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EDITED BY
Igor Kondrashuk,
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Sarfraz Ahmad,
COMSATS University Islamabad, Lahore Campus, Pakistan
Ugur Kadak,
Gazi University, Türkiye

*CORRESPONDENCE
Saira Hameed,
saira.math@pu.edu.pk

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A spherical fuzzy planar graph approach to optimize wire configuration in transformers

Hao Guan[1,2], Saira Hameed[3]*, Sadaf[3], Aysha Khan[4] and Jana Shafi[5]

[1]Institute of Computing Science and Technology, Guangzhou University, Guangzhou, China, [2]Institute of Computer Science of Information Technology, Qiannan Normal University for Nationalities, Duyun, China, [3]Department of Mathematics, University of the Punjab, Lahore, Pakistan, [4]University of Technology and Applied Sciences, Musanna, Oman, [5]Department of Computer Engineering and Information, College of Engineering in Wadi Alddawasir, Prince Sattam Bin Abdul Aziz University, Wadi Alddawasir, Saudi Arabia

In this modern era, graph theory has become an integral part of science and technology. It has enormous applications in handling various design-based problems. In this study, we present a new approach that increases the ability of graph theory to deal with uncertain challenges. The spherical fuzzy graph addresses the uncertainty domain more broadly, going beyond classical fuzzy graphs other than generalized fuzzy graph types. The concept of the spherical fuzzy planarity value provides a new method to evaluate the edge intersection and order in the graph. The ability of the spherical fuzzy planar graph ($\mathcal{SFPG}$) to model complex relationships can enable more precise and reliable network designs and analyses. In this work, the concepts of spherical fuzzy multigraph ($\mathcal{SFMG}$), spherical strong and weak edges, and the value of planarity for spherical fuzzy graphs ($\mathcal{SFG}s$) are introduced. Moreover, the concept of the degree of planarity within the context of spherical fuzzy planar graphs and the notion of strong and weak faces are introduced. Additionally, we delve into the construction of spherical fuzzy dual graphs, which can be realized in cases where the fuzzy graph is planar or possesses a degree of planarity ≥0.67. This notion also serves as the foundation for certain basic theorems. We talk about some significant findings related to this subject. We discuss some major results linked to this topic. To show the worth and importance of our work, we also provide a real-world application.

KEYWORDS

spherical fuzzy multigraph, spherical fuzzy planar graph, spherical fuzzy planarity value, spherical fuzzy dual graph, fuzzy graph theory

1 Introduction

One of the most advanced fields of science is "graph theory," which plays a vital role in the applications of other branches of science like chemistry, biology, physics, electrical engineering, computer science, discrete mathematics, astronomy, and operations research. Graph theory research has experienced significant advancements recently due to its diverse range of applications. It is also helpful in image segmentation, networking, data mining, structuring, organizing, communication, etc. For instance, a data set can be represented graphically in the form of a model, like a tree containing vertices and edges. Similarly, the concept of a graph can be used to organize network design. In many different types of graph structure applications, including design problems for electrical transmission lines, utility lines, subways, circuits, and trains, crossing through edges can be problematic. It is actually essential to run paths of communication at different levels for it to cross them. Although the positioning of the electrical cables is not extremely difficult, the cost of some types of lines can increase if subway tunnels are constructed beneath them. Specifically, circuits with a few layers in their structure are simpler to produce. Planar graphs become a framework for these applications. All the abovementioned applications utilize the concept of planar graphs. Crossing has certain benefits, such as saving space and being affordable, but it also has some disadvantages. In city road planning, due to crossing, there are increased chances of accidents because of the increasing rush of vehicles day by day. Moreover, the expenditure on crossing routes underground is high, but traffic jams are reduced on underground routes. In urban planning, it would be protective for human lives not to cross the routes. However, due to a lack of space, such crossing of routes is permitted. Generally, we use such linguistic terms as "congested, "very congested," and "high congested". The word congested has no definite meaning. All the abovementioned terms have some membership degrees. The choice of navigating between a congested road and a non-congested one is more favorable than navigating between two congested routes. In a fuzzy planar graph, the "congested" edges would represent strong connections between vertices, possibly indicating a high level of interaction or influence, while the "low congested" edges would represent weaker connections, perhaps suggesting a lower level of interaction or influence.

These days, science and technology deal with complicated models for which appropriate data are limited. To deal with such a phenomenon, we use mathematical models to tackle different kinds of systems that contain elements of uncertainty and vagueness. The generalization of the ordinary set theory, namely, fuzzy sets, is the foundation for handling such types of models. The concept of the fuzzy set, introduced by Lotif A. Zadeh [1] in 1965, revolutionized how we handle ambiguity and partial data in various fields. Numerous implications for fuzzy sets extend the range of investigation areas across various educational fields. In 1983, Atanassov presented the idea of an intuitionistic fuzzy set ($\mathcal{IFS}$) to overcome the lack of fuzzy sets as it provides a degree of truthfulness ($\mathfrak{f}$) as well as degree of falseness ($\mathfrak{h}$) with the constraint that $\mathfrak{f} + \mathfrak{h} \leq 1$. To deal with such types of models, which require more space, $\mathcal{IFS}$ did not support. In order to address this requirement, Yager [2] introduced the concept of Pythagorean fuzzy sets ($\mathcal{P}\dagger\mathcal{FS}$), which extends the space by introducing additional limitations $0 \leq \mathfrak{f}^2 + \mathfrak{h}^2 \leq 1$. People often have a variety of opinions such as Yes, Abstain, No, and Refusal; a picture fuzzy set has been supported, as initiated by Cuong [3, 4], which is the generalization of $\mathcal{P}\dagger\mathcal{FS}$, which extends the space with the new limitations $0 \leq \mathfrak{f} + \mathfrak{g} + \mathfrak{h} \leq 1$, where $\mathfrak{f}:\mathcal{V} \to [0,1]$, $\mathfrak{g}:\mathcal{V} \to [0,1]$, and $\mathfrak{h}:\mathcal{V} \to [0,1]$ denote the degree of truthfulness, degree of abstinence, and degree of falseness, respectively. Furthermore, $\mathfrak{I} = 1 - (\mathfrak{f} + \mathfrak{g} + \mathfrak{h})$ represents the degree of refusal.

However, aspects such as being young, smart, tall, short, healthy, and successful in a certain field cannot be easily quantified. It is possible to express these qualitative and vague predicates by defining appropriate boundaries. To enlarge the space for uncertain and vague information, Gundogdu and Kahraman [5] expanded the concept of $\mathcal{P}\dagger\mathcal{FS}$ by introducing the notion of a spherical fuzzy set ($\mathcal{SFS}$) with the new limitations. $0 \leq \mathfrak{f}^2\mathfrak{g}^2 + \mathfrak{h}^2 \leq 1$, and $\mathfrak{I} = \sqrt{1 - (\mathfrak{f}^2 + \mathfrak{g}^2 + \mathfrak{h}^2)}$. The notion of $\mathcal{SFS}$ with various practical applications in decision-making problems was investigated by Ashraf et al. [6]. Ashraf et al. [7] likely extended the Dombi aggregation operators to the context of $\mathcal{SFS}$, providing a framework for aggregating information or making decisions in scenarios where uncertainty is represented using $\mathcal{SFS}$.

Based on Zadeh's fuzzy relation [8], Kaufmann [9] introduced the concept of fuzzy graphs in 1973. Then, in fuzzy graphs, various graph-based theoretical concepts were initiated by Rosenfeld [10]. In his study of fuzzy graphs, Bhattacharya [11] made lots of remarkable perspectives about how they differ from classical graph theory. He proved that not all ideas in the field of fuzzy graphs have a direct equivalent or parallel in classical graph theory. In Mordeson and Nair [12], the idea of the complement of a fuzzy graph was introduced. A few operations on fuzzy graphs were also presented. Subsequently, the original fuzzy graph was redefined as the complement of the complement, following this change in the definition of complement. Nagoorgani and Malarvizhi [13] introduced the notion of isomorphism on fuzzy graphs. One of the most important tools of graphing is its dual graph. Abdul-Jabbar et al. [14] introduced the concept of fuzzy dual graphs. Shannon and Atanassov [15] put forward the concepts of intuitionistic fuzzy relations and intuitionistic fuzzy graphs. Several operations on intuitionistic fuzzy graphs were introduced by Parvathi et al. [16]. Akram et al. [17–20] proposed some additional advanced concepts: intuitionistic fuzzy hypergraphs, intuitionistic fuzzy cycles, strong intuitionistic fuzzy graphs, and intuitionistic fuzzy trees. Alshehri and Akram [21] gave an idea of intuitionistic fuzzy planar graphs. The concept of Pythagorean fuzzy graphs $\mathcal{P}\dagger\mathcal{FG}$ was introduced by Naz et al. [22] along with some applications. Naz and Akram [23] presented the idea of the Pythagorean fuzzy energy of $\mathcal{P}\dagger\mathcal{FG}$. Moreover, some operations of $\mathcal{P}\dagger\mathcal{FG}$ were defined by Akram et al. [24]. Akram et al. [25] suggested many graphs in a Pythagorean fuzzy environment.

Akram et al. [26] developed the concept of a spherical fuzzy graph $\mathcal{SFG}$. In addition, some operators on $\mathcal{SFG}$, namely, symmetric difference and rejection, were defined. The term energy of $\mathcal{SFG}$ defined by Yahya and Mohamed [27] and some bounds of energy of $\mathcal{SFG}$ were extracted too. Yager [28] proposed the idea of the fuzzy multiset. Akram et al. [26] developed the concept of a $\mathcal{SFG}$. Recently, the idea of a fuzzy planar graph was introduced by Pal et al. [29] and Samanta et al. [30]. Furthermore, some properties were discussed too. Pramanik et al. [31] looked into a few unique planar fuzzy graphs. Moreover, planar fuzzy graphs have some extensions discussed in [32, 33]. Their ability to capture and represent uncertainty makes them a valuable tool in scenarios where traditional crisp graphs may fall short; refer [34–49].

The item that follows is the subscription to our suggested research project:

- This research project aims to introduce the concept of a $\mathcal{SFMG}$ based on a spherical fuzzy multiset.
- The concept of the degree of vertex in $\mathcal{SFMG}$ is defined and elaborated with an example.

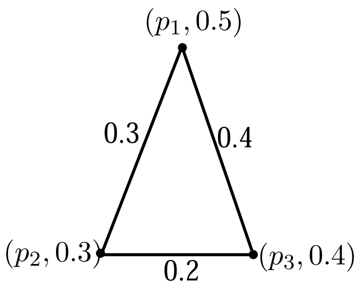

FIGURE 1
Fuzzy graph.

- An idea of a strong edge in $\mathcal{SFMG}$ as well as of complete $\mathcal{SFMG}$ is initiated.
- Under a spherical fuzzy environment, planar graphs are discussed, and the planarity of $\mathcal{SFPG}$ is defined.
- We discuss the concept of duality in $\mathcal{SFPG}$.
- Some results regarding $\mathcal{SFPG}$ as well as the planarity of $\mathcal{SFPG}$ are presented.
- Finally, to utilize an idea of $\mathcal{SFPG}$ in an MCDM problem.

