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State estimation for Markovian jump Hopfield neural networks with mixed time delays

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Markovian jump Hopfield NNs (MJHNNs) have received considerable attention due to their potential for application in various areas. This paper deals with the issue of state estimation concerning a category of MJHNNs with discrete and distributed delays. Both time-invariant and time-variant discrete delay cases are taken into account. The objective is to design full-order state estimators such that the filtering error systems exhibit exponential stability in the mean-square sense. Two sufficient conditions on the mean-square exponential stability of MJHNNs are established utilizing augmented Lyapunov−Krasovskii functionals, the Wirtinger−based integral inequality, the Bessel-Legendre inequality, and the convex combination inequality. Then, linear matrix inequalities-based design methods for the required estimators are developed through eliminating nonlinear coupling terms. The feasibility of these linear matrix inequalities can be readily verified via available Matlab software, thus enabling numerically tractable implementation of the proposed design methods. Finally, two numerical examples with simulations are provided to demonstrate the applicability and less conservatism of the proposed stability criteria and estimators. Lastly, two numerical examples are given to demonstrate the applicability and reduced conservatism of the proposed stability criteria and estimator design methods. Future research could explore further refinement of these analysis and design results, and exporing their extention to more complex neural network models.

KEYWORDS

Hopfield neural networks, Markovian jump, mean-square exponential stability, state estimation, time delays

1 Introduction

Neural networks (NNs) are composed of numerous interconnected neurons, providing them the ability to process large amounts of data simultaneously. Recently, NNs have been widely utilized in speech recognition [1], tracking control [2], associative memory [3], image restoration [4], and various other fields. As we all know, time delays are inevitable in the above practical applications, which may cause NNs to oscillate or become unstable [5]. This phenomenon has prompted extensive research on the stability analysis of NNs with time delays [6–9]. Notably, [10] designed an output feedback controller for NNs with time-invariant discrete delay to ensure the asymptotic stability of the closed-loop control system. The time delay considered in this study was constant. In cases where the time delay varies over time, [11] utilized a Lyapunov–Krasovskii functional (LKF) and subsequently established a stability criterion based on linear matrix inequalities (LMIs). However, these studies primarily focus on discrete delays, which may oversimplify these scenarios. In NNs, numerous interconnections between neurons form various parallel paths. Due to the varying sizes and complexities of these paths, signal transmission times are distributed within a certain period of time, resulting in distributed delays [12, 13].

On the other hand, in practical scenarios, environmental fluctuations can induce changes in the parameters of NNs. Researchers have recognized the significant advantages of Markovian jump processes in handling random changes in parameters [14–19]. Over the past few decades, many Markovian jump NN models, either in discrete-time form [20–23] or continuous-time form [24–27], have been proposed and studied. Among these models, Markovian jump Hopfield NNs (MJHNNs) have received considerable attention due to their potential for application. Notably, for an MJHNN, the neuron states are often not completely available. Thus, to achieve a given control goal, it is necessary to estimate the neuron states based on available output data, which has led to a growing focus on the topic of state estimation. [28] examined the finite-time state estimation for MJHNNs with discrete delays and presented a discontinuous estimator design method. [29] focused on the design of state estimators for continuous-time MJHNNs with both discrete and distributed delays and proposed a mean-square exponential stability (MSES) criterion and an LMI-based state estimation strategy.

It should be noted that the criterion derived in [29] was based on Jensen's inequality, and the LKF used therein omitted some items involving time-delay-related integrals, thus leaving room for further improvement. Another discovery is that the discrete delay considered therein was assumed to be time-invariant, which restricts the application scope of the state estimation strategy since the magnitude of delays may vary over time in practice. Inspired by the observations above, we re-examine the state estimation in MJHNNs with discrete and distributed delays. Unlike the assumption of time invariance made in [29], the discrete delay under consideration here is allowed to be time-variant. The primary contributions of this study are as follows:

(1) Establishing MSES criteria for MJHNNs by integrating augmented LKFs, the Wirtinger-based integral inequality (WBII), the Bessel–Legendre inequality (BLI), and the convex combination inequality (CCI). Compared to the criteria proposed in [29], those proposed in this study are less conservative.

(2) Developing design methods for the required estimators to ensure the MSES of the filtering error systems (FESs) by eliminating nonlinear coupling terms. The estimator gain matrices can be easily determined by solving a set of LMIs.

Notation: In this paper, $\mathbb{R}^p$ and $\mathbb{S}_+^q$ denote the set of p-dimensional real matrices and q-dimensional symmetric positive definite matrices, respectively. $I_{m\times n}$ and $0_{p\times q}$ represent the $m\times n$ unit matrix and the $p\times q$ zero matrix, respectively. We denote ϑ^T as the transpose of the matrix ϑ, ϑ^{-1} as the inverse of the matrix ϑ, $col\{\mu_1,\ldots,\mu_l\}$ as the column vector with elements $\{\mu_1^T,\ldots,\mu_l\}$, $diag\{\Phi_1,\ldots,\Phi_m\}$ as the diagonal matrix with diagonal elements $\{\Phi_1,\ldots,\Phi_m\}$, $*$ as the symmetry term of a symmetric matrix, and $\|\cdot\|$ as the Euclidean norm. The operator $\mathcal{F}(\cdot)$ denotes the expectation, and $He(\Pi)=\Pi+\Pi^T$. The notation $\Upsilon>0$ $(\Upsilon<0)$ indicates that the matrix Υ is positive definite (negative definite).

2 Preliminaries

The MJHNN with time-invariant discrete and distributed delays is modeled as follows:

$$\begin{aligned}\dot{\vartheta}(t)&=-A(\kappa_t)\vartheta(t)+C(\kappa_t)\varphi(\vartheta(t))+D(\kappa_t)\varphi(\vartheta(t-\xi))\\&\quad+G(\kappa_t)\int_{t-\varsigma}^{t}\varphi(\vartheta(z))dz,\quad t\geq 0,\\ \alpha(t)&=B(\kappa_t)\vartheta(t)+\delta(\vartheta(t)),\vartheta(s)=\vartheta_0(s),\quad s\in[-2\zeta,0],\\ \zeta&=\max\{\xi,\varsigma\},\end{aligned}\tag{1}$$

where $\vartheta(t)\in\mathbb{R}^n$ and $\alpha(t)\in\mathbb{R}^n$ represent the system state and measurement output, respectively. The positive scalars ξ and ς denote the time-invariant discrete delay and the distributed delay, respectively. $\vartheta_0(s)$ is an initial function defined on $[-2\zeta, 0]$. $A(\kappa_t)$, $B(\kappa_t)$, $C(\kappa_t)$, $D(\kappa_t)$, and $G(\kappa_t)$ are matrix functions of the random jump process κ_t. To simplify the notation, we denote $A(\kappa_t)$ as A_i and use a similar notation for the other matrices. In addition, κ_t takes values in the set $\mathcal{D}=\{1,2,\ldots,d\}$. The transition probability matrix is given by

$$\Pr\left(\kappa_{t+e}=j\mid\kappa_t=i\right)=\begin{cases}\eta_{ij}(e)e+o(e), & j\neq i\\ 1+\eta_{ii}(e)e+o(e), & j=i,\end{cases}$$

where $e>0$, $\lim_{e\to 0}\frac{o(e)}{e}=0$, and $\eta_{ij}(e)\geq 0$ $(j\neq i)$ denotes the rate at which the system transitions from mode i at time t to mode j at time $t+e$, and $\eta_{ii}=-\sum_{j=1,j\neq i}^{d}\eta_{ij}$ [30].

In the MJHNN Equation 1, $\varphi(\cdot)$ and $\delta(\cdot)$ represent the activation and perturbation functions, respectively. These functions satisfy the following assumptions.

Assumption 1. For given matrices μ_1 and $\mu_2\in\mathbb{R}^{n\times n}$, $\varphi(\cdot)$ satisfies

$$[\varphi(\psi)-\varphi(\hat{\psi})-\mu_1(\psi-\hat{\psi})]^T[\varphi(\psi)-\varphi(\hat{\psi})-\mu_2(\psi-\hat{\psi})]\leq 0,\ \forall\psi,\ \hat{\psi}\in\mathbb{R}^n.\tag{2}$$

Assumption 2. For given matrices λ_1 and $\lambda_2\in\mathbb{R}^{m\times n}$, $\delta(\cdot)$ satisfies

$$[\delta(\psi)-\delta(\hat{\psi})-\lambda_1(\psi-\hat{\psi})]^T[\delta(\psi)-\delta(\hat{\psi})-\lambda_2(\psi-\hat{\psi})]\leq 0,\ \forall\psi,\ \hat{\psi}\in\mathbb{R}^n.\tag{3}$$

Remark 1. Equations 2 and 3 satisfy sector-bounded conditions [31], which have broader applicability than the standard Lipschitz conditions [32, 33] and are widely utilized in the study of NNs [9, 34].

The full-order state estimator for the MJHNN Equation 1 is designed as follows:

$$\begin{aligned}\dot{\hat{\vartheta}}(t)&=-A_i\hat{\vartheta}(t)+C_i\varphi\left(\hat{\vartheta}(t)\right)+D_i\varphi\left(\hat{\vartheta}(t-\xi)\right)+G_i\int_{t-\varsigma}^{t}\varphi\left(\hat{\vartheta}(z)\right)dz\\&\quad+K_i\left[\alpha(t)-B_i\hat{\vartheta}(t)-\delta\left(\hat{\vartheta}(t)\right)\right],\end{aligned}\tag{4}$$

where $\hat{\vartheta}(t)\in\mathbb{R}^n$ and $K_i\in\mathbb{R}^{n\times m}$ $(i\in\mathcal{D})$ represent the state estimate and gain matrices, respectively.

Define $\epsilon(t)=\vartheta(t)-\hat{\vartheta}(t)$. Then, we obtain the following FES:

$$\begin{aligned}\dot{\epsilon}(t)&=-(A_i+K_iB_i)\epsilon(t)+C_i\varrho(t)+D_{1i}\varrho(t-\xi)+G_i\int_{t-\varsigma}^{t}\varrho(z)dz\\&\quad-K_i\phi(t),\end{aligned}\tag{5}$$

where

$$\begin{aligned} \varrho(t) &= \varphi(\vartheta(t)) - \varphi\big(\hat{\vartheta}(t)\big), \\ \phi(t) &= \delta(\vartheta(t)) - \delta\big(\hat{\vartheta}(t)\big). \end{aligned} \tag{6}$$

Next, we introduce the definition of the MSES.

