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Action-angle variables for the Lie–Poisson Hamiltonian systems associated with the Hirota–Satsuma modified Boussinesq equation

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In this work, we present two finite-dimensional Lie–Poisson Hamiltonian systems associated with the Hirota–Satsuma modified Boussinesq equation by using the nonlinearization method. Moreover, the separation of variables on the common level set of Casimir functions is introduced to study these systems which are associated with a non-hyperelliptic algebraic curve. Finally, in light of the Hamilton–Jacobi theory, the action-angle variables for these systems are constructed, and the Jacobi inversion problem associated with the Hirota–Satsuma modified Boussinesq equation is obtained.

KEYWORDS

Hirota–Satsuma modified Boussinesq equation, non-hyperelliptic algebraic curve, separation of variables, action-angle variables, Jacobi inversion problem

1 Introduction

The Boussinesq-type equations are typical nonlinear integrable equations in mathematical physics and mechanics. We consider the Hirota–Satsuma modified Boussinesq equation

$$u_{tt} + \frac{1}{3}\left(u_{xxx} - \frac{2}{3}u^2 u_x - 2u_x \partial_x^{-1} u_t\right)_x = 0, \tag{1}$$

introduced in Hirota and Satsuma [1], which is derived from

$$u_t = -u_{xx} + \frac{2}{3}uu_x + 2v_x, \quad v_t = \frac{2}{3}\left(-u_{xxx} + uu_{xx} + u_x v - uv_x\right) \tag{2}$$

by canceling the variable v. Here, ∂_x^{-1} stands for an inverse operator of $\partial = \partial/\partial x$ under conditions $\partial\partial_x^{-1} = \partial_x^{-1}\partial = 1$. This equation was initially proposed by Hirota and Satsuma [1] from a Bäcklund transformation of the Boussinesq equation

$$w_{tt} + \frac{1}{3}\left(w_{xx} - 4w^2\right)_{xx} = 0,$$

which describes the motion of long waves which are propagated in both directions in shallow water under gravity. Similarity solutions to Eq. 1 are discussed in Quispel et al. [2]; Clarkson [3]. It is shown that this equation has a Lax pair associated with the 3 × 3 matrix spectral problem, from which the Darboux transformation is derived with the help of gage transformation Geng [4]. The corresponding finite-dimensional completely integrable systems in the Liouville sense were derived. As an application, solutions to Eq. 1 are decomposed into solving two compatible Hamiltonian systems of ordinary differential equations Dai and Geng [5]. The explicit Riemann theta function representations of solutions for the Hirota–Satsuma modified Boussinesq hierarchy were studied in He et al. [6].

The separation of variables for finite-dimensional integrable systems is important for constructing action-angle variables. A series of literature studies shows research on finite-dimensional integrable systems associated with hyperelliptic spectral curves (see, e.g., Kuznetsov [7]; Babelon and Talon [8]; Kalnins et al; [9]; Eilbeck et al; [10]; Harnad and Winternitz [11]; Ragnisco [12]; Kulish et al; [13]; Qiao [14]; Zeng [15]; Zhou [16]; Zeng and Lin [17]; Cao et al; [18]; Derkachev [19]; Du and Geng [20]; Du and Yang [21]). However, the study on integrable systems associated with non-hyperelliptic spectral curves is much more complicated (see, e.g., Sklyanin [22]; Adams et al; [23]; Buchstaber et al; [24]; Dickey [25]; Derkachov and Valinevich [26]).

Sklyanin introduced a powerful method of constructing the separated variables for the classical integrable $SL(3)$ magnetic chain, which is associated with a non-hyperelliptic algebraic curve Sklyanin [22]. By this effective way, more general cases are studied Scott [27]; Gekhtman [28]; Dubrovin and Skrypnyk [29]. We follow this method to construct the separable variables for the Lie–Poisson Hamiltonian associated with the Hirota–Satsuma modified Boussinesq Eq. 1 on the common level set of Casimir functions and define action-angle variables with the help of the Hamilton–Jacobi equation. Furthermore, the Jacobi inversion problem for the Hirota–Satsuma modified Boussinesq equation is obtained with action-angle variables.

This paper is organized as follows. In the following section, we will review the Lie–Poisson structure associated with $\mathfrak{sl}(3)$. In Section 3, in the framework of the Lie–Poisson structure on $\mathfrak{sl}(3)$, two Lie–Poisson Hamiltonian systems associated with the Hirota–Satsuma modified Boussinesq Eq. 1 are presented by using the nonlinearization of the adjoint representations of the 3 × 3 spectral problem and auxiliary spectral one. Moreover, the involution property of conserved integrals is discussed by using the generating function method. In Section 4, on the common level set of Casimir functions, the separated variables are introduced to study these Lie–Poisson Hamiltonian systems. In Section 5, in light of the Hamilton–Jacobi theory, the generating function S for obtaining the canonical transformation from separated variables to action-angle variables is obtained. In Section 6, in terms of the evolution of action-angle variables, the functional independence of conserved integrals is elucidated. Finally, the Jacobi inversion problems for those Lie–Poisson Hamiltonian systems and the Hirota–Satsuma modified Boussinesq Eq. 1 are built.

2 Preliminary

In this section, we introduce some basic notations of Lie–Poisson structures associated with Lie algebra $\mathfrak{sl}(3)$.

The Lie algebra $\mathfrak{sl}(3)$ has an invariant nondegenerate symmetric form $<A,B> = \mathrm{tr}(AB)$ by means of which we can make an identification $\mathfrak{sl}(3) \cong \mathfrak{sl}(3)^*$. For convenience, we choose

$$\mathfrak{sl}(3) = \left\{ \alpha | \alpha = \sum_{i,j=1}^{3} \alpha_{ij} e_{ij}, \mathrm{tr}(\alpha) = 0 \right\},$$
$$\mathfrak{sl}(3)^* = \left\{ y | y = \sum_{i,j=1}^{3} y^{ij} E_{ij}, \mathrm{tr}(y) = 0 \right\},$$

where

$$E_{ij} = \left(\delta_{mi}\delta_{nj}\right),\ 1 \le i, j \le 3,$$

are the basis of Lie algebra $\mathfrak{sl}(3)^*$, and the dual bases are given by $\{e_{ij} = E_{ji},\ 1 \le i, j \le 3\}$. We can confirm that these bases satisfy the commutation relation

