 The λ-symmetry reduction method of DDEs

In this section, we extend the λ-symmetry reduction method to the case of (1+1)-dimensional DDEs.

Definition 1. *For the following (1+1)-dimensional DDE with a discrete variable n and a continuous variable x,*

$$\Delta_2 \equiv \Delta\left(x, u_{n-1}^{(0)}, u_n^{(0)}, u_{n+1}^{(0)}, \ldots, u_{n-1}^{(m)}, u_n^{(m)}, u_{n+1}^{(m)}\right) = 0, \tag{15}$$

where $u_n^{(i)} = \frac{\mathrm{d}^i u_n(x)}{\mathrm{d}x^i}$, the vector field

$$\begin{aligned} v = X(x,u_n)\frac{\partial}{\partial x} + U_{n-1}(x,u_{n-1})\frac{\partial}{\partial u_{n-1}} + U_n(x,u_n)\frac{\partial}{\partial u_n} \\ + U_{n+1}(x,u_{n+1})\frac{\partial}{\partial u_{n+1}} \end{aligned}$$

is said to be λ-symmetry for this equation if there exists a differential function λ such that the mth λ-prolongation of the vector field satisfies.

$$v^{[\lambda,(m)]}(\Delta_2)\big|_{\Delta_2=0} = 0. \tag{16}$$

Particularly, for the following (1+1)-dimensional DDE

$$u_n^{(m)} = F_n\left(x, u_{n-1}^{(m-1)}, u_n^{(m-1)}, u_{n+1}^{(m-1)}\right), \tag{17}$$

we can set a vector field

$$\begin{aligned} A = \frac{d}{dx} + \sum_{n=-1}^{1} u_{n+k}^{(1)}\frac{d}{du_{n+k}} + \cdots \\ + \sum_{k=-1}^{1} F_{n+k}\left(\lambda, u_{n+k-1}^{(m-1)}, u_{n+k}^{(m-1)}, u_{n+k+1}^{(m-1)}\right)\frac{\partial}{u_{n+k}^{(m-1)}}. \end{aligned} \tag{18}$$

Here $F_n(x, u_{n-1}^{(m-1)}, u_n^{(m-1)}, u_{n+1}^{(m-1)}) = F_n(x, u_{n-1}^{(0)}, u_n^{(0)}, u_{n+1}^{(0)}, \ldots, u_{n-1}^{(m-1)}, u_n^{(m-1)}, u_{n+1}^{(m-1)})$ is for ease of writing. So we have Theorem 2.

Theorem 2. *Let us suppose that, for some differential functions λ, the vector field v is a λ-symmetry of the following DDE*

$$u_n^{(m)} = F_n(x, u_{n-1}^{(m-1)}, u_n^{(m-1)}, u_{n+1}^{(m-1)}), \tag{19}$$

Then

$$\left[v^{[\lambda,(m-1)]}, A\right] = \lambda \cdot v^{[\lambda,(m-1)]} + \mu \cdot A. \tag{20}$$

for some differential functions μ. Here A is the vector field of Eq. 19,

$$A = \frac{d}{dx} + \sum_{n=-1}^{1} u_{n+k}^{(1)} \frac{d}{du_{n+k}} + \cdots + \sum_{k=-1}^{1} F_{n+k}\left(\lambda, u_{n+k-1}^{(m-1)}, u_{n+k}^{(m-1)}, u_{n+k+1}^{(m-1)}\right) \frac{\partial}{u_{n+k}^{(m-1)}}. \tag{21}$$

Conversely, if

$$K = X(x, u_n)\frac{\partial}{\partial x} + \sum_{k=-1}^{1}\sum_{i=0}^{m-1} U_{n+k}^{(i)}\left(x, u_{n+k}^{(i)}\right)\frac{\partial}{\partial u_{n+k}^{(i)}}, \tag{22}$$

is a vector field such that

$$[K, A] = \lambda \cdot K + \mu \cdot A, \tag{23}$$

for some differential functions λ and μ, then the vector field

$$v = X(x,u)\frac{\partial}{\partial x} + U_{n-1}^{(0)}(x,u)\frac{\partial}{\partial u_{n-1}} + U_n^{(0)}(x,u)\frac{\partial}{\partial u_n} + U_{n+1}^{(0)}(x,u)\frac{\partial}{\partial u_{n+1}}, \tag{24}$$

is a λ-symmetry of Eq. 19 *and* $K = v^{[\lambda,(m-1)]}$.