Definition 1. The FES Equation 5 has MSES if there exist positive scalars ι and τ such that

$$\mathcal{F}\{\|\epsilon(t),\ \epsilon_0\|^2\} < \iota\exp(-\tau t)\sup_{-\xi\le s\le 0}\mathcal{F}\{\|\epsilon_0(s)\|^2\},\ \forall t\ge 0$$

holds, where $\epsilon_0(t)\in\mathbb{R}^n$ is the initial function of $\epsilon(t)$.

The primary objective of this paper is to establish the MSES criterion for the MJHNN and design the state estimator to ensure that the FES achieves MSES. To facilitate subsequent derivations, we have prepared four lemmas.

Lemma 1. [35, 36] (BLI) For a differentiable function ϑ: $[\mu_1,\ \mu_2]\rightarrow\mathbb{R}^n$, the inequality

$$\int_{\mu_1}^{\mu_2}\dot{\vartheta}^T(u)Z\dot{\vartheta}(u)du \ge \frac{1}{\mu_2-\mu_1}\Gamma^T diag\{Z, 3Z, 5Z\}\Gamma$$

holds, where $Z\in\mathbb{S}_+^n$, $\upsilon_{\mu_1,\mu_2}(u) = 2\left(\frac{u-\mu_1}{\mu_2-\mu_1}\right)-1$, and

$$\Gamma = \begin{bmatrix} \vartheta(\mu_2)-\vartheta(\mu_1) \\ \vartheta(\mu_2)+\vartheta(\mu_1)-\dfrac{2}{\mu_2-\mu_1}\displaystyle\int_{\mu_1}^{\mu_2}\vartheta(u)du \\ \vartheta(\mu_2)-\vartheta(\mu_1)-\dfrac{6}{\mu_2-\mu_1}\displaystyle\int_{\mu_1}^{\mu_2}\upsilon_{\mu_1,\mu_2}(u)\vartheta(u)du \end{bmatrix}.$$

Lemma 2. [37] (WBII) For any matrix $Q\in\mathbb{S}_+^{n\times n}$, scalars $\eta_1<\eta_2$, and function μ: $[\eta_1,\ \eta_2]\rightarrow\mathbb{R}^n$, the inequality

$$\int_{\eta_1}^{\eta_2}\mu^T(z)Q\mu(z)dz \ge \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\mu^T(z)dzQ\int_{\eta_1}^{\eta_2}\mu(z)dz + \frac{3}{\eta_2-\eta_1}\Xi^T Q\Xi$$

holds, where

$$\Xi = \int_{\eta_1}^{\eta_2}\mu(z)dz - \frac{2}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\int_{\lambda}^{\eta_2}\mu(z)dzd\lambda.$$

Lemma 3. [38, 39] (CCI) If there exist matrices $Z\in\mathbb{S}_+^n$ and $\Pi\in\mathbb{R}^{n\times n}$ such that $\begin{bmatrix} Z & \Pi \\ \Pi^T & Z\end{bmatrix}\ge 0$ holds, one has

$$\begin{bmatrix} \frac{1}{\psi}Z & 0 \\ 0 & \frac{1}{1-\psi}Z \end{bmatrix} \ge \begin{bmatrix} Z & \Pi \\ \Pi^T & Z\end{bmatrix},\ \forall\psi\in(0,\ 1).$$

Lemma 4. [40] Given matrices $\Xi, W_r, U_r,$ and Z_r $(r=1,\ldots,R)$ of appropriate dimensions, if there exist scalars $\mu_r>0$ such that

$$\Xi + \sum_{i=1}^{R} He\left(W_r Z_r^{-1}U_r^T\right) < 0$$

holds, then we obtain

$$\left[\begin{array}{c:c} \Xi & W_1+\mu_1U_1\cdots W_R+\mu_RU_R \\ \hdashline * & diag\{-He(\mu_1Z_1),\ldots,-He(\mu_RZ_R)\} \end{array}\right] < 0.$$

3 Stability analysis and state estimation of MJHNNs

In this section, we establish an MSES criterion and design a state estimator for MJHNNs with time-invariant discrete and distributed delays.

It can be deduced from Equations 2, 3, 6 that

$$\begin{aligned} -\varpi_1^T(t)\hat{R}\varpi_1(t) &\ge 0, \\ -\varpi_1^T(t-\xi)\hat{R}\varpi_1(t-\xi) &\ge 0, \end{aligned} \tag{7}$$

and

$$\begin{aligned} -\varpi_2^T(t)\hat{R}\varpi_2(t) &\ge 0, \\ -\varpi_2^T(t-\xi)\hat{R}\varpi_2(t-\xi) &\ge 0, \\ -\varpi_3^T(t)\hat{H}\varpi_3(t) &\ge 0, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \varpi_1(t) &= col\{\vartheta(t),\ \varphi(\vartheta(t))\}, \\ \varpi_2(t) &= col\{\epsilon(t),\ \varrho(t)\}, \\ \varpi_3(t) &= col\{\epsilon(t),\ \phi(t)\}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} R_1 &= \frac{\mu_1^T\mu_2+\mu_2^T\mu_1}{2}, & R_2 &= -\frac{\mu_1^T+\mu_2^T}{2}, \\ H_1 &= \frac{\lambda_1^T\lambda_2+\lambda_2^T\lambda_1}{2}, & H_2 &= -\frac{\lambda_1^T+\lambda_2^T}{2}, \\ \hat{R} &= \begin{bmatrix} R_1 & R_2 \\ * & I\end{bmatrix}, & \hat{H} &= \begin{bmatrix} H_1 & H_2 \\ * & I\end{bmatrix}. \end{aligned} \tag{10}$$

Next, the MSES criterion for the MJHNN Equation 1 is established.

Theorem 1. For given positive scalars ξ and ς and matrices μ_1 and μ_2, if there exist positive scalars ρ_{1i}, and ρ_{2i} and matrices $E_i\in\mathbb{S}_+^{3n}$, Z and $W\in\mathbb{S}_+^n$, T_i and $U\in\mathbb{S}_+^{2n}$, and $N_1, N_2\in\mathbb{R}^n$ such that

$$\sum_{j=1}^{d}\eta_{ij}T_j \le U, \tag{11}$$

$$\Lambda_i \le 0 \tag{12}$$

hold, for $i\in\mathcal{D}$ where

$$\begin{aligned} \Lambda_i &= He\left[F_1^TE_iF_0+F_5^T\hat{N}F_{6i}\right]+F_1^T\left(\sum_{j=1}^{d}\eta_{ij}E_j\right)F_1+F_3^TT_iF_3 \\ &\quad -F_4^TT_iF_4+F_3^T(\xi U)F_3+c_8^T(\xi Z)c_8-\frac{1}{\xi}F_2^T\hat{Z}F_2+c_5^T(\varsigma W)c_5 \\ &\quad -\frac{1}{\varsigma}c_7^TWc_7-\frac{3}{\varsigma}F_7^TWF_7-\rho_{1i}F_3^T\hat{R}F_3-\rho_{2i}F_4^T\hat{R}F_4, \\ c_i &= \left[\,0_{n\times(i-1)n}\ \ I_n\ \ 0_{n\times(9-i)n}\,\right],\quad i=1,\ldots,9, \\ F_0 &= \left[\,c_8^T\ \ c_1^T-c_2^T\ \ c_1^T+c_2^T-2c_3^T\,\right]^T, \\ F_1 &= \left[\,c_1^T\ \ \xi c_3^T\ \ \xi c_4^T\,\right]^T, \\ F_2 &= \left[\,c_1^T-c_2^T\ \ c_1^T+c_2^T-2c_3^T\ \ c_1^T-c_2^T-6c_4^T\,\right]^T, \\ F_3 &= \left[\,c_1^T\ \ c_5^T\,\right]^T,\ F_4=\left[\,c_2^T\ \ c_6^T\,\right]^T, \\ F_5 &= \left[\,c_1^T\ \ c_8^T\,\right]^T,\ \hat{N}=\left[\,N_1^T\ \ N_2^T\,\right]^T, \\ F_{6i} &= -c_8-A_ic_1+C_ic_5+D_ic_6+G_ic_7, \\ F_7 &= c_7-2c_9, \\ \hat{Z} &= diag\{Z,\ 3Z,\ 5Z\}, \end{aligned} \tag{13}$$

and the remaining symbols are defined in Equation 10. Then, the MJHNN Equation 1 achieves MSES.