2 Motivation

The motivation behind spherical fuzzy sets lies in their ability to provide a richer, more flexible, and accurate framework for handling uncertainty. By addressing the limitations of classical fuzzy sets and $\mathcal{IFS}s$, $\mathcal{SFS}s$ open new avenues for research and application in various fields, ultimately leading to reliable and effective decision-making processes. Despite significant progress in the study of $\mathcal{SFS}s$, there has been limited effort directed toward the exploration and development of $\mathcal{SFG}s$. Inspired by the potential of fuzzy planar graphs and intuitionistic planar graphs, we developed the concept of $\mathcal{SFPG}s$. This paper is organized as follows.

In Section 3, some basic definitions are presented. In Section 4, we define the concept of $\mathcal{SFMG}$, degree of the vertex in $\mathcal{SFMG}$, strong edge in $\mathcal{SFMG}$, complete $\mathcal{SFMG}$, strength of the spherical fuzzy edge, planar spherical fuzzy graph, strong $\mathcal{SFPG}$, and strong and weak spherical fuzzy faces of $\mathcal{SFPG}$ illustrated with examples. In addition, some results regarding planarity are also present. In Section 5, we introduce the notion of $\mathcal{SFDG}$ of $\mathcal{SFPG}$ and illustrate with the example. In Section 6, the application of $\mathcal{SFPG}$ is presented. In Section 8, we put our recommended task to its conclusion.

Some symbols and notations are used, which are presented in Table 2 along with their meanings.

Symbol	Meaning
$\mathcal{V}$	An underlying set of vertices
$\mathfrak{f}$	Degree of truthfulness
$\mathfrak{g}$	Degree of abstinence
$\mathfrak{h}$	Degree of falseness
$\mathfrak{I}$	Degree of refusal
$\mathcal{A}$	Spherical fuzzy multiset.
$\mathcal{B}$	Spherical fuzzy multiedge set.
T_X	Intersecting value at the point X
R	Spherical fuzzy planarity value
Notation	**Meaning**
$deg(v)$	Degree of the vertex
$\mathcal{SFG}$	Spherical fuzzy graphs
$\mathcal{SFMG}$	Spherical fuzzy multigraph
$\mathcal{SFPG}$	Spherical fuzzy planar graph
$\mathcal{SFF}$	Spherical fuzzy face
$\mathcal{SFDG}$	Spherical fuzzy dual graph

3 Preliminaries

Definition 3.1: [1] Let $\mathcal{V}$ be an underlying set of vertices. A fuzzy set M is characterized by a membership function $\xi:\mathcal{V}\to[0,1]$ and is defined as $M=\{<p,\xi_M(p)>:v\in\mathcal{V}\}$. The fuzzy binary relation is a fuzzy subset ξ on $\mathcal{V}\times\mathcal{V}$ given as $\eta:\mathcal{V}\times\mathcal{V}\to[0,1]$. A fuzzy graph $\vec{\mathcal{G}}=(\mathcal{V},\xi,\eta)$ is a pair of mappings $\xi:\mathcal{V}\to[0,1]$ and $\eta:\mathcal{V}\times\mathcal{V}\to[0,1]$, such that $\eta(p_1,p_2)\le\min\{\xi(p_1),\xi(p_2)\}$ for all $p_1,p_2\in\mathcal{V}$, where $\xi(p)$ and $\eta(p_1,p_2)$ denote the membership degrees of the vertex and of the edge (p_1,p_2) in $\vec{\mathcal{G}}$, respectively.

Example 3.2: Given the set $\mathcal{V}=\{p_1,p_2,p_3\}$, the fuzzy set and fuzzy relation are defined as follows: the fuzzy set ξ is

$$\xi=\{(p_1,0.5),(p_2,0.3),(p_3,0.4)\}$$

The fuzzy relation η is

$$\eta=\{\langle(p_1,p_2),0.3\rangle,\langle(p_1,p_3),0.4\rangle,\langle(p_2,p_3),0.2\rangle\}.$$

The graph is shown in Figure 1.

Definition 3.3: [28] Let $\mathcal{V}$ be a non-empty set of vertices. A fuzzy set M is said to be a fuzzy multiset if the fuzzy set M is characterized by a membership function named as count membership, such that $\mathfrak{f}:\mathcal{V}\to L$, where L is defined as the collection of every crisp multiset taken from the interval [0, 1]. The value of $\mathfrak{f}(p)$ in L is a crisp multiset taken from [0, 1] for all $p\in\mathcal{V}$. Furthermore, the entries $\mathfrak{f}^1(p)$, $\mathfrak{f}^2(p)$, $\mathfrak{f}^3(p),\ldots,\mathfrak{f}^s(p)$ in $\mathfrak{f}(p)$ construct a decreasingly ordered sequence, i.e., $\mathfrak{f}^1(p)\ge\mathfrak{f}^2(p)\ge\mathfrak{f}^3(p)\cdots\ge\mathfrak{f}^s(p)$, for all $v\in\mathcal{V}$.

Definition 3.4: Let $\xi:\mathcal{V}\to[0,1]$ be a mapping on an underlying set of vertices $\mathcal{V}$ and $\mathcal{S}=\{(p_1,p_2),\eta(p_1,p_2)_k,\ k=1,2,3,\ldots,q_{p_1p_2}\,|\,(p_1,p_2)\in\mathcal{V}\times\mathcal{V}\}$ be a fuzzy multiset of $\mathcal{V}\times\mathcal{V}$, such that $\eta(p_1,p_2)_k\le\min\{\xi(p_1),\xi(p_2)\}$ for all $k=1,2,\ldots,q_{p_1p_2}$, where

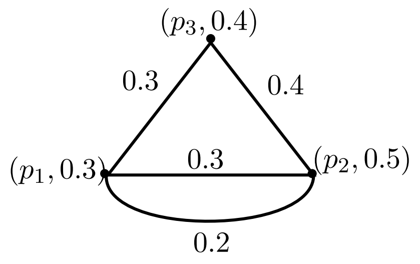

FIGURE 2
Fuzzy multigraph.

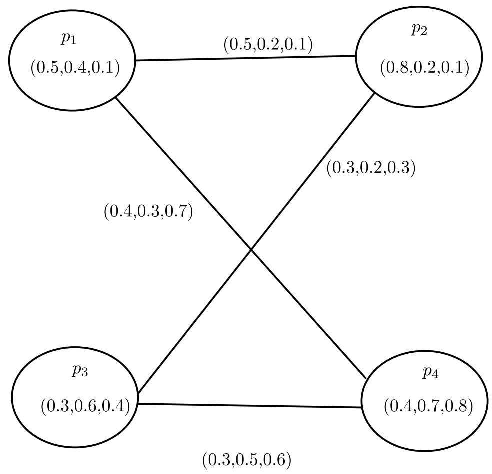

FIGURE 3
Spherical fuzzy graph $\mathbb{S}$.

$q_{p_1p_2} = \max\{k \mid \eta(p_1,p_2)_k \neq 0\}$. Then, $\tilde{\mathcal{G}} = (\mathcal{V},\xi,\mathcal{S})$ is called a fuzzy multigraph, where $\xi(p)$ and $\eta(p_1,p_2)_k$ denote the membership degrees of the vertex p and of the edge (p_1,p_2) in $\tilde{\mathcal{G}}$, respectively.

Example 3.5: Let $\mathcal{V} = \{p_1,p_2,p_3\}$ be a non-empty set. Then, a fuzzy multiset is given in Equation 1

$$\xi = \{(p_1,0.3),(p_2,0.5),(p_3,0.4)\}.. \tag{1}$$

The fuzzy multigraph is shown in Figure 2. There are two edges between p_1 and p_2, which is called the fuzzy multigraph.

Definition 3.6: [26] Let the underlying set of vertices be $\mathcal{V}$. A spherical fuzzy set T on a universe $\mathcal{V}$ is an object having a form as

$$T = \{< p, \mathfrak{f}_T(p), \mathfrak{g}_T(p), \mathfrak{h}_T(p) > \mid p \in \mathcal{V}\},$$

where $\mathfrak{f}_T(p) \in [0,1]$ denote the degree of truthfulness of p in T, $\mathfrak{g}_T(p) \in [0,1]$ denote the degree of abstinence of p in T, and $\mathfrak{h}_T(p) \in [0, 1]$ denote the degree of falseness of p in T with the following condition

$$0 \leq \mathfrak{f}_T^2(p) + \mathfrak{g}_T^2(p) + \mathfrak{h}_T^2(p) \leq 1.$$

Moreover, $\mathfrak{I}_T(p) = \sqrt{1-(\mathfrak{f}_T^2(p)+\mathfrak{g}_T^2(p)+\mathfrak{h}_T^2(p))}$ is the degree of refusal of p in T for all $p \in \mathcal{V}$. A spherical fuzzy relation on $\mathcal{V}\times\mathcal{V}$.

Example 3.7: Let $A = \{p, \mathfrak{f}_A(p), \mathfrak{g}_A(p), \mathfrak{h}_A(p) \mid p \in \mathcal{V}\}$ and $B = \{(p_1,p_2), \mathfrak{f}_B(p_1,p_2)_i, \mathfrak{g}_B(p_1,p_2)_i, \mathfrak{h}_B(p_1,p_2)_i \mid (p_1,p_2) \in \mathcal{V}\times\mathcal{V}\}$ for all $i = 1,2,\ldots,k$ be a spherical fuzzy set and spherical fuzzy edge set in Spherical fuzzy graph shown in Figure 3, and defined by Tables 1, 2, respectively.

Definition 3.8: [26] Let the underlying set of vertices be $\mathcal{V}$. A spherical fuzzy multiset U is characterized by functions named as "count truthness membership" of $P(CT_U)$, "count abstinence membership" of $P(CA_U)$, and "count falseness membership" of $P(CF_U)$ given by $CT_U{:}\mathcal{V}\to W$, $CA_U{:}\mathcal{V}\to W$, and $CF_U{:}\mathcal{V}\to W$, respectively, where W is the set of all crisp multisets drawn from the unit interval [0, 1], such that, for every $p \in \mathcal{V}$, the truthfulness membership sequence and abstinence membership sequence denoted as $(\mathfrak{f}_U^1(p), \mathfrak{f}_U^2(p), \mathfrak{f}_U^3(p), \ldots, \mathfrak{f}_U^r(p))$ and $(\mathfrak{g}_U^1(p), \mathfrak{g}_U^2(p), \mathfrak{g}_U^3(p), \ldots, \mathfrak{g}_U^r(p))$, respectively, construct decreasing sequences. Similarly, the terms of $CF_U(p)$ form a sequence represented as $(\mathfrak{h}_U^1(p), \mathfrak{h}_U^2(p), \mathfrak{h}_U^3(p), \ldots, \mathfrak{h}_U^r(p))$ such that $0 \leq \mathfrak{f}_U^{2(i)}(p) + \leq \mathfrak{g}_U^{2(i)}(p) + \leq \mathfrak{h}_U^{2(i)}(p) \leq 1$; $i = 1,2,3,\ldots,r$. An $\mathcal{SFMS}$ U is denoted by

$$\Big\{ < p, \left(\mathfrak{f}_U^1(p), \mathfrak{f}_U^2(p), \mathfrak{f}_U^3(p), \ldots, \mathfrak{f}_U^r(p)\right), \left(\mathfrak{g}_U^1(p), \mathfrak{g}_U^2(p), \mathfrak{g}_U^3(p), \ldots, \mathfrak{g}_U^r(p)\right), \left(\mathfrak{h}_U^1(p), \mathfrak{h}_U^2(p), \mathfrak{h}_U^3(p), \ldots, \mathfrak{h}_U^r(p)\right) > : p \in \mathcal{V} \Big\}.$$

The orders of membership for truthfulness and abstinence are listed in descending order, but the sequence for falsehood may not follow the same order.