$$\left[E_{ij}, E_{kl}\right] = \delta_{jk} E_{il} - \delta_{li} E_{kj}.$$

Thus, for any functions $F(y),\ G(y) \in C^\infty(\mathfrak{sl}(3)^*)$, the corresponding Lie–Poisson bracket at the point $y \in \mathfrak{sl}(3)^*$ is

$$\{F, G\}(y) = \langle y, [\nabla F, \nabla G]\rangle = \mathrm{tr}\left(y[\nabla F, \nabla G]\right), \tag{3}$$

with the gradient $\nabla F \in \mathfrak{sl}(3)$ defined as

$$\nabla F = \sum_{k,l=1}^{3} \frac{\partial F}{\partial y^{kl}} e_{kl}.$$

The Hamiltonian vector field associated with (3) by a smooth function $F(y) \in C^\infty(\mathfrak{sl}(3)^*)$ is represented as

$$X_F = [\nabla F, y].$$

The Lie–Poisson structure equations in terms of variables $\{y^{ij},\ 1 \le i, j \le 3\}$ are

$$\left\{y^{lk}, y^{mn}\right\} = < y, [E_{kl}, E_{nm}] > = \delta_{ln} y^{mk} - \delta_{mk} y^{ln},\ 1 \le n, m, l, k \le 3. \tag{4}$$

The two Casimir functions of the Lie–Poisson structure Eq. 3 are

$$\mathrm{tr}\left(y^2\right),\ \mathrm{tr}\left(y^3\right).$$

If we take the direct product of N copies of $\mathfrak{sl}(3)^*$, the Lie–Poisson structure becomes

$$\{F, G\}\left(y_j\right) = \sum_{j=1}^{N} \langle y_j, \left[\nabla_j F, \nabla_j G\right]\rangle,\ \nabla_j F = \sum_{k,l=1}^{3} \frac{\partial F}{\partial y_j^{kl}} e_{kl}, \tag{5}$$

and the Hamiltonian vector field associated with a smooth function F is

$$X_{jF} = \left[\nabla_j F, y_j\right], \quad j = 1, \ldots, N,$$

and the $2N$ Casimir functions

$$\mathrm{tr}\left(y_j^2\right),\ \ \mathrm{tr}\left(y_j^3\right),\ \ j = 1, \ldots, N.$$

3 The Lie–Poisson Hamiltonian systems for the Hirota–Satsuma modified Boussinesq equation

According to the Lie–Poisson bracket Eq. 5 on N copies of $\mathfrak{sl}(3)^*$, we discuss the finite-dimensional Lie–Poisson Hamiltonian systems associated with the Hirota–Satsuma modified Boussinesq Eq. 1:

$$y_{jx} = \left[\nabla_j H, y_j\right], \quad j = 1, \ldots, N, \tag{6}$$

and

$$y_{jt} = \left[\nabla_j H_1, y_j\right], \quad j = 1, \ldots, N, \tag{7}$$

with Hamiltonians

$$H = r_0^{21} + r_0^{13} + r_1^{32} + 3r_0^{12}r_0^{22} - 2\left(r_0^{12}\right)^3, \tag{8}$$

and

$$H_1 = \left(r_0^{11}\right)^2 + r_0^{22}r_0^{11} + \left(r_0^{22}\right)^2 + r_0^{23} + r_0^{12}r_0^{21} + r_1^{31} + r_1^{12} - 2r_0^{13}r_0^{12} + r_0^{12}r_1^{32}, \tag{9}$$

with $\lambda_1, \ldots, \lambda_N$ being N distinct parameters and $r_m^{kl} = \sum_{j=1}^N \lambda_j^m y_j^{kl}$.

In fact, the Lie–Poisson Hamiltonian systems Eqs 6, 7 are derived from the 3×3 matrix spectral problem

$$\varphi_x = U\varphi,\ \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, U = \begin{pmatrix} 0 & 1 & 0 \\ v & u & \lambda \\ 1 & 0 & 0 \end{pmatrix}, \tag{10}$$

and the auxiliary spectral problem

$$\varphi_t = V\varphi,\ V = \begin{pmatrix} -\frac{2}{3}u_x + v & \frac{1}{3}u & \lambda \\ \lambda - \frac{2}{3}u_{xx} + v_x + \frac{1}{3}uv & -\frac{1}{3}u_x + \frac{1}{3}u^2 + v & \frac{1}{3}\lambda u \\ -\frac{2}{3}u & 1 & 0 \end{pmatrix}, \tag{11}$$

where u, v are the potentials and λ is a constant spectral parameter. The adjoint representations of the spectral problems Eqs 10, 11 are given by

$$y_x = [U, y], \tag{12}$$

and

$$y_t = [V, y], \tag{13}$$

respectively. In order to obtain the Lie–Poisson Hamiltonian systems associated with the Hirota–Satsuma modified Boussinesq Eq. 1, we take N copies of (12)

$$y_{jx} = \left[U\left(\lambda_j\right), y_j\right], \quad j = 1, \ldots, N, \tag{14}$$

and N copies of (13)

$$y_{jt} = \left[V\left(\lambda_j\right), y_j\right], \quad j = 1, \ldots, N. \tag{15}$$

Now, under the constraint

$$u = 3r_0^{12}, \quad v = 3r_0^{22} - 6\left(r_0^{12}\right)^2, \tag{16}$$

Eqs 14, 15 are nonlinearized into the Lie–Poisson Hamiltonian systems Eqs 6, 7, respectively.

The Lax representation and the involution property of conserved integrals are also given by using the generating function method.

Since the Lie–Poisson structure Eq. 5 has $2N$ Casimir functions

$$\mathrm{tr}\left(y_j^2\right),\ \mathrm{tr}\left(y_j^3\right), \quad j = 1, \ldots, N,$$

thus to prove the integrability of the Lie–Poisson Hamiltonian systems Eqs 6, 7, it is necessary to find $3N$ functionally independent Poisson commuting integrals. By using the constraint Eq. 16, after a direct calculation, we can get the following proposition.

