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Some novel exact solutions for the generalized time-space fractional coupled Hirota-Satsuma KdV equation

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In this paper, two efficient methods, namely the modified $(G'/G,1/G)$-expansion method and the $G'/(bG'+G+a)$-expansion method, are employed to obtain novel exact wave solutions for the generalized time-space fractional coupled Hirota-Satsuma KdV equation. Various types of analytical explicit solutions, including well-known bell-shape solitons, mixed solitary wave solutions, and periodic wave solutions are obtained. These solutions are of great significance for revealing the nonlinear interaction between two long waves with different dispersion effects. The two-dimensional and three-dimensional distribution maps and contour plots corresponding to partial solutions are simulated to visually display the evolution process of relevant physical quantities over time. Moreover, the potential applications of these solutions in nano/micro devices and systems, especially in MEMS (Micro-Electro-Mechanical Systems) are discussed. It is demonstrated that the methods and processes utilized have strong applicability for constructing analytical solutions of nonlinear evolution equations.

KEYWORDS

time-space fractional coupled Hirota-Saturma KdV equation, the modified (G′/G,1/G)-expansion method, the G′/(bG′+G+a)-expansion method, blow-up, analytical solutions

1 Introduction

As we all know, nonlinear partial differential equation models are applied to almost every corner of social life. For example, various nonlinear soliton equations can be used to describe wave phenomena in many natural sciences and engineering fields such as fluid physics, solid state physics, laser physics, astrophysics, geophysics, lattice vibration, optical fiber communication, quantum mechanics, geomechanics, oceanography, superconductivity, field theory, transportation, etc. Since fractional order nonlinear systems can describe these nonlinear processes more accurately than integer order systems, studying the solution of fractional order nonlinear systems is particularly important in the development of natural sciences, and has always been a hot topic for domestic and foreign scholars. Till now, people have given many forms of fractional derivative definitions for different situations. For example, Riemann-Liouville definition [1], Caputo definition [2], Jumaries's definition [3], Atangana's definition [4], Atangana-Baleanu-Riemann definition [5], conformable definition [6], Abu-Shady-Kaabar definition [7], He's definition [8], etc. [9–15]. Each definition has its own advantages and disadvantages, for example, the Riemann-Liouville definition is to consider the derivative of the integral factor, the Caputo definition is to consider the integration of the derivative factor, the Jumaries's definition is to consider the influence of the initial value and the He's definition takes into account the more general problem of initial values, etc., and their efficiency varies in the process of solving some specific problems. In order to find the analytical solutions of nonlinear fractional partial differential equations, many domestic and foreign scholars have made great efforts, and the existing methods mainly include the Bäcklund transformation method [16], the homogeneous equilibrium method [17], the Riccati equation expansion method [18], the F-expansion method [19], the Jacobi elliptic function expansion method [20], the generalized expansion method [21], the Darboux transform method [22], the Lie symmetry method [23], the Adomian decomposition method [24], The homotopy perturbation method [25], the variational iterative method [26], and so on [27–29]. These methods have their own advantages and characteristics, providing powerful tools for exploring the solutions of complex nonlinear equations.

Furthermore, the applications of these solutions are not limited to traditional fields. In the era of rapid development of nano/micro technology, they have potential applications in nano/micro devices and systems, especially in MEMS (Micro-Electro-Mechanical Systems) [30]. MEMS technology combines mechanical elements, sensors, actuators, and electronics on a microscale, and understanding the behavior of fractional partial differential equations can contribute to the design, optimization, and performance improvement of MEMS devices [31]. For example, in the field of sensors, fractional order models can help analyze and predict the response of micro-sensors to various stimuli more accurately. In actuators, the solutions of fractional equations can provide insights into the dynamic behavior and control strategies [32]. Additionally, in integrated micro-systems, the understanding of fractional order phenomena can enhance the functionality and reliability of the overall system [33, 34].

The study of fractional order nonlinear partial differential equations and their solutions is not only of theoretical significance but also has practical applications in a wide range of fields, especially in the emerging field of nano/micro devices and systems such as MEMS. Our research is based on the definition of M-fractional derivative proposed by Sousa and Oliveira recently [35], this new fractional derivative definition generalizes the conformable derivative by a truncated Mittag-Leffler function of one parameter [6]. By adopting the ideas of generalized Jacobi elliptic function method [36], modified $(G'/G, 1/G)$-expansion method [38, 39] and $G'/(bG' + G + a)$-expansion method [40], using the homogeneous equilibrium principle [41] and mathematical symbolic calculation software, to study a class of generalized time-space fractional coupled Hirota-Satsuma KdV system arising in interaction of two long waves with different dispersion effects under the definition of M-fractional derivative,

$$\begin{cases} u_t^\alpha = \dfrac{1}{4}u_{xxx}^{3\beta} + 3uu_x^\beta + 3\left(-v^2 + w\right)_x^\beta, \\ v_t^\alpha = -\dfrac{1}{2}v_{xxx}^{3\beta} - 3uv_x^\beta, \\ w_t^\alpha = -\dfrac{1}{2}w_{xxx}^{3\beta} - 3uw_x^\beta, 0 < \alpha, \beta \le 1. \end{cases} \tag{1}$$

where $(\cdot)_t^\alpha = D_{M,t}^{\gamma_1,\alpha}(\cdot), (\cdot)_x^\beta = D_{M,x}^{\gamma_2,\beta}(\cdot), (\cdot)_{xxx}^{3\beta} = D_{M,x}^{\gamma_2,\beta}\left(D_{M,x}^{\gamma_2,\beta}\left(D_{M,x}^{\gamma_2,\beta}(\cdot)\right)\right)$ mean the M-fractional derivative [35, 39, 42, 43], $u = u(x,t), v = v(x,t), w = w(x,t)$. If we select $\alpha = 1, \beta = 1$, we get the well-known integer order coupled Hirota-Satsuma KdV equation. The equation is mainly used to describe the interaction between two columns of long waves with different dispersion relations [44]. Ref. [45] studies the case when $w = 0$, and Ref. [46] obtains the general form when $w \neq 0$ through a matrix spectrum problem. Ref. [47] studies its elliptic sine function solution by direct expansion method. Refs. [48, 49] use the modified Riccati expansion method and the extended elliptic function expansion method to study its exact solutions in various forms. Ref. [50] studies the Darboux transformation of the equation, and the branch structure of the equation is studied by the theory of plane dynamic system in reference [51]. Under the definition of conformable fractional derivative with $\beta = 1$, Ref. [52] studies the analytical solutions of system (1) by using auxiliary equation method and series expansion method, and Ref. [53] uses (G'/G) expansion method to study the solitary wave solution and trigonometric function periodic solution of system (1). Additional relevant studies on the system can be referred to Refs. [54–58]. Let's first introduce several relevant definitions and properties.

Definition 1: For a function $f(t)$:$[0,\infty) \to R$, We defined the M-fractional derivative operator of $f(t)$ of order α as [35].

$$D_{M,t}^{\gamma,\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(tE_\gamma(\varepsilon t^{-\alpha})\right) - f(t)}{\varepsilon}, \gamma \ge 0, 0 < \alpha \le 1. \tag{2}$$

where $E_\gamma(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma k+1)}$ is a Mittag-Leffler function of parameter γ.

Property 1: The M-fractional derivative operator of $f(t)$ of order α have the following important properties [35, 39, 42, 43]:

(1) $D_{M,t}^{\gamma,\alpha} f(t) = \frac{t^{-\alpha}}{\Gamma(\gamma+1)} \frac{df(t)}{dt}$.
(2) $D_{M,t}^{\gamma,\alpha}(af(t) + bg(t)) = aD_{M,t}^{\gamma,\alpha}f(t) + bD_{M,t}^{\gamma,\alpha}g(t), \forall a, b \in R$.
(3) $D_{M,t}^{\gamma,\alpha}(f(t)g(t)) = f(t)D_{M,t}^{\gamma,\alpha}g(t) + g(t)D_{M,t}^{\gamma,\alpha}f(t)$.
(4) $D_{M,t}^{\gamma,\alpha}(f(t)/g(t)) = \left[g(t)D_{M,t}^{\gamma,\alpha}f(t) - f(t)D_{M,t}^{\gamma,\alpha}g(t)\right]/g^2(t)$.
(5) $D_{M,t}^{\gamma,\alpha}(f \circ g)(t) = f'(g(t))D_{M,t}^{\gamma,\alpha}g(t) = \frac{t^{-\alpha}}{\Gamma(\gamma+1)} f'(g(t)) \frac{dg(t)}{dt}$.

Definition 2: The M-fractional derivative system (1) determined by Definition 1 and Equation 2 performs the following travelling wave transformation:

$$u = u(x,t) = u(\xi), v = v(x,t) = v(\xi), w = w(x,t) = w(\xi) \tag{3}$$

$$\xi = \frac{\Gamma(\gamma_2+1)}{\beta}2kx + \frac{\Gamma(\gamma_1+1)}{\alpha}Ct^{\alpha} + \xi_0, \gamma_1, \gamma_2 \geq 0, 0 < \alpha, \beta \leq 1 \tag{4}$$

where k, C are undetermined constants, $\gamma_1, \gamma_2 \xi_0$ are arbitrary constants.

Substituting Equations 3, 4 into Equation 1, we obtain the following system of ordinary differential equations:

$$\begin{cases} Cu' = 2k^3u''' + 6kuu' - 12kvv' + 6kw', \\ Cv' = -4k^3v''' - 6kuv', \\ Cw' = -4k^3w''' - 6kuw'. \end{cases} \tag{5}$$

where

$$u' = \frac{du}{d\xi}, u''' = \frac{d^3u}{d\xi^3}, v' = \frac{dv}{d\xi}, v''' = \frac{d^3v}{d\xi^3}, w' = \frac{dw}{d\xi}, w''' = \frac{d^3w}{d\xi^3}.$$

2 Description of the two methods

2.1 The modified $(G'/G, 1/G)$-expansion method

Consider the following nonlinear M-fractional nonlinear partial differential equations:

$$\begin{cases} E_1\left(u, u_t^{\alpha}, u_x^{\beta}, uu_x^{\beta}, v, v_t^{\alpha}, v_x^{\beta}, vv_x^{\beta}, w, w_t^{\alpha}, w_x^{\beta}, ww_x^{\beta}, uv_x^{\beta}, uw_x^{\beta}, vw_x^{\beta}, \cdots\right) = 0, \\ E_2\left(u, u_t^{\alpha}, u_x^{\beta}, uu_x^{\beta}, v, v_t^{\alpha}, v_x^{\beta}, vv_x^{\beta}, w, w_t^{\alpha}, w_x^{\beta}, ww_x^{\beta}, uv_x^{\beta}, uw_x^{\beta}, vw_x^{\beta}, \cdots\right) = 0, \\ E_3\left(u, u_t^{\alpha}, u_x^{\beta}, uu_x^{\beta}, v, v_t^{\alpha}, v_x^{\beta}, vv_x^{\beta}, w, w_t^{\alpha}, w_x^{\beta}, ww_x^{\beta}, uv_x^{\beta}, uw_x^{\beta}, vw_x^{\beta}, \cdots\right) = 0. \end{cases} \tag{6}$$

By using the wave transformation (4), Equation 6 is converted into a nonlinear ordinary differential equations (ODE):

$$\begin{cases} O_1(u, u', uu', v, v', vv', w, w', ww', uv', uw', vw', \cdots) = 0, \\ O_2(u, u', uu', v, v', vv', w, w', ww', uv', uw', vw', \cdots) = 0, \\ O_3(u, u', uu', v, v', vv', w, w', ww', uv', uw', vw', \cdots) = 0. \end{cases} \tag{7}$$

Assume that Equation 7 has the following solution:

$$\begin{cases} u = \sum_{i=0}^{M} a_i\psi^i + \sum_{j=1}^{M} b_j\phi^j + \sum_{i=1}^{M-1} c_i\psi^i\phi, \\ v = \sum_{i=0}^{N} d_i\psi^i + \sum_{j=1}^{N} e_j\phi^j + \sum_{i=1}^{N-1} f_i\psi^i\phi, \\ w = \sum_{i=0}^{P} g_i\psi^i + \sum_{j=1}^{P} h_j\phi^j + \sum_{i=1}^{P-1} l_i\psi^i\phi. \end{cases} \tag{8}$$

where M, N, P are balance numbers, $\phi = \phi(\xi) = \frac{G'}{G}, \psi = \psi(\xi) = \frac{1}{G}$, $a_i, b_j, c_i, d_i, e_j, f_i, g_i, h_j, l_i$ and variable function $\xi = \xi(x,t)$ are determined later. The G is a solution of the following auxiliary ODE:

$$G'' = \varepsilon G - \varepsilon\mu. \tag{9}$$

where $\varepsilon = \pm 1, \mu$ is an arbitrary real number. It satisfies the following constrained conditions:

$$\phi' = \varepsilon - \varepsilon\mu\psi - \phi^2, \ \psi' = -\phi\psi, \ \phi^2 = \varepsilon - 2\varepsilon\mu\psi - \varepsilon(b^2 - \varepsilon c^2 - \mu^2)\psi^2. \tag{10}$$

where arbitrary constants μ, b, c satisfied the relation $c^2 + b^2 + \mu^2 \neq 0$. Equations 9, 10 admit the following solutions

Case 1: When $\varepsilon = 1$, we have $G = b\cosh\xi + c\sinh\xi + \mu$, thus

$$\phi = \frac{G'}{G} = \frac{b\sinh\xi + c\cosh\xi}{b\cosh\xi + c\sinh\xi + \mu}, \psi = \frac{1}{G} = \frac{1}{b\cosh\xi + c\sinh\xi + \mu}. \tag{11}$$

Case 2: When $\varepsilon = -1$, we have $G = b\cos\xi + c\sin\xi + \mu$, thus

$$\phi = \frac{G'}{G} = \frac{-b\sin\xi + c\cos\xi}{b\cos\xi + c\sin\xi + \mu}, \psi = \frac{1}{G} = \frac{1}{b\cos\xi + c\sin\xi + \mu}. \tag{12}$$

Substituting Equations 8, 10 into Equation 7 and setting coefficients of $\phi^i\psi^j(i = 0,1, j = 0,1,2,3,4,\cdots)$ to zero yield a set of algebraic equations (AEs) for $a_i, b_j, c_i, d_i, e_j, f_i, g_i, h_j, l_i, b, c, \mu, k, C$. After solving the AEs and substituting each of the solutions $\phi(\xi), \psi(\xi)$ from Equations 11, 12 along with (4) into Equation 1, we can get the analytical solutions of Equation 1.

Remark 1: Due to the arbitrariness of taking values of b, c, μ, this method can encompass a lot of other methods, for example, when $\mu = 0$, it includes the results of (G′/G) method and generalized (G′/G)-expansion method in Refs. [59, 60], if selecting special values of b, c, μ, we can easily obtain all results of the Riccati equation method generalized in literature and the Kudryashov method generalized in Refs. [48, 61, 62]. Therefore, the method is highly applicable.