Proof. Define

$$\hat{\psi}(t) = col\{\psi_0(t), \psi_1(t), \psi_2(t)\, \psi_3(t)\},$$

where

$$\begin{aligned}
\psi_0(t) &= \left[\vartheta^T(t) \;\; \vartheta^T(t-\xi)\right]^T,\\
\psi_1(t) &= \frac{1}{\xi}\left[\int_{-\xi}^{0}\vartheta_t^T(z)dz \int_{-\xi}^{0}\upsilon(z)\vartheta_t^T(z)dz\right]^T,\\
\psi_2(t) &= \left[\varphi^T(\vartheta(t)) \;\; \varphi^T(\vartheta(t-\xi)) \;\; \int_{-\varsigma}^{0}\varphi_t^T(\vartheta_t(z))dz\right]^T,\\
\psi_3(t) &= \left[\dot{\vartheta}^T(t)\tfrac{1}{\varsigma}\int_{-\varsigma}^{0}\int_{\sigma}^{0}\varphi^T(\vartheta_t(z))dzd\sigma\right]^T,\\
\vartheta_t(z) &= \vartheta(t+z),\; \varphi_t(z) = \varphi_t(t+z),\; \upsilon(z) = 2\frac{z+\xi}{\xi} - 1.
\end{aligned}$$

The LKF is constructed as follows:

$$\begin{aligned}
\mathcal{V}(\vartheta_t, \dot{\vartheta}_t, i) &= \mathcal{V}_1(\vartheta_t, i) + \mathcal{V}_2(\vartheta_t, i) + \mathcal{V}_3(\vartheta_t, \dot{\vartheta}_t) + \mathcal{V}_4(\vartheta_t),\\
\mathcal{V}_1(\vartheta_t, i) &= \hat{\vartheta}^T(t)E_i\hat{\vartheta}(t),\\
\mathcal{V}_2(\vartheta_t, i) &= \int_{t-\xi}^{t}\varpi_1^T(z)T_i\varpi_1(z)dz,\\
\mathcal{V}_3(\vartheta_t, \dot{\vartheta}_t) &= \int_{-\xi}^{0}\int_{t+\beta}^{t}\varpi_1^T(z)U\varpi_1(z)dzd\beta + \int_{-\xi}^{0}\int_{t+\beta}^{t}\dot{\vartheta}^T(z)Z\dot{\vartheta}(z)dzd\beta,\\
\mathcal{V}_4(\vartheta_t) &= \int_{-\varsigma}^{0}\int_{t+\beta}^{t}\varphi^T(\vartheta(z))W\varphi(\vartheta(z))dzd\beta,
\end{aligned}$$

where $\varpi_1(t)$ is defined in Equation 9 and

$$\hat{\vartheta}(t) = col\{\vartheta(t), \xi\psi_1(t)\}.$$

The infinitesimal generator $\mathcal{L}$ is defined as

$$\begin{aligned}
\mathcal{L}\mathcal{V}_1(\vartheta_t, i) &= \lim_{\iota\to 0^+}\frac{1}{\iota}\left[\mathcal{F}\{\mathcal{V}_1(\vartheta(t+\iota), \kappa(t+\iota)) \mid \vartheta(t), \kappa(t) = i\} - \mathcal{V}_\infty(\vartheta(t), \kappa(t) = i)\right]\\
&= \frac{\partial\mathcal{V}_1}{\partial t} + \dot{\vartheta}^T(t)\frac{\partial\mathcal{V}_1}{\partial\vartheta}\Big|_{\kappa(t)=i} + \sum_{j=1}^{d}\eta_{ij}\mathcal{V}_1(\vartheta_t, j)\\
&= He\left(\hat{\vartheta}^T(t)E_i\dot{\hat{\vartheta}}(t)\right) + \hat{\vartheta}^T(t)\left(\sum_{j=1}^{d}\eta_{ij}E_j\right)\hat{\vartheta}(t).
\end{aligned} \tag{14}$$

$\hat{\psi}(t)$ is abbreviated as $\hat{\psi}$. Then, $\hat{\vartheta}(t)$ and $\dot{\hat{\vartheta}}(t)$ have the following first components:

$$\vartheta(t) = c_1\hat{\psi},\; \dot{\vartheta}(t) = c_8\hat{\psi}.$$

The second components are given by

$$\begin{aligned}
\xi\psi_1(t) &= \xi\left[c_3^T c_4^T\right]^T\hat{\psi},\\
\xi\dot{\psi}_1(t) &= \left[c_1^T - c_2^T \;\; c_1^T + c_2^T - 2c_3^T\right]^T\hat{\psi},
\end{aligned}$$

and these enable us to deduce that

$$\hat{\vartheta}(t) = F_1\hat{\psi},\; \dot{\hat{\vartheta}}(t) = F_0\hat{\psi},$$

where F_0 and F_1 are defined in Equation 13. It follows from Equation 14 that

$$\begin{aligned}
\mathcal{L}\mathcal{V}_1(\vartheta_t, i) &= \hat{\psi}^T\left(He\left(F_1^TE_iF_0\right) + F_1^T\left(\sum_{j=1}^{d}\eta_{ij}E_j\right)F_1\right)\hat{\psi},\\
\mathcal{L}\mathcal{V}_2(\vartheta_t, i) &= \varpi_1^T(t)T_i\varpi_1(t) - \varpi_1^T(t-\xi)T_i\varpi_1(t-\xi)\\
&\quad + \sum_{j=1}^{d}\eta_{ij}\int_{t-\xi}^{t}\varpi_1^T(z)T_j\varpi_1(z)dz,\\
\mathcal{L}\mathcal{V}_3(\vartheta_t, \dot{\vartheta}_t) &= \varpi_1^T(t)(\xi U)\varpi_1(t) + \dot{\vartheta}^T(t)(\xi Z)\dot{\vartheta}(t)\\
&\quad - \int_{t-\xi}^{t}\dot{\vartheta}^T(z)Z\dot{\vartheta}(z)dz + \int_{t-\xi}^{t}\varpi_1^T(z)U\varpi_1(z)dz,\\
\mathcal{L}\mathcal{V}_4(\vartheta_t) &= \varphi^T(\vartheta(t))(\varsigma W)\varphi(\vartheta(t)) - \int_{-\varsigma}^{0}\varphi_t^T(\vartheta(z))W\varphi_t(\vartheta(z))dz.
\end{aligned} \tag{15}$$

Under Equation 11, the integral terms $\sum_{j=1}^{d}\eta_{ij}\int_{t-\xi}^{t}\varpi_1^T(z)T_j\varpi_1(z)dz$ in $\mathcal{L}\mathcal{V}_2$ and $\int_{t-\xi}^{t}\varpi_1^T(z)U\varpi_1(z)dz$ in $\mathcal{L}\mathcal{V}_3$ can be eliminated to obtain

$$\begin{aligned}
\mathcal{L}\mathcal{V}_2(\vartheta_t, i) &\le \varpi_1^T(t)T_i\varpi_1(t) - \varpi_1^T(t-\xi)T_i\varpi_1(t-\xi)\\
&= \hat{\psi}^T(F_3^TT_iF_3 - F_4^TT_iF_4)\hat{\psi},\\
\mathcal{L}\mathcal{V}_3(\vartheta_t, \dot{\vartheta}_t) &\le \varpi_1^T(t)(\xi U)\varpi_1(t)\\
&\quad + \dot{\vartheta}^T(t)(\xi Z)\dot{\vartheta}(t) - \int_{t-\xi}^{t}\dot{\vartheta}^T(z)Z\dot{\vartheta}(z)dz.
\end{aligned} \tag{16}$$

Utilizing Lemma 1, the following inequality holds:

$$-\int_{t-\xi}^{t}\dot{\vartheta}^T(z)Z\dot{\vartheta}(z)dz \le -\frac{1}{\xi}F_2^T\hat{Z}F_2. \tag{17}$$

Combining Equations 16 and 17 yields

$$\mathcal{L}\mathcal{V}_3(\vartheta_t, \dot{\vartheta}_t) \le \hat{\psi}^T\left(F_3^T(\xi U)F_3 + c_8^T(\xi Z)c_8 - \frac{1}{\xi}F_2^T\hat{Z}F_2\right)\hat{\psi}. \tag{18}$$

For the integral term $\int_{-\varsigma}^{0}\varphi_t^T(\vartheta(z))W\varphi_t(\vartheta(z))dz$ in $\mathcal{L}\mathcal{V}_4$, by applying Lemma 2, we can establish

$$\begin{aligned}
\mathcal{L}\mathcal{V}_4(\vartheta_t) &\le \varphi^T(\vartheta(t))(\varsigma W)\varphi(\vartheta(t))\\
&\quad - \frac{1}{\varsigma}\int_{-\varsigma}^{0}\varphi_t^T(\vartheta(z))dzW\int_{-\varsigma}^{0}\varphi_t(\vartheta(z))dz - \frac{3}{\varsigma}\Xi^TW\Xi,
\end{aligned}$$

where

$$\Xi = \int_{-\varsigma}^{0}\varphi_t(\vartheta(z))dz - \frac{2}{\varsigma}\int_{-\varsigma}^{0}\int_{\sigma}^{0}\varphi_t(\vartheta(z))dzd\sigma.$$

Then, it is easy to obtain

$$\mathcal{L}\mathcal{V}_4(\vartheta_t) \le \hat{\psi}^T\left(c_5^T(\varsigma W)c_5 - \frac{1}{\varsigma}c_7^TWc_7 - \frac{3}{\varsigma}F_7^TWF_7\right)\hat{\psi}. \tag{19}$$

Additionally, for the system Equation 1, utilizing the free-weight matrix technique yields

$$\hat{\psi}^THe\left(F_5^T\hat{N}F_{6i}\right)\hat{\psi} = 0. \tag{20}$$

From Equations 7 and 15–20, for any scalars $\rho_{1i} > 0$ and $\rho_{2i} > 0$, we can obtain

$$\mathcal{LV}(\vartheta_t, \dot{\vartheta}_t, i) \le \hat{\psi}^T\Bigg(He\big[F_1^T E_i F_0 + F_5^T \hat{N} F_{6i}\big] + F_1^T\Bigg(\sum_{j=1}^{d}\eta_{ij}E_j\Bigg)F_1 + F_3^T T_i F_3 - F_4^T T_i F_4 + F_3^T(\xi U)F_3 + c_8^T(\xi Z)c_8 - \frac{1}{\xi}F_2^T\hat{Z}F_2 + c_5^T(\varsigma W)c_5 - \frac{1}{\varsigma}c_7^T W c_7 - \frac{3}{\varsigma}F_7^T W F_7 \quad - \rho_{1i}F_3^T\hat{R}F_3 - \rho_{2i}F_4^T\hat{R}F_4\Bigg)\hat{\psi} = \hat{\psi}^T\Lambda_i\hat{\psi}.$$

Applying the Schur complement to Equation 12 yields

$$\mathcal{LV}(\vartheta_t, \dot{\vartheta}_t, i) \le -b\left(\|\vartheta(t)\|^2 + \|\vartheta(t-\xi)\|^2 + \|\vartheta(t-\varsigma)\|^2\right),$$

where $b = \lambda_{min}\{-\Lambda_i\} > 0$. From the definitions of $\mathcal{V}(\vartheta_t, \dot{\vartheta}_t, i)$, $\dot{\vartheta}(t)$, and $\varphi(\vartheta(t))$, there exist positive scalars a_1, a_2, a_3, a_4, and ι such that the following inequalities hold:

$$\mathcal{LV}(\vartheta_t, \dot{\vartheta}_t, i) \le a_1\|\vartheta(t)\|^2 + a_2\int_{t-\zeta}^{t}\|\vartheta(t)\|^2 dt + a_3\int_{t-\zeta}^{t}\|\vartheta(z-\xi)\|^2 dz + a_4\int_{t-\zeta}^{t}\|\vartheta(z-\varsigma)\|^2 dz, \quad (21)$$

$$\iota\max\left\{a_1 + a_2\zeta e^{\iota\zeta}, a_3\zeta e^{\iota\zeta}, a_4\zeta e^{\iota\zeta}\right\} \le b. \quad (22)$$