Proof. Compute $[v^{[\lambda,(m-1)]}, A]$ as a function of $\{x, u_{n-1}, u_n, u_{n+1}, \ldots, u_{n-1}^{(m-1)}, u_n^{(m-1)}, u_{n+1}^{(m-1)}\}$ at each lattice point, with

$$\begin{aligned}
\left[v^{[\lambda,(m-1)]}, A\right](x) &= -A(X(x)),\\
\left[v^{[\lambda,(m-1)]}, A\right](u_{n+k}) &= U_{n+k}^{[\lambda,(1)]}\left(x, u_{n+k}^{(1)}\right) - A\left(U_{n+k}^{[\lambda,(0)]}(x, u_{n+k})\right)\\
&= -A(X(x))u_{n+k}^{(1)} + \lambda\left(U_{n+k}^{[\lambda,(0)]}(x, u_{n+k}) - X(x)u_{n+k}^{(1)}\right),\\
\left[v^{[\lambda,(m-1)]}, A\right]\left(u_{n+k}^{(1)}\right) &= U_{n+k}^{[\lambda,(2)]}\left(x, u_{n+k}^{(2)}\right) - A\left(U_{n+k}^{[\lambda,(1)]}\left(x, u_{n+k}^{(1)}\right)\right)\\
&= -A(X(x))u_{n+k}^{(2)} + \lambda\left(U_{n+k}^{[\lambda,(1)]}\left(x, u_{n+k}^{(1)}\right) - X(x)u_{n+k}^{(2)}\right),\\
&\vdots\\
\left[v^{[\lambda,(m-1)]}, A\right]\left(u_{n+k}^{(i)}\right) &= U_{n+k}^{[\lambda,(i+1)]}\left(x, u_{n+k}^{(i+1)}\right) - A\left(U_{n+k}^{[\lambda,(i)]}\left(x, u_{n+k}^{(i)}\right)\right)\\
&= -A(X(x))u_{n+k}^{(i+1)} + \lambda\left(U_{n+k}^{[\lambda,(i)]}\left(x, u_{n+k}^{(i)}\right) - X(x)u_{n+k}^{(i+1)}\right),\\
&\vdots\\
\left[v^{[\lambda,(m-1)]}, A\right]\left(u_{n+k}^{(m-1)}\right) &= v^{[\lambda,(m-1)]}\left(F_{n+k}\left(\lambda, u_{n+k-1}^{(m-1)}, u_{n+k}^{(m-1)}, u_{n+k+1}^{(m-1)}\right)\right)\\
&\quad -A\left(U_{n+k}^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right)\right),
\end{aligned} \tag{25}$$

and

$$v^{[\lambda,(m)]}\left(u_{n+k}^{(m)}\right) = D_x\left(U_{n+k}^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right)\right) - D_x(X(x))u_{n+k}^{(m)} + \lambda\left(U_{n+k}^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right)\right) - \lambda(X(x))u_{n+k}^{(m)}. \tag{26}$$

Since v is a λ-symmetry,

$$\begin{aligned}
v^{[\lambda,(m-1)]}\left(F_{n+k}\left(\lambda, u_{n+k-1}^{(m-1)}, u_{n+k}^{(m-1)}, u_{n+k+1}^{(m-1)}\right)\right) &= A\left(U_{n+k}^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right)\right)\\
-A(X(x))u_{n+k}^{(m)} &+ \lambda\left(U_{n+k}^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right)\right) - \lambda(X(x))u_{n+k}^{(m)}.
\end{aligned} \tag{27}$$

Hence, if $u_n^{(m)} = F_n(x, u_{n-1}^{(m-1)}, u_n^{(m-1)}, u_{n+1}^{(m-1)})$, Eq. 26 says that

$$\begin{aligned}
[\lambda,(m-1)]\left(F_{n+k}\left(\lambda, u_{n+k-1}^{(m-1)}, u_{n+k}^{(m-1)}, u_{n+k+1}^{(m-1)}\right)\right) &= A\left(U_{n+k}^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right)\right)\\
-A(X(x))u_{n+k}^{(m)} &+ \lambda\left(U_{n+k}^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right)\right) - \lambda(X(x))u_{n+k}^{(m)}.
\end{aligned} \tag{28}$$