2.2 The $G'/(bG' + G + a)$-expansion method

Assume that Equation 7 has the following solution

$$\begin{cases} u = \sum_{i=0}^{M} a_i \left(\frac{G'}{bG' + G + a}\right)^i, \\ v = \sum_{i=0}^{N} b_i \left(\frac{G'}{bG' + G + a}\right)^i, \\ w = \sum_{i=0}^{P} c_i \left(\frac{G'}{bG' + G + a}\right)^i. \end{cases} \tag{13}$$

where M, N, P are balance numbers. Parameters a_i, b_i, c_i are determined later. $G = G(\xi)$ satisfies the following second-order ordinary differential equation:

$$G'' = -\frac{\lambda}{b}G' - \frac{\mu}{b^2}G - \frac{\mu}{b^2}a \tag{14}$$

Set $F = \frac{G'}{bG'+G+a}$, then F satisfies

$$F' = (\lambda - \mu - 1)F^2 + \frac{1}{b}(2\mu - \lambda)F - \frac{1}{b^2}\mu \tag{15}$$

Equations 14, 15 admit the following solutions.

Case 1: When $\Delta = \lambda^2 - 4\mu > 0$, we have the solitary wave solution

$$G = -a + C_1 e^{\frac{1}{2b}(-\lambda - \sqrt{\Delta})\xi} + C_2 e^{\frac{1}{2b}(-\lambda + \sqrt{\Delta})\xi} \tag{16}$$

$$F = \frac{C_1(\lambda + \sqrt{\Delta}) + C_2(\lambda - \sqrt{\Delta})e^{\frac{\sqrt{\Delta}}{b}\xi}}{bC_1(\lambda - 2 + \sqrt{\Delta}) + bC_2(\lambda - 2 - \sqrt{\Delta})e^{\frac{\sqrt{\Delta}}{b}\xi}} \tag{17}$$

Case 2: When $\Delta = \lambda^2 - 4\mu < 0$, we have the periodic wave solution

$$G = e^{-\frac{\lambda}{2b}\xi}\left(C_1 \cos\left(\frac{\sqrt{-\Delta}}{2b}\xi\right) + C_2 \sin\left(\frac{\sqrt{-\Delta}}{2b}\xi\right)\right) - a \tag{18}$$

$$F = \frac{(\lambda C_1 - \sqrt{-\Delta}C_2)\cos\left(\frac{\sqrt{-\Delta}}{2b}\xi\right) + (\lambda C_2 + \sqrt{-\Delta}C_1)\sin\left(\frac{\sqrt{-\Delta}}{2b}\xi\right)}{b((\lambda - 2)C_1 - \sqrt{-\Delta}C_2)\cos\left(\frac{\sqrt{-\Delta}}{2b}\xi\right) + b((\lambda - 2)C_2 + \sqrt{-\Delta}C_1)\sin\left(\frac{\sqrt{-\Delta}}{2b}\xi\right)} \tag{19}$$

Case 3: When $\Delta = \lambda^2 - 4\mu = 0$, we have the rational function solution

$$G = C_1 e^{-\frac{\lambda}{2b}\xi} + C_2 \xi e^{-\frac{\lambda}{2b}\xi} - a \tag{20}$$

$$F = \frac{2bC_2 - \lambda C_1 - \lambda C_2 \xi}{2b^2 C_2 + b(2-\lambda)(C_1 + C_2)\xi} \tag{21}$$

Substituting Equations 13, 15 into Equation 7 and setting coefficients of $F^i (i = 0, 1, 2, 3, 4, \cdots)$ zero yield a set of AEs for $a_i, b_i, c_i, b, \lambda, \mu, k, C$, after solving the AEs with the aid of mathematical software, the wave solutions of Equation 1 can be obtained by these solutions and Equations 4, 16–21.

3 Exact solutions to the Hirota-Satsuma KdV equation

3.1 Solving Eq. (1) by the modified $\left(G'/G, 1/G\right)$-expansion method

From the homogeneous equilibrium principle [41], we can get $M = 2, N = 2, P = 2$ in Equation 8. We assume that Equation 5 has solutions in the following form

$$\begin{cases} u = a_0 + a_1\psi + a_2\psi^2 + a_3\phi + a_4\psi\phi + a_5\phi^2, \\ v = b_0 + b_1\psi + b_2\psi^2 + b_3\phi + b_4\psi\phi + b_5\phi^2, \\ w = c_0 + c_1\psi + c_2\psi^2 + c_3\phi + c_4\psi\phi + c_5\phi^2. \end{cases} \tag{22}$$

where $a_i, b_i, c_i (i = 0, \cdots, 5)$ are undetermined constants.

Substituting Equations 22, 10 into Equation 5, and setting the coefficients of $\phi^i \psi^j (i = 0, 1, j = 0, 1, 2, 3, 4, \cdots)$ to zero yield a set of algebraic equations (AEs) for $a_i, b_i, c_i, b, c, \mu, k, C$.

$$\begin{aligned}
&\psi{:}C\varepsilon\mu a_3 - 2k^3\varepsilon^2\mu a_3 - 6k\varepsilon\mu a_0 a_3 + 6k\varepsilon a_1 a_3 - C\epsilon a_4 + 2k^3\varepsilon^2 a_4 + 6k\varepsilon a_0 a_4 - 18k\varepsilon^2\mu a_3 a_5 + 6k\varepsilon^2 a_4 a_5 + 12k\varepsilon\mu b_0 b_3 \\
&\quad - 12k\varepsilon b_1 b_3 - 12k\varepsilon b_0 b_4 + 36k\varepsilon^2\mu b_3 b_5 - 12k\varepsilon^2 b_4 b_5 - 6k\varepsilon\mu c_3 + 6k\varepsilon c_4 = 0, \\
&\phi\psi{:} - Ca_1 + 2k^3\varepsilon a_1 + 6ka_0 a_1 - 6k\varepsilon\mu a_3^2 + 6k\varepsilon a_3 a_4 + 2C\varepsilon\mu a_5 - 4k^3\varepsilon^2\mu a_5 - 12k\varepsilon\mu a_0 a_5 + 6k\varepsilon a_1 a_5 - 12k\varepsilon^2\mu a_5^2 \\
&\quad - 12kb_0 b_1 + 12k\varepsilon\mu b_3^2 - 12k\varepsilon b_3 b_4 + 24k\varepsilon\mu b_0 b_5 - 12k\varepsilon b_1 b_5 + 24k\varepsilon^2\mu b_5^2 + 6kc_1 - 12k\varepsilon\mu c_5 = 0, \\
&\psi^2{:}b^2 C\varepsilon a_3 - c^2 C\varepsilon^2 a_3 - 8b^2k^3\varepsilon^2 a_3 + 8c^2k^3\varepsilon^3 a_3 - C\varepsilon\mu^2 a_3 + 14k^3\varepsilon^2\mu^2 a_3 - 6b^2k\varepsilon a_0 a_3 + 6c^2k\varepsilon^2 a_0 a_3 + 6k\varepsilon\mu^2 a_0 a_3 \\
&\quad - 18k\varepsilon\mu a_1 a_3 + 12k\varepsilon a_2 a_3 + 3C\varepsilon\mu a_4 - 30k^3\varepsilon^2\mu a_4 - 18k\varepsilon\mu a_0 a_4 + 12k\varepsilon a_1 a_4 - 18b^2k\varepsilon^2 a_3 a_5 + 18c^2k\varepsilon^3 a_3 a_5 + 54k\varepsilon^2\mu^2 a_3 a_5 \\
&\quad - 42k\varepsilon^2\mu a_4 a_5 + 12b^2k\varepsilon b_0 b_3 - 12c^2k\varepsilon^2 b_0 b_3 - 12k\varepsilon\mu^2 b_0 b_3 + 36k\varepsilon\mu b_1 b_3 - 24k\varepsilon b_2 b_3 + 36k\varepsilon\mu b_0 b_4 - 24k\varepsilon b_1 b_4 + 36b^2k\varepsilon^2 b_3 b_5 \\
&\quad - 36c^2k\varepsilon^3 b_3 b_5 - 108k\varepsilon^2\mu^2 b_3 b_5 + 84k\varepsilon^2\mu b_4 b_5 - 6b^2k\varepsilon c_3 + 6c^2k\varepsilon^2 c_3 + 6k\varepsilon\mu^2 c_3 - 18k\varepsilon\mu c_4 = 0, \\
&\phi\psi^2{:} - 12k^3\varepsilon\mu a_1 + 6ka_1^2 - 2Ca_2 + 16k^3\varepsilon a_2 + 12ka_0 a_2 - 6b^2k\varepsilon a_3^2 + 6c^2k\varepsilon^2 a_3^2 + 6k\varepsilon\mu^2 a_3^2 - 24k\varepsilon\mu a_3 a_4 + 6k\varepsilon a_4^2 \\
&\quad + 2b^2 C\varepsilon a_5 - 2c^2 C\varepsilon^2 a_5 - 16b^2k^3\varepsilon^2 a_5 + 16c^2k^3\varepsilon^3 a_5 - 2C\varepsilon\mu^2 a_5 + 40k^3\varepsilon^2\mu^2 a_5 - 12b^2k\varepsilon a_0 a_5 + 12c^2k\varepsilon^2 a_0 a_5 \\
&\quad + 12k\varepsilon\mu^2 a_0 a_5 - 24k\varepsilon\mu a_1 a_5 + 12k\varepsilon a_2 a_5 - 12b^2k\varepsilon^2 a_5^2 + 12c^2k\varepsilon^3 a_5^2 + 36k\varepsilon^2\mu^2 a_5^2 - 12kb_1^2 - 24kb_0 b_2 + 12b^2k\varepsilon b_3^2 \\
&\quad - 12c^2k\varepsilon^2 b_3^2 - 12k\varepsilon\mu^2 b_3^2 + 48k\varepsilon\mu b_3 b_4 - 12k\varepsilon b_4^2 + 24b^2k\varepsilon b_0 b_5 - 24c^2k\varepsilon^2 b_0 b_5 - 24k\varepsilon\mu^2 b_0 b_5 + 48k\varepsilon\mu b_1 b_5 - 24k\varepsilon b_2 b_5 \\
&\quad + 24b^2k\varepsilon^2 b_5^2 - 24c^2k\varepsilon^3 b_5^2 - 72k\varepsilon^2\mu^2 b_5^2 + 12kc_2 - 12b^2k\varepsilon c_5 + 12c^2k\varepsilon^2 c_5 + 12k\varepsilon\mu^2 c_5 = 0, \\
&\psi^3{:}24b^2k^3\varepsilon^2\mu a_3 - 24c^2k^3\varepsilon^3\mu a_3 - 24k^3\varepsilon^2\mu^3 a_3 - 12b^2k\varepsilon a_1 a_3 + 12c^2k\varepsilon^2 a_1 a_3 + 12k\varepsilon\mu^2 a_1 a_3 - 30k\varepsilon\mu a_2 a_3 + 2b^2 C\varepsilon a_4 \\
&\quad - 2c^2 C\varepsilon^2 a_4 - 40b^2k^3\varepsilon^2 a_4 + 40c^2k^3\varepsilon^3 a_4 - 2C\varepsilon\mu^2 a_4 + 100k^3\varepsilon^2\mu^2 a_4 - 12b^2k\varepsilon a_0 a_4 + 12c^2k\varepsilon^2 a_0 a_4 + 12k\varepsilon\mu^2 a_0 a_4 \\
&\quad - 30k\varepsilon\mu a_1 a_4 + 18k\varepsilon a_2 a_4 + 54b^2k\varepsilon^2\mu a_3 a_5 - 54c^2k\varepsilon^3\mu a_3 a_5 - 54k\varepsilon^2\mu^3 a_3 a_5 - 30b^2k\varepsilon^2 a_4 a_5 + 30c^2k\varepsilon^3 a_4 a_5 \\
&\quad + 90k\varepsilon^2\mu^2 a_4 a_5 + 24b^2k\varepsilon b_1 b_3 - 24c^2k\varepsilon^2 b_1 b_3 - 24k\varepsilon\mu^2 b_1 b_3 + 60k\varepsilon\mu b_2 b_3 + 24b^2k\varepsilon b_0 b_4 - 24c^2k\varepsilon^2 b_0 b_4 - 24k\varepsilon\mu^2 b_0 b_4 \\
&\quad + 60k\varepsilon\mu b_1 b_4 - 36k\varepsilon b_2 b_4 - 108b^2k\varepsilon^2\mu b_3 b_5 + 108c^2k\varepsilon^3\mu b_3 b_5 + 108k\varepsilon^2\mu^3 b_3 b_5 + 60b^2k\varepsilon^2 b_4 b_5 - 60c^2k\varepsilon^3 b_4 b_5 \\
&\quad - 180k\varepsilon^2\mu^2 b_4 b_5 - 12b^2k\varepsilon c_4 + 12c^2k\varepsilon^2 c_4 + 12k\varepsilon\mu^2 c_4 = 0, \\
&\phi\psi^3{:} - 12b^2k^3\varepsilon a_1 + 12c^2k^3\varepsilon^2 a_1 + 12k^3\varepsilon\mu^2 a_1 - 60k^3\varepsilon\mu a_2 + 18ka_1 a_2 - 18b^2k\varepsilon a_3 a_4 + 18c^2k\varepsilon^2 a_3 a_4 + 18k\varepsilon\mu^2 a_3 a_4 - 18k\varepsilon\mu a_4^2 \\
&\quad + 84b^2k^3\varepsilon^2\mu a_5 - 84c^2k^3\varepsilon^3\mu a_5 - 84k^3\varepsilon^2\mu^3 a_5 - 18b^2k\varepsilon a_1 a_5 + 18c^2k\varepsilon^2 a_1 a_5 + 18k\varepsilon\mu^2 a_1 a_5 - 36k\varepsilon\mu a_2 a_5 + 36b^2k\varepsilon^2\mu a_5^2 \\
&\quad - 36c^2k\varepsilon^3\mu a_5^2 - 36k\varepsilon^2\mu^3 a_5^2 - 36kb_1 b_2 + 36b^2k\varepsilon b_3 b_4 - 36c^2k\varepsilon^2 b_3 b_4 - 36k\varepsilon\mu^2 b_3 b_4 + 36k\varepsilon\mu b_4^2 + 36b^2k\varepsilon b_1 b_5 - 36c^2k\varepsilon^2 b_1 b_5 \\
&\quad - 36k\varepsilon\mu^2 b_1 b_5 + 72k\varepsilon\mu b_2 b_5 - 72b^2k\varepsilon^2\mu b_5^2 + 72c^2k\varepsilon^3\mu b_5^2 + 72k\varepsilon^2\mu^3 b_5^2 = 0,
\end{aligned}$$

$$\psi^4{:}12b^4k^3\varepsilon^2a_3-24b^2c^2k^3\varepsilon^3a_3+12c^4k^3\varepsilon^4a_3-24b^2k^3\varepsilon^2\mu^2a_3+24c^2k^3\varepsilon^3\mu^2a_3+12k^3\varepsilon^2\mu^4a_3-18b^2k\varepsilon a_2a_3+18c^2k\varepsilon^2a_2a_3$$
$$+18k\varepsilon\mu^2a_2a_3+120b^2k^3\varepsilon^2\mu a_4-120c^2k^3\varepsilon^3\mu a_4-120k^3\varepsilon^2\mu^3a_4-18b^2k\varepsilon a_1a_4+18c^2k\varepsilon^2a_1a_4+18k\varepsilon\mu^2a_1a_4-42k\varepsilon\mu a_2a_4$$
$$+18b^4k\varepsilon^2a_3a_5-36b^2c^2k\varepsilon^3a_3a_5+18c^4k\varepsilon^4a_3a_5-36b^2k\varepsilon^2\mu^2a_3a_5+36c^2k\varepsilon^3\mu^2a_3a_5+18k\varepsilon^2\mu^4a_3a_5+78b^2k\varepsilon^2\mu a_4a_5$$
$$-78c^2k\varepsilon^3\mu a_4a_5-78k\varepsilon^2\mu^3a_4a_5+36b^2k\varepsilon b_2b_3-36c^2k\varepsilon^2b_2b_3-36k\varepsilon\mu^2b_2b_3+36b^2k\varepsilon b_1b_4-36c^2k\varepsilon^2b_1b_4-36k\varepsilon\mu^2b_1b_4$$
$$+84k\varepsilon\mu b_2b_4-36b^4k\varepsilon^2b_3b_5+72b^2c^2k\varepsilon^3b_3b_5-36c^4k\varepsilon^4b_3b_5+72b^2k\varepsilon^2\mu^2b_3b_5-72c^2k\varepsilon^3\mu^2b_3b_5-36k\varepsilon^2\mu^4b_3b_5$$
$$-156b^2k\varepsilon^2\mu b_4b_5+156c^2k\varepsilon^3\mu b_4b_5+156k\varepsilon^2\mu^3b_4b_5=0,$$