Proposition 1. *The Lie–Poisson Hamiltonian systems* Eqs 6, 7 *admit the Lax representations*

$$\frac{\mathrm{d}}{\mathrm{d}x}V_\lambda = [U, V_\lambda],$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}V_\lambda = [V, V_\lambda],$$

respectively, where

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 3r_0^{22} - 6\left(r_0^{12}\right)^2 & 3r_0^{12} & \lambda \\ 1 & 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} 2r_0^{11} + r_0^{22} & r_0^{12} & \lambda \\ \lambda + r_0^{21} - 2r_0^{13} + r_1^{32} & 2r_0^{22} + r_0^{11} & \lambda r_0^{12} \\ -2r_0^{12} & 1 & 0 \end{pmatrix},$$

and

$$V_\lambda = \left(V_{ij}(\lambda)\right)_{3\times 3} = \beta(\lambda) + \sum_{j=1}^N \frac{y_j}{\lambda - \lambda_j}, \tag{17}$$

with

$$\beta(\lambda) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & r_0^{12} \\ -\lambda^{-1}\left(r_0^{31} + 2r_0^{12}\right) & \lambda^{-1}\left(1 - r_0^{32}\right) & 0 \end{pmatrix}.$$

It follows that the integrals of motion for the Lie–Poisson Hamiltonian systems Eqs 6, 7 are provided by the spectral invariants of Lax matrix V_λ. Therefore, one has the generating function of integrals for systems Eqs 6, 7:

$$\mathcal{F}_2(\lambda) = \frac{1}{2}\mathrm{tr}\left(V_\lambda^2\right),\ \mathcal{F}_3(\lambda) = \frac{1}{3}\mathrm{tr}\left(V_\lambda^3\right). \tag{18}$$

Furthermore, substituting Eqs 17, 18, we have

$$\mathcal{F}_2(\lambda) = \frac{1}{2}\mathrm{tr}\left(V_\lambda^2\right) = \frac{1}{2}\mathrm{tr}\left(\beta(\lambda)^2\right) + \sum_{j=1}^N \frac{q_{1j}}{\lambda - \lambda_j} + \sum_{j=1}^N \frac{h_{2j}}{\left(\lambda - \lambda_j\right)^2} := \sum_{l=1}^\infty \frac{F_l^S}{\lambda^{l+1}}, \tag{19}$$

where

$$q_{1j} = \mathrm{tr}\left(\beta(\lambda)y_j\right) + \sum_{k\neq j}^N \frac{\mathrm{tr}\left(y_j y_k\right)}{\lambda_j - \lambda_k},\ h_{2j} = \frac{1}{2}\mathrm{tr}\left(y_j^2\right),$$

$$F_l^S = \sum_{j=1}^N \lambda_j^l q_{1j} + l\sum_{j=1}^N \lambda_j^{l-1}h_{2j},\ l = 1, \ldots,$$

and

$$\mathcal{F}_3(\lambda) = \frac{1}{3}\mathrm{tr}\left(V_\lambda^3\right) = \frac{1}{3}\mathrm{tr}\left(\beta(\lambda)^3\right) + \sum_{j=1}^N \frac{q_{2j}}{\lambda - \lambda_j} + \sum_{j=1}^N \frac{q_{3j}}{\left(\lambda - \lambda_j\right)^2} + \sum_{j=1}^N \frac{h_{3j}}{\left(\lambda - \lambda_j\right)^3} := \sum_{l=0}^\infty \frac{F_l^T}{\lambda^{l+1}}, \tag{20}$$

where

$$q_{2j} = \sum_{k=1}^{N} \frac{1}{\lambda_j - \lambda_k} \left[\mathrm{tr}\left(\beta(\lambda) y_j y_k + \beta(\lambda) y_k y_j\right) + \sum_{i\neq k,j}^{N} \frac{\mathrm{tr}\left(y_j y_k y_i + y_j y_i y_k\right)}{\lambda_k - \lambda_i} + \sum_{i\neq k,j}^{N} \frac{\mathrm{tr}\left(y_j y_k y_i + y_j y_i y_k\right)}{\lambda_j - \lambda_i} + \mathrm{tr}\left(\beta(\lambda)^2 y_j\right) + \sum_{k\neq j}^{N} \frac{\mathrm{tr}\left(y_k^2 y_j - y_j^2 y_k\right)}{\left(\lambda_j - \lambda_k\right)^2}, \right.$$

$$q_{3j} = \mathrm{tr}\left(\beta(\lambda) y_j^2\right) + \sum_{k\neq j} \frac{\mathrm{tr}\left(y_j^2 y_k\right)}{\lambda_j - \lambda_k}, \quad h_{3j} = \frac{1}{3}\mathrm{tr}\left(y_j^3\right),$$

$$F_l^T = \sum_{j=1}^{N} \lambda_j^l q_{2j} + l \sum_{j=1}^{N} \lambda_j^{l-1} q_{3j} + \frac{1}{2} l(l-1) \sum_{j=1}^{N} \lambda_j^{l-2} h_{3j}, \; l = 0, 1, \ldots.$$

From the expressions of $\mathcal{F}_2(\lambda)$ and $\mathcal{F}_3(\lambda)$ in (19) and 20, we know that for $j = 1, \ldots, N$, q_{1j}, q_{2j}, q_{3j} provide $3N$ generators of conserved integrals for systems Eqs 6, 7. The Hamiltonian functions Eqs 8, 9 can also be written as

$$H = F_1^T \tag{21}$$

and

$$H_1 = F_1^S, \tag{22}$$

respectively.

Denoting the variables of $\mathcal{F}_2(\lambda)$-flow and $\mathcal{F}_3(\lambda)$-flow by $t_{2\lambda}$ and $t_{3\lambda}$, respectively, let $V_\lambda^2 = (v_{ij}(\lambda))_{3\times 3}$; then, the Hamiltonian equations for $\mathcal{F}_2(\lambda)$ and $\mathcal{F}_3(\lambda)$ are

$$y_{jt_{k\lambda}} = \left[\nabla_j \mathcal{F}_k(\lambda), y_j\right] = \frac{1}{\lambda - \lambda_j}\left[V_\lambda^{k-1}, y_j\right] + \left[\Delta_{k-1}, y_j\right], \; k = 2, 3, \; j = 1, \ldots, N, \tag{23}$$

where

$$\Delta_1 = \begin{pmatrix} 0 & 0 & -\lambda^{-1} V_{13}(\lambda) \\ V_{32}(\lambda) - 2\lambda^{-1} V_{13}(\lambda) & 0 & -\lambda^{-1} V_{23}(\lambda) \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Delta_2 = \begin{pmatrix} 0 & 0 & -\lambda^{-1} v_{13}(\lambda) \\ v_{32}(\lambda) - 2\lambda^{-1} v_{13}(\lambda) & 0 & -\lambda^{-1} v_{23}(\lambda) \\ 0 & 0 & 0 \end{pmatrix}.$$