4 Spherical fuzzy planar graphs

Definition 4.1: [26] Let the underlying set of vertices be $\mathcal{V}$ and A be a spherical fuzzy multiset on $\mathcal{V}$; further, let B be a spherical fuzzy multiset of $\mathcal{V}\times\mathcal{V}$ such that

$$\begin{aligned}
\mathfrak{f}_B(p_1,p_2)_i &\leq \min\{\mathfrak{f}_A(p_1), \mathfrak{f}_A(p_2)\} \\
\mathfrak{g}_B(p_1,p_2)_i &\leq \min\{\mathfrak{g}_A(p_1), \mathfrak{g}_A(p_2)\} \\
\mathfrak{h}_B(p_1,p_2)_i &\leq \max\{\mathfrak{h}_A(p_1), \mathfrak{h}_A(p_2)\},
\end{aligned}$$

for all $i = 1,2,\ldots,k$. Then, $\tilde{\mathcal{G}}$ is called $\mathcal{SFMG}$. By noting that, between two vertices (say) p_1 and p_2, more than one edge may be present. Furthermore, $\mathfrak{f}_B(p_1,p_2)_i$ denotes the degree of truthfulness, $\mathfrak{g}_B(p_1,p_2)_i$ denotes the degree of abstinence, and $\mathfrak{h}_B(p_1,p_2)_i$ denotes the degree of falseness of an edge (p_1,p_2) and i represents the number of edges between two vertices.

Example 4.2: In Figure 4, a $\mathcal{SFMG}$ $\tilde{\mathcal{G}} = (\mathcal{V},\mathcal{E})$ such that $\mathcal{V} = \{p_1,p_2,p_3,p_4\}$ and $\mathcal{E} = \{p_1p_2, p_1p_3, p_1p_4, p_1p_4, p_2p_3, p_2p_4, p_3p_4, p_3p_4\}$. Let $A = \{p, \mathfrak{f}_A(p), \mathfrak{g}_A(p), \mathfrak{h}_A(p) \mid p \in \mathcal{V}\}$ and $B = \{(p_1,p_2), \mathfrak{f}_B(p_1,p_2)_i, \mathfrak{g}_B(p_1,p_2)_i, \mathfrak{h}_B(p_1,p_2)_i \mid (p_1,p_2) \in \mathcal{V}\times\mathcal{V}$ for all $i = 1,2,\ldots,k$ be $\mathcal{SFS}$ and $\mathcal{SFMS}$ defined by Table 3 and Table 4, respectively.

Definition 4.3: Let A be $\mathcal{SFMS}$ and B be a spherical fuzzy multiedge set in $\mathcal{SFMG}$. The degree of a vertex $p_1 \in \mathcal{V}$ is

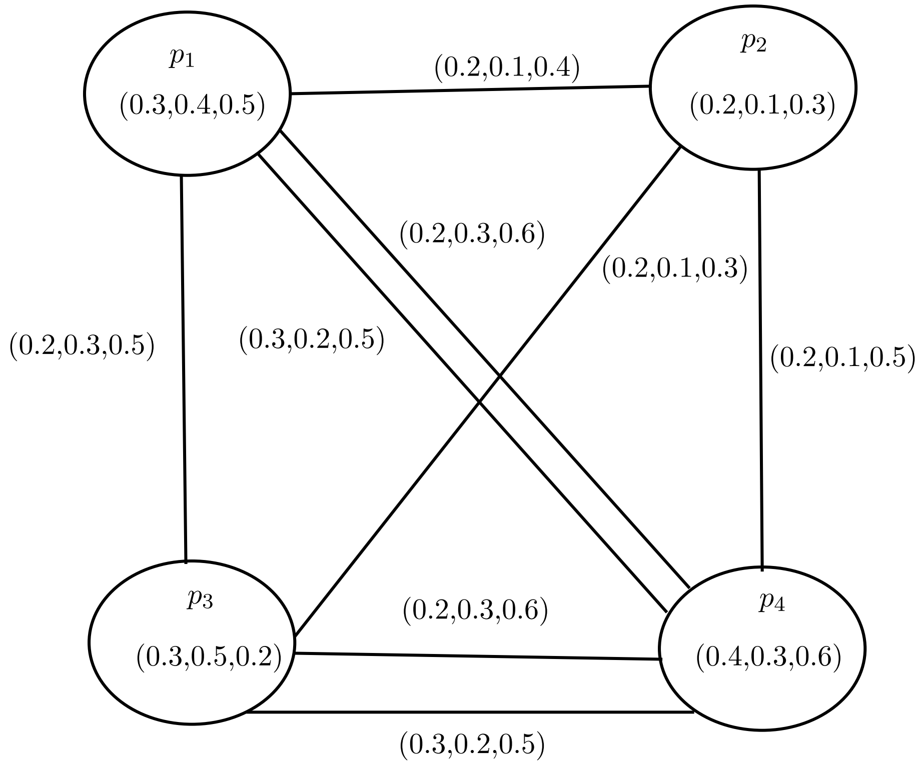

FIGURE 4
$\mathcal{SFMG}$ $\mathbb{C}$.

TABLE 1 Spherical fuzzy set *A*.

	p_1	p_2	p_3	p_4
$\mathfrak{f}$	0.5	0.8	0.3	0.4
$\mathfrak{g}$	0.4	0.2	0.6	0.7
$\mathfrak{h}$	0.1	0.1	0.4	0.8

TABLE 2 Spherical fuzzy edge set *B*.

	p_1p_2	p_2p_3	p_1p_4	p_3p_4
$\mathfrak{f}$	0.5	0.3	0.3	0.4
$\mathfrak{g}$	0.2	0.5	0.2	0.3
$\mathfrak{h}$	0.1	0.6	0.3	0.7

represented by $deg(p_1)$ and is defined as

$$deg(p_1) = \left(\sum_{i=1}^{k} \mathfrak{f}_B(p_1,p_2)_i, \sum_{i=1}^{k} \mathfrak{g}_B(p_1,p_2)_i, \sum_{i=1}^{k} \mathfrak{h}_B(p_1,p_2)_i \right),$$

TABLE 3 $\mathcal{SFS}$ *A*.

	p_1	p_2	p_3	p_4
$\mathfrak{f}$	0.3	0.2	0.3	0.4
$\mathfrak{g}$	0.4	0.1	0.5	0.3
$\mathfrak{h}$	0.5	0.3	0.2	0.6

TABLE 4 $\mathcal{SFMS}$ *B*.

	p_1p_2	p_1p_3	p_1p_4	p_1p_4	p_2p_3	p_2p_4	p_3p_4	p_3p_4
$\mathfrak{f}$	0.2	0.2	0.3	0.2	0.2	0.2	0.3	0.2
$\mathfrak{g}$	0.1	0.3	0.2	0.3	0.1	0.1	0.2	0.3
$\mathfrak{h}$	0.4	0.5	0.5	0.6	0.3	0.5	0.5	0.6

for all $p_1,p_2 \in \mathcal{V}$.

Example 4.4: In Example 4.2, the degrees of the vertices of $\mathcal{SFMG}$ $\mathbb{C}$ are computed as follows:

- $\deg(p_1)$=(0.9,0.9,2.0).
- $\deg(p_2)$=(0.6,0.3,1.2).

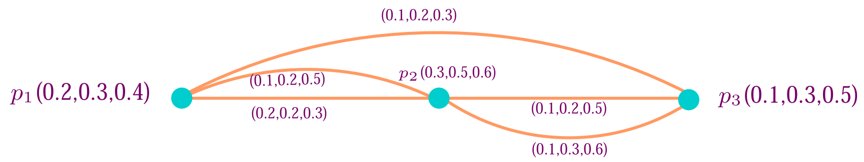

FIGURE 5
$\mathcal{SFMG}\ \mathcal{J}$.

TABLE 5 $\mathcal{SFS}\ \mathcal{X}$.

	p_1	p_2	p_3
$\mathfrak{f}$	0.2	0.3	0.1
$\mathfrak{g}$	0.3	0.5	0.3
$\mathfrak{h}$	0.4	0.6	0.5

TABLE 6 Spherical fuzzy multiedge set $\mathcal{Y}$.

	p_1p_2	p_1p_2	p_1p_3	p_2p_3	p_2p_3
$\mathfrak{f}$	0.1	0.2	0.1	0.1	0.1
$\mathfrak{g}$	0.2	0.2	0.2	0.2	0.3
$\mathfrak{h}$	0.5	0.3	0.3	0.5	0.6

- deg (p_3)=(0.9,0.9,1.9).
- deg(p_4)=(1.2,1.1,2.7).

Definition 4.5: Let a spherical fuzzy multiedge set B be defined as

$$B=\{((p_1,p_2),\mathfrak{f}_B(p_1,p_2)_i,\mathfrak{g}_B(p_1,p_2)_i,\mathfrak{h}_B(p_1,p_2)_i)\ i=1,\ldots,k|\ (p_1,p_2)\in\mathcal{V}\times\mathcal{V}\}$$

in $\mathcal{SFMG}$. A multiedge (p_1,p_2) of $\mathcal{SFMG}$ is strong if

$$\frac{1}{2}\min\{\mathfrak{f}_A(p_1),\mathfrak{f}_A(p_2)\}\ \le \mathfrak{f}_B(p_1,p_2)_i,$$
$$\frac{1}{2}\min\{\mathfrak{g}_A(p_1),\mathfrak{g}_A(p_2)\}\le \mathfrak{h}_B(p_1,p_2)_i$$

and

$$\frac{1}{2}\max\{\mathfrak{h}_A(p_1),\mathfrak{h}_A(p_2)\}\le \mathfrak{h}_B(p_1,p_2)_i,$$

for all $i=1,2,\ldots,k$ and for all $p_1,p_2\ \in\ \mathcal{V}$.

Example 4.6: Consider a $\mathcal{SFMG}\ \mathcal{J}=(\mathcal{X},\mathcal{Y})$ as shown in Figure 5, where $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{SFS}$ and spherical fuzzy multiedge set are defined in Table 5 and in Table 6, respectively.

By direct computations, we can easily summarize that in this $\mathcal{SFMG}\ \mathcal{J}$, these are (p_1p_2), (p_2p_3) strong edges.

Definition 4 7: Let

$$B=\{((p_1,p_2),\mathfrak{f}_B(p_1,p_2)_i,\mathfrak{g}_B(p_1,p_2)_i,\mathfrak{h}_B(p_1,p_2)_i)\ i=1,\ldots,k|\ (p_1,p_2)\in\mathcal{V}\times\mathcal{V}\},$$

be a spherical fuzzy multiedge set in $\mathcal{SFMG}\ \vec{\mathcal{G}}$. A $\mathcal{SFMG}\ \vec{\mathcal{G}}$ is complete if

$$\min\{\mathfrak{f}_A(p_1),\mathfrak{f}_A(p_2)\}=\mathfrak{f}_B(p_1,p_2)_i,$$
$$\min\{\mathfrak{g}_A(p_1),\mathfrak{g}_A(p_2)\}=\mathfrak{g}_B(p_1,p_2)_i,$$
$$\max\{\mathfrak{h}_A(p_1),\mathfrak{h}_A(p_2)\}=\mathfrak{h}_B(p_1,p_2)_i,$$

for all i = 1, 2, …, k and for all $p_1,p_2\in\mathcal{V}$.