$$\phi\psi^4{:}-48b^2k^3\varepsilon a_2+48c^2k^3\varepsilon^2a_2+48k^3\varepsilon\mu^2a_2+12ka_2^2-12b^2k\varepsilon a_4^2+12c^2k\varepsilon^2a_4^2+12k\varepsilon\mu^2a_4^2+48b^4k^3\varepsilon^2a_5-96b^2c^2k^3\varepsilon^3a_5$$
$$+48c^4k^3\varepsilon^4a_5-96b^2k^3\varepsilon^2\mu^2a_5+96c^2k^3\varepsilon^3\mu^2a_5+48k^3\varepsilon^2\mu^4a_5-24b^2k\varepsilon a_2a_5+24c^2k\varepsilon^2a_2a_5+24k\varepsilon\mu^2a_2a_5+12b^4k\varepsilon^2a_5^2$$
$$-24b^2c^2k\varepsilon^3a_5^2+12c^4k\varepsilon^4a_5^2-24b^2k\varepsilon^2\mu^2a_5^2+24c^2k\varepsilon^3\mu^2a_5^2+12k\varepsilon^2\mu^4a_5^2-24kb_2^2+24b^2k\varepsilon b_4^2-24c^2k\varepsilon^2b_4^2-24k\varepsilon\mu^2b_4^2$$
$$+48b^2k\varepsilon b_2b_5-48c^2k\varepsilon^2b_2b_5-48k\varepsilon\mu^2b_2b_5-24b^4k\varepsilon^2b_5^2+48b^2c^2k\varepsilon^3b_5^2-24c^4k\varepsilon^4b_5^2+48b^2k\varepsilon^2\mu^2b_5^2-48c^2k\varepsilon^3\mu^2b_5^2$$
$$-24k\varepsilon^2\mu^4b_5^2=0,$$

$$\psi^5{:}48b^4k^3\varepsilon^2a_4-96b^2c^2k^3\varepsilon^3a_4+48c^4k^3\varepsilon^4a_4-96b^2k^3\varepsilon^2\mu^2a_4+96c^2k^3\varepsilon^3\mu^2a_4+48k^3\varepsilon^2\mu^4a_4-24b^2k\varepsilon a_2a_4+24c^2k\varepsilon^2a_2a_4$$
$$+24k\varepsilon\mu^2a_2a_4+24b^4k\varepsilon^2a_4a_5-48b^2c^2k\varepsilon^3a_4a_5+24c^4k\varepsilon^4a_4a_5-48b^2k\varepsilon^2\mu^2a_4a_5+48c^2k\varepsilon^3\mu^2a_4a_5+24k\varepsilon^2\mu^4a_4a_5$$
$$+48b^2k\varepsilon b_2b_4-48c^2k\varepsilon^2b_2b_4-48k\varepsilon\mu^2b_2b_4-48b^4k\varepsilon^2b_4b_5+96b^2c^2k\varepsilon^3b_4b_5-48c^4k\varepsilon^4b_4b_5+96b^2k\varepsilon^2\mu^2b_4b_5-96c^2k\varepsilon^3\mu^2b_4b_5$$
$$-48k\varepsilon^2\mu^4b_4b_5=0,$$

$$\psi{:}-6k\varepsilon a_3b_1+C\varepsilon\mu b_3+4k^3\varepsilon^2\mu b_3+6k\varepsilon\mu a_0b_3+6k\varepsilon^2\mu a_5b_3-C\varepsilon b_4-4k^3\varepsilon^2b_4-6k\varepsilon a_0b_4-6k\varepsilon^2a_5b_4+12k\varepsilon^2\mu a_3b_5=0,$$

$$\phi\psi{:}-Cb_1-4k^3\varepsilon b_1-6ka_0b_1-6k\varepsilon a_5b_1+6k\varepsilon\mu a_3b_3-6k\varepsilon a_3b_4+2C\varepsilon\mu b_5+8k^3\varepsilon^2\mu b_5+12k\varepsilon\mu a_0b_5+12k\varepsilon^2\mu a_5b_5=0,$$

$$\psi^2{:}12k\varepsilon\mu a_3b_1-6k\varepsilon a_4b_1-12k\varepsilon a_3b_2+b^2C\varepsilon b_3-c^2C\varepsilon^2b_3+16b^2k^3\varepsilon^2b_3-16c^2k^3\varepsilon^3b_3-C\varepsilon\mu^2b_3-28k^3\varepsilon^2\mu^2b_3+6b^2k\varepsilon a_0b_3$$
$$-6c^2k\varepsilon^2a_0b_3-6k\varepsilon\mu^2a_0b_3+6k\varepsilon\mu a_1b_3+6b^2k\varepsilon^2a_5b_3-6c^2k\varepsilon^3a_5b_3-18k\varepsilon^2\mu^2a_5b_3+3C\varepsilon\mu b_4+60k^3\varepsilon^2\mu b_4+18k\varepsilon\mu a_0b_4$$
$$-6k\varepsilon a_1b_4+30k\varepsilon^2\mu a_5b_4+12b^2k\varepsilon^2a_3b_5-12c^2k\varepsilon^3a_3b_5-36k\varepsilon^2\mu^2a_3b_5+12k\varepsilon^2\mu a_4b_5=0,$$

$$\phi\psi^2{:}24k^3\varepsilon\mu b_1-6ka_1b_1+12k\varepsilon\mu a_5b_1-2Cb_2-32k^3\varepsilon b_2-12ka_0b_2-12k\varepsilon a_5b_2+6b^2k\varepsilon a_3b_3-6c^2k\varepsilon^2a_3b_3-6k\varepsilon\mu^2a_3b_3$$
$$+6k\varepsilon\mu a_4b_3+18k\varepsilon\mu a_3b_4-6k\varepsilon a_4b_4+2b^2C\varepsilon b_5-2c^2C\varepsilon^2b_5+32b^2k^3\varepsilon^2b_5-32c^2k^3\varepsilon^3b_5-2C\varepsilon\mu^2b_5-80k^3\varepsilon^2\mu^2b_5$$
$$+12b^2k\varepsilon a_0b_5-12c^2k\varepsilon^2a_0b_5-12k\varepsilon\mu^2a_0b_5+12k\varepsilon\mu a_1b_5+12b^2k\varepsilon^2a_5b_5-12c^2k\varepsilon^3a_5b_5-36k\varepsilon^2\mu^2a_5b_5=0,$$

$$\psi^3{:}6b^2k\varepsilon a_3b_1-6c^2k\varepsilon^2a_3b_1-6k\varepsilon\mu^2a_3b_1+12k\varepsilon\mu a_4b_1+24k\varepsilon\mu a_3b_2-12k\varepsilon a_4b_2-48b^2k^3\varepsilon^2\mu b_3+48c^2k^3\varepsilon^3\mu b_3+48k^3\varepsilon^2\mu^3b_3$$
$$+6b^2k\varepsilon a_1b_3-6c^2k\varepsilon^2a_1b_3-6k\varepsilon\mu^2a_1b_3+6k\varepsilon\mu a_2b_3-18b^2k\varepsilon^2\mu a_5b_3+18c^2k\varepsilon^3\mu a_5b_3+18k\varepsilon^2\mu^3a_5b_3+2b^2C\varepsilon b_4-2c^2C\varepsilon^2b_4$$
$$+80b^2k^3\varepsilon^2b_4-80c^2k^3\varepsilon^3b_4-2C\varepsilon\mu^2b_4-200k^3\varepsilon^2\mu^2b_4+12b^2k\varepsilon a_0b_4-12c^2k\varepsilon^2a_0b_4-12k\varepsilon\mu^2a_0b_4+18k\varepsilon\mu a_1b_4-6k\varepsilon a_2b_4$$
$$+18b^2k\varepsilon^2a_5b_4-18c^2k\varepsilon^3a_5b_4-54k\varepsilon^2\mu^2a_5b_4-36b^2k\varepsilon^2\mu a_3b_5+36c^2k\varepsilon^3\mu a_3b_5+36k\varepsilon^2\mu^3a_3b_5+12b^2k\varepsilon^2a_4b_5-12c^2k\varepsilon^3a_4b_5$$
$$-36k\varepsilon^2\mu^2a_4b_5=0,$$

$$\phi\psi^3{:}24b^2k^3\varepsilon b_1-24c^2k^3\varepsilon^2b_1-24k^3\varepsilon\mu^2b_1-6ka_2b_1+6b^2k\varepsilon a_5b_1-6c^2k\varepsilon^2a_5b_1-6k\varepsilon\mu^2a_5b_1+120k^3\varepsilon\mu b_2-12ka_1b_2+24k\varepsilon\mu a_5b_2$$
$$+6b^2k\varepsilon a_4b_3-6c^2k\varepsilon^2a_4b_3-6k\varepsilon\mu^2a_4b_3+12b^2k\varepsilon a_3b_4-12c^2k\varepsilon^2a_3b_4-12k\varepsilon\mu^2a_3b_4+18k\varepsilon\mu a_4b_4-168b^2k^3\varepsilon^2\mu b_5+168c^2k^3\varepsilon^3\mu b_5$$
$$+168k^3\varepsilon^2\mu^3b_5+12b^2k\varepsilon a_1b_5-12c^2k\varepsilon^2a_1b_5-12k\varepsilon\mu^2a_1b_5+12k\varepsilon\mu a_2b_5-36b^2k\varepsilon^2\mu a_5b_5+36c^2k\varepsilon^3\mu a_5b_5+36k\varepsilon^2\mu^3a_5b_5=0,$$

$$\psi^4{:}6b^2k\varepsilon a_4b_1-6c^2k\varepsilon^2a_4b_1-6k\varepsilon\mu^2a_4b_1+12b^2k\varepsilon a_3b_2-12c^2k\varepsilon^2a_3b_2-12k\varepsilon\mu^2a_3b_2+24k\varepsilon\mu a_4b_2-24b^4k^3\varepsilon^2b_3+48b^2c^2k^3\varepsilon^3b_3$$
$$-24c^4k^3\varepsilon^4b_3+48b^2k^3\varepsilon^2\mu^2b_3-48c^2k^3\varepsilon^3\mu^2b_3-24k^3\varepsilon^2\mu^4b_3+6b^2k\varepsilon a_2b_3-6c^2k\varepsilon^2a_2b_3-6k\varepsilon\mu^2a_2b_3-6b^4k\varepsilon^2a_5b_3$$
$$+12b^2c^2k\varepsilon^3a_5b_3-6c^4k\varepsilon^4a_5b_3+12b^2k\varepsilon^2\mu^2a_5b_3-12c^2k\varepsilon^3\mu^2a_5b_3-6k\varepsilon^2\mu^4a_5b_3-240b^2k^3\varepsilon^2\mu b_4+240c^2k^3\varepsilon^3\mu b_4$$
$$+240k^3\varepsilon^2\mu^3b_4+12b^2k\varepsilon a_1b_4-12c^2k\varepsilon^2a_1b_4-12k\varepsilon\mu^2a_1b_4+18k\varepsilon\mu a_2b_4-42b^2k\varepsilon^2\mu a_5b_4+42c^2k\varepsilon^3\mu a_5b_4+42k\varepsilon^2\mu^3a_5b_4$$
$$-12b^4k\varepsilon^2a_3b_5+24b^2c^2k\varepsilon^3a_3b_5-12c^4k\varepsilon^4a_3b_5+24b^2k\varepsilon^2\mu^2a_3b_5-24c^2k\varepsilon^3\mu^2a_3b_5-12k\varepsilon^2\mu^4a_3b_5-36b^2k\varepsilon^2\mu a_4b_5$$
$$+36c^2k\varepsilon^3\mu a_4b_5+36k\varepsilon^2\mu^3a_4b_5=0,$$

$$\phi\psi^4{:}96b^2k^3\varepsilon b_2-96c^2k^3\varepsilon^2b_2-96k^3\varepsilon\mu^2b_2-12ka_2b_2+12b^2k\varepsilon a_5b_2-12c^2k\varepsilon^2a_5b_2-12k\varepsilon\mu^2a_5b_2+12b^2k\varepsilon a_4b_4$$
$$-12c^2k\varepsilon^2a_4b_4-12k\varepsilon\mu^2a_4b_4-96b^4k^3\varepsilon^2b_5+192b^2c^2k^3\varepsilon^3b_5-96c^4k^3\varepsilon^4b_5+192b^2k^3\varepsilon^2\mu^2b_5-192c^2k^3\varepsilon^3\mu^2b_5$$
$$-96k^3\varepsilon^2\mu^4b_5+12b^2k\varepsilon a_2b_5-12c^2k\varepsilon^2a_2b_5-12k\varepsilon\mu^2a_2b_5-12b^4k\varepsilon^2a_5b_5+24b^2c^2k\varepsilon^3a_5b_5-12c^4k\varepsilon^4a_5b_5$$
$$+24b^2k\varepsilon^2\mu^2a_5b_5-24c^2k\varepsilon^3\mu^2a_5b_5-12k\varepsilon^2\mu^4a_5b_5=0,$$

$$\psi^5{:}12b^2k\varepsilon a_4b_2-12c^2k\varepsilon^2a_4b_2-12k\varepsilon\mu^2a_4b_2-96b^4k^3\varepsilon^2b_4+192b^2c^2k^3\varepsilon^3b_4-96c^4k^3\varepsilon^4b_4+192b^2k^3\varepsilon^2\mu^2b_4-192c^2k^3\varepsilon^3\mu^2b_4$$
$$-96k^3\varepsilon^2\mu^4b_4+12b^2k\varepsilon a_2b_4-12c^2k\varepsilon^2a_2b_4-12k\varepsilon\mu^2a_2b_4-12b^4k\varepsilon^2a_5b_4+24b^2c^2k\varepsilon^3a_5b_4-12c^4k\varepsilon^4a_5b_4+24b^2k\varepsilon^2\mu^2a_5b_4$$
$$-24c^2k\varepsilon^3\mu^2a_5b_4-12k\varepsilon^2\mu^4a_5b_4-12b^4k\varepsilon^2a_4b_5+24b^2c^2k\varepsilon^3a_4b_5-12c^4k\varepsilon^4a_4b_5+24b^2k\varepsilon^2\mu^2a_4b_5-24c^2k\varepsilon^3\mu^2a_4b_5$$
$$-12k\varepsilon^2\mu^4a_4b_5=0,$$