Taking the sum of Eq. 23 with respect to j from 1 to N, we have

$$\sum_{j=1}^{N} y_{jt_{k\lambda}} = \left[\Delta_{k-1}, \sum_{j=1}^{N} y_j\right] - \left[V_\lambda^{k-1}, \beta(\lambda)\right],$$

from which we arrive at

$$\beta(\tau)_{t_{k\lambda}} = \frac{1}{\lambda - \tau}\left[V_\lambda^{k-1}, \beta(\tau) - \beta(\lambda)\right] + \left[\Delta_{k-1}, \beta(\tau)\right], \; k = 2, 3. \tag{24}$$

For Casimir functions tr (y_j), $1 \le j \le N$, it is evident that

$$\mathrm{tr}\left(y_j\right)_{t_{k\lambda}} = 0, \; k = 2, 3. \tag{25}$$

Proposition 2. *The Lax matrix V_τ satisfies the Lax equations along the $\mathcal{F}_k(\lambda)$-flows:*

$$\frac{\mathrm{d}}{\mathrm{d}t_{k\lambda}} V_\tau = \left[\frac{1}{\lambda - \tau} V_\lambda^{k-1} + \Delta_{k-1}, V_\tau\right], \; k = 2, 3.$$

Proof. By using (23), (24), and (25), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t_{k\lambda}} V_\tau &= \sum_{j=}^{N} \frac{1}{\tau - \lambda_j} y_{jt_{k\lambda}} + \beta(\tau)_{t_{k\lambda}} \\ &= \left[\frac{1}{\lambda - \tau} V_\lambda^{k-1} + \Delta_{k-1}, V_\tau\right] + \beta(\tau)_{t_{k\lambda}} \\ &\quad - \frac{1}{\lambda - \tau}\left[V_\lambda^{k-1}, \beta(\tau) - \beta(\lambda)\right] - \left[\Delta_{k-1}, \beta(\tau)\right] \\ &= \left[\frac{1}{\lambda - \mu} V_\lambda^{k-1} + \Delta_{k-1}, V_\mu\right]. \end{aligned} \tag{26}$$

Based on Proposition 2, for any λ, τ, it is easy to verify that for l, $k = 2, 3$,

$$\begin{aligned} \{\mathcal{F}_l(\tau), \mathcal{F}_k(\lambda)\} &= \frac{\mathrm{d}}{\mathrm{d}t_{k\lambda}} \mathcal{F}_l(\tau) = \frac{1}{l}\mathrm{tr}\left(\frac{\mathrm{d}}{\mathrm{d}t_{k\lambda}} V_\tau^l\right) \\ &= \frac{1}{l}\mathrm{tr}\left(\left[\frac{1}{\lambda - \tau} V_\lambda^{k-1}, V_\tau^l\right]\right) = 0, \end{aligned}$$

from which we have $\{q_{hj}, q_{im}\} = 0$, $h, i = 1, 2, 3$, $j, m = 1, \ldots, N$.

Corollary 1. *F_l^S, F_l^T, $l \ge 1$ are in involution in pairs with respect to the Lie–Poisson bracket* Eq. 5.

By observing Eqs 21, 22, we know that $\{H, H_1\} = 0$. Thus, some solutions of the Hirota–Satsuma modified Boussinesq Eq. 1 can be obtained by solving two compatible Hamiltonian systems of ordinary differential equations.

Proposition 3. *Let y_j be a compatible solution of the Lie–Poisson Hamiltonian systems* Eqs 6, 7, *then*

$$u = 3r_0^{12}, \quad v = 3r_0^{22} - 6\left(r_0^{12}\right)^2$$

solves the Hirota–Satsuma modified Boussinesq Eq. (1).

4 Separation of variables

In this section, we construct the separable variables on the common level set of the Casimir functions

$$\left\{y_1, \ldots, y_j, \ldots, y_N \middle| \mathrm{tr}\left(y_j^2\right) = c_{2j}, \; \mathrm{tr}\left(y_j^3\right) = c_{3j}, \; j = 1, \ldots, N\right\} \tag{27}$$

to deal with the Lie–Poisson Hamiltonian systems. The characteristic polynomial of Lax matrix V_λ for the Hirota–Satsuma modified Boussinesq Eq. 1 is an independent constant with variables x and t in the expansion

$$\det(zI - V_\lambda) = z^3 - \mathcal{F}_2(\lambda) z - \mathcal{F}_3(\lambda), \tag{28}$$

which defines a non-hyperelliptic algebraic curve of genus $\mathcal{G} = 3N - 2$ by introducing variable $\zeta = a(\lambda)z$:

$$\zeta^3 + a^2(\lambda)\mathcal{F}_2(\lambda)\zeta - a^3(\lambda)\mathcal{F}_3(\lambda) = 0,$$

where

$$a(\lambda) = \prod_{j=1}^{N}\left(\lambda - \lambda_j\right).$$

With the application of Sklyanin's method given in Sklyanin [22], a half of the variables of separation μ_i ($i = 1, \ldots, 3N-2$) should be defined as zeros of some polynomial $B(\lambda)$ with degree $3N-2$, and the corresponding conjugate variables ν_i ($i = 1, \ldots, 3N-2$) are related to μ_i by the secular equation

$$\nu_i^3 - \mathcal{F}_2(\mu_i)\nu_i - \mathcal{F}_3(\mu_i) = 0. \tag{29}$$

It follows from (28) that ν_i should be an eigenvalue of the matrix V_{μ_i}. Therefore, there must exist such a similarity transformation

$$V_{\mu_i} \to \tilde{V}_{\mu_i} = K_i V_{\mu_i} K_i^{-1}$$

for each i that the matrix $\tilde{V}_{\mu_i}$ is block-triangular

$$\tilde{V}_{21}(\mu_i) = \tilde{V}_{31}(\mu_i) = 0, \tag{30}$$

and ν_i is the eigenvalue of V_{μ_i} split from the upper block

$$\nu_i = \tilde{V}_{11}(\mu_i). \tag{31}$$

Therefore, the problem is reduced to a determination of the matrix K_i and polynomial $B(\lambda)$. Let us consider $K(k)$ to be as follows:

$$K(k) = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the matrix

$$\begin{aligned}\tilde{V}_\lambda &\equiv K(k)V_\lambda K^{-1}(k)\\ &= \begin{pmatrix} V_{11}(\lambda) - kV_{12}(\lambda) & V_{12}(\lambda) & V_{13}(\lambda) \\ V_{21}(\lambda) + kV_{11}(\lambda) - k(kV_{12}(\lambda) + V_{22}(\lambda)) & V_{22}(\lambda) + kV_{12}(\lambda) & V_{23} + kV_{13}(\lambda) \\ V_{31}(\lambda) - kV_{32}(\lambda) & V_{32}(\lambda) & V_{33}(\lambda) \end{pmatrix}\end{aligned}$$