Example 4.8: Let $\mathrm{H}=(\mathrm{D},\mathrm{R})$ be a $\mathcal{SFMG}$ as shown in Figure 6, where D and R are $\mathcal{SFS}$ and spherical fuzzy multiedge set defined in Table 7 and in Table 8, respectively.

Using basic calculations, from Figure 6, it is clear that it is complete $\mathcal{SFMG}$.

Definition 4.9: Strength of the spherical fuzzy edge (p_1,p_2) is denoted by

$$I_{(p_1,p_2)}=\left(M_{(p_1,p_2)},N_{(p_1,p_2)},O_{(p_1,p_2)}\right)$$

and can be calculated as given in Equation 2.

$$\left(\frac{\mathfrak{f}_B(p_1,p_2)_i}{\min(\mathfrak{f}_A(p_1),\mathfrak{f}_A(p_2))},\frac{\mathfrak{g}_B(p_1,p_2)_i}{\min(\mathfrak{g}_A(p_1),\mathfrak{g}_A(p_2))},\frac{\mathfrak{h}_B(p_1,p_2)_i}{\max(\mathfrak{h}_A(p_1),\mathfrak{h}_A(p_2))}\right),\tag{2}$$

where i = 1, 2, …, k and for all $(p_1,p_2)\in\mathcal{V}\times\mathcal{V}$.

Example 4.10: Let $\mathcal{L}=(\mathcal{U},\mathcal{P})$ be a $\mathcal{SFMG}$ as shown in Figure 7, where $\mathcal{U}$ and $\mathcal{P}$ are $\mathcal{SFS}$ and spherical fuzzy multiedge set defined in Table 9 and in Table 10, respectively.

From Equation 2, the strength of the edges is computed as follows.

- For an edge $(p_1p_2,0.1,0.1,0.2)$, $I_{(p_1,p_2)}$=(1, 0.5, 0.6).
- For an edge $(p_1p_2,0,0.1,0.1)$, $I_{(p_1,p_2)}$=(0, 0.5, 0.3).
- For an edge $(p_1p_2,0.1,0.2,0.3)$, $I_{(p_1,p_2)}$=(1, 1, 1).
- For an edge $(p_2p_3,0.1,0.2,0.4)$, $I_{(p_2,p_3)}$=(1, 1, 0.8).
- For an edge $(p_2p_4,0.1,0.2,0.2)$, $I_{(p_2,p_4)}$=(1, 1, 0.6).

Definition 4.11: Let $\vec{\mathcal{G}}$ be a $\mathcal{SFMG}$. A spherical fuzzy strong edge is an edge (p_1,p_2) if $M_{(p_1,p_2)}\ \ge 0.5$, $N_{(p_1,p_2)}$ or $O_{(p_1,p_2)}$, otherwise weak.

Definition 4.12: Let $\vec{\mathcal{G}}=(A,B)$ be a $\mathcal{SFMG}$, and the spherical fuzzy multiedge set B contains two edges $((p_1,p_2),\mathfrak{f}_B(p_1,p_2)_a,\mathfrak{g}_B(p_1,p_2)_a,\mathfrak{h}_B(p_1,p_2)_a)$ and $((p_3,p_4),\mathfrak{f}_B(p_3,p_4)_b,\mathfrak{g}_B(p_3,p_4)_b,\mathfrak{h}_B(p_3,p_4)_b)$ which are intersected

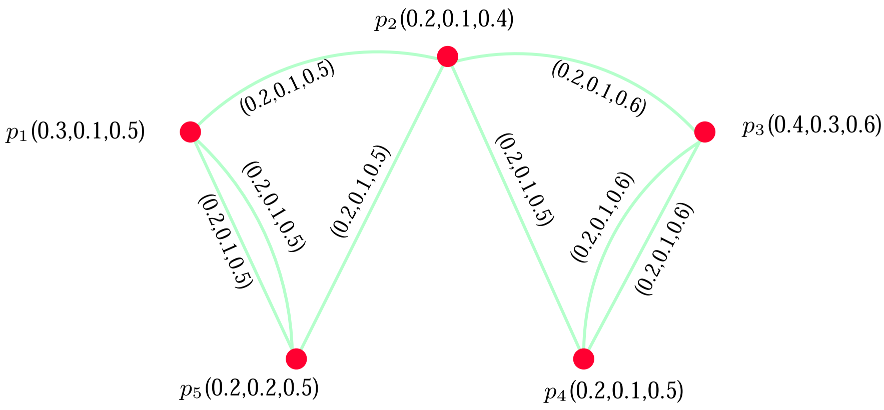

FIGURE 6
Complete $\mathcal{SFMG}$ H.

TABLE 7 $\mathcal{SFS}$ D.

	p_1	p_2	p_3	p_4	p_5
$\mathfrak{f}$	0.3	0.2	0.4	0.2	0.2
$\mathfrak{g}$	0.1	0.1	0.3	0.1	0.2
$\mathfrak{h}$	0.5	0.4	0.6	0.5	0.5

TABLE 8 Spherical fuzzy multiedge set R.

	p_1p_2	p_1p_5	p_1p_5	p_2p_3	p_2p_4	p_2p_5	p_3p_4	p_3p_4
$\mathfrak{f}$	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
$\mathfrak{g}$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
$\mathfrak{h}$	0.5	0.5	0.5	0.6	0.5	0.5	0.6	0.6

at a point X, where a and b are fixed integers. The intersecting value at the point X is defined as given in Equation 3.

$$T_X = (M_X, N_X, O_X),$$

$$T_X = \left(\frac{M_{(p_1,p_2)} + M_{(p_3,p_4)}}{2}, \frac{N_{(p_1,p_2)} + N_{(p_3,p_4)}}{2}, \frac{O_{(p_1,p_2)} + O_{(p_3,p_4)}}{2} \right). \tag{3}$$

If the number of intersecting points increases, then the value of the planarity decreases. Thus, for $\mathcal{SFMG}$ T_X is inversely proportional to the value of spherical planarity.

Example 4.13: Let $\mathfrak{S} = (\mathfrak{M}, \mathfrak{N})$ be a $\mathcal{SFMG}$, as shown in Figure 8, where $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathcal{SFS}$ and spherical fuzzy multiedge set, which are defined in Table 11 and in Table 12, respectively, as below.

The intersecting value at the point X is given by Equation 4.

$$T_X = (M_X, N_X, O_X),$$

$$T_X = \left(\frac{M_{(p_1,p_3)} + M_{(p_2,p_4)}}{2}, \frac{N_{(p_1,p_3)} + N_{(p_2,p_4)}}{2}, \frac{O_{(p_1,p_3)} + O_{(p_2,p_4)}}{2} \right). \tag{4}$$

For an edge $((p_1, p_3), 0.3, 0.1, 0.3)$, we can write from Equation 2 as follows:

$$\left(M_{(p_1,p_3)}, N_{(p_1,p_3)}, O_{(p_1,p_3)}\right) = \left(\frac{\mathfrak{f}(p_1,p_3)}{\min\{\mathfrak{f}(p_1), \mathfrak{f}(p_3)\}}, \frac{\mathfrak{g}(p_1,p_3)}{\min\{\mathfrak{g}(p_1), \mathfrak{g}(p_3)\}}, \frac{\mathfrak{h}(p_1,p_3)}{\max\{\mathfrak{h}(p_1), \mathfrak{h}(p_3)\}} \right).$$

By substituting the values from Tables 9, 10, we get Equation 5.

$$\left(M_{(p_1,p_3)}, N_{(p_1,p_3)}, O_{(p_1,p_3)}\right) = (0.75, 0.5, 0.75). \tag{5}$$

Similarly, for an edge $((p_2, p_4), 0.1, 0.2, 0.2)$,

$$\left(M_{(p_2,p_4)}, N_{(p_2,p_4)}, O_{(p_2,p_4)}\right) = \left(\frac{\mathfrak{f}(p_2,p_4)}{\min\{\mathfrak{f}(p_2), \mathfrak{f}(p_4)\}}, \frac{\mathfrak{g}(p_2,p_4)}{\min\{\mathfrak{g}(p_2), \mathfrak{g}(p_4)\}}, \frac{\mathfrak{h}(p_2,p_4)}{\max\{\mathfrak{h}(p_2), \mathfrak{h}(p_4)\}} \right).$$

By putting the values from Tables 9, 10, we get Equation 6.

$$\left(M_{(p_2,p_4)}, N_{(p_2,p_4)}, O_{(p_2,p_4)}\right) = (0.5, 1, 0.66). \tag{6}$$

Substituting the values from Equations 8, 9 in Equation 7, we get the intersecting value as

$$T_X = (0.625, 0.75, 0.705).$$

Definition 4.14: For $\mathcal{SFMG}$, $J_1 = (R_{M_1}, R_{N_1}, R_{O_1})$, $J_2 = (R_{M_2}, R_{N_2}, R_{O_2})$, …, $J_u = (R_{M_u}, R_{N_u}, R_{O_u})$ are intersecting points between the edges of $\vec{\mathcal{G}}$; $\vec{\mathcal{G}}$ is said to be $\mathcal{SFPG}$ having the value of spherical planarity defined as

$$R = (R_M, R_N, R_O),$$

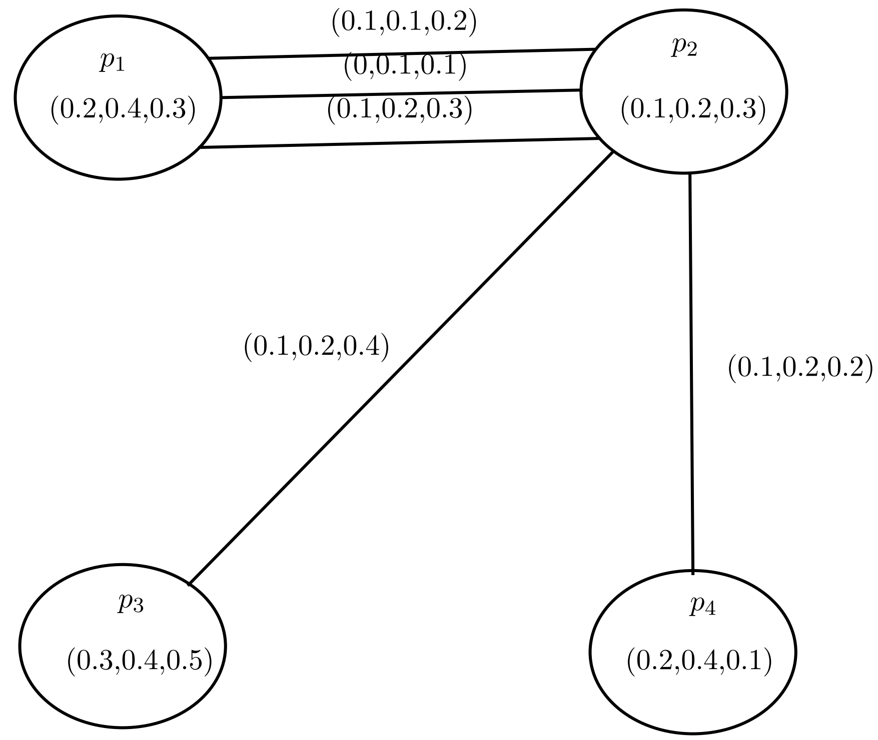

FIGURE 7
$\mathcal{SFMG}$ $\mathcal{L}$.