If we set $\mu = -A(X(x)) - \lambda X(x)$, then we can write

$$\begin{aligned}
\left[v^{[\lambda,(m-1)]}, A\right](x) &= \lambda X(x) + \mu,\\
\left[v^{[\lambda,(m-1)]}, A\right](u_{n+k}) &= \lambda U_{n+k}^{[\lambda,(0)]}(x, u_{n+k}) + \mu u_{n+k}^{(1)},\\
\left[v^{[\lambda,(m-1)]}, A\right]\left(u_{n+k}^{(1)}\right) &= \lambda U_{n+k}^{[\lambda,(1)]}\left(x, u_{n+k}^{(1)}\right) + \mu u_{n+k}^{(2)},\\
&\vdots\\
\left[v^{[\lambda,(m-1)]}, A\right]\left(u_{n+k}^{(i)}\right) &= \lambda U_{n+k}^{[\lambda,(i)]}\left(x, u_{n+k}^{(i)}\right) + \mu u_{n+k}^{(i+1)},\\
&\vdots\\
\left[v^{[\lambda,(m-1)]}, A\right]\left(u_{n+k}^{(m-1)}\right) &= \lambda U_{n+k}^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right) + \mu u_{n+k}^{(m)}
\end{aligned} \tag{29}$$

Therefore, we conclude that $[v^{[\lambda,(m-1)]}, A] = \lambda \cdot v^{[\lambda,(m-1)]} + \mu \cdot A$. The vector field

$$K = X(x, u_n)\frac{\partial}{\partial x} + \sum_{k=-1}^{1}\sum_{i=0}^{m-1} U_{n+k}^{(i)}\left(x, u_{n+k}^{(i)}\right)\frac{\partial}{\partial u_{n+k}^{(i)}}, \tag{30}$$

depends on three lattice points with $n-1$, n and $n+1$. If we apply both elements of this equation to each coordinate function, we obtain

$$\mu = -A(X(x,u)) - \lambda X(x,u), \tag{31}$$

and, for $0 \le i \le m-2$, the coordinate $U_{n+k}^{(\mathrm{i})}(x, u_{n+k}^{(\mathrm{i})})$ of K must satisfy

$$U_{n+k}^{[\lambda,(i+1)]}\left(x, u_{n+k}^{(i+1)}\right) = D_x\left(U_{n+k}^{[\lambda,(i)]}\left(x, u_{n+k}^{(i)}\right)\right) - D_x(X(x))u_{n+k}^{(i+1)} + \lambda\left(U_{n+k}^{[\lambda,(i)]}\left(x, u_{n+k}^{(i)}\right)\right) - \lambda(X(x))u_{n+k}^{(i+1)}. \tag{32}$$

Hence

$$K = v^{[\lambda,(m-1)]}. \tag{33}$$

Then we apply both elements of $[K, A] = \lambda K + \mu A$, to the coordinate function $u_{n-1}^{(m-1)}$, $u_n^{(m-1)}$ and $u_{n+1}^{(m-1)}$, we obtain

$$\begin{aligned}
[K, A]\left(u_{n+k}^{(m-1)}\right) &= K\left(F_{n+k}\left(\lambda, u_{n+k-1}^{(m-1)}, u_{n+k}^{(m-1)}, u_{n+k+1}^{(m-1)}\right)\right) - A\left(U_n^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right)\right)\\
&= \lambda U_{n+k}^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right)\\
&\quad -(A(X(x)) + \lambda X(x)) \cdot \left(F_{n+k}\left(\lambda, u_{n+k-1}^{(m-1)}, u_{n+k}^{(m-1)}, u_{n+k+1}^{(m-1)}\right)\right),
\end{aligned} \tag{34}$$

where $k = -1, 0, 1$. The above equation yields

$$\begin{aligned}
&K\left(F_{n+k}\left(\lambda, u_{n+k-1}^{(m-1)}, u_{n+k}^{(m-1)}, u_{n+k+1}^{(m-1)}\right)\right)\\
&= A\left(U_{n+k}^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right)\right) + \lambda U_{n+k}^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right)\\
&\quad -(A(X(x)) + \lambda X(x)) \cdot \left(F_{n+k}\left(\lambda, u_{n+k-1}^{(m-1)}, u_{n+k}^{(m-1)}, u_{n+k+1}^{(m-1)}\right)\right).
\end{aligned} \tag{35}$$