depends on two parameters λ and k. Hence, we can consider condition Eq. 30 as the set of two algebraic equations

$$\begin{cases} \tilde{V}_{21}(\lambda) & = V_{21}(\lambda) + kV_{11}(\lambda) - k(kV_{12}(\lambda) + V_{22}(\lambda)) = 0, \\ \tilde{V}_{31}(\lambda) & = V_{31}(\lambda) - kV_{32}(\lambda) = 0 \end{cases} \tag{32}$$

for two variables λ and k. By eliminating k from (32) yields the polynomial equation for λ:

$$\begin{aligned} &V_{32}(\lambda)V_{31}(\lambda)[V_{11}(\lambda) - V_{22}(\lambda)] + V_{32}(\lambda)^2 V_{21}(\lambda) - V_{31}(\lambda)^2 V_{12}(\lambda). \\ &= 0. \end{aligned} \tag{33}$$

Based on (33), we can define the polynomial $B(\lambda)$ of degree $3N$ as

$$\begin{aligned} B(\lambda) &= V_{32}(\lambda)V_{31}(\lambda)[V_{11}(\lambda) - V_{22}(\lambda)] + V_{32}(\lambda)^2 V_{21}(\lambda) \\ &\quad - V_{31}(\lambda)^2 V_{12}(\lambda). := \frac{n(\lambda)}{a(\lambda)^3}, \end{aligned} \tag{34}$$

where

$$n(\lambda) = \prod_{i=1}^{3N-2}(\lambda - \mu_i),\ a(\lambda) = \prod_{j=1}^{N}(\lambda - \lambda_j) := \sum_{j=0}^{N} a_j \lambda^{N-j}\ (a_0 = 1). \tag{35}$$

Expressing k from $\tilde{V}_{31}(\lambda) = 0$ as $k = V_{31}(\lambda)/V_{32}(\lambda)$ and substituting it into the definition Eq. 31 of ν_i yields

$$\nu_i = \tilde{V}_{11}(\mu_i) = V_{11}(\mu_i) - \frac{V_{12}(\mu_i)V_{31}(\mu_i)}{V_{32}(\mu_i)}, \quad i = 1, \ldots, 3N-2, \tag{36}$$

thereby giving rise to $3N$ pairs of variables μ_i, ν_i. Let

$$A(\lambda) = V_{11}(\lambda) - \frac{V_{12}(\lambda)V_{31}(\lambda)}{V_{32}(\lambda)}, \tag{37}$$

with the help of (4) and (17), it is easy to see that

$$\{V_{lk}(\tau), V_{mn}(\lambda)\} = \frac{1}{\lambda - \tau}[(V_{mk}(\tau) - V_{mk}(\lambda))\delta_{ln} - (V_{ln}(\tau) - V_{ln}(\lambda))\delta_{mk}],$$

from which, together with the definitions of B by (34) and A by (37), the Lie–Poisson brackets for $B(\lambda)$ and $A(\tau)$ satisfy

$$\begin{cases} \{A(\tau), A(\lambda)\} = 0, \\ \{B(\tau), B(\lambda)\} = 0, \\ \{A(\tau), B(\lambda)\} = \dfrac{1}{\lambda - \tau}\left(B(\lambda) - \dfrac{V_{32}^2(\lambda)}{V_{32}^2(\tau)}B(\tau)\right). \end{cases} \tag{38}$$

Proposition 4. *$\{\mu_i, \nu_i, 1 \le i \le 3N\}$ are canonical coordinates, that is,*

$$\{\mu_i, \mu_j\} = 0,\ \{\nu_i, \nu_j\} = 0,\ \{\nu_i, \mu_j\} = \delta_{ij}.$$

Proof. The commutativity of Bs Eq. 38 obviously entrains the commutativity of μ_j (zeros of $B(\lambda)$). The Poisson brackets including ν_j can be calculated by using the implicit definition of μ_j. From $B(\mu_j) = 0$, for $j = 1, \ldots, 3N$, it follows that

$$0 = \{F, B(\mu_j)\} = \{F, B(\lambda)\}|_{\lambda=\mu_j} + B'(\mu_j)\{F, \mu_j\}$$

or

$$\{F, \mu_j\} = -\frac{\{F, B(\lambda)\}|_{\lambda=\mu_j}}{B'(\mu_j)}, \tag{39}$$

for any function F, in the same way, we have

$$\{\nu_i, F\} = \{A(\mu_i), F\} = \{A(\mu), F\}|_{\mu=\mu_i} + A'(\mu_i)\{\mu_i, F\}.$$

Now, we turn to prove $\{\nu_i, \mu_j\} = \delta_{ij}$. Starting with

$$\{\nu_i, \mu_j\} = \{A(\mu), \mu_j\}|_{\mu=\mu_i} + A'(\mu_i)\{\mu_i, \mu_j\} = \{A(\mu), \mu_j\}|_{\mu=\mu_i},$$

using (39) and the third equation of (38), we arrive at

$$\{\nu_i, \mu_j\} = -\frac{\{A(\mu), B(\lambda)\}|_{\lambda=\mu_j}^{\mu=\mu_i}}{B'(\mu_j)} = \frac{1}{\mu_i - \mu_j}\left(\frac{V_{32}^2(\mu_j)}{V_{32}^2(\mu_i)}B(\mu_i) - B(\mu_j)\right).$$

The last expression vanishes for $\mu_i \neq \mu_j$ due to $B(\mu_i) = B(\mu_j) = 0$ and is evaluated via L'Hôpital's rule for $\mu_i = \mu_j$ to produce the proclaimed result. The commutativity of νs can be shown in the same way, starting from the first equation of (38).