TABLE 9 $\mathcal{SFS}$ $\mathcal{U}$.

	p_1	p_2	p_3	p_4
$\mathfrak{f}$	0.2	0.1	0.3	0.2
$\mathfrak{g}$	0.4	0.2	0.4	0.4
$\mathfrak{h}$	0.3	0.3	0.5	0.1

TABLE 10 Spherical fuzzy multiedge set $\mathcal{P}$.

	p_1p_2	p_1p_2	p_1p_2	p_2p_3	p_2p_4
$\mathfrak{f}$	0.1	0	0.1	0.1	0.1
$\mathfrak{g}$	0.1	0.1	0.2	0.2	0.2
$\mathfrak{h}$	0.2	0.1	0.3	0.4	0.2

$$\left(\frac{1}{1+\left\{R_{M_1}+R_{M_2}+R_{M_3}+,,,+R_{M_n}\right\}},\frac{1}{1+\left\{R_{N_1}+R_{N_2}+R_{N_3}+,,,+R_{N_n}\right\}},\frac{1}{1+\left\{R_{O_1}+R_{O_2}+R_{O_3}+,,,+R_{O_n}\right\}}\right).$$

Vividly, $R=(R_M,R_N,R_O)$ is bounded, and $0<R_M\leq 1$; $0<R_N\leq 1$; and $0<R_O\leq 1$. For certain geometrical representations of a $\mathcal{SFPG}$, if there is no intersecting points, then its value of spherical planarity is (1, 1, 1). Thus, the crisp graph of this $\mathcal{SFMG}$ becomes a crisp planar graph. If R_M decreases and R_N increases, the number of intersecting points between edges increases and the nature of spherical planarity decreases. Conversely, if R_M increases and R_N decreases, the number of intersecting points between edges decreases and the nature of spherical planarity increases. We summarize that every $\mathcal{SFMG}$ is a $\mathcal{SFPG}$ with a certain value of spherical planarity.

Example 4.15: An $\mathcal{G}=(\xi,\eta)$ is said to be a crisp graph where $\xi=\{e,f,g,h\}$ and $\eta=\{(e,f),(e,g),(f,g),(f,h),(g,h),(h,e)\}$. Then, $\mathcal{SPG}$, $G^*=(E,F)$ as shown in Figure 9.

The intersecting value at the point X is given by

$$T_X=(M_X,N_X,O_X),$$

$$T_X=\left(\frac{M_{(e,g)}+M_{(f,h)}}{2},\frac{N_{(e,g)}+N_{(f,h)}}{2},\frac{O_{(e,g)}+O_{(f,h)}}{2}\right). \tag{7}$$

For an edge $((e,g),0.5,0.3,0.4)$, we can write from Equation 2 as follows:

$$\left(M_{(e,g)},N_{(e,g)},O_{(e,g)}\right)$$
$$=\left(\frac{\mathfrak{f}(e,g)}{\min\{\mathfrak{f}(e),\mathfrak{f}(g)\}},\frac{\mathfrak{g}(e,g)}{\min\{\mathfrak{g}(e),\mathfrak{g}(g)\}},\frac{\mathfrak{h}(e,g)}{\max\{\mathfrak{h}(e),\mathfrak{h}(g)\}}\right).$$

By substituting the values from Tables 13, 14, we get

$$\left(M_{(e,g)},N_{(e,g)},O_{(e,g)}\right)=(0.714,0.75,0.66). \tag{8}$$

Similarly, for an edge $((f,h),0.4,0.3,0.6)$,

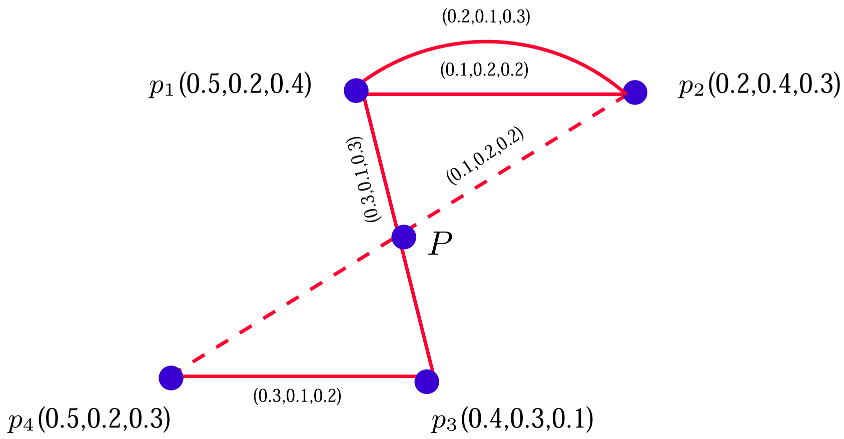

FIGURE 8
$\mathcal{SPMG}$ $\mathfrak{G}$.

TABLE 11 $\mathcal{SFS}$ $\mathfrak{M}$.

	p_1	p_2	p_3	p_4
$\mathfrak{f}$	0.5	0.2	0.4	0.5
$\mathfrak{g}$	0.2	0.4	0.3	0.2
$\mathfrak{h}$	0.4	0.3	0.1	0.3

TABLE 12 Spherical fuzzy multiedge set $\mathfrak{N}$.

	(p_1,p_2)	(p_1,p_2)	(p_1,p_3)	(p_2,p_4)	(p_3,p_4)
$\mathfrak{f}$	0.1	0.2	0.3	0.1	0.3
$\mathfrak{g}$	0.2	0.1	0.1	0.2	0.1
$\mathfrak{h}$	0.2	0.3	0.3	0.2	0.2

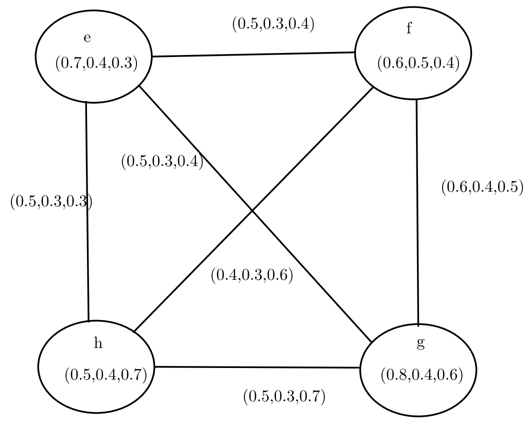

FIGURE 9
$\mathcal{SFPG}$ $\mathcal{G}$.

$$\left(M_{(f,h)},N_{(f,h)},O_{(f,h)}\right)=\left(\frac{\mathfrak{f}(f,h)}{\min\{\mathfrak{f}(f),\mathfrak{f}(h)\}},\frac{\mathfrak{g}(f,h)}{\min\{\mathfrak{g}(f),\mathfrak{g}(h)\}},\frac{\mathfrak{h}(f,h)}{\max\{\mathfrak{h}(f),\mathfrak{h}(h)\}}\right).$$

By putting the values from Tables 13, 14, we get

$$\left(M_{(f,h)},N_{(f,h)},O_{(f,h)}\right)=(0.8,0.75,0.86). \tag{9}$$

Substituting the values from Equations 8, 9 in Equation 7, we get intersecting value as

$$T_X=(0.76,0.75,0.76),$$

$$R_X=\left(\frac{1}{1+\{M_1+M_2+M_3+,,,+M_u\}},\frac{1}{1+\{N_1+N_2+N_3+,,,+N_u\}},\frac{1}{1+\{O_1+O_2+O_3+,,,+O_u\}}\right),$$

$$R_X=\left(\frac{1}{1+0.76},\frac{1}{1+0.75},\frac{1}{1+0.76}\right),$$

$$R_X=(0.56,0.571,0.56).$$

Theorem 4.16: Let $\vec{\mathcal{G}}=(\mathrm{A},\mathrm{B})$ be complete $\mathcal{SFPG}$. Then, the spherical planarity value R of $\vec{\mathcal{G}}$ is defined by $R=(R_M,R_N,R_O)$ where $R_M=\frac{1}{1+J_n}$, having a value less than 1, $R_N=\frac{1}{1+J_n}$ and $R_O=\frac{1}{1+J_n}$, where J_n is the number of intersecting points for edges in $\vec{\mathcal{G}}$.

Proof. For complete $\mathcal{SFMG}$, we have

$$\begin{aligned}\mathfrak{f}_{\mathrm{B}}(p_1,p_2)&=\min\{\mathfrak{f}_{\mathrm{A}}(p_1),\mathfrak{f}_{\mathrm{A}}(p_2)\}\\ \mathfrak{g}_{\mathrm{B}}(p_1,p_2)&=\min\{\mathfrak{g}_{\mathrm{A}}(p_1),\mathfrak{g}_{\mathrm{A}}(p_2)\}\\ \mathfrak{h}_{\mathrm{B}}(p_1,p_2)&=\max\{\mathfrak{h}_{\mathrm{A}}(p_1),\mathfrak{h}_{\mathrm{A}}(p_2)\}\end{aligned}$$

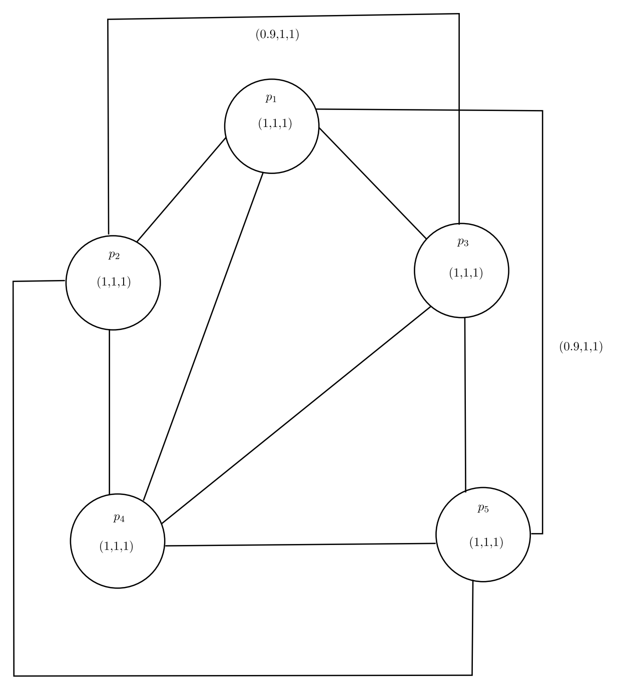

FIGURE 10
Example of strong $\mathcal{SFPG}$.

TABLE 13 $\mathcal{SFS}$ *E*.

	e	f	g	h
$\mathfrak{f}$	0.7	0.6	0.8	0.5
$\mathfrak{g}$	0.4	0.5	0.4	0.4
$\mathfrak{h}$	0.3	0.4	0.6	0.7

TABLE 14 Spherical fuzzy multiedge set *F*.