Combining Equation 21 with Equation 22 results in

$$\mathcal{F}\left\{e^{\iota t}\mathcal{V}(\vartheta_t, \dot{\vartheta}_t, i)\right\} \le e^{\iota t}\Big[(\iota a_1 - b)\|\vartheta(t)\|^2 - b\|\vartheta(t-\xi)\|^2 - b\|\vartheta(t-\varsigma)\|^2 + \iota a_2\int_{t-\zeta}^{t}\|\vartheta(z)\|^2 dz + \iota a_3\int_{t-\zeta}^{t}\|\vartheta(z-\xi)\|^2 dz + \iota a_4\int_{t-\zeta}^{t}\|\vartheta(z-\varsigma)\|^2 dz\Big].$$

By utilizing Dynkin's formula, for $h > 0$, it follows that

$$\begin{aligned}\mathcal{F}\left\{e^{\iota t}\mathcal{V}(\vartheta_h, \dot{\vartheta}_h, i)\right\} \le{}& J_1 + (\iota a_1 - b)\mathcal{F}\left\{\int_0^h e^{\iota t}\|\vartheta(t)\|^2 dt\right\} - b\mathcal{F}\left\{\int_0^h e^{\iota t}\|\vartheta(t-\xi)\|^2 dt\right\} \quad - b\mathcal{F}\left\{\int_0^h e^{\iota t}\|\vartheta(t-\varsigma)\|^2 dt\right\}\\ &+ \iota a_2\mathcal{F}\left\{\int_0^h\int_{t-\zeta}^{t} e^{\iota t}\|\vartheta(z)\|^2 dz\, dt\right\} \quad + \iota a_3\mathcal{F}\left\{\int_0^h\int_{t-\zeta}^{t} e^{\iota t}\|\vartheta(z-\xi)\|^2 dz\, dt\right\}\\ &+ \iota a_4\mathcal{F}\left\{\int_0^h\int_{t-\zeta}^{t} e^{\iota t}\|\vartheta(z-\varsigma)\|^2 dz\, dt\right\},\end{aligned} \quad (23)$$

where

$$J_1 = [a_1 + \zeta a_2 + \zeta a_3 + \zeta a_4]\sup_{-2\zeta\le s\le 0}\mathcal{F}\|\omega(s)\|^2.$$

By changing the order of integration, we can write

$$\begin{aligned}\int_0^h\int_{t-\zeta}^{t} e^{\iota t}\|\vartheta(z)\|^2 dz dt &\le \int_{-\zeta}^{h}\left(\int_{z\vee 0}^{(z+\zeta)\wedge h} e^{\iota t}dt\right)\|\vartheta(z)\|^2 dz\\ &\le \int_{-\zeta}^{h}\zeta e^{\iota(z+\zeta)}\|\vartheta(z)\|^2 dz\\ &\le \zeta e^{\iota\zeta}\int_0^h e^{\iota t}\|\vartheta(t)\|^2 dt + \zeta e^{\iota\zeta}\int_{-\zeta}^{0}\|\vartheta_0(s)\|^2 ds\\ &\le \zeta e^{\iota\zeta}\int_0^h e^{\iota t}\|\vartheta(t)\|^2 dt + \zeta^2 e^{\iota\zeta}\sup_{-\zeta\le s\le 0}\|\vartheta_0(s)\|^2\\ &\le \zeta e^{\iota\zeta}\int_0^h e^{\iota t}\|\vartheta(t)\|^2 dt + \zeta^2 e^{\iota\zeta}\sup_{-2\zeta\le s\le 0}\|\vartheta_0(s)\|^2,\end{aligned} \quad (24)$$

$$\int_0^h\int_{t-\zeta}^{t} e^{\iota t}\|\vartheta(z-\xi)\|^2 dz dt \le \zeta e^{\iota\zeta}\int_0^h e^{\iota t}\|\vartheta(t-\xi)\|^2 dt + \zeta^2 e^{\iota\zeta}\sup_{-2\zeta\le s\le 0}\|\vartheta_0(s)\|^2, \quad (25)$$

$$\int_0^h\int_{t-\zeta}^{t} e^{\iota t}\|\vartheta(z-\zeta(z))\|^2 dz dt \le \zeta e^{\iota\zeta}\int_0^h e^{\iota t}\|\vartheta(t-\varsigma)\|^2 dt + \zeta^2 e^{\iota\zeta}\sup_{-2\zeta\le s\le 0}\|\vartheta_0(s)\|^2. \quad (26)$$

Substituting Equations 24–26 into Equation 23 and combining it with Equation 21, we obtain

$$\mathcal{F}\left(e^{\iota t}\mathcal{V}(\vartheta_h, \dot{\vartheta}_h, i)\right) \le J_1 + J_2,$$

where

$$J_2 = \left(\iota a_2\zeta^2 e^{\iota\zeta} + \iota a_3\zeta^2 e^{\iota\zeta} + \iota a_4\zeta^2 e^{\iota\zeta}\right)\sup_{-2\zeta\le s\le 0}\mathcal{F}\|\vartheta_0(s)\|^2.$$

One can write the following inequality:

$$\mathcal{F}\{\|\vartheta(h, \vartheta_0)\|^2\} \le \frac{J_1 + J_2}{\lambda_{\min}(P_i)}e^{-\iota t}.$$

Then, we can prove that for any $h > 0$,

$$\mathcal{F}\{\|x(h, \vartheta_0)\|^2\} \le \beta e^{-\iota t}\sup_{-2\zeta\le s\le 0}\mathcal{F}\|\vartheta_0(s)\|^2$$

holds, where

$$\beta = \frac{1}{\lambda_{\min}(P_i)}\left[a_1 + \zeta a_2 + \zeta a_3 + \zeta a_4 + \iota a_2\zeta^2 e^{\iota\zeta} + \iota a_3\zeta^2 e^{\iota\zeta} + \iota a_4\zeta^2 e^{\iota\zeta}\right].$$

Therefore, following a similar approach as in [29], system Equation 1 achieves MSES according to Definition 1.

Remark 2. [39] considered the BLI and proved that $\mathcal{V}_1$ needs to include $\int_{-\xi}^{0}\vartheta_t^T(z)dz$ and $\int_{-\xi}^{0}v(z)\vartheta_t^T(z)dz$ to fully benefit from the BLI. Therefore, we consider the state augmentation of $\mathcal{V}_1$ and demonstrate its conservative reduction through Example 1.

Remark 3. As shown in Equation 17, the term $-\int_{t-\xi}^{t}\dot{\vartheta}^T(z)Z_1\dot{\vartheta}(z)dz$ is processed using the BLI, instead of scaling it up by $-\frac{1}{\xi}[\vartheta(t)-\vartheta(t-\xi)]^T Z_1[\vartheta(t)-\vartheta(t-\xi)]$ as in [29]. This approach helps further reduce conservatism.

Next, the state estimator design method is as follows.

Theorem 2. For given positive scalars ξ, ε, and ς, and matrices μ_1, μ_2, λ_1, and λ_2, there exist positive scalars ρ_{1i}, ρ_{2i}, and ρ_{3i} and matrices $E_i \in \mathbb{S}_+^{3n}$, Z and $W \in \mathbb{S}_+^{n}$, T_i and $U \in \mathbb{S}_+^{2n}$, and L_i, M_i, N_1, and $N_2 \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{d}\eta_{ij}T_j \le U, \quad (27)$$

$$\begin{bmatrix}\Lambda_i & \Omega_i\\ * & -\varepsilon\left(L + L^T\right)\end{bmatrix} \le 0 \quad (28)$$

hold, for $i \in \mathcal{D}$ where

$$\begin{aligned}\Lambda_i &= He\left[F_1^T E_i F_0 + F_5^T \hat{N} F_{6i}\right] + F_1^T\left(\sum_{j=1}^d \eta_{ij} E_j\right)F_1 + F_3^T T_i F_3 \\ &\quad - F_4^T T_i F_4 + F_3^T(\xi U)F_3 + c_8^T(\xi Z)c_8 - \frac{1}{\xi}F_2^T \hat{Z} F_2 + c_5^T(\varsigma W)c_5 \\ &\quad -\frac{1}{\varsigma}c_7^T W c_7 - \frac{3}{\varsigma}F_7^T W F_7 - \rho_{1i}F_3^T \hat{R} F_3 - \rho_{2i}F_4^T \hat{R} F_4 - \rho_{3i}F_8^T \hat{H} F_8, \\ \Omega_i &= \left[\, N_1^T - \varepsilon M_i B_i 0_{n\times 6n} N_2^T 0_{n\times n} \quad -\varepsilon M_i{}^T \right], \\ c_i &= \left[\, 0_{n\times(i-1)n} \quad I_n \quad 0_{n\times(10-i)n} \,\right], \quad i = 1, \ldots, 10, \\ F_8 &= \left[\, c_1^T \quad c_{10}^T \,\right]^T,\end{aligned} \tag{29}$$

The remaining symbols coincide with those defined in Theorem 1. For the MJHNN Equation 1, the estimator Equation 4 with gain matrices

$$K_i = L_i^{-1} M_i \tag{30}$$

guarantees the MSES of the FES Equation 5.