5 Action-angle variables and Jacobi inversion problems

Let us start with

$$\begin{aligned} &\frac{1}{2}\mathrm{tr}(\beta(\lambda)^2) + \sum_{j=1}^{N}\frac{q_{1j}}{\lambda - \lambda_j} := \frac{b_2(\lambda)}{a(\lambda)} := \sum_{l=1}^{\infty}\frac{f_l^S}{\lambda^{l+1}}, \\ &\frac{1}{3}\mathrm{tr}(\beta(\lambda)^3) + \sum_{j=1}^{N}\frac{I_{2j}}{\lambda - \lambda_j} + \sum_{j=1}^{N}\frac{I_{3j}}{(\lambda - \lambda_j)^2} := \frac{b_3(\lambda)}{a^2(\lambda)} := \sum_{l=0}^{\infty}\frac{f_l^T}{\lambda^{l+1}}, \end{aligned}$$

where

$$\begin{aligned} b_2(\lambda) &= I_1\lambda^{N-2} + I_2\lambda^{N-3}\cdots + I_{N-3}\lambda^2 + I_{N-2}\lambda + I_{N-1}, \\ b_3(\lambda) &= \lambda^{2N-1} + I_N\lambda^{2N-2} + \cdots + I_{3N-3}\lambda + I_{3N-2}, \end{aligned} \tag{40}$$

from which we can rewrite the generating functions $\mathcal{F}_2(\lambda)$, $\mathcal{F}_3(\lambda)$ as

$$\mathcal{F}_2(\lambda) = \frac{b_2(\lambda)}{a(\lambda)} + \sum_{j=1}^{N}\frac{C_{2j}}{(\lambda-\lambda_j)^2} = \sum_{l=1}^{\infty}\frac{f_l^S}{\lambda^{l+1}} + \sum_{j=1}^{N}\frac{C_{2j}}{(\lambda-\lambda_j)^2} := \frac{R_2(\lambda)}{a^2(\lambda)}, \tag{41}$$

$$\mathcal{F}_3(\lambda) = \frac{b_3(\lambda)}{a^2(\lambda)} + \sum_{j=1}^{N}\frac{C_{3j}}{(\lambda-\lambda_j)^3} = \sum_{l=0}^{\infty}\frac{f_l^T}{\lambda^{l+1}} + \sum_{j=1}^{N}\frac{C_{3j}}{(\lambda-\lambda_j)^3}, \tag{42}$$

with $R_2(\lambda) = a(\lambda)b_2(\lambda) + a^2(\lambda)\sum_{j=1}^{N}\frac{C_{2j}}{(\lambda-\lambda_j)^2}$, $C_{2j} = \frac{1}{2}c_{2j}$, $C_{3j} = \frac{1}{3}c_{3j}$.

The comparison of the coefficients of λ^l $(l = 0, \ldots, N-1)$ in equation

$$b_2(\lambda) = a(\lambda)\left(\sum_{l=1}^{\infty}\frac{f_l^S}{\lambda^{l+1}}\right)$$

and the comparison of the coefficients of λ^l $(l = 0, 1, \ldots, 2N-1)$ in equation

$$b_3(\lambda) = a^2(\lambda)\left(\sum_{l=0}^{\infty}\frac{f_l^T}{\lambda^{l+1}}\right),$$

respectively, yield

$$I_j = \sum_{i=1}^{j} a_i f_{j-i}^S, \quad j = 1,\ldots,N-1,$$
$$I_{N+k} = \sum_{l=0}^{k+1}\left(\sum_{\substack{i,j\geq 0\\ i+j=l}} a_i a_j\right) f_{k+1-l}^T, \quad k = 0,\ldots,2N-2.$$

Let

$$\nu_i = \frac{\partial S}{\partial \mu_i}, \quad i = 1,\ldots,3N-2,$$

with the help of Eq. 29, we have the completely separable Hamilton–Jacobi equations:

$$\left(\frac{\partial S}{\partial \mu_i}\right)^3 - \left(\frac{b_2(\mu_i)}{a(\mu_i)} + \sum_{j=1}^{N}\frac{C_{2j}}{(\mu_i-\lambda_j)^2}\right)\frac{\partial S}{\partial \mu_i} - \left(\frac{b_3(\mu_i)}{a^2(\mu_i)} + \sum_{j=1}^{N}\frac{C_{3j}}{(\mu_i-\lambda_j)^3}\right) = 0,$$

for $i = 1, \ldots, 3N-2$, from which we can obtain an implicit complete integral of Hamilton–Jacobi equations for the generating functions $\mathcal{F}_2(\lambda)$ and $\mathcal{F}_3(\lambda)$:

$$S = \sum_{j=1}^{3N-2} S_j(\mu_j) = S(\mu_1,\ldots,\mu_{3N-2}; I_1,\ldots,I_{3N-2}) = \sum_{j=1}^{3N-2}\int_0^{\mu_j} z\,\mathrm{d}\lambda, \tag{43}$$

where z satisfies Eq. 28.

Now, let us consider a canonical transformation from (μ, ν) to (ϕ, I) generated by the generating function S:

$$\sum_{i=1}^{3N-2}\nu_i \mathrm{d}\mu_i + \sum_{i=1}^{3N-2}\phi_i \mathrm{d}I_i = \mathrm{d}S,$$

which satisfies

$$\nu_i = \frac{\partial S}{\partial \mu_i}, \quad \phi_i = \frac{\partial S}{\partial I_i}. \tag{44}$$

From Eqs 28, 41–44, we have

$$\phi_i = \frac{\partial S}{\partial I_i} = \sum_{j=1}^{3N-2}\int_0^{\mu_j}\frac{\partial z}{\partial I_i}\mathrm{d}\lambda = \begin{cases} \sum_{j=1}^{3N-2}\int_0^{\mu_j}\frac{a(\lambda)z\lambda^{N-i-1}}{R(\lambda)}\mathrm{d}\lambda, & i = 1,\ldots,N-1, \\ \sum_{j=1}^{3N-2}\int_0^{\mu_j}\frac{\lambda^{3N-i-2}}{R(\lambda)}\mathrm{d}\lambda, & i = N,\ldots,3N-2, \end{cases} \tag{45}$$

where $R(\lambda) = 3a^2(\lambda)z^2 - R_2(\lambda)$. Thus, by using (40), (41), and (42), the generating functions of integrals can be rewritten as

$$\begin{aligned}\mathcal{F}_2(\lambda) &= \sum_{j=1}^{N}\frac{C_{2j}}{(\lambda-\lambda_j)^2} + \frac{I_1\lambda^{N-2} + \cdots + I_{N-1}}{a(\lambda)} := K_2(I_1,\ldots,I_{N-1},\lambda), \mathcal{F}_3(\lambda) \\ &= \sum_{j=1}^{N}\frac{C_{3j}}{(\lambda-\lambda_j)^3} + \frac{\lambda^{2N-1} + I_N\lambda^{2N-2} + \cdots + I_{3N-2}}{a^2(\lambda)} := K_3(I_N,\ldots,I_{3N-2},\lambda).\end{aligned}$$

The variables $I_1, \ldots, I_{3N-2}$ will be variables of action type, and the conjugate variables $\phi_1, \ldots, \phi_{3N-2}$ will be the corresponding angles.