	(e, f)	(e,g)	(f,g)	(f,h)	(g,h)	(h,e)
$\mathfrak{f}$	0.5	0.5	0.6	0.4	0.5	0.5
$\mathfrak{g}$	0.3	0.3	0.4	0.3	0.3	0.3
$\mathfrak{h}$	0.4	0.4	0.5	0.6	0.7	0.3

for all $p_1, p_2 \in$ A. Let $J_1, J_2, \ldots, J_u$ be the intersecting points along the edges in $\tilde{\mathcal{G}}$. For any edge (p_3, p_4) in complete $\mathcal{SFPG}$ $\tilde{\mathcal{G}} = (\mathrm{A}, \mathrm{B})$

$$M_{(p_3,p_4)} = \frac{\mathfrak{f}(p_3,p_4)}{\min\{\mathfrak{f}(p_3), \mathfrak{f}(p_4)\}} \leq 1,$$

$$N_{(p_3,p_4)} = \frac{\mathfrak{g}(p_3,p_4)}{\min\{\mathfrak{g}(p_3), \mathfrak{g}(p_4)\}} = 1$$

and

$$O_{(p_3,p_4)} = \frac{\mathfrak{h}(p_3,p_4)}{\max\{\mathfrak{h}(p_3), \mathfrak{h}(p_4)\}} = 1.$$

In this way, for point p, the intersecting value at point p along edges (p_1, p_2) and (p_3, p_4), $M_1 = \frac{1+1}{2} \leq 1$, and $O_1 = \frac{1+1}{2} = 1$, hence $M_n \leq 1$, $N_n \leq 1$, and $O_n = 1$, where $n = 1, 2, 3, \ldots, u$. Then,

$$M_n = \frac{1}{1 + M_1 + M_2 + M_3 + \cdots + M_u} = \frac{1}{1 + (1 + 1 + \cdots + 1)} = \frac{1}{1 + J_n},$$

$$O_n = \frac{1}{1 + O_1 + O_2 + \cdots + O_u} = \frac{1}{1 + (1 + 1 + 1 + \cdots + 1)} = \frac{1}{1 + J_n}.$$

Here, J_n represents the number of intersecting for edge in $\tilde{\mathcal{G}}$. So, p has a value less than 1.

Definition 4.17: A $\mathcal{SFPG}$ $\tilde{\mathcal{G}}$ is called strong $\mathcal{SFPG}$ if the spherical fuzzy planarity value $R = (R_M, R_N, R_O)$ of the graph is $R_M \geq 0.5$, $R_N \geq 0.5$, or $R_O \leq 0.5$.

Theorem 4.18: For $\mathcal{SMG}$, $\tilde{\mathcal{G}}$ with $R > (0.5, 0.5, 0.5)$ then spherical-valued strong edges in $\tilde{\mathcal{G}}$ containing the number of intersecting value are at most 1.

Proof. Let $\tilde{\mathcal{G}} = (\mathrm{A}, \mathrm{B})$ be $\mathcal{SMG}$ with $R = (R_M, R_N, R_O)$, where $R_M > 0.5$, $R_N > 0.5$, and $R_O > 0.5$. Let us assume that $\tilde{\mathcal{G}}$ contains two intersecting values, J_1 and J_2, which correspond to two spherical, strongly valued edges. For a strong edge $[(p_1, p_2), (\mathfrak{f}, \mathfrak{g}, \mathfrak{h})]$, $M_{(p_1,p_2)} \geq 0.5$, $N_{(p_1,p_2)} \geq 0.5$, and $O_{(p_1,p_2)} \geq 0.5$. Accordingly, for two intersecting spherical-valued strong edges $[(p_1, p_2), (\mathfrak{f}.\mathfrak{g}, \mathfrak{h})]$ and $[(p_3, p_4), (\mathfrak{f}, \mathfrak{g}, \mathfrak{h})]$.

$$\frac{M_{(p_1,p_2)} + M_{(p_3,p_4)}}{2} \geq 0.5,$$
$$\frac{N_{(p_1,p_2)} + N_{(p_3,p_4)}}{2} \geq 0.5,$$

and

$$\frac{O_{(p_1,p_2)} + _{(p_3,p_4)}}{2} \geq 0.5,$$

that is, $M_1 \geq 0.5$, $N_1 \geq 0.5$ and $O_1 \geq 0.5$, $M_2 \geq 0.5$, $N_2 \geq 0.5$ and $O_2 \geq 0.5$. Then, $1 + M_1 + M_2 \geq 2$, $1 + N_1 + N_2 \geq 2$ and $1 + O_1 + O_2 \geq 2$; therefore, $R_M = \frac{1}{1+(M_1+M_2)} \leq 0.5$, $R_N = \frac{1}{1+(N_1+N_2)} \leq 0.5$, and $R_O = \frac{1}{1+(O_1+O_2)} \leq 0.5$ which is contradiction as $R > (0.5, 0.5, 0.5)$. So, the intersecting value among spherical strong edges can never be 2. The level of planarity diminishes as the number of cutting spherical strong edges increases. Moreover, if the number of intersecting points of strong-valued strong edges is 1, then in this case, the level of planarity is assumed as $R = (0.5, 0.5, 0.5)$. Accordingly, we found that spherical-valued strong edges in $\tilde{\mathcal{G}}$ containing the number of intersecting point are at most 1.

Example 4.19: Two $\mathcal{SFPG}$ are shown in Figures 10, 11. In Figure 10, a $\mathcal{SFPG}$ with one crossing among two strong edges that there is one point in $\mathcal{SFPG}$ where two strong edges (p_1, p_3) and (p_2, p_5) intersect or cross each other. The value of the spherical planarity of $\mathcal{SFPG}$ is (0.52,0.5,0.5). Hence, this $\mathcal{SFPG}$ is strong and the number of intersecting values at point p is 1. In Figure 3.8, a $\mathcal{SFPG}$ is considered with two intersecting points among strong edges $(p_1, p_3)(p_2, p_5)$ and $(p_1, p_3)(p_2, p_4)$. The value of the spherical planarity of this graph is (0.5, 0.5, 0.5). Hence, the $\mathcal{SFG}$ is not strong. So, the graph has no intersecting point, and then the $\mathcal{SPG}$ must be strong.

Theorem 4.20: Let $\tilde{\mathcal{G}}$ be a $\mathcal{SFPG}$ having a value of spherical fuzzy planarity R. If $R \geq (0.67, 0.67, 0.67)$, then two spherical strong multi-valued edges in $\tilde{\mathcal{G}}$ do not have any crossing between them.

Proof. A $\tilde{\mathcal{G}}$ is called $\mathcal{SFPG}$ with spherical fuzzy planarity $R = (0.67, 0.67, 0.67)$. Take the value of spherical fuzzy planarity R where two spherical strong-valued edges $((p_1,p_2),(\mathfrak{f},\mathfrak{g},\mathfrak{h}))$ $((p_3,p_4),(\mathfrak{f},\mathfrak{g},\mathfrak{h}))$ intersect. For any spherical-valued strong edge $((p_1,p_2),(\mathfrak{f}(p_1,p_2),\mathfrak{g}(p_1,p_2),\mathfrak{h}(p_1,p_2)))$, $M_1 \geq 0.5$, $N_1 \geq 0.5$, and $O_1 \leq 0.5$. For a minimum value of $M_{(p_1,p_2)}$, $N_{(p_1,p_2)}$, $O_{(p_1,p_2)}$, $M_{(p_3,p_4)}$, $N_{(p_3,p_4)}$, and $O_{(p_3,p_4)}$. $M_1 = 0.5$, $N_1 = 0.5$, and $O_1 = 0.5$. Then, the value of $M_n = \frac{1}{1+0.5} \leq 0.67$, $N_n = \frac{1}{1+0.5} \leq 0.67$, and $O_n = \frac{1}{1+0.5} \leq 0.67$. Hence, $\tilde{\mathcal{G}}$ contains no intersecting point between spherical-valued strong edges.

The above theorem motivated to define the term strong planar graph.

Definition 4.21: A $\mathcal{SFPG}$ is called strong if $R \geq (0.67, 0.67, 0.67)$.

Theorem 4.22: A $\mathcal{SMG}$ containing complete $\tilde{K_5}$ or $\tilde{K}_{3,3}$ spherical graph is not strong $\mathcal{SFG}$.

Proof. The $\tilde{\mathcal{G}}$=($\mathcal{A}$, $\mathcal{B}$) is said to be complete $\mathcal{SFG}$ corresponding to the crisp graph $\mathfrak{G} = (\xi, \eta)$ with vertices such that $\xi = \{p_1, p_2, p_3, p_4, p_5\}$ and the set B and values of $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ are given in Equations 10-13.

$$B = \{(p_1,p_2),(\mathfrak{f}(p_1,p_2),\mathfrak{g}(p_1,p_2),\mathfrak{h}(p_1,p_2)) \,|\, p_1,p_2 \in \xi\}. \tag{10}$$

$$\mathfrak{f}(p_1,p_2) = min\{\mathfrak{f}(p_1),\mathfrak{f}(p_2)\}, \tag{11}$$

$$\mathfrak{g}(p_1,p_2) = min\{\mathfrak{g}(p_1),\mathfrak{g}(p_2)\}, \tag{12}$$

$$\mathfrak{h}(p_1,p_2) = max\{\mathfrak{h}(p_1),\mathfrak{h}(p_2)\}. \tag{13}$$

From Theorem 4.16, the value of planarity for complete $\mathcal{SFG}$ is the number of intersecting values for edges in $\tilde{\mathcal{G}}$. So, it has only one intersecting point that it can be desisted from $\tilde{\mathcal{G}}$. Then, $R = \frac{1}{1+J_u} = \frac{1}{1+1} = 0.5$. Hence, $\tilde{\mathcal{G}}$ is not strong.

Definition 4.23: Let $\tilde{\mathcal{G}}$ be $\mathcal{SFG}$ and $0 \leq h \leq 0.5$ be a rational number. An edge (say) (p_1,p_2) is said to be the considerable edge if the following condition must be held.

$$\frac{\mathfrak{f}(p_1,p_2)}{\min\{\mathfrak{f}(p_1),\mathfrak{f}(p_2)\}} \geq h,$$
$$\frac{\mathfrak{g}(p_1,p_2)}{\min\{\mathfrak{g}(p_1),\mathfrak{g}(p_2)\}} \geq h$$

and

$$\frac{\mathfrak{h}(p_1,p_2)}{\max\{\mathfrak{h}(p_1),\mathfrak{h}(p_2)\}} \leq h.$$

Otherwise, it is not a considerable edge. For $\mathcal{SFMG}$ $\tilde{\mathcal{G}}$, a multiedge $(p_1,p_2) \in \mathcal{V}\times\mathcal{V}$ is said to be considerable edge if $M_{(p_1,p_2)} \geq h$, $N_{(p_1,p_2)} \geq h$, $O_{(p_1,p_2)} \leq h$, for each edge (p_1,p_2) in $\tilde{\mathcal{G}}$.

Theorem 4.24: A $\tilde{\mathcal{G}} = (\tilde{A}, \tilde{B})$ is said to be a strong $\mathcal{SPG}$, where h be a considerable number. Then, the considerable edges in $\tilde{\mathcal{G}}$ have at most $[\frac{0.49}{h}]$ intersecting points (here $[x]$ is the greatest integer not exceeding x).