Calculate

$$\begin{aligned}
&v^{[\lambda,(m)]}\left(u_{n+k}^{(m)} - F_{n+k}\left(x, u_{n+k-1}^{(m-1)}, u_{n+k}^{(m-1)}, u_{n+k+1}^{(m-1)}\right)\right)\\
&= D_x\left(U_{n+k}^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right)\right) - D_x(X(x))u_{n+k}^{(m)}\\
&\quad + \lambda\left(U_{n+k}^{[\lambda,(m-1)]}\left(x, u_{n+k}^{(m-1)}\right) - X(x)u_{n+k}^{(m)}\right)\\
&\quad -K\left(F_{n+k}\left(x, u_{n+k-1}^{(m-1)}, u_{n+k}^{(m-1)}, u_{n+k+1}^{(m-1)}\right)\right)
\end{aligned} \tag{36}$$

when $u_n^{(m)} = F_n(x, u_{n-1}^{(m-1)}, u_n^{(m-1)}, u_{n+1}^{(m-1)})$, we obtain, by Eq. 35, that

$$\begin{aligned}
&v^{[\lambda,(m)]}\left(\Delta\left(x, u_{n-1}^{(0)}, u_n^{(0)}, u_{n+1}^{(0)}, \ldots, u_{n-1}^{(m)}, u_n^{(m)}, u_{n+1}^{(m)}\right)\right) = 0,\\
&\textit{when} \quad u_{n+k}^{(m)} = F_{n+k}\left(\lambda, u_{n+k-1}^{(m-1)}, u_{n+k}^{(m-1)}, u_{n+k+1}^{(m-1)}\right).
\end{aligned} \tag{37}$$

Therefore v is a λ-symmetry of Eq. 19.

In order to reduce the mth-order DDEs to $(m-1)$th-order DDEs and first-order DDEs, we can determine invariants for the λ-prolongation of v by deriving invariants of lower order. This can be achieved through the application of the main tools, Theorem 2.

Theorem 3. *Let v be a vector field defined on M and let λ is a differential function, If*

$$\alpha = \alpha\left(x, u_{n-1}^{(k)}, u_n^{(k)}, u_{n+1}^{(k)}\right), \quad \beta = \beta\left(x, u_{n-1}^{(k)}, u_n^{(k)}, u_{n+1}^{(k)}\right), \tag{38}$$

are such that

$$v^{[\lambda,(k)]}\left(\alpha\left(x, u_{n-1}^{(k)}, u_n^{(k)}, u_{n+1}^{(k)}\right)\right) = v^{[\lambda,(k)]}\left(\beta\left(x, u_{n-1}^{(k)}, u_n^{(k)}, u_{n+1}^{(k)}\right)\right) = 0, \tag{39}$$

then

$$v^{[\lambda,(k+1)]}\left(\frac{D_x\alpha\left(x, u_{n-1}^{(k)}, u_n^{(k)}, u_{n+1}^{(k)}\right)}{D_x\beta\left(x, u_{n-1}^{(k)}, u_n^{(k)}, u_{n+1}^{(k)}\right)}\right) = 0. \tag{40}$$

Proof 3. By Theorem 2, we have

$$\left[v^{[\lambda,(k+1)]}, D_x\right] = \lambda v^{[\lambda,(k+1)]} + \mu D_x, \tag{41}$$

where $\mu = -D_x(v(x)) - \lambda v(x)$. Therefore,

$$\begin{aligned} v^{[\lambda,(k+1)]}\left(\frac{D_x\alpha}{D_x\beta}\right) &= \frac{1}{(D_x\beta)^2}\left(D_x\beta \cdot v^{[\lambda,(k+1)]}(D_x\alpha) - D_x\alpha \cdot v^{[\lambda,(k+1)]}(D_x\beta)\right) \\ &= \frac{1}{(D_x\beta)^2}\left(D_x\beta \cdot \left[v^{[\lambda,(k+1)]}, D_x\right](\alpha) - D_x\alpha \cdot \left[v^{[\lambda,(k+1)]}, D_x\right](\beta)\right) \\ &= \frac{1}{(D_x\beta)^2}\left(D_x\beta \cdot (\mu \cdot D_x\alpha) - D_x\alpha \cdot (\mu \cdot D_x\beta)\right) = 0. \end{aligned} \tag{42}$$