Proof. Define

$$\hat{\psi}(t) = col\{\psi_0(t),\ \psi_1(t),\ \psi_2(t),\ \psi_3(t)\},$$

where

$$\begin{aligned}\psi_0(t) &= \left[\, \epsilon^T(t) \quad \epsilon^T(t-\xi) \,\right]^T, \\ \psi_1(t) &= \frac{1}{\xi}\left[\, \int_{-\xi}^0 \epsilon_t^T(z)dz \quad \int_{-\xi}^0 \upsilon(z)\epsilon_t^T(z)dz \,\right]^T, \\ \psi_2(t) &= \left[\, \varrho^T(t) \quad \varrho^T(t-\xi) \quad \int_{-\varsigma}^0 \varrho_t^T(z)dz \,\right]^T, \\ \psi_3(t) &= \left[\, \dot{\epsilon}^T(t) \quad \tfrac{1}{\varsigma}\int_{-\varsigma}^0\int_\sigma^0 \varrho_t^T(z)dzd\sigma \quad \phi^T(t) \,\right]^T, \\ \epsilon_t(z) &= \epsilon(t+z),\ \varrho_t(z) = \varrho(t+z),\ \upsilon(z) = 2\frac{z+\xi}{\xi} - 1.\end{aligned}$$

The LKF is constructed as follows:

$$\begin{aligned}\mathcal{V}(\epsilon_t, \dot{\epsilon}_t, i) &= \mathcal{V}_1(\epsilon_t, i) + \mathcal{V}_2(\epsilon_t, i) + \mathcal{V}_3(\epsilon_t, \dot{\epsilon}_t) + \mathcal{V}_4(\epsilon_t), \\ \mathcal{V}_1(\epsilon_t, i) &= \hat{\epsilon}^T(t) E_i \hat{\epsilon}(t), \\ \mathcal{V}_2(\epsilon_t, i) &= \int_{t-\xi}^t \varpi_2^T(z) T_i \varpi_2(z)dz, \\ \mathcal{V}_3(\epsilon_t, \dot{\epsilon}_t) &= \int_{-\xi}^0\int_{t+\beta}^t \varpi_2^T(z) U \varpi_2(z)dzd\beta + \int_{-\xi}^0\int_{t+\beta}^t \dot{\epsilon}^T(z) Z \dot{\epsilon}(z)dzd\beta, \\ \mathcal{V}_4(\epsilon_t) &= \int_{-\varsigma}^0\int_{t+\beta}^t \varrho^T(z) W \varrho(z)dzd\beta,\end{aligned}$$

where $\varpi_2(t)$ is defined in Equation 9 and

$$\hat{\epsilon}(t) = col\{\epsilon(t),\ \xi\psi_1(t)\}.$$

Along similar lines to Theorem 1, by combining Equations 5 and 8, we can obtain $\mathcal{LV}(\epsilon_t, \dot{\epsilon}_t, i)$. Then, under Equation 30, one has

$$\begin{aligned}\mathcal{LV}(\epsilon_t, \dot{\epsilon}_t, i) \leq\ & \Lambda_i + He\left(\Upsilon_N K \Upsilon_B^T\right) \\ =\ & \Lambda_i + He\left(\Upsilon_N L_i^{-1} M_i \Upsilon_B^T\right),\end{aligned}$$

where Λ_i $(i \in \mathcal{D})$ are defined in Equation 29 and

$$\begin{aligned}\Upsilon_N &= \left[N_1^T \quad 0_{n\times 6n} \quad N_2^T \quad 0_{n\times 2n}\right]^T, \\ \Upsilon_B &= \left[-B \quad 0_{n\times 8n} \quad -I\right]^T.\end{aligned}$$

In light of Lemma 4, Equations 27 and 28 can ensure the MSES of the FES Equation 5.

4 Extension to the time-variant discrete-delay situation

Consider the MJHNN with time-variant discrete and distributed delays as follows:

$$\begin{aligned}\dot{\vartheta}(t) &= -A_i\vartheta(t) + C_i\varphi(\vartheta(t)) + D_i\varphi(\vartheta(t-\xi)) \\ &\quad + G_i\int_{t-\varsigma}^t \varphi(\vartheta(z))dz, \quad t \geq 0, \\ \alpha(t) &= B_i\vartheta(t) + \delta(\vartheta(t)),\ \vartheta(s) = \vartheta_0(s), \quad s \in [-2\zeta, 0],\ \zeta = \max\{\xi, \varsigma\},\end{aligned} \tag{31}$$

where ξ represents a time-variant discrete delay and satisfies

$$0 < \xi_1 < \xi < \xi_2, \quad \xi_{12} \triangleq \xi_2 - \xi_1.$$

Then, the estimator and the FES for the MJHNN Equation 31 are considered as follows:

$$\begin{aligned}\dot{\hat{\vartheta}}(t) &= -A_i\hat{\vartheta}(t) + C_i\varphi\left(\hat{\vartheta}(t)\right) + D_i\varphi\left(\hat{\vartheta}(t-\xi)\right) + G_i\int_{t-\varsigma}^t \varphi\left(\hat{\vartheta}(z)\right)dz \\ &\quad + K_i\left[\alpha(t) - B_i\hat{\vartheta}(t) - \delta\left(\hat{\vartheta}(t)\right)\right], \\ \dot{\epsilon}(t) &= -(A_i + K_iB_i)\epsilon(t) + C_i\varrho(t) + D_{1i}\varrho(t-\xi) + G_i\int_{t-\varsigma}^t \varrho(z)dz \\ &\quad - K_i\phi(t).\end{aligned} \tag{32}$$

We can provide the condition of the MSES for the MJHNN Equation 31 as follows.

Theorem 3. For given positive scalars ξ_1, ξ_2, and ς and matrices μ_1 and μ_2, there exist positive scalars ρ_{1i} and ρ_{2i} and matrices $E_i \in \mathbb{S}_+^{5n}$, W, Z_1, and $Z_2 \in \mathbb{S}_+^n$, T_{1i}, T_{2i}, U_1, and $U_2 \in \mathbb{S}_+^{2n}$, P_1 and $P_2 \in \mathbb{R}^{21n\times 2n}$, $S \in \mathbb{R}^{3n}$, and N_1 and $N_2 \in \mathbb{R}^n$ such that

$$\sum_{j=1}^d \eta_{ij}T_{1j} \leq U_1, \quad \sum_{j=1}^d \eta_{ij}T_{2j} \leq U_2, \tag{33}$$

$$\Gamma = \begin{bmatrix} \hat{Z}_2 & S \\ S^T & \hat{Z}_2 \end{bmatrix} \geq 0, \quad \Lambda_i(\xi) < 0 \tag{34}$$

hold, for $i \in \mathcal{D}$ and $\xi \in [\xi_1, \xi_2]$, where

$$\begin{aligned}
\Lambda_i(\xi) &= He\left[F_1^T(\xi)E_iF_0 + P_1u_{1i}(\xi) + P_2u_{2i}(\xi) + F_9^T\hat{N}X_{1i}\right] \\
&\quad + F_1^T(\xi)\left(\sum_{j=1}^{d}\eta_{ij}E_j\right)F_1(\xi) + \hat{T}_i + F_5^T(\xi_1U_1 + \xi_{12}U_2)F_5 \\
&\quad + c_{20}^T(\xi_1Z_1 + \xi_{12}Z_2)c_{20} - \frac{1}{\xi_1}F_2^T\hat{Z}_1F_2 - \frac{1}{\xi_{12}}\Sigma^T\Gamma\Sigma \\
&\quad + c_{15}^T(\varsigma W)c_{15} - \frac{1}{\varsigma}c_{17}^TWc_{17} - \frac{3}{\varsigma}X_2^TWX_2 \\
&\quad - \rho_{1i}F_5^T\hat{R}F_5 - \rho_{2i}F_6^T\hat{R}F_6, \\
\hat{T}_i &= F_5^TT_{1i}F_5 - F_7^TT_{1i}F_7 + F_7^TT_{2i}F_7 - F_8^TT_{2i}F_8, \\
\hat{Z}_i &= diag\{Z_i, 3Z_i, 5Z_i\}, \quad i = 1, 2, \\
F_0 &= \left[\, c_{20}^T \;\; c_1^T - c_2^T \;\; c_1^T + c_2^T - 2c_5^T \;\; c_2^T - c_4^T \;\; \hat{F}_0^T \,\right]^T, \\
\hat{F}_0 &= \xi_{12}(c_2 + c_4) - 2(c_{11} + c_{13}), \\
F_1(\xi) &= \left[\, c_1^T \;\; \xi_1c_5^T \;\; \xi_1c_6^T \;\; c_{11}^T + c_{13}^T \;\; \hat{F}_1^T(\xi) \,\right]^T, \\
\hat{F}_1(\xi) &= (\xi_2 - \xi)(c_{11} + c_{14}) + (\xi - \xi_1)(c_{12} - c_{13}), \\
F_2 &= \left[\, c_1^T - c_2^T \;\; c_1^T + c_2^T - 2c_5^T \;\; c_1^T - c_2^T - 6c_6^T \,\right]^T, \\
F_3 &= \left[\, c_2^T - c_3^T \;\; c_2^T + c_3^T - 2c_7^T \;\; c_2^T - c_3^T - 6c_8^T \,\right]^T, \\
F_4 &= \left[\, c_3^T - c_4^T \;\; c_3^T + c_4^T - 2c_9^T \;\; c_3^T - c_4^T - 6c_{10}^T \,\right]^T, \\
F_5 &= \left[\, c_1^T \;\; c_{15}^T \,\right]^T, \quad F_6 = \left[\, c_3^T \;\; c_{16}^T \,\right]^T, \\
F_7 &= \left[\, c_2^T \;\; c_{18}^T \,\right]^T, \quad F_8 = \left[\, c_4^T \;\; c_{19}^T \,\right]^T, \\
F_9 &= \left[\, c_1^T \;\; c_{20}^T \,\right]^T, \quad \Sigma = \left[\, F_3^T \;\; F_4^T \,\right]^T, \\
X_{1i} &= -c_{20} - A_ic_1 + C_ic_{15} + D_ic_{16} + G_ic_{17}, \\
X_2 &= c_{17} - 2c_{21}, \\
c_i &= \left[\, 0_{n\times(i-1)n} \;\; I_n \;\; 0_{n\times(21-i)n} \,\right], \quad i = 1, \ldots, 21, \\
u_{1i}(\xi) &= (\xi - \xi_1)\begin{bmatrix} c_7 \\ c_8 \end{bmatrix} - \begin{bmatrix} c_{11} \\ c_{12} \end{bmatrix}, \\
u_{2i}(\xi) &= (\xi_2 - \xi)\begin{bmatrix} c_9 \\ c_{10} \end{bmatrix} - \begin{bmatrix} c_{13} \\ c_{14} \end{bmatrix},
\end{aligned} \tag{35}$$

and the remaining symbols coincide with those defined in theorem 1. Then, the MJHNN Equation 31 achieves MSES.