$$\psi{:}-6k\varepsilon a_3c_1+C\varepsilon\mu c_3+4k^3\varepsilon^2\mu c_3+6k\varepsilon\mu a_0c_3+6k\varepsilon^2\mu a_5c_3-C\varepsilon c_4-4k^3\varepsilon^2c_4-6k\varepsilon a_0c_4-6k\varepsilon^2a_5c_4+12k\varepsilon^2\mu a_3c_5=0,$$

$$\phi\psi{:}-Cc_1-4k^3\varepsilon c_1-6ka_0c_1-6k\varepsilon a_5c_1+6k\varepsilon\mu a_3c_3-6k\varepsilon a_3c_4+2C\varepsilon\mu c_5+8k^3\varepsilon^2\mu c_5+12k\varepsilon\mu a_0c_5+12k\varepsilon^2\mu a_5c_5=0,$$

$$\psi^2{:}12k\varepsilon\mu a_3c_1 - 6k\varepsilon a_4c_1 - 12k\varepsilon a_3c_2 + b^2C\varepsilon c_3 - c^2C\varepsilon^2c_3 + 16b^2k^3\varepsilon^2c_3 - 16c^2k^3\varepsilon^3c_3 - C\varepsilon\mu^2c_3 - 28k^3\varepsilon^2\mu^2c_3 + 6b^2k\varepsilon a_0c_3 - 6c^2k\varepsilon^2a_0c_3 - 6k\varepsilon\mu^2a_0c_3 + 6k\varepsilon\mu a_1c_3 + 6b^2k\varepsilon^2a_5c_3 - 6c^2k\varepsilon^3a_5c_3 - 18k\varepsilon^2\mu^2a_5c_3 + 3C\varepsilon\mu c_4 + 60k^3\varepsilon^2\mu c_4 + 18k\varepsilon\mu a_0c_4 - 6k\varepsilon a_1c_4 + 30k\varepsilon^2\mu a_5c_4 + 12b^2k\varepsilon^2a_3c_5 - 12c^2k\varepsilon^3a_3c_5 - 36k\varepsilon^2\mu^2a_3c_5 + 12k\varepsilon^2\mu a_4c_5 = 0,$$

$$\phi\psi^2{:}24k^3\varepsilon\mu c_1 - 6ka_1c_1 + 12k\varepsilon\mu a_5c_1 - 2Cc_2 - 32k^3\varepsilon c_2 - 12ka_0c_2 - 12k\varepsilon a_5c_2 + 6b^2k\varepsilon a_3c_3 - 6c^2k\varepsilon^2a_3c_3 - 6k\varepsilon\mu^2a_3c_3 + 6k\varepsilon\mu a_4c_3 + 18k\varepsilon\mu a_3c_4 - 6k\varepsilon a_4c_4 + 2b^2C\varepsilon c_5 - 2c^2C\varepsilon^2c_5 + 32b^2k^3\varepsilon^2c_5 - 32c^2k^3\varepsilon^3c_5 - 2C\varepsilon\mu^2c_5 - 80k^3\varepsilon^2\mu^2c_5 + 12b^2k\varepsilon a_0c_5 - 12c^2k\varepsilon^2a_0c_5 - 12k\varepsilon\mu^2a_0c_5 + 12k\varepsilon\mu a_1c_5 + 12b^2k\varepsilon^2a_5c_5 - 12c^2k\varepsilon^3a_5c_5 - 36k\varepsilon^2\mu^2a_5c_5 = 0,$$

$$\psi^3{:}6b^2k\varepsilon a_3c_1 - 6c^2k\varepsilon^2a_3c_1 - 6k\varepsilon\mu^2a_3c_1 + 12k\varepsilon\mu a_4c_1 + 24k\varepsilon\mu a_3c_2 - 12k\varepsilon a_4c_2 - 48b^2k^3\varepsilon^2\mu c_3 + 48c^2k^3\varepsilon^3\mu c_3 + 48k^3\varepsilon^2\mu^3c_3 + 6b^2k\varepsilon a_1c_3 - 6c^2k\varepsilon^2a_1c_3 - 6k\varepsilon\mu^2a_1c_3 + 6k\varepsilon\mu a_2c_3 - 18b^2k\varepsilon^2\mu a_5c_3 + 18c^2k\varepsilon^3\mu a_5c_3 + 18k\varepsilon^2\mu^3a_5c_3 + 2b^2C\varepsilon c_4 - 2c^2C\varepsilon^2c_4 + 80b^2k^3\varepsilon^2c_4 - 80c^2k^3\varepsilon^3c_4 - 2C\varepsilon\mu^2c_4 - 200k^3\varepsilon^2\mu^2c_4 + 12b^2k\varepsilon a_0c_4 - 12c^2k\varepsilon^2a_0c_4 - 12k\varepsilon\mu^2a_0c_4 + 18k\varepsilon\mu a_1c_4 - 6k\varepsilon a_2c_4 + 18b^2k\varepsilon^2a_5c_4 - 18c^2k\varepsilon^3a_5c_4 - 54k\varepsilon^2\mu^2a_5c_4 - 36b^2k\varepsilon^2\mu a_3c_5 + 36c^2k\varepsilon^3\mu a_3c_5 + 36k\varepsilon^2\mu^3a_3c_5 + 12b^2k\varepsilon^2a_4c_5 - 12c^2k\varepsilon^3a_4c_5 - 36k\varepsilon^2\mu^2a_4c_5 = 0,$$

$$\phi\psi^3{:}24b^2k^3\varepsilon c_1 - 24c^2k^3\varepsilon^2c_1 - 24k^3\varepsilon\mu^2c_1 - 6ka_2c_1 + 6b^2k\varepsilon a_5c_1 - 6c^2k\varepsilon^2a_5c_1 - 6k\varepsilon\mu^2a_5c_1 + 120k^3\varepsilon\mu c_2 - 12ka_1c_2 + 24k\varepsilon\mu a_5c_2 + 6b^2k\varepsilon a_4c_3 - 6c^2k\varepsilon^2a_4c_3 - 6k\varepsilon\mu^2a_4c_3 + 12b^2k\varepsilon a_3c_4 - 12c^2k\varepsilon^2a_3c_4 - 12k\varepsilon\mu^2a_3c_4 + 18k\varepsilon\mu a_4c_4 - 168b^2k^3\varepsilon^2\mu c_5 + 168c^2k^3\varepsilon^3\mu c_5 + 168k^3\varepsilon^2\mu^3c_5 + 12b^2k\varepsilon a_1c_5 - 12c^2k\varepsilon^2a_1c_5 - 12k\varepsilon\mu^2a_1c_5 + 12k\varepsilon\mu a_2c_5 - 36b^2k\varepsilon^2\mu a_5c_5 + 36c^2k\varepsilon^3\mu a_5c_5 + 36k\varepsilon^2\mu^3a_5c_5 = 0,$$

$$\psi^4{:}6b^2k\varepsilon a_4c_1 - 6c^2k\varepsilon^2a_4c_1 - 6k\varepsilon\mu^2a_4c_1 + 12b^2k\varepsilon a_3c_2 - 12c^2k\varepsilon^2a_3c_2 - 12k\varepsilon\mu^2a_3c_2 + 24k\varepsilon\mu a_4c_2 - 24b^4k^3\varepsilon^2c_3 + 48b^2c^2k^3\varepsilon^3c_3 - 24c^4k^3\varepsilon^4c_3 + 48b^2k^3\varepsilon^2\mu^2c_3 - 48c^2k^3\varepsilon^3\mu^2c_3 - 24k^3\varepsilon^2\mu^4c_3 + 6b^2k\varepsilon a_2c_3 - 6c^2k\varepsilon^2a_2c_3 - 6k\varepsilon\mu^2a_2c_3 - 6b^4k\varepsilon^2a_5c_3 + 12b^2c^2k\varepsilon^3a_5c_3 - 6c^4k\varepsilon^4a_5c_3 + 12b^2k\varepsilon^2\mu^2a_5c_3 - 12c^2k\varepsilon^3\mu^2a_5c_3 - 6k\varepsilon^2\mu^4a_5c_3 - 240b^2k^3\varepsilon^2\mu c_4 + 240c^2k^3\varepsilon^3\mu c_4 + 240k^3\varepsilon^2\mu^3c_4 + 12b^2k\varepsilon a_1c_4 - 12c^2k\varepsilon^2a_1c_4 - 12k\varepsilon\mu^2a_1c_4 + 18k\varepsilon\mu a_2c_4 - 42b^2k\varepsilon^2\mu a_5c_4 + 42c^2k\varepsilon^3\mu a_5c_4 + 42k\varepsilon^2\mu^3a_5c_4 - 12b^4k\varepsilon^2a_3c_5 + 24b^2c^2k\varepsilon^3a_3c_5 - 12c^4k\varepsilon^4a_3c_5 + 24b^2k\varepsilon^2\mu^2a_3c_5 - 24c^2k\varepsilon^3\mu^2a_3c_5 - 12k\varepsilon^2\mu^4a_3c_5 - 36b^2k\varepsilon^2\mu a_4c_5 + 36c^2k\varepsilon^3\mu a_4c_5 + 36k\varepsilon^2\mu^3a_4c_5 = 0,$$

$$\phi\psi^4{:}96b^2k^3\varepsilon c_2 - 96c^2k^3\varepsilon^2c_2 - 96k^3\varepsilon\mu^2c_2 - 12ka_2c_2 + 12b^2k\varepsilon a_5c_2 - 12c^2k\varepsilon^2a_5c_2 - 12k\varepsilon\mu^2a_5c_2 + 12b^2k\varepsilon a_4c_4 - 12c^2k\varepsilon^2a_4c_4 - 12k\varepsilon\mu^2a_4c_4 - 96b^4k^3\varepsilon^2c_5 + 192b^2c^2k^3\varepsilon^3c_5 - 96c^4k^3\varepsilon^4c_5 + 192b^2k^3\varepsilon^2\mu^2c_5 - 192c^2k^3\varepsilon^3\mu^2c_5 - 96k^3\varepsilon^2\mu^4c_5 + 12b^2k\varepsilon a_2c_5 - 12c^2k\varepsilon^2a_2c_5 - 12k\varepsilon\mu^2a_2c_5 - 12b^4k\varepsilon^2a_5c_5 + 24b^2c^2k\varepsilon^3a_5c_5 - 12c^4k\varepsilon^4a_5c_5 + 24b^2k\varepsilon^2\mu^2a_5c_5 - 24c^2k\varepsilon^3\mu^2a_5c_5 - 12k\varepsilon^2\mu^4a_5c_5 = 0,$$

$$\psi^5{:}12b^2k\varepsilon a_4c_2 - 12c^2k\varepsilon^2a_4c_2 - 12k\varepsilon\mu^2a_4c_2 - 96b^4k^3\varepsilon^2c_4 + 192b^2c^2k^3\varepsilon^3c_4 - 96c^4k^3\varepsilon^4c_4 + 192b^2k^3\varepsilon^2\mu^2c_4 - 192c^2k^3\varepsilon^3\mu^2c_4 - 96k^3\varepsilon^2\mu^4c_4 + 12b^2k\varepsilon a_2c_4 - 12c^2k\varepsilon^2a_2c_4 - 12k\varepsilon\mu^2a_2c_4 - 12b^4k\varepsilon^2a_5c_4 + 24b^2c^2k\varepsilon^3a_5c_4 - 12c^4k\varepsilon^4a_5c_4 + 24b^2k\varepsilon^2\mu^2a_5c_4 - 24c^2k\varepsilon^3\mu^2a_5c_4 - 12k\varepsilon^2\mu^4a_5c_4 - 12b^4k\varepsilon^2a_4c_5 + 24b^2c^2k\varepsilon^3a_4c_5 - 12c^4k\varepsilon^4a_4c_5 + 24b^2k\varepsilon^2\mu^2a_4c_5 - 24c^2k\varepsilon^3\mu^2a_4c_5 - 12k\varepsilon^2\mu^4a_4c_5 = 0.$$

Solving the equations by Mathematical software can get the following solutions, where the unstated parameters are taking any constant.
Case 1: When $\varepsilon = 1$,

(1) $a_1 = 4k^2\mu, a_2 = a_3 = a_4 = a_5 = 0, b_1 = 2k^2\mu, b_2 = b_3 = b_4 = b_5 = 0, c_1 = -4(2a_0k^2\mu - b_0k^2\mu + k^4\mu),$
$c_2 = c_3 = c_4 = c_5 = 0, b = \pm\sqrt{c^2 + \mu^2}, C = -2(3a_0k + 2k^3).$

(2) $a_1 = 4k^2\mu, a_2 = 4k^2(b^2 - c^2 - \mu^2), a_3 = a_4 = a_5 = 0, b_1 = \pm k\sqrt{4k^2(2b^2 - 2c^2 - \mu^2) + 8a_0(b^2 - c^2 - \mu^2)},$
$b_2 = b_3 = b_4 = b_5 = 0, c_1 = -4k\left(2a_0k\mu + k^3\mu \pm b_0\sqrt{k^2(2b^2 - 2c^2 - \mu^2) + 2a_0(b^2 - c^2 - \mu^2)}\right),$
$c_2 = c_3 = c_4 = c_5 = 0, C = -2(3a_0k + 2k^3).$

(3) $a_1 = 8k^2\mu, a_2 = -8k^2\mu^2, a_3 = a_4 = a_5 = 0, b_1 = -4k^2\mu, b_2 = 4k^2\mu^2, b_3 = b_4 = b_5 = 0, c_1 = -8k^2(2a_0 + b_0 + k^2)\mu,$
$c_2 = 8k^2(2a_0 + b_0 + k^2)\mu^2, c_3 = c_4 = c_5 = 0, b = \pm c, C = -2(3a_0k + 2k^3).$

(4) $a_1 = a_2 = a_3 = a_4 = 0, a_5 = -2k^2, b_1 = b_2 = b_3 = b_4 = 0, b_5 = \pm k^2, c_1 = c_2 = c_3 = c_4 = 0,$
$c_5 = 2(2a_0k^2 \pm b_0k^2 - 2k^4), b = \pm\sqrt{c^2 + \mu^2}, C = -2(3a_0k - 4k^3).$

(5) $a_1 = a_3 = a_5 = 0, a_2 = 2k^2(b^2 - c^2), a_4 = 2\sqrt{-b^2k^4 + c^2k^4}, b_2 = b_4 = b_5 = 0, b_1 = \sqrt{(b^2 - c^2)k^2(2a_0 + k^2)},$
$b_3 = \pm\sqrt{-k^2(2a_0 + k^2)}, c_1 = 2b_0\sqrt{(b^2 - c^2)k^2(2a_0 + k^2)}, c_3 = \pm 2b_0\sqrt{-k^2(2a_0 + k^2)}, c_2 = c_4 = c_5 = 0,$
$\mu = 0, C = -6a_0k - 4k^3.$

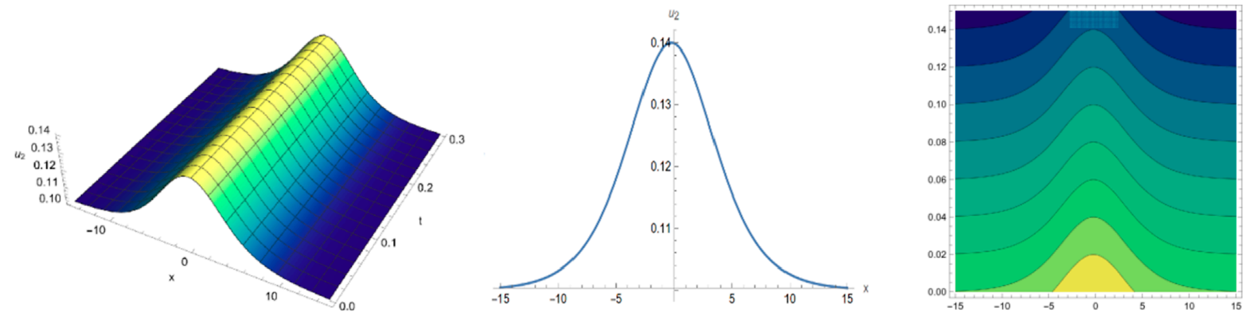

FIGURE 1
When $\alpha = 0.9, \beta = 1, k = 0.2, b = 45, c = 45, \mu = 40, a_0 = 0.1, \gamma_1 = 1, \xi_0 = 0$, $t = 0.1$, the 3D plot, 2D plot and contour plot of the solitary wave pulse u_2.