Proof. Let R be the value of spherical fuzzy planarity and $0 \leq h \leq 0.5$. Let (p_1,p_2) be a considerable edge; it is seen that $(\frac{\mathfrak{f}(p_1,p_2)}{\min\{\mathfrak{f}(p_1),\mathfrak{f}(p_2)\}} \geq h$ and $(\frac{\mathfrak{h}(p_1,p_2)}{\max\{\mathfrak{h}(p_1),\mathfrak{h}(p_2)\}} \geq h$. So, $\mathfrak{f}(p_1,p_2) \geq h \times \{min\{\mathfrak{f}(p_1),\mathfrak{h}(p_2)\}$ and $\mathfrak{h}(p_1,p_2) \leq h \times \{max\{\mathfrak{h}(p_1),\mathfrak{h}(p_2)\}$. In this case, $M_{(p_1,p_2)} \geq h$, $N_{(p_1,p_2)} \geq h$, and $O_{(p_1,p_2)} \geq h$. Let J_1, J_2,…, J_u be the intersecting points among the considerable edges. Let J_1 be the intersecting point between the considerable edges (p_1,p_2) and (p_3,p_4). Then, $M_1 = \frac{M_{(p_1,p_2)}+(M_{(p_3,p_4)}}{2}$ and $O_1 = \frac{O_{(p_1,p_2)}+(O_{(p_3,p_4)}}{2}$.

$$M_1 + M_2 + \cdots + M_u \geq uh$$
$$N_1 + N_2 + \cdots + N_u \geq uh$$
$$O_1 + O_2 + \cdots + O_u \leq uh.$$

$R_M, R_N \leq \frac{1}{1+uh}$, and $R_N \geq \frac{1}{1+uh}$.

Since $\tilde{\mathcal{G}}$ is a strong $\mathcal{SFPG}$, we have,

$$(0.67, 0.67) \leq R \leq \left(\frac{1}{1+uh}, \frac{1}{1+uh}\right)$$
$$(0.67) \leq \frac{1}{1+uh}$$
$$u \leq \left[\frac{0.49}{h}\right].$$

This implies that

$$u = \left[\frac{0.49}{h}\right]$$

and $R_N \geq \frac{1}{1+uh}$.

$$0.67 \geq R \geq \frac{1}{1+uh}$$
$$0.67 \geq \frac{1}{1+uh}$$
$$u \geq \left[\frac{0.49}{h}\right].$$

This implies that

$$u = \left[\frac{0.49}{h}\right].$$

The crucial parameter of a $\mathcal{SFPG}$ is its face. The face of a $\mathcal{SFG}$ is a region bounded by spherical fuzzy edges. Every $\mathcal{SFF}$is characterized by spherical fuzzy edges at its boundary. If all the edges in the boundary of a $\mathcal{SFF}$ have membership values of truthfulness, abstinence, and falseness (1,1,1), respectively, it becomes a crisp face. If one of these edges is eliminated, $\mathcal{SFF}$ will not exist. So, the existence of a $\mathcal{SFF}$ depends on the minimum value of the strength of the spherical fuzzy edges in its boundary.

Definition 4.25: Let $\tilde{\mathcal{G}}$ be a $\mathcal{SFPG}$ and

$$B = \left\{ (p_1,p_2), \mathfrak{f}_B(p_1,p_2)_i, \mathfrak{g}_B(p_1,p_2)_i, \mathfrak{h}_B(p_1,p_2)_i, \quad i = 1,2,\ldots,k| \right.$$
$$\left. (p_1,p_2) \in \mathcal{V}\times\mathcal{V} \right\},$$

for all $(p_1,p_2) \in \mathcal{V}\times\mathcal{V}$. A $\mathcal{SFF}$ of $\tilde{\mathcal{G}}$ is a region bounded by the set of spherical fuzzy edges $\check{E} \subset E$, of a geometric representation of $\tilde{\mathcal{G}}$. The truthfulness degree, abstinence degree, and falseness degree of the $\mathcal{SFF}$ are defined as follows: $min\{\frac{\mathfrak{f}_B(p_1,p_2)_i}{\min\{\mathfrak{f}_B(p_1),\mathfrak{f}_B(p_2)\}}, \quad i = 1,2,3,\ldots,k\,|\,(p_1,p_2) \in \check{E}\}$, $min\{\frac{\mathfrak{g}_B(p_1,p_2)_i}{\min\{\mathfrak{g}_B(p_1),\mathfrak{g}_B(p_2)\}}, \quad i = 1,2,3,\ldots,k\,|\,(p_1,p_2) \in \check{E}\}$, *and*, $max\{\frac{\mathfrak{h}_B(p_1,p_2)_i}{\max\{\mathfrak{h}_B(p_1),\mathfrak{h}_B(p_2)\}}, \quad i = 1,2,3,\ldots,k\,|\,(p_1,p_2) \in \check{E}\}$ respectively.

Definition 4.26: A $\mathcal{SFF}$ is called strong $\mathcal{SFF}$ if its value of truthfulness and value of abstinence are greater than 0.5 and degree

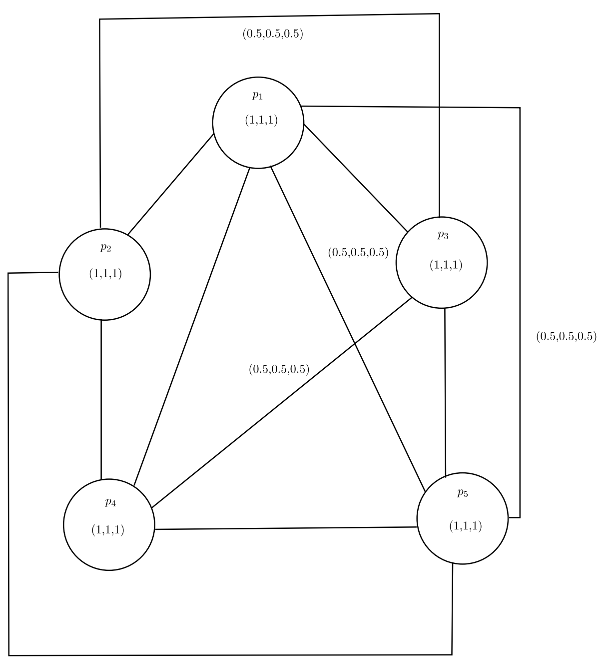

FIGURE 11
Example of not strong $\mathcal{SFPG}$.

of falseness is less than 0.5; otherwise, the face is weak. Every $\mathcal{SFPG}$ has an infinite region, which is called outer $\mathcal{SFF}$. Other faces are called inner $\mathcal{SFF}$.

Example 4.27: Suppose $\mathcal{SFPG}$ as shown in Figure 12. The $\mathcal{SFPG}$ has the following faces:

- $\mathcal{SFF}$ $\mathcal{F}_1$ is bounded by the edges $((p_1,p_2),0.1,0.3,0.5)$, $((p_2,p_3),0.2,0.3,0.2)$, and $((p_1,p_3),0.2,0.3,0.4)$.
- $\mathcal{SFF}$ $\mathcal{F}_2$ is bounded by the edges $((p_1,p_3),0.2,0.3,0.4)$, $((p_1,p_4),0.1,0.2,0.3)$, and $((p_3,p_4),0.3,0.2,0.2)$.
- $\mathcal{SFF}$ $\mathcal{F}_3$ is surrounded by the edges $((p_1,p_2),0.1,0.3,0.5)$, $((p_1,p_4),0.1,0.2,0.3)$, and $((p_2,p_4),0.1,0.2,0.1)$.
- Outer $\mathcal{SFF}$ $\mathcal{F}_4$ is surrounded by the edges $((p_2,p_3),0.2,0.3,0.2)$, $((p_3,p_4),0.3,0.2,0.2)$, and $((p_2,p_4),0.1,0.2,0.1)$.

Clearly, the values of truthfulness, abstinence, and falsehood of $\mathcal{SFF}$ $\mathcal{F}_1$ are 0.5, 0.75, and 0.83, respectively. The values of truthfulness, abstinence, and falseness of $\mathcal{SFF}$ $\mathcal{F}_2$ are 0.5, 0.66, and 0.666, respectively. The values of truthfulness, abstinence, and falseness of $\mathcal{SFF}$ $\mathcal{F}_3$ are 0.333, 0.666, and 0.833, respectively. The values of truthfulness, abstinence, and falseness of $\mathcal{SFF}$ $\mathcal{F}_4$ are 0.333, 0.666, and 0.666, respectively. Now, the observation shows that in this $\mathcal{SFPG}$ every face is weak.

5 Spherical fuzzy dual graph

The concept of duality is very useful in elaborating many models, like integrated circuits, drainage systems for basins, and others. A graph is planar if and only it has a dual graph. This means that for any planar graph, there exists a dual graph, and for any graph with dual graph, it must be planar. This concept works well to deal with a wide range of complicated and significant circumstances. By inspiring this concept, we are going to present the idea of a ($\mathcal{SFDG}$) of $\mathcal{SFPG}$. In a $\mathcal{SFDG}$, vertices correspond to the strong $\mathcal{SFF}$ of the $\mathcal{SFPG}$, and between two vertices, every spherical fuzzy edge corresponds to every edge in the boundary between two faces of the $\mathcal{SFPG}$. The formal definition is given below.

Definition 5.1: Let $\vec{\mathcal{G}}$ be $\mathcal{SFPG}$ and spherical fuzzy multiedge set

$$B = \{((p_1,p_2),\mathfrak{f}_B(p_1,p_2)_i,\mathfrak{g}_B(p_1,p_2)_i,\mathfrak{h}_B(p_1,p_2)_i),\ i = 1,2,\ldots,k \mid (p_1,p_2) \in \check{\mathcal{V}} \times \check{\mathcal{V}}\}.$$

Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ be the strong $\mathcal{SFF}$ of $\vec{\mathcal{G}}$. The $\mathcal{SFDG}$ of $\vec{\mathcal{G}}$ is a $\mathcal{SFPG}$ $\tilde{\mathcal{G}} = (\check{V},\check{C},\check{D})$, where $\check{V} = \{\check{u}_i, i = 1,2,\ldots,m\}$, and the vertex $\check{u}_i$ of $\tilde{\mathcal{G}}$ is considered for face $\mathcal{F}_i$ of $\vec{\mathcal{G}}$. The values of truthfulness, abstinence, and falseness of vertex $\check{u}_i$ are given by the mapping $\check{C} = (\mathfrak{f}_{\check{C}},\mathfrak{g}_{\check{C}},\mathfrak{h}_{\check{C}})$: $\check{V} \to [0,1]\times[0,1]\times[0,1]$ such that:

$$\mathfrak{f}_{\check{C}}(\check{u}_i) = \max\left\{ \mathfrak{f}_D(p_1,p_2)_i,\quad i = 1,2,3,\ldots,t, \mid (p_1,p_2) \text{ is an edge of the boundary of the strong } \mathcal{SFFF}_i \right\},$$

$$\mathfrak{g}_{\check{C}}(\check{u}_i) = \max\left\{ \mathfrak{g}_D(p_1,p_2)_i,\quad i = 1,2,3,\ldots,t, \mid (p_1,p_2) \text{ is an edge of the boundary of the strong } \mathcal{SFF}\,\mathcal{F}_i \right\}$$

and

$$\mathfrak{h}_{\check{C}}(\check{u}_i) = \min\left\{ \mathfrak{h}_D(p_1,p_2)_i,\quad i = 1,2,3,\ldots,t, \mid (p_1,p_2) \text{ is an edge of the boundary of the strong } \mathcal{SFF}\,\mathcal{F}_i \right\},$$

respectively. Between two faces $\mathcal{F}_i$ and $\mathcal{F}_j$ of $\vec{\mathcal{G}}$, there may exist more than one same edge. Hence, between two vertices $\check{u}_i$ and $\check{u}_j$ in $\mathcal{SFDG}$ $\tilde{\mathcal{G}}$, there may exist more than one edge. Let $\mathfrak{f}_D(\check{u}_i,\check{x}_j)$, $\mathfrak{g}_D(\check{u}_i,\check{x}_j)$, and $\mathfrak{h}_D(\check{u}_i,\check{u}_j)$ represent the degree of truthfulness, degree of abstinence, and degree of falseness of the path edge between $\check{u}_i$ and $\check{u}_j$, respectively.