Proof. Define

$$\hat{\psi} = col\{\psi_0(t), \ldots, \psi_8(t)\},$$

where

$$\begin{aligned}
\psi_0(t) &= \left[\, \vartheta^T(t) \;\; \vartheta^T(t-\xi_1) \;\; \vartheta^T(t-\xi) \;\; \vartheta^T(t-\xi_2) \,\right]^T, \\
\psi_1(t) &= \frac{1}{\xi_1}\left[\, \int_{-\xi_1}^{0}\vartheta_t^T(z)dz \;\; \int_{-\xi_1}^{0}v_1(z)\vartheta_t^T(z)dz \,\right]^T, \\
\psi_2(t) &= \frac{1}{\xi - \xi_1}\left[\, \int_{-\xi}^{-\xi_1}\vartheta_t^T(z)dz \;\; \int_{-\xi}^{-\xi_1}v_2(z)\vartheta_t^T(z)dz \,\right]^T, \\
\psi_3(t) &= \frac{1}{\xi_2 - \xi}\left[\, \int_{-\xi_2}^{-\xi}\vartheta_t^T(z)dz \;\; \int_{-\xi_2}^{-\xi}v_3(z)\vartheta_t^T(z)dz \,\right]^T, \\
\psi_4(t) &= (\xi - \xi_1)\psi_2(t), \quad \psi_5(t) = (\xi_2 - \xi)\psi_3(t), \\
\psi_6(t) &= \left[\, \varphi^T(\vartheta(t)) \;\; \varphi^T(\vartheta(t-\xi)) \;\; \int_{-\varsigma}^{0}\varphi_t^T(\vartheta(z))dz \,\right]^T, \\
\psi_7(t) &= \left[\, \varphi^T(\vartheta(t-\xi_1)) \;\; \varphi^T(\vartheta(t-\xi_2)) \,\right]^T, \\
\psi_8(t) &= \left[\, \dot{\vartheta}^T(t) \;\; \tfrac{1}{-\varsigma}\int_{-\varsigma}^{0}\int_{\sigma}^{0}\varphi_t^T(\vartheta(z))dzd\sigma \,\right]^T, \\
\psi_9(t) &= \left[\, \int_{-\xi_2}^{-\xi_1}\vartheta_t^T(z)dz \;\; \xi_{12}\int_{-\xi_2}^{-\xi_1}v_4(z)\vartheta_t^T(z)dz \,\right]^T, \\
v_1(z) &= 2\frac{z + \xi_1}{\xi_1} - 1, \quad v_2(z) = 2\frac{z + \xi}{\xi - \xi_1} - 1, \\
v_3(z) &= 2\frac{z + \xi_2}{\xi_2 - \xi} - 1, \quad v_4(z) = 2\frac{z + \xi_2}{\xi_{12}} - 1, \\
\vartheta_t(z) &= \vartheta(t + z), \quad \varphi_t(z) = \varphi(t + z).
\end{aligned}$$

The LKF is constructed as follows:

$$\begin{aligned}
\mathcal{V}\left(\vartheta_t, \dot{\vartheta}_t, i\right) &= \mathcal{V}_1(\vartheta_t, i) + \mathcal{V}_2(\vartheta_t, i) + \mathcal{V}_3\left(\vartheta_t, \dot{\vartheta}_t\right) + \mathcal{V}_4(\vartheta_t), \\
\mathcal{V}_1(\vartheta_t, i) &= \hat{\vartheta}^T(t)E_i\hat{\vartheta}(t), \mathcal{V}_2(\vartheta_t, i) \\
&= \int_{t-\xi_1}^{t}\varpi_1^T(z)T_{1i}\varpi_1(z)dz + \int_{t-\xi_2}^{t-\xi_1}\varpi_1^T(z)T_{2i}\varpi_1(z)dz, \\
\mathcal{V}_3\left(\vartheta_t, \dot{\vartheta}_t\right) &= \int_{-\xi_1}^{0}\int_{t+\beta}^{t}\varpi_1^T(z)U_1\varpi_1(z)dzd\beta \\
&\quad + \int_{-\xi_1}^{-\xi_2}\int_{t+\beta}^{t}\varpi_1^T(z)U_2\varpi_1(z)dzd\beta \\
&\quad + \int_{-\xi_1}^{0}\int_{t+\beta}^{t}\dot{\vartheta}^T(z)Z_1\dot{\vartheta}(z)dzd\beta \\
&\quad + \int_{-\xi_1}^{-\xi_2}\int_{t+\beta}^{t}\dot{\vartheta}^T(z)Z_2\dot{\vartheta}(z)dzd\beta, \\
\mathcal{V}_4(\vartheta_t) &= \int_{-\varsigma}^{0}\int_{t+\beta}^{t}\varphi^T(\vartheta(z))W\varphi(\vartheta(z))dzd\beta,
\end{aligned}$$

where $\varpi_1(t)$ is defined in Equation 9 and

$$\hat{\vartheta}(t) = \text{col}\{\vartheta(t), \xi_1\psi_1(t), \psi_9(t)\}.$$

By the definition of $\mathcal{L}$ in Equation 14, along the MJHNN Equation 31, we have

$$\mathcal{L}\mathcal{V}_1(\vartheta_t, i) = He\left[\hat{\vartheta}^T(t)E_i\dot{\hat{\vartheta}}(t)\right] + \hat{\vartheta}^T(t)\left(\sum_{j=1}^{d}\eta_{ij}E_j\right)\hat{\vartheta}(t), \tag{36}$$

where $\dot{\hat{\vartheta}}(t)$ includes the following components:

$$\begin{aligned}
\vartheta(t) &= c_1\hat{\psi}, \\
\xi_1\psi_1(t) &= \xi_1\left[\, c_5^T \;\; c_6^T \,\right]^T\hat{\psi}, \\
\psi_9(t) &= \begin{bmatrix} \int_{-\xi}^{-\xi_1}\vartheta_t(z)dz + \int_{-\xi_2}^{-\xi}\vartheta_t(z)dz \\ \xi_{12}\left(\int_{-\xi}^{-\xi_1}v_4(z)\vartheta_t(z)dz + \int_{-\xi_2}^{-\xi}v_4(z)\vartheta_t(z)dz\right) \end{bmatrix}.
\end{aligned} \tag{37}$$

The initial n components are $(c_{11} + c_{13})\hat{\psi}$, while the subsequent n components demand two distinct expressions for $v_4(z)$, depending on $v_2(z)$ and $v_3(z)$. It can be attained that

$$\begin{aligned}
\xi_{12}v_4(z) &= (\xi - \xi_1)v_2(z) + (\xi_2 - \xi) \\
&= (\xi_2 - \xi)v_3(z) - (\xi - \xi_1).
\end{aligned} \tag{38}$$

Substituting Equation 38 into Equation 37, we obtain

$$\begin{aligned}
&\xi_{12}\left(\int_{-\xi}^{-\xi_1}v_4(z)\vartheta_t(z)dz + \int_{-\xi_2}^{-\xi}v_4(z)\vartheta_t(z)dz\right) \\
&= \int_{-\xi}^{-\xi_1}[(\xi - \xi_1)v_2(z) + (\xi_2 - \xi)]\vartheta_t(z)dz \\
&\quad + \int_{-\xi_2}^{-\xi}[(\xi_2 - \xi)v_3(z) - (\xi - \xi_1)]\vartheta_t(z)dz \\
&= \hat{F}_{1i}(\xi)\hat{\psi}.
\end{aligned}$$

Therefore, we obtain $\psi_9(t) = \left[\, c_{11}^T + c_{13}^T \;\; \hat{F}_{1i}^T(\xi) \,\right]\hat{\psi}$ and

$$\hat{\vartheta}(t) = F_1(\xi)\hat{\psi}. \tag{39}$$

The components of $\dot{\hat{\vartheta}}(t)$ are as follows:

$$\begin{aligned}\dot{\vartheta}(t) &= c_{20}\hat{\psi},\\ \xi_1\dot{\psi}_1(t) &= \left[\, c_1^T - c_2^T \;\; c_1^T + c_2^T - 2c_5^T \,\right]^T\hat{\psi},\\ \dot{\psi}_9(t) &= \left[\, c_2^T - c_4^T \;\; \xi_{12}c_2^T + \xi_{12}c_4^T - 2c_{11}^T - 2c_{13}^T \,\right]^T\hat{\psi}\\ &= \left[\, c_2^T - c_4^T \;\; \hat{G}_0^T \,\right]^T\hat{\psi},\end{aligned}$$

and it is easy to obtain

$$\dot{\hat{\vartheta}}(t) = F_0\hat{\psi}. \tag{40}$$

Then, we deduce from Equation 35 that

$$\begin{aligned}(\xi - \xi_1)\psi_2(t) - \psi_4(t) = 0,\\ (\xi_2 - \xi)\psi_3(t) - \psi_5(t) = 0.\end{aligned}$$

Hence, leveraging the matrices u_1 and u_2 defined in Equation 35, we can derive the following equation for matrices P_1 and $P_2 \in \mathbb{R}^{21n\times 2n}$:

$$\hat{\psi}^T He(P_1u_{1i}(\xi) + P_2u_{2i}(\xi))\hat{\psi} = 0. \tag{41}$$

One can derive from Equations 36–41 that

$$\begin{aligned}\mathcal{LV}_1(\vartheta_t, i) = \hat{\psi}^T (He(F_1^T(\xi)E_iF_0 + P_1u_{1i}(\xi)\\ + P_2u_{2i}(\xi)) + F_1^T(\xi)\left(\sum_{j=1}^{d}\eta_{ij}E_j\right)F_1(\xi))\hat{\psi}.\end{aligned} \tag{42}$$