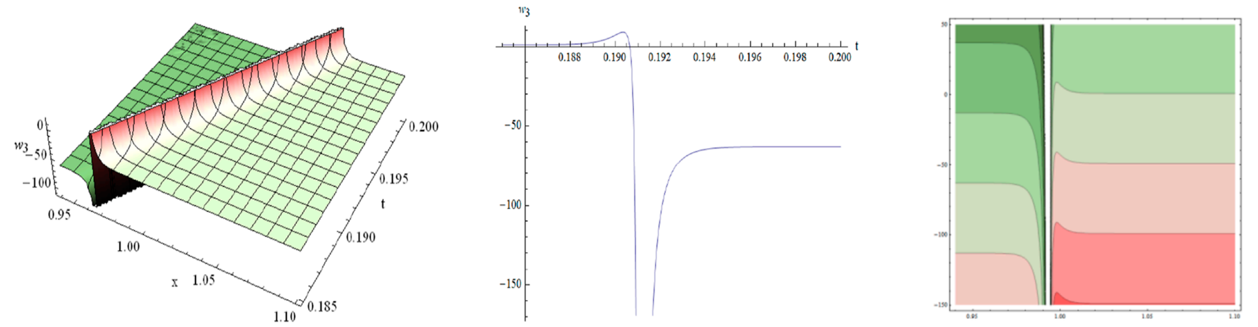

FIGURE 2
When $\alpha = 0.9, \beta = 0.8, k = 1, b = c = -2, \mu = 1$, $a_0 = b_0 = c_0 = 1$, $\gamma_1 = \gamma_2 = 5, \xi_0 = 0, x = 1$, the 3D plot, 2D plot and contour plot of the solitary wave pulse w_3.

According to Equations 4, 11, 22, we can get the solutions of Equation 1 when $\varepsilon = 1$

$$\begin{cases} u_1 = a_0 + \dfrac{4k^2\mu}{\pm\sqrt{c^2+\mu^2}\cosh\xi_1 + c\sinh\xi_1 + \mu}, \\ v_1 = b_0 + \dfrac{2k^2\mu}{\pm\sqrt{c^2+\mu^2}\cosh\xi_1 + c\sinh\xi_1 + \mu}, \\ w_1 = c_0 - \dfrac{4\left(2a_0k^2\mu - b_0k^2\mu + k^4\mu\right)}{\pm\sqrt{c^2+\mu^2}\cosh\xi_1 + c\sinh\xi_1 + \mu}, \\ \xi_1 = \dfrac{2k\Gamma(\gamma_2+1)}{\beta}x^\beta - \dfrac{2\Gamma(\gamma_1+1)}{\alpha}\left(3a_0k + 2k^3\right)t^\alpha + \xi_0. \end{cases}$$

Remark 2: When $\alpha = \beta = 1, \gamma_1 = \gamma_2 = 0$, the system of solutions (u_1, v_1, w_1) converted to the first set of solutions in the Ref. [48].

$$\begin{cases} u_2 = a_0 + \dfrac{4k^2\mu}{b\cosh\xi_2 + c\sinh\xi_2 + \mu} + \dfrac{4k^2\left(b^2 - c^2 - \mu^2\right)}{\left(b\cosh\xi_2 + c\sinh\xi_2 + \mu\right)^2}, \\ v_2 = b_0 \pm \dfrac{k\sqrt{4k^2\left(2b^2 - 2c^2 - \mu^2\right) + 8a_0\left(b^2 - c^2 - \mu^2\right)}}{b\cosh\xi_2 + c\sinh\xi_2 + \mu}, \\ w_2 = c_0 - \dfrac{4k\left(2a_0k\mu + k^3\mu \pm b_0\sqrt{k^2\left(2b^2 - 2c^2 - \mu^2\right) + 2a_0\left(b^2 - c^2 - \mu^2\right)}\right)}{b\cosh\xi_2 + c\sinh\xi_2 + \mu}, \\ \xi_2 = \dfrac{2k\Gamma(\gamma_2+1)}{\beta}x^\beta - \dfrac{2\Gamma(\gamma_1+1)}{\alpha}\left(3a_0k + 2k^3\right)t^\alpha + \xi_0. \end{cases}$$

Remark 3: If we set $b = \sqrt{c^2 + \mu^2}$, then the system of solutions (u_2, v_2, w_2) converted to the system of solutions (u_1, v_1, w_1).

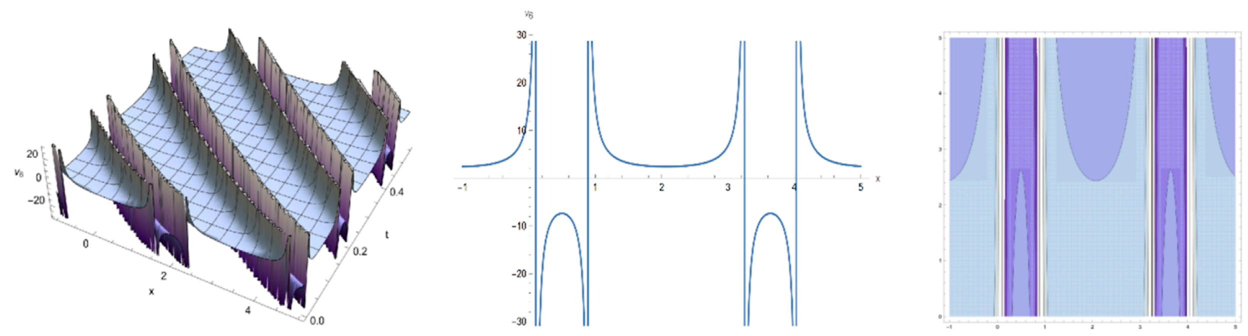

FIGURE 3
The 3D plot, 2D plot and contour plot of v_6 with $\alpha = 0.8, \beta = 1, k = 0.2, b = 45$,

$$\begin{cases} u_3 = a_0 + \dfrac{8k^2\mu}{\pm c\cosh\xi_3 + c\sinh\xi_3 + \mu} - \dfrac{8k^2\mu^2}{(\pm c\cosh\xi_3 + c\sinh\xi_3 + \mu)^2}, \\ v_3 = b_0 - \dfrac{4k^2\mu}{\pm c\cosh\xi_3 + c\sinh\xi_3 + \mu} + \dfrac{4k^2\mu^2}{(\pm c\cosh\xi_3 + c\sinh\xi_3 + \mu)^2}, \\ w_3 = c_0 - \dfrac{8k^2(2a_0 + b_0 + k^2)\mu}{\pm c\cosh\xi_3 + c\sinh\xi_3 + \mu} + \dfrac{8k^2(2a_0 + b_0 + k^2)\mu^2}{(\pm c\cosh\xi_3 + c\sinh\xi_3 + \mu)^2}, \\ \xi_3 = \dfrac{2k\Gamma(\gamma_2 + 1)}{\beta}x^\beta - \dfrac{2\Gamma(\gamma_1 + 1)}{\alpha}(3a_0k + 2k^3)t^\alpha + \xi_0. \end{cases}$$

$$\begin{cases} u_4 = a_0 - 2k^2\left(\dfrac{\pm\sqrt{c^2+\mu^2}\sinh\xi_4 + c\cosh\xi_4}{\pm\sqrt{c^2+\mu^2}\cosh\xi_4 + c\sinh\xi_4 + \mu}\right)^2, \\ v_4 = b_0 \pm k\left(\dfrac{\pm\sqrt{c^2+\mu^2}\sinh\xi_4 + c\cosh\xi_4}{\pm\sqrt{c^2+\mu^2}\cosh\xi_4 + c\sinh\xi_4 + \mu}\right)^2, \\ w_4 = c_0 + 2(2a_0k^2 \pm b_0k^2 - 2k^4)\left(\dfrac{\pm\sqrt{c^2+\mu^2}\sinh\xi_4 + c\cosh\xi_4}{\pm\sqrt{c^2+\mu^2}\cosh\xi_4 + c\sinh\xi_4 + \mu}\right)^2, \\ \xi_4 = \dfrac{2k\Gamma(\gamma_2 + 1)}{\beta}x^\beta - \dfrac{2\Gamma(\gamma_1 + 1)}{\alpha}(3a_0k - 4k^3)t^\alpha + \xi_0. \end{cases}$$

$$\begin{cases} u_5 = a_0 + \dfrac{2k^2(b^2 - c^2)}{(b\cosh\xi_5 + c\sinh\xi_5 + \mu)^2} + 2\sqrt{-b^2k^4 + c^2k^4}\dfrac{b\sinh\xi_5 + c\cosh\xi_5}{(b\cosh\xi_5 + c\sinh\xi_5 + \mu)^2}, \\ v_5 = b_0 + \dfrac{\sqrt{(b^2 - c^2)k^2(2a_0 + k^2)}}{b\cosh\xi_5 + c\sinh\xi_5 + \mu} \pm \sqrt{-k^2(2a_0 + k^2)}\dfrac{b\sinh\xi_5 + c\cosh\xi_5}{b\cosh\xi_5 + c\sinh\xi_5 + \mu}, \\ w_5 = c_0 + \dfrac{2b_0\sqrt{(b^2 - c^2)k^2(2a_0 + k^2)}}{b\cosh\xi_5 + c\sinh\xi_5 + \mu} \pm 2b_0\sqrt{-k^2(2a_0 + k^2)}\dfrac{b\sinh\xi_5 + c\cosh\xi_5}{b\cosh\xi_5 + c\sinh\xi_5 + \mu}, \\ \xi_5 = \dfrac{2k\Gamma(\gamma_2 + 1)}{\beta}x^\beta - \dfrac{2\Gamma(\gamma_1 + 1)}{\alpha}(3a_0k + 2k^3)t^\alpha + \xi_0. \end{cases}$$

Case 2: when $\varepsilon = -1$,

(6) $a_0 = 0, a_1 = -4k^2\mu, a_2 = -4k^2(b^2 + c^2 - \mu^2), a_3 = a_4 = 0, b_1 = \pm 2k^2\sqrt{2b^2 + 2c^2 - \mu^2}$,
$b_2 = b_3 = b_4 = 0, c_1 = -4\left(k^4\mu \pm b_0k^2\sqrt{2b^2 + 2c^2 - \mu^2}\right), c_2 = c_3 = c_4 = 0, C = 2(3a_0k + 2k^3)$.

(7) $a_1 = \pm 4\sqrt{b^2 + c^2}k^2, a_2 = a_3 = a_4 = 0, b_1 = \pm 2\sqrt{b^2 + c^2}k^2, b_2 = b_3 = b_4 = 0, \mu = \pm\sqrt{c^2 + b^2}$,
$c_1 = \pm\left(8a_0\sqrt{c^2 + b^2}k^2 - 4\sqrt{c^2 + b^2}k^4 + 4b_0\sqrt{(b^2 + c^2)k^4}\right), c_2 = c_3 = c_4 = 0, C = -6a_0k + 4k^3$.

(8) $a_1 = -8k^2\mu, a_2 = -8k^2(b^2 + c^2 - \mu^2), a_3 = a_4 = 0, b_1 = \pm 4k^2\mu, b_2 = \mp 4k^2\mu^2, b_3 = b_4 = 0, b = \pm ic$,
$c_1 = 8(2a_0k^2\mu \pm b_0k^2\mu - k^4\mu), c_2 = 8k^2(-2a_0 \mp b_0 + k^2)\mu^2, c_3 = c_4 = 0, C = -6a_0k + 4k^3$.

(9) $a_1 = a_2 = a_3 = a_4 = 0, a_5 = -2k^2, b_1 = b_2 = b_3 = b_4 = 0, b_5 = \pm k^2, c_1 = c_2 = c_3 = c_4 = 0,$
$c_5 = 2(2a_0k^2 \pm b_0k^2 + 2k^4), \mu = \pm\sqrt{c^2 + b^2}, C = -2(3a_0k + 4k^3).$

(10) $a_1 = a_3 = a_5 = 0, a_2 = -2k^2(b^2 + c^2), a_4 = -2\sqrt{(b^2 + c^2)k^4}, b_2 = b_4 = b_5 = 0, c_2 = c_4 = c_5 = 0,$
$b_1 = \sqrt{(b^2 + c^2)k^2(-2a_0 + k^2)}, b_3 = \pm\sqrt{k^2(-2a_0 + k^2)}, c_1 = \pm 2b_0\sqrt{(b^2 + c^2)k^2(-2a_0 + k^2)},$
$c_3 = \pm 2b_0\sqrt{k^2(-2a_0 + k^2)}, \mu = 0, C = -6a_0k + 4k^3.$

According to Equations 4, 11, 22, we could get solutions of the system (1) as $\varepsilon = -1$

$$\begin{cases} u_6 = -\dfrac{4k^2\mu}{b\cos\xi_6 + c\sin\xi_6 + \mu} - \dfrac{4k^2(b^2 + c^2 - \mu^2)}{(b\cos\xi_6 + c\sin\xi_6 + \mu)^2}, \\ v_6 = b_0 \pm \dfrac{2k^2\sqrt{2c^2 + 2b^2 - \mu^2}}{b\cos\xi_6 + c\sin\xi_6 + \mu}, \\ w_6 = c_0 - \dfrac{4\left(k^2\mu \pm b_0k^2\sqrt{2c^2 + 2b^2 - \mu^2}\right)}{b\cos\xi_6 + c\sin\xi_6 + \mu}, \\ \xi_6 = \dfrac{2k\Gamma(\gamma_2 + 1)}{\beta}x^\beta + \dfrac{2\Gamma(\gamma_1 + 1)}{\alpha}(3a_0k + 2k^3)t^\alpha + \xi_0. \end{cases}$$

Remark 4: When $\alpha = \beta = 1, \gamma_1 = \gamma_2 = 0$, if we let $\mu = \sqrt{c^2 + b^2}$, then the solution group (u_6, v_6, w_6) can be transformed into the fifth set of solutions in Ref. [48].