The Hamiltonian canonical equations for the generating functions $\mathcal{F}_2(\lambda)$, $\mathcal{F}_3(\lambda)$ in terms of action-angle variables I_j, ϕ_j, $j = 1, \ldots, 3N-2$ are

$$\phi_{jt_{2\lambda}} = \begin{cases} \frac{\partial K_2(\lambda)}{\partial I_j} = \frac{\lambda^{N-j-1}}{a(\lambda)}, & 1\leq j\leq N-1 \\ \frac{\partial K_2(\lambda)}{\partial I_j} = 0, & N\leq j\leq 3N-2 \end{cases}, \quad I_{jt_{2\lambda}} = -\frac{\partial K_2(\lambda)}{\partial \phi_j} = 0, \quad 1\leq j\leq 3N-2, \tag{46}$$

$$\phi_{jt_{3\lambda}} = \begin{cases} \frac{\partial K_3(\lambda)}{\partial I_j} = 0, & 1\leq j\leq N-1 \\ \frac{\partial K_3(\lambda)}{\partial I_j} = \frac{\lambda^{3N-j-2}}{a^2(\lambda)}, & N\leq j\leq 3N-2 \end{cases}, \quad I_{jt_{3\lambda}} = -\frac{\partial K_3(\lambda)}{\partial \phi_j} = 0, \quad 1\leq j\leq 3N-2. \tag{47}$$

Proposition 5. *Let $t_{2,l}$ and $t_{3,l}$ be the variables of F_l^S-flow and F_l^T-flow, respectively; then, we have*

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}t_{2,1}},\ldots,\frac{\mathrm{d}\phi}{\mathrm{d}t_{2,N-1}},\frac{\mathrm{d}\phi}{\mathrm{d}t_{3,1}},\ldots,\frac{\mathrm{d}\phi}{\mathrm{d}t_{3,2N-1}}\right) = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22}\end{pmatrix}, \tag{48}$$

where

$$Q_{11}=\begin{pmatrix}1 & A_1 & A_2 & \cdots & A_{N-2}\\ & 1 & A_1 & \cdots & A_{N-3}\\ & & \ddots & \ddots & \vdots\\ & & & 1 & A_1\\ & & & & 1\end{pmatrix},\ Q_{22}=\begin{pmatrix}1 & B_1 & B_2 & \cdots & B_{2N-2}\\ & 1 & B_1 & \cdots & B_{2N-3}\\ & & \ddots & \ddots & \vdots\\ & & & 1 & B_1\\ & & & & 1\end{pmatrix}$$

with A_ks being the coefficients in the expansion

$$\frac{\lambda^N}{a(\lambda)}=\sum_{k=0}^{\infty}\frac{A_k}{\lambda^k},$$

which could be represented through the power sums of λ_l, $\delta_k=\sum_{l=1}^{N}\lambda_l^k$,

$$A_0=1,\ A_1=\delta_1,\ A_2=\frac{1}{2}(\delta_2+\delta_1^2),$$

with the recursive formula

$$A_k=\frac{1}{k}\left(\delta_k+\sum_{\substack{i,\ j\geq 1\\ i+j=k}}\delta_i A_j\right),$$

and B_rs are the comparison of the coefficients of λ^r, $r=0,1,\ldots$ in

$$\frac{\lambda^{2N}}{a^2(\lambda)}=\left(\sum_{k=0}^{\infty}\frac{A_k}{\lambda^k}\right)^2=\sum_{r=0}^{\infty}\frac{B_r}{\lambda^r},$$

which can be written as $B_0=A_0^2=1$, $B_1=2A_1,\ldots,B_r=\sum_{\substack{i,j\geq 0\\ i+j=r}}A_iA_j$ with the supplementary definition $A_{-k}=B_{-k}=0$, $k=1,2,\ldots$.

Proof. According to the definition of the Lie–Poisson bracket,

$$\begin{aligned}I_{jt_{2\lambda}}&=\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\left\{I_j,F_l^S\right\}=\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\frac{\mathrm{d}I_j}{\mathrm{d}t_{2,l}},\ I_{jt_{3\lambda}}=\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\left\{I_j,F_l^T\right\}\\&=\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\frac{\mathrm{d}I_j}{\mathrm{d}t_{3,l}},\phi_{jt_{2\lambda}}=\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\left\{\phi_j,F_l^S\right\}=\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\frac{\mathrm{d}\phi_j}{\mathrm{d}t_{2,l}},\\ \phi_{jt_{3\lambda}}&=\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\left\{\phi_j,F_l^T\right\}=\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\frac{\mathrm{d}\phi_j}{\mathrm{d}t_{3,l}},\end{aligned}\tag{49}$$

for $j=1,\ldots,3N-2$. By using Eqs (46), (47), and 49, it is easy to see that

$$\begin{aligned}&\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\left\{I_j,F_l^S\right\}=\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\left\{I_j,F_l^T\right\}=0,\quad j=1,\ldots,3N-2,\\&\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\left\{\phi_j,F_l^S\right\}=\frac{\lambda^{N-j-1}}{a(\lambda)}=\sum_{k=0}^{\infty}\frac{A_k}{\lambda^{k+j+1}},\quad j=1,\ldots,N-1,\\&\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\left\{\phi_j,F_l^T\right\}=0,\quad j=1,\ldots,N-1,\\&\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\left\{\phi_j,F_l^S\right\}=0,\ j=N,\ldots,3N-2,\\&\sum_{l=0}^{\infty}\frac{1}{\lambda^{l+1}}\left\{\phi_j,F_l^T\right\}=\frac{\lambda^{3N-j-2}}{a^2(\lambda)}=\sum_{k=0}^{\infty}\frac{B_k}{\lambda^{k+j+2-N}},\ j=N\ldots,3N-2.\end{aligned}\tag{50}$$

By comparing the coefficients of λ^{-l-1} in (50), we get the Lie–Poisson brackets

$$\begin{aligned}&\left\{I_j,F_l^S\right\}=0,\ \left\{I_j,F_l^T\right\}=0,\ j=1,\ldots,3N-2,\\&\left\{\phi_j,F_l^S\right\}=A_{l-j},\ \left\{\phi_j,F_l^T\right\}=0,\ j=1,\ldots,N-1,\\&\left\{\phi_j,F_l^S\right\}=0,\ \left\{\phi_j,F_l^T\right\}=B_{l+N-j-1},\ j=N,\ldots,3N-2,\end{aligned}\tag{51}$$

thereby providing the nondegeneracy matrix Eq. 48.