It follows from Equation 33 that

$$\begin{aligned}\mathcal{LV}_2(\vartheta_t, i) &\le \hat{\psi}^T\hat{T}_i\hat{\psi},\\ \mathcal{LV}_3\left(\vartheta_t, \dot{\vartheta}_t\right) &\le \varpi_1^T(t)(\xi_1U_1 + \xi_{12}U_2)\varpi_1(t)\\ &\quad + \dot{\vartheta}^T(t)(\xi_1Z_1 + \xi_{12}Z_2)\dot{\vartheta}(t) - \int_{t-\xi_1}^{t}\dot{\vartheta}^T(z)Z_1\dot{\vartheta}(z)dz\\ &\quad - \int_{t-\xi_2}^{t-\xi_1}\dot{\vartheta}^T(z)Z_2\dot{\vartheta}(z)dz.\end{aligned} \tag{43}$$

By applying Lemma 1 to the integral term in $\mathcal{LV}_3$, one can obtain

$$-\int_{t-\xi_1}^{t}\dot{\vartheta}^T(z)Z_1\dot{\vartheta}(z)dz \le -\frac{1}{\xi_1}F_2^T\hat{Z}_1F_2, \tag{44}$$

and

$$\begin{aligned}-\int_{t-\xi_2}^{t-\xi_1}\dot{\vartheta}^T(z)Z_2\dot{\vartheta}(z)dz &\le -\frac{1}{\xi_2-\xi}F_4^T\hat{Z}_2F_4 - \frac{1}{\xi-\xi_1}F_3^T\hat{Z}_2F_3\\ &= -\Sigma^T\begin{bmatrix}\frac{1}{\xi-\xi_1}\hat{Z}_2 & 0\\ 0 & \frac{1}{\xi_2-\xi}\hat{Z}_2\end{bmatrix}\Sigma.\end{aligned} \tag{45}$$

Utilizing Lemma 2, we can deduce from Equation 45 that

$$\begin{bmatrix}\frac{1}{\xi-\xi_1}\hat{Z}_2 & 0\\ 0 & \frac{1}{\xi_2-\xi}\hat{Z}_2\end{bmatrix} \ge \frac{1}{\xi_{12}}\begin{bmatrix}\hat{Z}_2 & S\\ S^T & \hat{Z}_2\end{bmatrix}, \tag{46}$$

leading to

$$\begin{aligned}\mathcal{V}_3\left(\vartheta_t, \dot{\vartheta}_t\right) \le \hat{\psi}^T\Big(F_5^T(\xi_1U_1 + \xi_{12}U_2)F_5 - \frac{1}{\xi_1}F_2^T\hat{Z}_1F_2\\ + c_{20}^T(\xi_1Z_1 + \xi_{12}Z_2)c_{20} - \frac{1}{\xi_{12}}\Sigma^T\Gamma\Sigma\Big)\hat{\psi}.\end{aligned} \tag{47}$$

According to Lemma 2, $\mathcal{LV}_4(\vartheta_t)$ can be computed as follows:

$$\mathcal{LV}_4(\vartheta_t) \le \hat{\psi}^T\left(c_{15}^T(\varsigma W)c_{15} - \frac{1}{\varsigma}c_{17}^TWc_{17} - \frac{3}{\varsigma}X_2^TWX_2\right)\hat{\psi}. \tag{48}$$

Similar to Theorem 1, Equation 20 can be written as

$$\hat{\psi}^T He\left(F_9^T\hat{N}X_{1i}\right)\hat{\psi} = 0. \tag{49}$$

From Equation 7 and Equations 42–49, for any scalars $\rho_{1i} > 0$ and $\rho_{2i} > 0$, we can obtain

$$\begin{aligned}\mathcal{LV}\left(\vartheta_t, \dot{\vartheta}_t, i\right) &\le \hat{\psi}^T\Big(He\left[F_1^T(\xi)E_iF_0 + P_1u_{1i}(\xi) + P_2u_{2i}(\xi) + F_9^T\hat{N}X_{1i}\right]\\ &\quad + F_1^T(\xi)\left(\sum_{j=1}^{d}\eta_{ij}E_j\right)F_1(\xi) + F_5^T(\xi_1U_1 + \xi_{12}U_2)F_5\\ &\quad - \frac{1}{\xi_1}F_2^T\hat{Z}_1F_2 + c_{20}^T(\xi_1Z_1 + \xi_{12}Z_2)c_{20} - \frac{1}{\xi_{12}}\Sigma^T\Gamma\Sigma + \hat{T}_i\\ &\quad + c_{15}^T(\varsigma W)c_{15} - \frac{1}{\varsigma}c_{17}^TWc_{17} - \rho_{1i}F_5^T\hat{R}F_5 - \rho_{2i}F_6^T\hat{R}F_6\Big)\hat{\psi}\\ &= \hat{\psi}^T\Lambda_i(\xi)\hat{\psi},\end{aligned}$$

where Λ_i $(i \in \mathcal{D})$ are defined in Equation 35. Using a similar approach to Theorem 1, we can derive that Equations 34 and 35 can ensure the MSES of the MJHNN Equation 31.

Then, based on Theorem 2, we can establish the following estimator design method.

Theorem 4. For given positive scalars ξ_1, ξ_2, ε, and ς and matrices μ_1, μ_2, λ_1, and λ_2, there exist positive scalars ρ_{1i}, ρ_{2i}, and ρ_{3i} and matrices $E_i \in \mathbb{S}_+^{5n}$, W, Z_1, and $Z_2 \in \mathbb{S}_+^n$, T_{1i}, T_{2i}, U_1, and $U_2 \in \mathbb{S}_+^{2n}$, P_1 and $P_2 \in \mathbb{R}^{22n\times 2n}$, $S \in \mathbb{R}^{3n}$, and L_i, M_i, N_1, and $N_2 \in \mathbb{R}^n$ such that

$$\begin{aligned}\sum_{j=1}^{d}\eta_{ij}T_{1j} \le U_1, \quad \sum_{j=1}^{d}\eta_{ij}T_{2j} \le U_2,\\ \Gamma = \begin{bmatrix}\hat{Z}_2 & S\\ S^T & \hat{Z}_2\end{bmatrix} \ge 0, \begin{bmatrix}\Lambda_i(\xi) & \Omega\\ * & -\varepsilon\left(L + L^T\right)\end{bmatrix} \le 0\end{aligned}$$

hold, for $i \in \mathcal{D}$ and $\xi \in [\xi_1, \xi_2]$, where

$$\begin{aligned}\Lambda_i(\xi) &= He\left[F_1(\xi)^TE_iF_0 + P_1u_{1i}(\xi) + P_2u_{2i}(\xi) + F_9^T\hat{N}X_{1i}\right]\\ &\quad + F_1^T(\xi)\left(\sum_{j=1}^{d}\eta_{ij}E_j\right)F_1(\xi) + \hat{T}_i + F_5^T(\xi_1U_1 + \xi_{12}U_2)F_5\\ &\quad + c_{20}^T(\xi_1Z_1 + \xi_{12}Z_2)c_{20} - \frac{1}{\xi_1}F_2^T\hat{Z}_1F_2 - \frac{1}{\xi_{12}}\Sigma^T\Gamma\Sigma\\ &\quad + c_{15}^T(\varsigma W)c_{15} - \frac{1}{\varsigma}c_{17}^TWc_{17} - \frac{3}{\varsigma}X_2^TWX_2\\ &\quad - \rho_{1i}F_5^T\hat{R}F_5 - \rho_{2i}F_6^T\hat{R}F_6 - \rho_{3i}X_3^T\hat{H}X_3,\\ \Omega &= \left[\, N_1^T - \varepsilon M_iB_i \;\; 0_{n\times 18n} \;\; N_2^T \;\; 0_{n\times n} \;\; -\varepsilon M_i \,\right]^T,\\ c_i &= \left[\, 0_{n\times(i-1)n} \;\; I_n \;\; 0_{n\times(22-i)n} \,\right], \quad i = 1, \ldots, 22,\\ X_3 &= \left[\, c_1{}^T \;\; c_{22}{}^T \,\right]^T,\end{aligned}$$

and the remaining symbols coincide with those defined in theorem 3. For the MJHNN Equation 31, the estimator Equation 32 with gain matrices

$$K_i = L_i^{-1}M_i$$

guarantees the MSES of the FES Equation 32.

5 Numerical example

Example 1. Consider MJHNNs Equation 5 and Equation 32 with

TABLE 1 Maximum allowable delay upper bound ς_{max} for various ξ and μ_2.

Method	ξ	ς_{max}		
		$\mu_2 = 0.8I$	$\mu_2 = 0.9I$	$\mu_2 = I$
Theorem 2 in [29]	0.5	2.0222	1.7123	1.2863
Theorem 1		2.0777	1.7473	1.3058
Theorem 3		1.9167	1.5911	1.1169
($\xi_1 = 0.45$, $\xi_2 = 0.55$)				
Theorem 2 in [29]	1.0	1.9535	1.6741	1.2739
Theorem 1		2.0453	1.7347	1.2983
Theorem 3		1.8992	1.5776	1.1081
($\xi_1 = 0.95$, $\xi_2 = 1.05$)				
Theorem 2 in [29]	1.5	1.9161	1.6462	1.2675
Theorem 1		2.0094	1.7137	1.2882
Theorem 3		1.8743	1.5585	1.0979
($\xi_1 = 1.45$, $\xi_2 = 1.55$)				
Theorem 2 in [29]	2.0	1.9015	1.6358	1.2659
Theorem 1		1.9795	1.6908	1.2800
Theorem 3		1.8423	1.5402	1.0952
($\xi_1 = 1.95$, $\xi_2 = 2.05$)				

TABLE 2 Maximum allowable delay upper bound ξ_{max} for various ς and μ_2.

Method	ς	ξ_{max}	ς	ξ_{max}	ς	ξ_{max}
		$\mu_2 = 0.8I$		$\mu_2 = 0.9I$		$\mu_2 = I$
Theorem 2 in [29]	1.97	0.8471	1.67	1.0571	1.27	1.2389
Theorem 1		2.1964		2.5566		3.7560
Theorem 2 in [29]	1.98	0.7668	1.68	0.9221	1.28	0.7310
Theorem 1		1.9912		2.2648		2.0045
Theorem 2 in [29]	1.99	0.6949	1.69	0.7948	1.29	0.3709
Theorem 1		1.8080		2.0199		1.4085
Theorem 2 in [29]	2.00	0.6298	1.70	0.6673	1.30	0.1085
Theorem 1		1.6431		1.7966		0.9128

$$A_1 = \begin{bmatrix} 2.2 & 0 \\ 0 & 2.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2.3 & 0 \\ 0 & 2.4 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.2 & 0.5 \\ 0.4 & 0.3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.3 & -0.1 \\ 0.2 & 0.4 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.3 & 0.8 \\ 0.5 & 0.4 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.1 & 0.9 \\ -0.8 & 1.2 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0.5 & -0.5 \\ 0.2 & 0.7 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.3 & 0.2 \\ -0.5 & 0.4 \end{bmatrix},$$

$$\eta_{11} = -2, \qquad \eta_{22} = -3, \qquad \mu_1 = -I.$$

TABLE 3 Maximum allowable delay upper bound ξ_{2max} for various ς, ξ_1, and μ_2 in Theorem 3.