$$\begin{cases} u_7 = a_0 \pm \dfrac{4\sqrt{b^2 + c^2}k^2}{b\cos\xi_7 + c\sin\xi_7 \pm \sqrt{c^2 + b^2}}, \\ v_7 = b_0 \pm \dfrac{2\sqrt{b^2 + c^2}k^2}{b\cos\xi_7 + c\sin\xi_7 \pm \sqrt{c^2 + b^2}}, \\ w_7 = c_0 \pm \dfrac{8a_0\sqrt{c^2 + b^2}k^2 - 4\sqrt{c^2 + b^2}k^4 + 4b_0\sqrt{(b^2 + c^2)k^4}}{b\cos\xi_7 + c\sin\xi_7 \pm \sqrt{c^2 + b^2}}, \\ \xi_7 = \dfrac{2k\Gamma(\gamma_2 + 1)}{\beta}x^\beta + \dfrac{2\Gamma(\gamma_1 + 1)}{\alpha}(-3a_0k + 2k^3)t^\alpha + \xi_0. \end{cases}$$

$$\begin{cases} u_8 = a_0 - \dfrac{8k^2\mu}{\pm ic\cos\xi_8 + c\sin\xi_8 + \mu} - \dfrac{8k^2(b^2 + c^2 - \mu^2)}{(\pm ic\cos\xi_8 + c\sin\xi_8 + \mu)^2}, \\ v_8 = b_0 \pm \dfrac{4k^2\mu}{\pm ic\cos\xi_8 + c\sin\xi_8 + \mu} \mp \dfrac{4k^2\mu^2}{(\pm ic\cos\xi_8 + c\sin\xi_8 + \mu)^2}, \\ w_8 = c_0 + \dfrac{8(2a_0k^2\mu \pm b_0k^2\mu - k^4\mu)}{\pm ic\cos\xi_8 + c\sin\xi_8 + \mu} + \dfrac{8k^2(-2a_0 \mp b_0 + k^2)\mu^2}{(\pm ic\cos\xi_8 + c\sin\xi_8 + \mu)^2}, \\ \xi_8 = \dfrac{2k\Gamma(\gamma_2 + 1)}{\beta}x^\beta + \dfrac{2\Gamma(\gamma_1 + 1)}{\alpha}(-3a_0k + 2k^3)t^\alpha + \xi_0. \end{cases}$$

$$\begin{cases} u_9 = a_0 - 2k^2\left(\dfrac{\pm\sqrt{c^2 + \mu^2}\sin\xi_9 + c\cos\xi_9}{\pm\sqrt{c^2 + \mu^2}\cos\xi_9 + c\sin\xi_9 + \mu}\right)^2, \\ v_9 = b_0 \pm k^2\left(\dfrac{\pm\sqrt{c^2 + \mu^2}\sin\xi_9 + c\cos\xi_9}{\pm\sqrt{c^2 + \mu^2}\cos\xi_9 + c\sin\xi_9 + \mu}\right)^2, \\ w_9 = c_0 + 2\left(2a_0k^2 \pm b_0k^2 + 2k^4\right)\left(\dfrac{\pm\sqrt{c^2 + \mu^2}\sin\xi_9 + c\cos\xi_9}{\pm\sqrt{c^2 + \mu^2}\cos\xi_9 + c\sin\xi_9 + \mu}\right)^2, \\ \xi_9 = \dfrac{2k\Gamma(\gamma_2 + 1)}{\beta}x^\beta - \dfrac{2\Gamma(\gamma_1 + 1)}{\alpha}\left(3a_0k + 4k^3\right)t^\alpha + \xi_0. \end{cases}$$

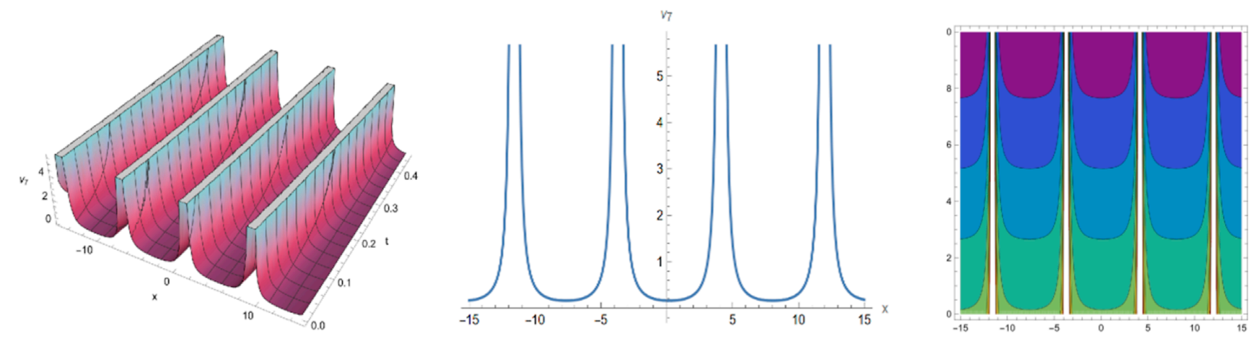

FIGURE 4
The 3D plot, 2D plot and contour plot of v_7 with $\alpha = 0.9, \beta = 1, k = 0.4, b = 2$,

$$\begin{cases} u_{10} = a_0 + \dfrac{-2k^2(b^2+c^2)}{(b\cos\xi_{10} + c\sin\xi_{10} + \mu)^2} - 2\sqrt{(b^2+c^2)k^4}\dfrac{b\sin\xi_{10} + c\cos\xi_{10}}{(b\cos\xi_{10} + c\sin\xi_{10} + \mu)^2}, \\ v_{10} = b_0 + \dfrac{\sqrt{(b^2+c^2)k^2(-2a_0+k^2)}}{b\cos\xi_{10} + c\sin\xi_{10} + \mu} \pm \sqrt{k^2(-2a_0+k^2)}\dfrac{b\sin\xi_{10} + c\cos\xi_{10}}{b\cos\xi_{10} + c\sin\xi_{10} + \mu}, \\ w_{10} = c_0 + \dfrac{\pm 2b_0\sqrt{(b^2+c^2)k^2(-2a_0+k^2)}}{b\cos\xi_{10} + c\sin\xi_{10} + \mu} \pm 2b_0\sqrt{k^2(-2a_0+k^2)}\dfrac{b\sin\xi_{10} + c\cos\xi_{10}}{b\cos\xi_{10} + c\sin\xi_{10} + \mu}, \\ \xi_{10} = \dfrac{2k\Gamma(\gamma_2+1)}{\beta}x^\beta - \dfrac{2\Gamma(\gamma_1+1)}{\alpha}(3a_0k - 2k^3)t^\alpha + \xi_0. \end{cases}$$

By selecting different parameters, we can get some graphical simulation including the famous bell-shape soliton solutions and blow-up pattern wave solution of system (1) in Figures 1, 2. Some simulation of the above periodic wave solutions are shown in Figures 3, 4.

$$c = 1, \mu = 1, a_0 = 1, \gamma_1 = 2, \xi_0 = 0, t = 0.55.$$

$$c = 1, \mu = 1, a_0 = 2, \gamma_1 = 1, \xi_0 = 100, t = 0.42.$$

3.2 Solving Eq. (1) by the $G'/(bG + G + a)$-expansion method

Suppose Equation 5 has solutions of the following form:

$$\begin{cases} u = a_0 + a_1\left(\dfrac{G'}{bG' + G + a}\right) + a_2\left(\dfrac{G'}{bG' + G + a}\right)^2 = a_0 + a_1F + a_2F^2, \\ v = b_0 + b_1\left(\dfrac{G'}{bG' + G + a}\right) + b_2\left(\dfrac{G'}{bG' + G + a}\right)^2 = b_0 + b_1F + b_2F^2, \\ w = c_0 + c_1\left(\dfrac{G'}{bG' + G + a}\right) + c_2\left(\dfrac{G'}{bG' + G + a}\right)^2 = c_0 + c_1F + c_2F^2. \end{cases} \tag{23}$$

Where $a_i, b_i, c_i (i = 0, 1, 2)$ are constants to be determined.

Substituting Equations 23, 15 into Equation 5, and setting the coefficients $F^i (i = 0, 1, 2, 3, 4, \cdots)$ to zero, we could get a set of algebraic equations about $a_i, b_i, c_i, b, \lambda, \mu, k, C$. Without losing generality, we let $b = 1$:

$$F^0: -C\mu a_1 + 2k^3\lambda^2\mu a_1 + 4k^3\mu^2 a_1 - 12k^3\lambda\mu^2 a_1 + 12k^3\mu^3 a_1 + 6k\mu a_0 a_1 + 12k^3\lambda\mu^2 a_2 - 24k^3\mu^3 a_2 - 12k\mu b_0 b_1 + 6k\mu c_1 = 0,$$
$$F: -C\lambda a_1 + 2k^3\lambda^3 a_1 + 2C\mu a_1 + 16k^3\lambda\mu a_1 - 28k^3\lambda^2\mu a_1 - 32k^3\mu^2 a_1 + 72k^3\lambda\mu^2 a_1 - 48k^3\mu^3 a_1 + 6k\lambda a_0 a_1 - 12k\mu a_0 a_1$$
$$+6k\mu a_1^2 - 2C\mu a_2 + 28k^3\lambda^2\mu a_2 + 32k^3\mu^2 a_2 - 144k^3\lambda\mu^2 a_2 + 144k^3\mu^3 a_2 + 12k\mu a_0 a_2 - 12k\lambda b_0 b_1 + 24k\mu b_0 b_1$$
$$-12k\mu b_1^2 - 24k\mu b_0 b_2 + 6k\lambda c_1 - 12k\mu c_1 + 12k\mu c_2 = 0,$$

$$F^2: -Ca_1+C\lambda a_1+14k^3\lambda^2a_1-14k^3\lambda^3a_1-C\mu a_1+16k^3\mu a_1-88k^3\lambda\mu a_1+86k^3\lambda^2\mu a_1+88k^3\mu^2a_1-144k^3\lambda\mu^2a_1$$
$$+72k^3\mu^3a_1+6ka_0a_1-6k\lambda a_0a_1+6k\mu a_0a_1+6k\lambda a_1^2-12k\mu a_1^2-2C\lambda a_2+16k^3\lambda^3a_2+4C\mu a_2+104k^3\lambda\mu a_2$$
$$-200k^3\lambda^2\mu a_2-208k^3\mu^2a_2+504k^3\lambda\mu^2a_2-336k^3\mu^3a_2+12k\lambda a_0a_2-24k\mu a_0a_2+18k\mu a_1a_2-12kb_0b_1$$
$$+12k\lambda b_0b_1-12k\mu b_0b_1-12k\lambda b_1^2+24k\mu b_1^2-24k\lambda b_0b_2+48k\mu b_1b_2-36k\mu b_1b_2+6kc_1-6k\lambda c_1+6k\mu c_1$$
$$+12k\lambda c_2-24k\mu c_2=0,$$
$$F^3: 24k^3\lambda a_1-48k^3\lambda^2a_1+24k^3\lambda^3a_1-48k^3\mu a_1+144k^3\lambda\mu a_1-96k^3\lambda^2\mu a_1-96k^3\mu^2a_1+120k^3\lambda\mu^2a_1-48k^3\mu^3a_1$$
$$+6ka_1^2-6k\lambda a_1^2+6k\mu a_1^2-2Ca_2+2C\lambda a_2+76k^3\lambda^2a_2-76k^3\lambda^3a_2-2C\mu a_2+80k^3\mu a_2-464k^3\lambda\mu a_2+460k^3\lambda^2\mu a_2$$
$$+464k^3\mu^2a_2-768k^3\lambda\mu^2a_2+384k^3\mu^3a_2+12ka_0a_2-12k\lambda a_0a_2+12k\mu a_0a_2+18k\lambda a_1a_2-36k\mu a_1a_2+12k\mu a_2^2$$
$$-12kb_1^2+12k\lambda b_1^2-12k\mu b_1^2-24kb_0b_2+24k\lambda b_0b_2-24k\mu b_0b_2-36k\lambda b_1b_2+72k\mu b_1b_2-24k\mu b_2^2+12kc_2$$
$$-12k\lambda c_2+12k\mu c_2=0,$$
$$F^4: 12k^3a_1-36k^3\lambda a_1+36k^3\lambda^2a_1-12k^3\lambda^3a_1+36k^3\mu a_1-72k^3\lambda\mu a_1+36k^3\lambda^2\mu a_1+36k^3\mu^2a_1-36k^3\lambda\mu^2a_1$$
$$+12k^3\mu^3a_1+108k^3\lambda a_2-216k^3\lambda^2a_2+108k^3\lambda^3a_2-216k^3\mu a_2+648k^3\lambda\mu a_2-432k^3\lambda^2\mu a_2-432k^3\mu^2a_2$$
$$+540k^3\lambda\mu^2a_2-216k^3\mu^3a_2+18ka_1a_2-18k\lambda a_1a_2+18k\mu a_1a_2+12k\lambda a_2^2-24k\mu a_2^2-36kb_1b_2+6k\lambda b_1b_2$$
$$-36k\mu b_1b_2-24k\lambda b_2^2+48k\mu b_2^2=0,$$
$$F^5: 48k^3a_2-144k^3\lambda a_2+144k^3\lambda^2a_2-48k^3\lambda^3a_2+144k^3\mu a_2-288k^3\lambda\mu a_2+144k^3\lambda^2\mu a_2+144k^3\mu^2a_2$$
$$-144k^3\lambda\mu^2a_2+48k^3\mu^3a_2+12ka_2^2-12k\lambda a_2^2+12k\mu a_2^2-24kb_2^2+24k\lambda b_2^2-24k\mu b_2^2=0,$$
$$F^0: -C\mu b_1-4k^3\lambda^2\mu b_1-8k^3\mu^2b_1+24k^3\lambda\mu^2b_1-24k^3\mu^3b_1-6k\mu a_0b_1-24k^3\lambda\mu^2b_2+48k^3\mu^3b_2=0,$$
$$F: -C\lambda b_1-4k^3\lambda^3b_1+2C\mu b_1-32k^3\lambda\mu b_1+56k^3\lambda^2\mu b_1+64k^3\mu^2b_1-144k^3\lambda\mu^2b_1+96k^3\mu^3b_1$$
$$-6k\lambda a_0b_1+12k\mu a_0b_1-6k\mu a_1b_1-2C\mu b_2-56k^3\lambda^2\mu b_2-64k^3\mu^2b_2+288k^3\lambda\mu^2b_2-288k^3\mu^3b_2$$
$$-12k\mu a_0b_2=0,$$
$$F^2: -Cb_1+C\lambda b_1-28k^3\lambda^2b_1+28k^3\lambda^3b_1-C\mu b_1-32k^3\mu b_1+176k^3\lambda\mu b_1-172k^3\lambda^2\mu b_1-176k^3\mu^2b_1$$
$$+288k^3\lambda\mu^2b_1-144k^3\mu^3b_1-6ka_0b_1+6k\lambda a_0b_1-6k\mu a_0b_1-6k\lambda a_1b_1+12k\mu a_1b_1-6k\mu a_2b_1$$
$$-2C\lambda b_2-32k^3\lambda^3b_2+4C\mu b_2-208k^3\lambda\mu b_2+400k^3\lambda^2\mu b_2+416k^3\mu^2b_2-1008k^3\lambda\mu^2b_2$$
$$+672k^3\mu^3b_2-12k\lambda a_0b_2+24k\mu a_0b_2-12k\mu a_1b_2=0,$$
$$F^3: -48k^3\lambda b_1+96k^3\lambda^2b_1-48k^3\lambda^3b_1+96k^3\mu b_1-288k^3\lambda\mu b_1+192k^3\lambda^2\mu b_1+192k^3\mu^2b_1$$
$$-240k^3\lambda\mu^2b_1+96k^3\mu^3b_1-6ka_1b_1+6k\lambda a_1b_1-6k\mu a_1b_1-6k\lambda a_2b_1+12k\mu a_2b_1-2Cb_2$$
$$+2C\lambda b_2-152k^3\lambda^2b_2+152k^3\lambda^3b_2-2C\mu b_2-160k^3\mu b_2+928k^3\lambda\mu b_2-920k^3\lambda^2\mu b_2$$
$$-928k^3\mu^2b_2+1536k^3\lambda\mu^2b_2-768k^3\mu^3b_2-12ka_0b_2+12k\lambda a_0b_2-12k\mu a_0b_2-12k\lambda a_1b_2$$
$$+24k\mu a_1b_2-12k\mu a_2b_2=0,$$
$$F^4: -24k^3b_1+72k^3\lambda b_1-72k^3\lambda^2b_1+24k^3\lambda^3b_1-72k^3\mu b_1+144k^3\lambda\mu b_1-72k^3\lambda^2\mu b_1-72k^3\mu^2b_1$$
$$+72k^3\lambda\mu^2b_1-24k^3\mu^3b_1-6ka_2b_1+6k\lambda a_2b_1-6k\mu a_2b_1-216k^3\lambda b_2+432k^3\lambda^2b_2$$
$$-216k^3\lambda^3b_2+432k^3\mu b_2-1296k^3\lambda\mu b_2+864k^3\lambda^2\mu b_2+864k^3\mu^2b_2-1080k^3\lambda\mu^2b_2$$
$$+432k^3\mu^3b_2-12ka_1b_2+12k\lambda a_1b_2-12k\mu a_1b_2-12k\lambda a_2b_2+24k\mu a_2b_2=0,$$
$$F^5: -96k^3b_2+288k^3\lambda b_2-288k^3\lambda^2b_2+96k^3\lambda^3b_2-288k^3\mu b_2+576k^3\lambda\mu b_2-288k^3\lambda^2\mu b_2$$
$$-288k^3\mu^2b_2+288k^3\lambda\mu^2b_2-96k^3\mu^3b_2-12ka_2b_2+12k\lambda a_2b_2-12k\mu a_2b_2=0,$$
$$F^0: -C\mu c_1-4k^3\lambda^2\mu c_1-8k^3\mu^2c_1+24k^3\lambda\mu^2c_1-24k^3\mu^3c_1-6k\mu a_0c_1-24k^3\lambda\mu^2c_2+48k^3\mu^3c_2=0,$$
$$F: -C\lambda c_1-4k^3\lambda^3c_1+2C\mu c_1-32k^3\lambda\mu c_1+56k^3\lambda^2\mu c_1+64k^3\mu^2c_1-144k^3\lambda\mu^2c_1+96k^3\mu^3c_1$$
$$-6k\lambda a_0c_1+12k\mu a_0c_1-6k\mu a_1c_1-2C\mu c_2-56k^3\lambda^2\mu c_2-64k^3\mu^2c_2+288k^3\lambda\mu^2c_2-288k^3\mu^3c_2$$
$$-12k\mu a_0c_2=0,$$
$$F^2: -Cc_1+C\lambda c_1-28k^3\lambda^2c_1+28k^3\lambda^3c_1-C\mu c_1-32k^3\mu c_1+176k^3\lambda\mu c_1-172k^3\lambda^2\mu c_1-176k^3\mu^2c_1$$
$$+288k^3\lambda\mu^2c_1-144k^3\mu^3c_1-6ka_0c_1+6k\lambda a_0c_1-6k\mu a_0c_1-6k\lambda a_1c_1+12k\mu a_1c_1-6k\mu a_2c_1$$
$$-2C\lambda c_2-32k^3\lambda^3c_2+4C\mu c_2-208k^3\lambda\mu c_2+400k^3\lambda^2\mu c_2+416k^3\mu^2c_2-1008k^3\lambda\mu^2c_2$$
$$+672k^3\mu^3c_2-12k\lambda a_0c_2+24k\mu a_0c_2-12k\mu a_1c_2=0,$$
$$F^3: -48k^3\lambda c_1+96k^3\lambda^2c_1-48k^3\lambda^3c_1+96k^3\mu c_1-288k^3\lambda\mu c_1+192k^3\lambda^2\mu c_1+192k^3\mu^2c_1$$
$$-240k^3\lambda\mu^2c_1+96k^3\mu^3c_1-6ka_1c_1+6k\lambda a_1c_1-6k\mu a_1c_1-6k\lambda a_2c_1+12k\mu a_2c_1-2Cc_2+2C\lambda c_2$$
$$-152k^3\lambda^2c_2+152k^3\lambda^3c_2-2C\mu c_2-160k^3\mu c_2+928k^3\lambda\mu c_2-920k^3\lambda^2\mu c_2-928k^3\mu^2c_2$$
$$+1536k^3\lambda\mu^2c_2-768k^3\mu^3c_2-12ka_0c_2+12k\lambda a_0c_2-12k\mu a_0c_2-12k\lambda a_1c_2+24k\mu a_1c_2$$
$$-12k\mu a_2c_2=0,$$