Proposition 6. *$F_1^S,\ldots,F_{N-1}^S$, $F_1^T,\ldots,F_{2N-1}^T$ given in Eqs 19, 20 are functionally independent.*

Proof. We only need to prove the linear independence of the gradients:

$$\nabla F_1^S,\ldots,\nabla F_{N-1}^S,\nabla F_1^T,\cdots\nabla F_{2N-1}^T.$$

Suppose

$$\sum_{k=1}^{N-1}c_k\nabla F_k^S+\sum_{m=1}^{2N-1}c_{N+m-1}\nabla F_m^T=0,$$

we have

$$\begin{aligned}0&=\sum_{k=1}^{N-1}c_k\left\{\phi_j,F_k^S\right\}+\sum_{m=1}^{2N-1}c_{N+m-1}\left\{\phi_j,F_m^T\right\}\\&=\sum_{k=1}^{N-1}c_k\frac{\mathrm{d}\phi_j}{\mathrm{d}t_{2,k}}+\sum_{m=1}^{2N-1}c_{N+m-1}\frac{\mathrm{d}\phi_j}{\mathrm{d}t_{3,m}}.\end{aligned}$$

Hence, $c_1=c_2=\cdots=c_{3N-2}=0$ since the coefficient determinant is equal to 1 by matrix Eq. 48.Remark. Corollary 1 and the present Proposition completely prove the Liouville integrability of the Lie–Poisson Hamiltonian systems Eqs 6, 7 with the Hamiltonians Eqs 21, 22, and $3N-2$ integrals $F_1^S,\ldots,F_{N-1}^S$, $F_1^T,\ldots,F_{2N-1}^T$, which are involutive in pairs and functionally independent.

After fixing the values of the $2N$ Casimir functions in (27), based on (51), using (21), the solution of system Eq. 6 in terms of action-angle variables ϕ_j, I_j is

$$I_j(x)=I_j(0),\ \phi_j(x)=\begin{cases}\phi_j(0), & j=1,\ldots,N-1,\\ \phi_j(0)+B_{N-j}x, & j=N,\ldots,3N-2.\end{cases}\tag{52}$$

Thus, combining Eq. 45 with (52) yields the Jacobi inversion problem for the Lie–Poisson Hamiltonian system Eq. 6

$$\begin{cases}\phi_j(0) &=\sum_{k=1}^{3N-2}\int_0^{\mu_k}\frac{a(\lambda)z\lambda^{N-j-1}}{R(\lambda)}\mathrm{d}\lambda,\ j=1,\ldots,N-1,\\ \phi_j(0)+B_{N-j}x &=\sum_{k=1}^{3N-2}\int_0^{\mu_k}\frac{\lambda^{3N-j-2}}{R(\lambda)}\mathrm{d}\lambda,\ j=N,\ldots,3N-2.\end{cases}$$

For the Lie–Poisson Hamiltonian system Eq. 7 with respect to Lie–Poisson bracket Eq. 51, using (22), we obtain the solution of system Eq. 7 in terms of action-angle variables ϕ_j, I_j

$$I_j(t)=I_j(0),\ \phi_j(t)=\begin{cases}\phi_j(0)+A_{1-j}t, & j=1,\ldots,N-1,\\ \phi_j(0), & j=N,\ldots,3N-2.\end{cases}\tag{53}$$

According to Eqs 45, 53, we have the Jacobi inversion problem for the Lie–Poisson Hamiltonian system Eq. 7

$$\begin{cases}\phi_j(0)+A_{1-j}t &=\sum_{k=1}^{3N-2}\int_0^{\mu_k}\frac{a(\lambda)z\lambda^{N-j-1}}{R(\lambda)}\mathrm{d}\lambda,\ j=1,\ldots,N-1,\\ \phi_j(0) &=\sum_{k=1}^{3N-2}\int_0^{\mu_k}\frac{\lambda^{3N-j-2}}{R(\lambda)}\mathrm{d}\lambda,\ j=N,\ldots,3N-2.\end{cases}$$

The compatible solution of systems Eqs 6, 7 in terms of action-angle variables I_j, ϕ_j is

$$I_j(x,t) = I_j(0,0),\ \phi_j(x,t) = \begin{cases} \phi_j(0,0) + A_{1-j}t, & j = 1,\ldots,N-1, \\ \phi_j(0,0) + B_{N-j}x, & j = N,\ldots,3N-2. \end{cases} \tag{54}$$

From (45) and (54), we finally obtain the Jacobi inversion problem for the Hirota–Satsuma modified Boussinesq Eq. 1:

$$\begin{cases} \phi_j(0,0) + A_{1-j}t = \sum_{k=1}^{3N-2} \int_0^{\mu_k} \frac{a(\lambda) z \lambda^{N-j-1}}{R(\lambda)} \mathrm{d}\lambda, & j = 1,\ldots,N-1, \\ \phi_j(0,0) + B_{N-j}x = \sum_{k=1}^{3N-2} \int_0^{\mu_k} \frac{\lambda^{3N-j-2}}{R(\lambda)} \mathrm{d}\lambda, & j = N,\ldots,3N-2. \end{cases}$$

6 Conclusion

In this paper, two finite-dimensional Lie–Poisson Hamiltonian systems associated with a 3 × 3 spectral problem related to the Hirota–Satsuma modified Boussinesq equation are presented. Separation of variables for the integrable systems with non-hyperelliptic spectral curves is constructed by using the method proposed by Sklyanin. Then, 3N-2 pairs of action-angle variables are introduced with the help of Hamilton–Jacobi theory. The Jacobi inversion problems for these Lie–Poisson Hamiltonian systems and the Hirota–Satsuma modified Boussinesq equation are discussed. Furthermore, based on the Jacobi inversion problems, we may use the algebro-geometric method to obtain the multi-variable sigma-function solutions, which will be left to future research. The methods in this paper can be applied to other systems of soliton hierarchies with 3 × 3 matrix spectral problems, even 4 × 4 matrix spectral problems.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

XuG: writing–original draft and writing–review and editing. DD: writing–review and editing. XiG: writing–review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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