Proposition 1. *Let v be a λ-symmetry. Let*

$$y = y(x, u_{n-1}, u_n, u_{n+1}) \quad and \quad w = w\left(x, u_{n-1}, u_n, u_{n+1}, u_{n-1}^{(1)}, u_n^{(1)}, u_{n+1}^{(1)}\right)$$

be two functionally independent first-order invariants of $v^{[\lambda,(m)]}$. By solving an equation of $\Delta_r(y, w^{(m-1)}) = 0$ and an auxiliary equation $w = w(x, u_{n-1}, u_n, u_{n+1}, u_{n-1}^{(1)}, u_n^{(1)}, u_{n+1}^{(1)})$, the general solution of the equation can be obtained.

With the help of independent first-order invariant, we demonstrate a simple application of λ-symmetry. Considering a (1+1)-dimensional DDE

$$u_n^{(2)} = \left[(x + x^2)e^{u_{n+1}}\right]_x, \tag{43}$$

Eq. 43 has the from

$$u_n^{(2)} = D_x\left(F_n(x, u_{n+1})\right), \tag{44}$$

which admits the obvious order reduction

$$u_n^{(1)} = F_n(x, u_{n+1}) + C, \quad C \in \mathbb{R}. \tag{45}$$

Letting $X(x) = 0$, $U_{n-1}(x, u_{n-1}) = 1$, $U_n(x, u_n) = 1$, $U_{n+1}(x, u_{n+1}) = 1$ and $\lambda = F_{n,u_{n+1}}(x, u_{n+1})$, we have the following λ-prolongation vector field

$$\begin{aligned} v^{[\lambda,(2)]} = \frac{\partial}{\partial u_{n-1}} + \frac{\partial}{\partial u_n} + \frac{\partial}{\partial u_{n+1}} + F_{n,u_{n+1}}\left(\frac{\partial}{\partial u_{n-1}^{(1)}} + \frac{\partial}{\partial u_n^{(1)}} + \frac{\partial}{\partial u_{n+1}^{(1)}}\right) \\ + \left(F_{n,u_{n+1}}^2 + u_{n+1}^{(1)} F_{n,u_{n+1}u_{n+1}} + F_{n,xu_{n+1}}\right)\left(\frac{\partial}{\partial u_{n-1}^{(2)}} + \frac{\partial}{\partial u_n^{(2)}} + \frac{\partial}{\partial u_{n+1}^{(2)}}\right), \end{aligned} \tag{46}$$

We can easily prove that the vector field v is the λ-symmetry of Eq. 43. The λ-symmetry generator has two obvious invariants $z = x$, $w = u_n^{(1)} - F_n(x, u_{n+1})$. Furthermore, the differential invariant $w_z = \frac{D_x w}{D_x Z} = u_n^{(2)} - D_x(F_n(x, u_{n+1}))$. Therefore, Eq. 43 can be reduced to Eq. 45.

4 Conclusion

λ-symmetry reduction method is useful in establishing effective alternative methods analyze ODEs without using Lie point symmetries. At present, there is no programmatic algorithm package to solve λ-symmetry directly. Therefore, it is difficult to determine the general form of λ.

There are many examples of DDEs, without Lie point symmetries, that can be completely integrated. So we have to study the reduction of these DDEs. In this paper, we have extended the λ-symmetry reduction method to the case of (1+1)-dimensional DDEs. We have obtained some theorems Theorem 2, 3 and Proposition 1 which can be used to reduce and solve DDEs in Section 3. By comparison, DDEs can be more complex. Here we have just listed a simple example to illustrate the method. How to combine the integrating factor method and the λ-symmetry reduction method of DDEs to construct more effective examples will be the next work.

Data availability statement

The raw data supporting the conclusion of this article will be made available by the authors, without undue reservation.

Author contributions

JL: Methodology, software, formal analysis, writing-original draft. KS: Investigation, formal analysis, writing-original draft. BR: software, formal analysis. YJ: Conceptualization, funding acquisition, resources, supervision, writing-review and editing. All authors contributed to the article and approved the submitted version.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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