$$
\begin{aligned}
F^4&: -24k^3c_1+72k^3\lambda c_1-72k^3\lambda^2c_1+24k^3\lambda^3c_1-72k^3\mu c_1+144k^3\lambda\mu c_1-72k^3\lambda^2\mu c_1-72k^3\mu^2c_1\\
&+72k^3\lambda\mu^2c_1-24k^3\mu^3c_1-6ka_2c_1+6k\lambda a_2c_1-6k\mu a_2c_1-216k^3\lambda c_2+432k^3\lambda^2c_2-216k^3\lambda^3c_2\\
&+432k^3\mu c_2-1296k^3\lambda\mu c_2+864k^3\lambda^2\mu c_2+864k^3\mu^2c_2-1080k^3\lambda\mu^2c_2+432k^3\mu^3c_2-12ka_1c_2\\
&+12k\lambda a_1c_2-12k\mu a_1c_2-12k\lambda a_2c_2+24k\mu a_2c_2=0,\\
F^5&: -96k^3c_2+288k^3\lambda c_2-288k^3\lambda^2c_2+96k^3\lambda^3c_2-288k^3\mu c_2+576k^3\lambda\mu c_2-288k^3\lambda^2\mu c_2\\
&-288k^3\mu^2c_2+288k^3\lambda\mu^2c_2-96k^3\mu^3c_2-12ka_2c_2+12k\lambda a_2c_2-12k\mu a_2c_2=0.
\end{aligned}
$$

After solving the above AEs, we can get the following solutions, where the parameters not specified are arbitrary constants.

(11) $a_1=0, a_2=-8k^2(-1+\mu)^2, b_1=0, b_2=4k^2(-1+\mu)^2, \lambda=2\mu, \Delta=\lambda^2-4\mu=4\mu(\mu-1)$,
$c_1=0, c_2=-8k^2(-1+\mu)^2(-2a_0-b_0-8k^2\mu+8k^2\mu^2), b=1, C=-2(3a_0k+16k^3\mu-16k^3\mu^2)$.

(12) $b=1, a_1=4k^2[\lambda^2+2\mu(1+\mu)-\lambda(1+3\mu)], a_2=-4k^2(1-\lambda+\mu)^2, b_0=0$,
$b_1=\pm 2k-(1-\lambda+\mu)^2[2a_0+k^2(\lambda^2-8\lambda\mu+4\mu(1+2\mu)]\sqrt{}, b_2=0, c_2=0$,
$c_1=-4k^2[\lambda^2+2\mu(\mu+1)-\lambda(1+3\mu)][2a_0+k^2(\lambda^2-8\lambda\mu+4\mu(1+2\mu)]$,
$C=-6a_0k-4k^3(\lambda^2+2\mu-6\lambda\mu+6\mu^2)$.

According to Equations 4, 17, 19, 21, 23, we obtain the following solutions for system (1):

$$
\begin{cases}
u_{11}=a_0-8k^2(-1+\mu)^2F^2,\\
v_{11}=b_0+4k^2(-1+\mu)^2F^2,\\
w_{11}=c_0-8k^2(-1+\mu)^2(-2a_0-b_0-8k^2\mu+8k^2\mu^2)F^2,\\
\xi_{11}=\dfrac{2k\Gamma(\gamma_2+1)}{\beta}x^\beta-\dfrac{2\Gamma(\gamma_1+1)}{\alpha}(3a_0k+16k^3\mu-16k^3\mu^2)t^\alpha+\xi_0.
\end{cases}
$$

$$
\begin{cases}
u_{12}=a_0+4k^2[\lambda^2+2\mu(1+\mu)-\lambda(1+3\mu)]F-4k^2(1-\lambda+\mu)^2F^2,\\
v_{12}=\pm 2k-(1-\lambda+\mu)^2[2a_0+k^2(\lambda^2-8\lambda\mu+4\mu(1+2\mu)]\sqrt{F},\\
w_{12}=c_0-4k^2[\lambda^2+2\mu(\mu+1)-\lambda(1+3\mu)][2a_0+k^2(\lambda^2-8\lambda\mu+4\mu(1+2\mu)]F,\\
\xi_{12}=\dfrac{2k\Gamma(\gamma_2+1)}{\beta}x^\beta-\dfrac{\Gamma(\gamma_1+1)}{\alpha}[6a_0k+4k^3(\lambda^2+2\mu-6\lambda\mu+6\mu^2)]t^\alpha+\xi_0.
\end{cases}
$$

We can determine the following solutions.
Case 1: $\Delta>0$, when $\mu<0$ or $\mu>1$.

$$
\begin{cases}
u_{11.1}=a_0-8k^2(-1+\mu)^2\left(1+\dfrac{2C_1+2C_2e^{\sqrt{\Delta}\xi_{11.1}}}{C_1\left(-2+\lambda+\sqrt{\Delta}\right)+C_2\left(-2+\lambda-\sqrt{\Delta}\right)e^{\sqrt{\Delta}\xi_{11.1}}}\right)^2,\\
v_{11.1}=b_0+4k^2(-1+\mu)^2\left(1+\dfrac{2C_1+2C_2e^{\sqrt{\Delta}\xi_{11.1}}}{C_1\left(-2+\lambda+\sqrt{\Delta}\right)+C_2\left(-2+\lambda-\sqrt{\Delta}\right)e^{\sqrt{\Delta}\xi_{11.1}}}\right)^2,\\
w_{11.1}=c_0-8k^2(-1+\mu)^2(-2a_0-b_0-8k^2\mu+8k^2\mu^2)\left(1+\dfrac{2C_1+2C_2e^{\sqrt{\Delta}\xi_{11.1}}}{C_1\left(-2+\lambda+\sqrt{\Delta}\right)+C_2\left(-2+\lambda-\sqrt{\Delta}\right)e^{\sqrt{\Delta}\xi_{11.1}}}\right)^2,\\
\xi_{11.1}=\dfrac{2k\Gamma(\gamma_2+1)}{\beta}x^\beta-\dfrac{2\Gamma(\gamma_1+1)}{\alpha}(3a_0k+16k^3\mu-16k^3\mu^2)t^\alpha+\xi_0.
\end{cases}
$$

$$
\begin{cases}
u_{12.1}=a_0+4k^2[\lambda^2+2\mu(1+\mu)-\lambda(1+3\mu)]\left(1+\dfrac{2C_1+2C_2e^{\sqrt{\Delta}\xi_{12.1}}}{C_1\left(-2+\lambda+\sqrt{\Delta}\right)+C_2\left(-2+\lambda-\sqrt{\Delta}\right)e^{\sqrt{\Delta}\xi_{12.1}}}\right)\\
-4k^2(1-\lambda+\mu)^2\left(1+\dfrac{2C_1+2C_2e^{\sqrt{\Delta}\xi_{12.1}}}{C_1\left(-2+\lambda+\sqrt{\Delta}\right)+C_2\left(-2+\lambda-\sqrt{\Delta}\right)e^{\sqrt{\Delta}\xi_{12.1}}}\right)^2,\\
v_{12.1}=\pm 2k-(1-\lambda+\mu)^2[2a_0+k^2(\lambda^2-8\lambda\mu+4\mu(1+2\mu)]\sqrt{\left(1+\dfrac{2C_1+2C_2e^{\sqrt{\Delta}\xi_{12.1}}}{C_1\left(-2+\lambda+\sqrt{\Delta}\right)+C_2\left(-2+\lambda-\sqrt{\Delta}\right)e^{\sqrt{\Delta}\xi_{12.1}}}\right)},\\
w_{12.1}=c_0-4k^2[\lambda^2+2\mu(\mu+1)-\lambda(1+3\mu)][2a_0+k^2(\lambda^2-8\lambda\mu+4\mu(1+2\mu)]\left(1+\dfrac{2C_1+2C_2e^{\sqrt{\Delta}\xi_{12.1}}}{C_1\left(-2+\lambda+\sqrt{\Delta}\right)+C_2\left(-2+\lambda-\sqrt{\Delta}\right)e^{\sqrt{\Delta}\xi_{12.1}}}\right),\\
\xi_{12.1}=\dfrac{2k\Gamma(\gamma_2+1)}{\beta}x^\beta-\dfrac{\Gamma(\gamma_1+1)}{\alpha}[6a_0k+4k^3(\lambda^2+2\mu-6\lambda\mu+6\mu^2)]t^\alpha+\xi_0.
\end{cases}
$$

Case 2: $\Delta < 0$, when $0 < \mu < 1$.

$$\begin{cases} u_{11.2} = a_0 - 8k^2(-1+\mu)^2\left(\dfrac{\left(\lambda C_1 - \sqrt{-\Delta}C_2\right)\cos\left(\frac{\sqrt{-\Delta}}{2}\xi_{11.2}\right) + \left(\lambda C_2 + \sqrt{-\Delta}C_1\right)\sin\left(\frac{\sqrt{-\Delta}}{2}\xi_{11.2}\right)}{\left((\lambda-2)C_1 - \sqrt{-\Delta}C_2\right)\cos\left(\frac{\sqrt{-\Delta}}{2}\xi_{11.2}\right) + \left((\lambda-2)C_2 + \sqrt{-\Delta}C_1\right)\sin\left(\frac{\sqrt{-\Delta}}{2}\xi_{11.2}\right)}\right)^2, \\ v_{11.2} = b_0 + 4k^2(-1+\mu)^2\left(\dfrac{\left(\lambda C_1 - \sqrt{-\Delta}C_2\right)\cos\left(\frac{\sqrt{-\Delta}}{2}\xi_{11.2}\right) + \left(\lambda C_2 + \sqrt{-\Delta}C_1\right)\sin\left(\frac{\sqrt{-\Delta}}{2}\xi_{11.2}\right)}{\left((\lambda-2)C_1 - \sqrt{-\Delta}C_2\right)\cos\left(\frac{\sqrt{-\Delta}}{2}\xi_{11.2}\right) + \left((\lambda-2)C_2 + \sqrt{-\Delta}C_1\right)\sin\left(\frac{\sqrt{-\Delta}}{2}\xi_{11.2}\right)}\right)^2, \\ w_{11.2} = c_0 - 8k^2(-1+\mu)^2(-2a_0 - b_0 - 8k^2\mu + 8k^2\mu^2)\left(\dfrac{\left(\lambda C_1 - \sqrt{-\Delta}C_2\right)\cos\left(\frac{\sqrt{-\Delta}}{2}\xi_{11.2}\right) + \left(\lambda C_2 + \sqrt{-\Delta}C_1\right)\sin\left(\frac{\sqrt{-\Delta}}{2}\xi_{11.2}\right)}{\left((\lambda-2)C_1 - \sqrt{-\Delta}C_2\right)\cos\left(\frac{\sqrt{-\Delta}}{2}\xi_{11.2}\right) + \left((\lambda-2)C_2 + \sqrt{-\Delta}C_1\right)\sin\left(\frac{\sqrt{-\Delta}}{2}\xi_{11.2}\right)}\right)^2, \\ \xi_{11.2} = \dfrac{2k\Gamma(\gamma_2+1)}{\beta}x^\beta - \dfrac{2\Gamma(\gamma_1+1)}{\alpha}(3a_0k + 16k^3\mu - 16k^3\mu^2)t^\alpha + \xi_0. \end{cases}$$

$$\begin{cases} u_{12.2} = a_0 + 4k^2[\lambda^2 + 2\mu(1+\mu) - \lambda(1+3\mu)]F_{12.2} - 4k^2(1-\lambda+\mu)^2F_{12.2}^2, \\ v_{12.2} = \pm 2k\sqrt{-(1-\lambda+\mu)^2\left[2a_0 + k^2(\lambda^2 - 8\lambda\mu + 4\mu(1+2\mu)\right]}F_{12.2}, \\ w_{12.2} = c_0 - 4k^2[\lambda^2 + 2\mu(\mu+1) - \lambda(1+3\mu)]\left[2a_0 + k^2\ (\lambda^2 - 8\lambda\mu + 4\mu(1+2\mu)\right]F_{12.2}, \\ \xi_{12.2} = \dfrac{2k\Gamma(\gamma_2+1)}{\beta}x^\beta - \dfrac{\Gamma(\gamma_1+1)}{\alpha}[6a_0k + 4k^3(\lambda^2 + 2\mu - 6\lambda\mu + 6\mu^2)]t^\alpha + \xi_0. \end{cases}$$

$$F_{12.2} = \frac{\left(\lambda C_1 - \sqrt{-\Delta}C_2\right)\cos\left(\frac{\sqrt{-\Delta}}{2}\xi_{12.2}\right) + \left(\lambda C_2 + \sqrt{-\Delta}C_1\right)\sin\left(\frac{\sqrt{-\Delta}}{2}\xi_{12.2}\right)}{\left((\lambda-2)C_1 - \sqrt{-\Delta}C_2\right)\cos\left(\frac{\sqrt{-\Delta}}{2}\xi_{12.2}\right) + \left((\lambda-2)C_2 + \sqrt{-\Delta}C_1\right)\sin\left(\frac{\sqrt{-\Delta}}{2}\xi_{12.2}\right)}.$$