ς	ξ_1	ξ_{2max}		
		$\mu_2 = 0.8I$	$\mu_2 = 0.9I$	$\mu_2 = I$
0.5	0.5	1.7005	1.5293	1.2861
	1.0	2.1101	1.9703	1.7635
	1.5	2.5509	2.4244	2.2457
1.0	0.5	1.5133	1.2680	0.8427
	1.0	1.9077	1.7082	1.3149
	1.5	2.3342	2.1532	1.7882

By solving LMIs in theorems 1 and 3 in this paper and Theorem 2 in [29], we can obtain the maximum admissible upper bound ς_{max} for different values of ξ and μ_2, as shown in Table 1, which clearly shows the effectiveness of theorems 1 and 3 in this paper. Compared to Theorem 2 in [29], theorem 1 in this study is less conservative.

The maximum admissible upper bound ξ_{max} for various ς and μ_2 is presented in Table 2. It can be observed that for the same ς and μ_2, ξ_{max} in theorem 1 is larger than in Theorem 2 of [29].

In addition, for different combinations of ς, ξ_1, and μ_2, solving Theorem 3 yields the maximum admissible upper bound ξ_{2max}, as shown in Table 3. As ς increases, ξ_{2max} decreases under the corresponding ξ_1 and μ_2 conditions.

Example 2. Consider MJHNNs Equation 5 and Equation 32 with

$$A_1 = \begin{bmatrix} 0.8 & 0 \\ 0 & 1.0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.5 & 0.2 \\ 0.6 & 0.3 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.1 & 0 \\ -0.1 & 0.2 \end{bmatrix}, \quad G_1 = \begin{bmatrix} -0.2 & -0.1 \\ 0.3 & 0.1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.6 & 0.3 \\ 0.7 & 0.6 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.3 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 0.1 & 0.2 \\ -0.2 & 0.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.8 & 0.5 \\ 0.6 & 0.9 \end{bmatrix}, \quad \eta_{11} = -0.2, \quad \eta_{22} = -0.8,$$

$$\mu_1 = -\mu_2 = -0.5I, \qquad \lambda_1 = -0.2I, \qquad \lambda_2 = 0.3I.$$

Next, we design state estimators to ensure the MSES of FESs with discrete and distributed delays, considering cases of time-invariant and time-variant discrete delays.

Case 1. Time-invariant discrete delay.

For the FES Equation 5, with $\varsigma = 1.5$, $\xi = 1.0$, and $\varepsilon = 0.9$, solving the LMIs in Theorem 2 yields the estimator Equation 4 with gain matrices

$$K_1 = \begin{bmatrix} 0.6317 & 0.0146 \\ 0.1439 & 0.6857 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.4783 & 0.2092 \\ 0.2351 & 0.5689 \end{bmatrix}.$$

Case 2. Time-variant discrete delay.

For the FES Equation 32, with $\varsigma = 1.5$, $\xi_1 = 0.5$, $\xi_2 = 1.5$, and $\varepsilon = 0.9$, solving the LMIs in Theorem 4 yields the estimator Equation 32 with gain matrices

$$K_1 = \begin{bmatrix} 0.6739 & -0.0045 \\ 0.1310 & 0.7854 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.5257 & 0.1694 \\ 0.1774 & 0.6153 \end{bmatrix}.$$

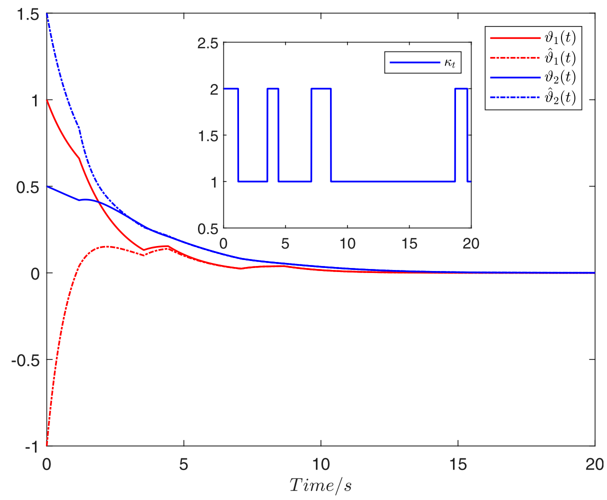

FIGURE 1
State trajectories $\vartheta_1(t)$ and $\vartheta_2(t)$, estimations $\hat{\vartheta}_1(t)$ and $\hat{\vartheta}_2(t)$, and Markovian chain κ_t (Case 1).

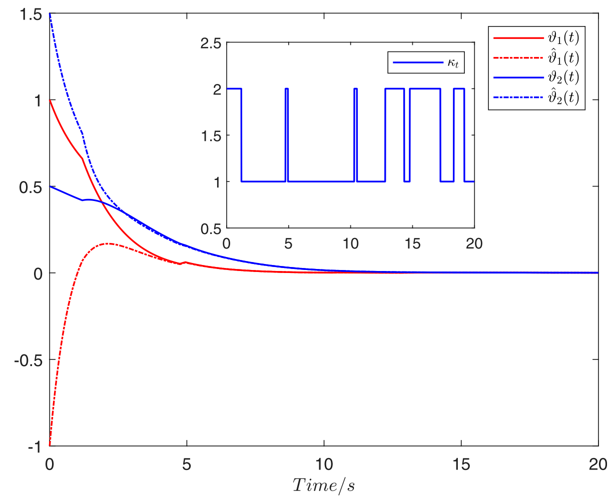

FIGURE 3
State trajectories $\vartheta_1(t)$ and $\vartheta_2(t)$, estimations $\hat{\vartheta}_1(t)$ and $\hat{\vartheta}_2(t)$, and Markovian chain κ_t (Case 2).

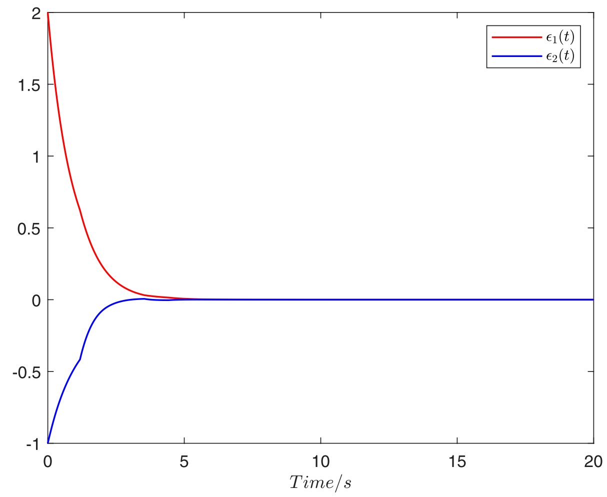

FIGURE 2
State trajectories of the FES $\epsilon_1(t)$ and $\epsilon_2(t)$ (Case 1).

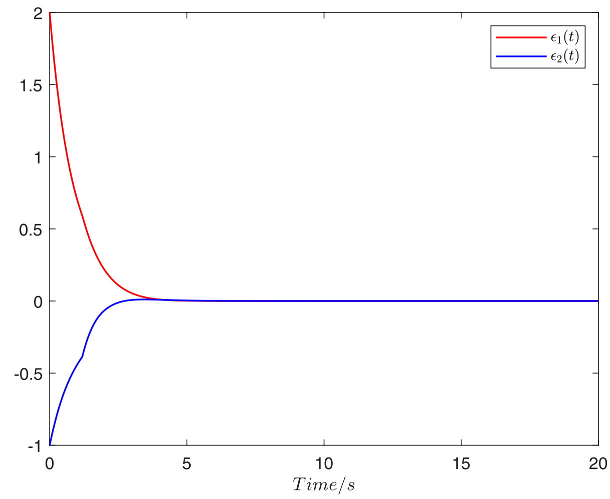

FIGURE 4
State trajectories of the FES $\epsilon_1(t)$ and $\epsilon_2(t)$ (Case 2).

In the simulations, we set $\delta(\vartheta) = 0.25 \sin\vartheta + 0.05\vartheta$, $\varphi(\vartheta) = 0.5 \sin\vartheta$, and

$$\vartheta(s) = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad \hat{\vartheta}(s) = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}, \quad s \in [-2 \ \ 0].$$

The simulations of the state estimators are shown in Figures 1–4. For the MJHNN with time-invariant discrete and distributed delays, Figure 1 shows the state trajectories and their corresponding estimations, while Figure 2 shows the trajectories of the FES. Figures 3, 4 show the corresponding trajectories under time-variant discrete delay scenarios. The proposed state estimator design methods are effective for MJHNNs with discrete and distributed delays in both cases of time-invariant and time-variant discrete delays.

6 Conclusion

This paper has investigated the state estimation for MJHNNs with discrete and distributed delays. Specifically, both time-invariant and time-variant discrete delay cases are considered. Two conditions for the MSES of MJHNNs have been proposed utilizing augmented LKFs, the WBII, the BLI, and the CCI. The LMIs-based design methods for the required estimators have been developed by eliminating nonlinear coupling terms. Lastly, two numerical examples are given to demonstrate the applicability and reduced conservatism of the proposed stability criteria and estimator design methods. Future research could explore further refinement of these analysis and design results, and exporing their extention to more complex neural network models.

Data availability statement

The raw data supporting the conclusion of this article will be made available by the authors, without undue reservation.

Author contributions

LG: Investigation, Methodology, Writing–review and editing. WH: Writing–original draft.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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