Case 3: $\Delta = 0$, when $\mu = 0$.

$$\begin{cases} u_{11.3} = a_0 - 8k^2\left(\dfrac{2C_2 - \lambda C_1 - \lambda C_2\xi_{11.3}}{2C_2 + (2-\lambda)(C_1+C_2)\xi_{11.3}}\right)^2, \\ v_{11.3} = b_0 + 4k^2\left(\dfrac{2C_2 - \lambda C_1 - \lambda C_2\xi_{11.3}}{2C_2 + (2-\lambda)(C_1+C_2)\xi_{11.3}}\right)^2, \\ w_{11.3} = c_0 - 8k^2(-2a_0 - b_0)\left(\dfrac{2C_2 - \lambda C_1 - \lambda C_2\xi_{11.3}}{2C_2 + (2-\lambda)(C_1+C_2)\xi_{11.3}}\right)^2, \\ \xi_{11.3} = \dfrac{2k\Gamma(\gamma_2+1)}{\beta}x^\beta - \dfrac{\Gamma(\gamma_1+1)}{\alpha}(6a_0k)t^\alpha + \xi_0. \end{cases}$$

$$\begin{cases} u_{12.3} = a_0 + 4k^2(\lambda^2-\lambda)\dfrac{2C_2 - \lambda C_1 - \lambda C_2\xi_{12.3}}{2C_2 + (2-\lambda)(C_1+C_2)\xi_{12.3}} - 4k^2(1-\lambda)^2\left(\dfrac{2C_2 - \lambda C_1 - \lambda C_2\xi_{12.3}}{2C_2 + (2-\lambda)(C_1+C_2)\xi_{12.3}}\right)^2, \\ v_{12.3} = \pm 2k\sqrt{-(1-\lambda)^2(2a_0 + k^2\lambda^2)}\dfrac{2C_2 - \lambda C_1 - \lambda C_2\xi_{12.3}}{2C_2 + (2-\lambda)(C_1+C_2)\xi_{12.3}}, \\ w_{12.3} = c_0 - 4k^2(\lambda^2-\lambda)(2a_0 + k^2\lambda^2)\dfrac{2C_2 - \lambda C_1 - \lambda C_2\xi}{2C_2 + (2-\lambda)(C_1+C_2)\xi_{12.3}}, \\ \xi_{12.3} = \dfrac{2k\Gamma(\gamma_2+1)}{\beta}x^\beta - \dfrac{\Gamma(\gamma_1+1)}{\alpha}(6a_0k + 4k^3\lambda^2)t^\alpha + \xi_0. \end{cases}$$

Selecting different parameters, we can get some graphical simulation of above solutions $\alpha = 1, \beta = 1$ (Figures 5, 6) and $\alpha = 0.8, \beta = 1$ (Figure 7) as follows:

$$\gamma_1 = 0, \xi_0 = 0, t = 0.02, C_1 = 1, C_2 = 2.$$

$$\gamma_1 = 0, \xi_0 = 0, t = 0.02, C_1 = 1, C_2 = 2.$$

$$\gamma_1 = 1, \xi_0 = 10, t = 0.12, C_1 = 3, C_2 = 0.3.$$

Remark 5: All of the above results have been checked by computer programs, and they are founded for the first time to our knowledge.

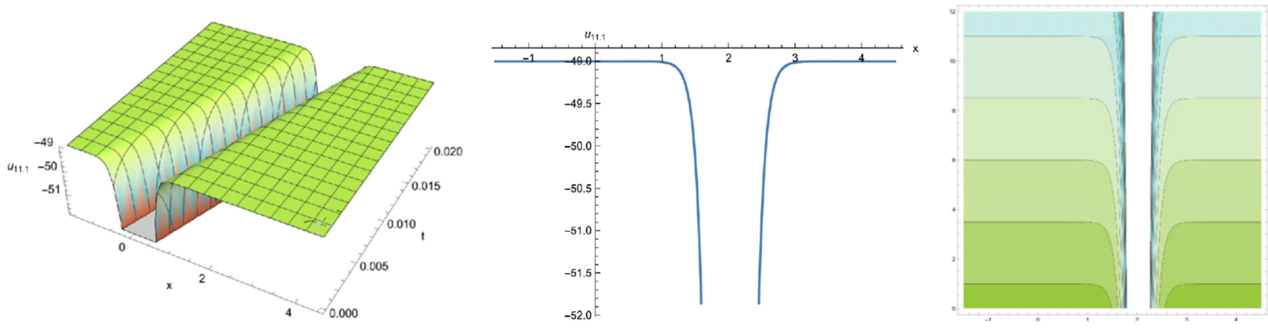

FIGURE 5
The 3D plot, 2D plot and contour plot of $u_{11.1}$ with $\mu = 3, \lambda = 6, k = 1, b_0 = 1, c_0 = 1, a_0 = 1,$

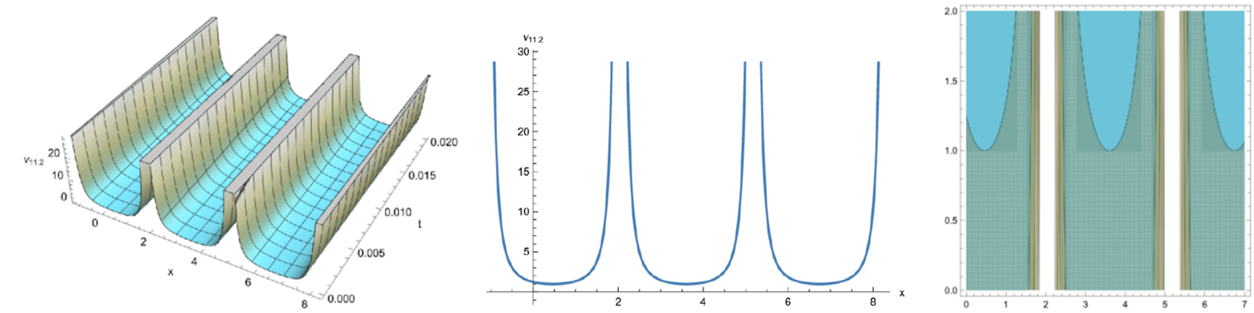

FIGURE 6
The 3D plot, 2D plot and contour plot of $v_{11.2}$ with $\mu = \frac{1}{2}, \lambda = 1, k = 1, b_0 = 1, c_0 = 1, a_0 = 1,$

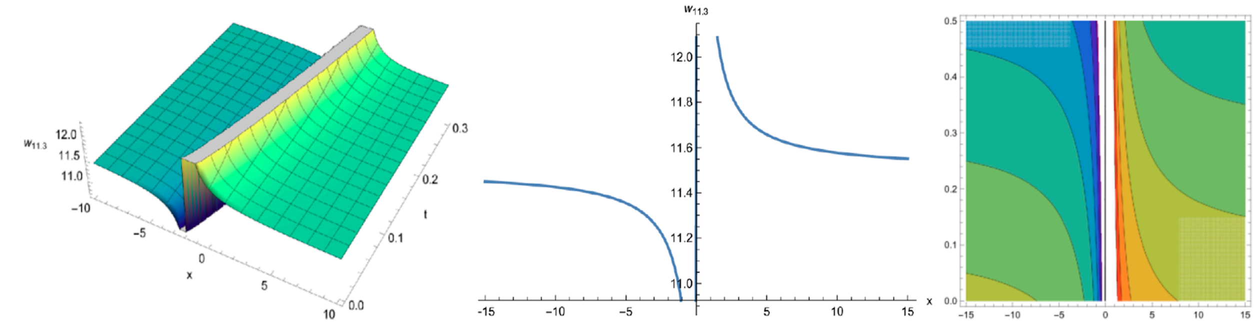

FIGURE 7
The 3D plot, 2D plot and contour plot of $w_{11.3}$ with $\mu = 0, \lambda = 1.5, k = 3, b_0 = 1, c_0 = 10, a_0 = 3,$

3.3 Results and discussion

We have obtained many types of new analytical solutions of the system (1) by two efficient methods, which include the famous bell-shaped solitary wave u_2, this smooth solution reveals a balance of nonlinear effects and dispersion effects, the blow-up wave w_3 which is distorted between the interval (0.190, 0.192) etc. There are also many forms of periodic waves, and these periodic wave solutions embody different

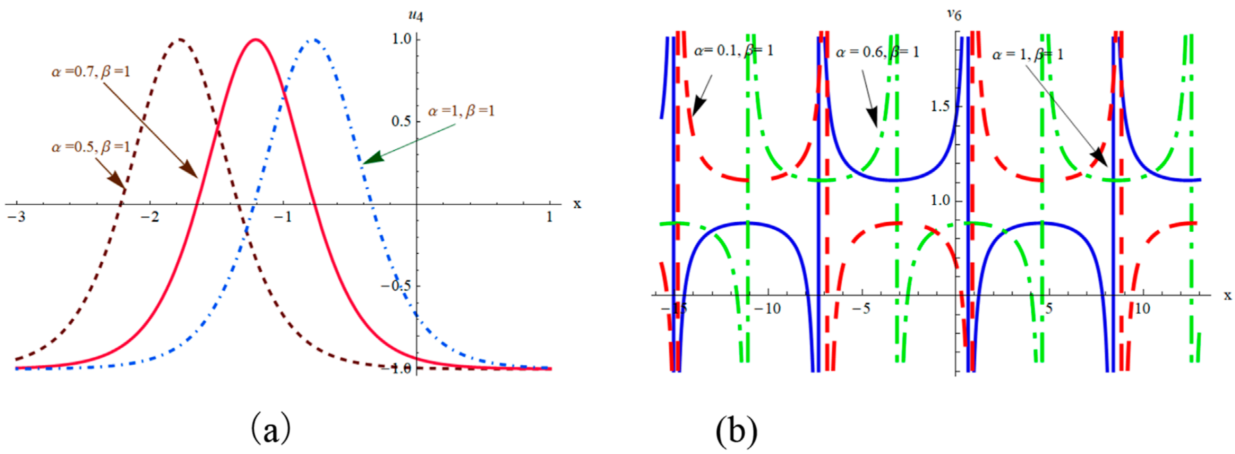

FIGURE 8
Fractional evolution of u_4 in **(A)** and v_6 in **(B)** over time under different parameters.

properties. For example, the waveform of v_6 alternates up and down in both directions, periodicity of v_7 and $v_{11.2}$ are only reflected in one direction. If we choose different parameters and orders, we could find that the waveform of system (1) will evolve with time t. The Figures (a) and (b) in Figure 8 show the evolutionary process of u_4, v_6 with time fractional order with parameters $a_0 = b_0 = c_0 = 1, b = 1, c = 1, \mu = 1, k = -1, t = 1, \gamma_1 = \gamma_2 = 2, \beta = 1$ and $a_0 = b_0 = c_0 = 1,\ b = 45, c = 1, \mu = 1, k = 0.2, t = 0.55, \gamma_1 = 2,\ \gamma_2 = 0, \beta = 1$ respectively. Numerical simulations show that the waveform shifts to the right as the time order increases, and these properties may be of great significance for revealing the internal structure of system (1). However, In MEMS, the understanding of nonlinear wave phenomena and the availability of exact solutions can contribute to the design and optimization of various components. For example, in MEMS sensors, these solutions can help analyze the response to external stimuli and improve the sensitivity and accuracy.

4 Conclusion

In conclusion, by utilizing the modified $(G'/G, 1/G)$-expansion method, the $G'/(bG' + G + a)$-expansion method and the travelling wave transform under the definition of M fractional derivative, twelve new types of exact solutions of the generalized time-space fractional coupled Hirota-Satsuma KdV system are obtained successfully. These solutions include complex solitary wave solutions, trigonometric periodic wave solutions, and rational function solutions. These solutions can be transformed into integer order cases under special parameter selection, and they have important theoretical guiding value for profoundly revealing the interaction between two nonlinear long waves with different dispersion effects. The waveforms of partial solutions and their characteristic images of time evolution are obtained by numerical simulation. It is proved by practice that these two methods can be applied to many other nonlinear equations including the MEMS. Additionally, in integrated MEMS systems, the knowledge of these solutions can enhance the functionality and reliability. However, the proposed definition of M-fractional derivatives still has some limitations, and it is difficult to characterize the necessary connection between two real number or complex number order derivatives. On the other hand, the unified definition of fractional derivatives definition needs to be further explored and developed for us in the future. Once our definition has been substantially refined, then we work on perturbation theory, dynamical system theory, soliton theory, etc., will be better developed [63–65]. How to extend this method to discretely-coupled nonlinear systems with arbitrary subhigher dimensions is still worth further study. This will open up new avenues for exploring more complex nonlinear phenomena and expanding the application scope of these methods in the field of nano/micro devices and systems.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

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Author contributions

YC: Funding acquisition, Investigation, Supervision, Writing–original draft, Formal Analysis. SH: Methodology, Resources, Writing–review and editing, Conceptualization, Project administration. SY: Conceptualization, Data curation, Writing–original draft, Formal Analysis, Visualization. XC: Formal Analysis, Investigation, Methodology, Resources, Writing–review and editing. JY: Conceptualization, Project administration, Visualization, Writing–original draft.

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Conflict of interest

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