Example 5.2: Let the underlying set of vertices $\mathcal{V} = \{p_1,p_2,p_3,p_4,p_5\}$, the $\mathcal{SFS}$ $C = \{(p_1,0.2,0.3,0.7),(p_2,0.4,0.3,0.2),(p_3,0.6,0.1,0.4),(p_4,0.7,0.4,0.3)$ and the spherical fuzzy multiedge set $D = \{((p_1,p_2),0.1,0.2,0.3),((p_1,p_2),0.1,0.2,0.4),((p_1,p_4),0.1,0.2,0.3),((p_2,p_3),0.2,0.1,0.3),\ ((p_2,p_3),0.3,0.1,0.4),((p_2,p_4),0.2,0.3,0.3),((p_2,p_4),0.3,0.3,0.2),((p_3,p_4),0.5,0.1,0.4)$. Then, the $\mathcal{SFPG}$ $\vec{\mathcal{G}} = (A,B)$ has the following faces.

- $\mathcal{SFF}$ $\mathcal{F}_1$ is bounded by the edges $((p_1,p_2),0.1,0.2,0.3)$ and $((p_1,p_2),0.1,0.2,0.4)$.
- $\mathcal{SFF}$ $\mathcal{F}_2$ is bounded by the edges $((p_2,p_4),0.2,0.3,0.3)$ and $((p_2,p_4),0.3,0.3,0.2)$.
- $\mathcal{SFF}$ $\mathcal{F}_3$ is bounded by the edges $((p_2,p_3),0.2,0.1,0.3)$ and $((p_2,p_3),0.3,0.1,0.4)$.
- $\mathcal{SFF}$ $\mathcal{F}_4$ is bounded by the edges $((p_1,p_2),0.1,0.1,0.4)$, $((p_1,p_4),0.1,0.2,0.3)$, and $((p_2,p_4),0.2,0.3,0.3)$.
- $\mathcal{SFF}$ $\mathcal{F}_5$ is surrounded by the edges $((p_2,p_4),0.3,0.3,0.2)$, $((p_2,p_3),0.2,0.1,0.3)$, and $((p_3,p_4),0.5,0.1,0.4)$.

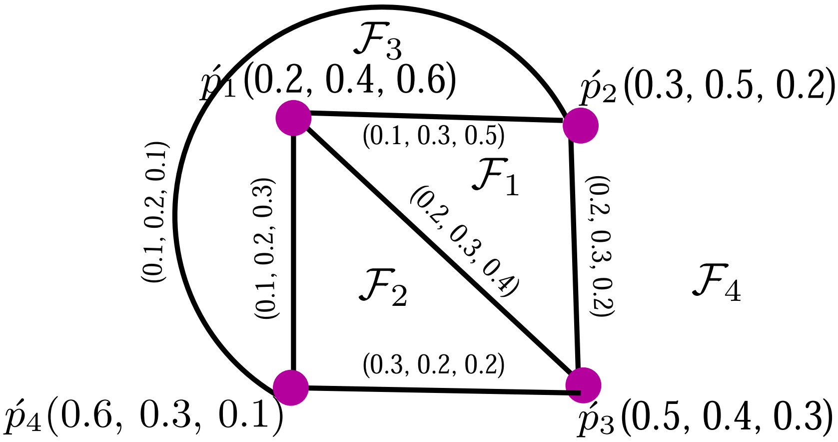

FIGURE 12
Faces of $\mathcal{SFPG}$.

- $\mathcal{SFF}$ $\mathcal{F}_6$ is bounded by the edges $((p_1,p_4),0.1,0.2,0.3)$, $((p_1,p_2),0.1,0.2,0.3)$, $((p_3,p_4),0.5,0.1,0.4)$, and $((p_2,p_3),0.3,0.1,0.4)$.

Here, F_6 is an outer $\mathcal{SFF}$ and remaining all are inner $\mathcal{SFF}$. By direct computations, it can be easily observed that all the $\mathcal{SFF}$ are strong $\mathcal{SFF}$. Suppose the vertex set

$$\check{V} = \{\breve{u}_1,\breve{u}_2,\breve{u}_3,\breve{u}_4,\breve{u}_5,\breve{u}_6\},$$

where the vertex $\breve{u}_j$ is taken corresponding to the strong $\mathcal{SFF}$ $\mathcal{F}_j$, j = 1, 2, 3, 4, 5, 6.

Thus, we get the values of truthfulness, abstinence, and falseness for the vertex $\breve{u}_j$, respectively, as follows:

For vertex $\breve{u}_1$:

$$\mathfrak{f}_{\check{C}}(\breve{u}_1) = \max\{0.1,0.1\} = 0.1, \mathfrak{g}_{\check{C}}(\breve{u}_1) = \max\{0.2,0.2\} = 0.2, \mathfrak{h}_{\check{C}}(\breve{u}_1) = \min\{0.3,0.4\} = 0.3.$$

For vertex $\breve{u}_2$:

$$\mathfrak{f}_{\check{C}}(\breve{u}_2) = \max\{0.2,0.3\} = 0.3, \mathfrak{g}_{\check{C}}(\breve{u}_2) = \max\{0.3,0.3\} = 0.3, \mathfrak{h}_{\check{A}}(\breve{u}_2) = \min\{0.2,0.3\} = 0.2.$$

For vertex $\breve{u}_3$:

$$\mathfrak{f}_{\check{C}}(\breve{u}_3) = \max\{0.2,0.3\} = 0.3, \mathfrak{g}_{\check{C}}(\breve{u}_3) = \max\{0.1,0.1\} = 0.1, \mathfrak{h}_{\check{C}}(\breve{u}_3) = \min\{0.3,0.4\} = 0.3.$$

For vertex $\breve{u}_4$:

$$\mathfrak{f}_{\check{C}}(\breve{u}_4) = \max\{0.1,0.1,0.2\} = 0.2, \mathfrak{g}_{\check{C}}(\breve{u}_4) = \max\{0.2,0.2,0.3\} = 0.3, \mathfrak{h}_{\check{C}}(\breve{u}_4) = \min\{0.3,0.3,0.4\} = 0.3.$$

For vertex $\breve{u}_5$:

$$\mathfrak{f}_{\check{C}}(\breve{u}_5) = \max\{0.2,0.3,0.5\} = 0.5, \mathfrak{g}_{\check{C}}(\breve{u}_5) = \max\{0.1,0.1,0.3\} = 0.3, \mathfrak{h}_{\check{C}}(\breve{u}_5) = \min\{0.2,0.3,0.4\} = 0.2.$$

Finally, for vertex $\breve{x}_6$:

$$\mathfrak{f}_{\check{C}}(\breve{u}_6) = \max\{0.1,0.1,0.3,0.5\} = 0.5, \mathfrak{g}_{\check{C}}(\breve{u}_6) = \max\{0.2,0.2,0.1,0.1\} = 0.2, \mathfrak{h}_{\check{C}}(\breve{u}_6) = \min\{0.3,0.3,0.4,0.4\} = 0.3.$$

Now, the values of truthfulness, abstinence, and falseness of edges of $\mathcal{SFDG}$ are given below.

For an edge $(\breve{u}_1,\breve{u}_6)$:

$$\mathfrak{f}(\breve{u}_1,\breve{u}_6) = \mathfrak{f}(p_1,p_2) = 0.1, \mathfrak{g}(\breve{u}_1,\breve{u}_6) = \mathfrak{g}(p_1,p_2) = 0.2, \mathfrak{h}(\breve{u}_1,\breve{u}_6) = \mathfrak{h}(p_1,p_2) = 0.3.$$

For an edge $(\breve{u}_1,\breve{u}_4)$:

$$\mathfrak{f}(\breve{u}_1,\breve{u}_4) = \mathfrak{f}(p_1,p_2) = 0.1, \mathfrak{g}(\breve{u}_1,\breve{u}_4) = \mathfrak{g}(p_1,p_2) = 0.2, \mathfrak{h}(\breve{u}_1,\breve{u}_4) = \mathfrak{h}(p_1,p_2) = 0.4.$$

For an edge $(\breve{u}_4,\breve{u}_2)$:

$$\mathfrak{f}(\breve{u}_4,\breve{u}_2) = \mathfrak{f}(p_2,p_4) = 0.2, \mathfrak{g}(\breve{u}_4,\breve{u}_2) = \mathfrak{g}(p_2,p_4) = 0.3, \mathfrak{h}(\breve{u}_4,\breve{u}_2) = \mathfrak{h}(p_2,p_4) = 0.3.$$

For an edge $(\breve{u}_2,\breve{u}_5)$:

$$\mathfrak{f}(\breve{u}_2,\breve{u}_5) = \mathfrak{f}(p_2,p_4) = 0.3, \mathfrak{g}(\breve{u}_2,\breve{u}_5) = \mathfrak{g}(p_2,p_4) = 0.3, \mathfrak{h}(\breve{u}_2,\breve{u}_5) = \mathfrak{h}(p_2,p_4) = 0.2.$$

For an edge $(\breve{u}_3,\breve{u}_5)$:

$$\mathfrak{f}(\breve{u}_3,\breve{u}_5) = \mathfrak{f}(p_2,p_3) = 0.2, \mathfrak{g}(\breve{u}_3,\breve{u}_5) = \mathfrak{g}(p_2,p_3) = 0.1, \mathfrak{h}(\breve{u}_3,\breve{u}_5) = \mathfrak{h}(p_2,p_3) = 0.3.$$

For an edge $(\breve{u}_3,\breve{u}_6)$:

$$\mathfrak{f}(\breve{u}_3,\breve{u}_6) = \mathfrak{f}(p_2,p_3) = 0.3, \mathfrak{g}(\breve{u}_2,\breve{u}_3) = \mathfrak{g}(p_2,p_3) = 0.1, \mathfrak{h}(\breve{u}_3,\breve{u}_6) = \mathfrak{h}(p_2,p_3) = 0.4.$$

For an edge $(\breve{u}_5,\breve{u}_6)$:

$$\mathfrak{f}(\breve{u}_5,\breve{u}_6) = \mathfrak{f}(p_3,p_4) = 0.5, \mathfrak{g}(\breve{u}_5,\breve{u}_6) = \mathfrak{g}(p_3,p_4) = 0.1, \mathfrak{h}(\breve{u}_5,\breve{u}_6) = \mathfrak{h}(p_3,p_4) = 0.4.$$

For an edge $(\breve{u}_6, \breve{u}_4)$:

$$\mathfrak{f}(\breve{u}_6, \breve{u}_4) = \mathfrak{f}(p_1, p_4) = 0.1, \mathfrak{g}(\breve{u}_6, \breve{u}_4) = \mathfrak{g}(p_1, p_4) = 0.2, \mathfrak{h}(\breve{u}_6, \breve{u}_4) = \mathfrak{h}(p_1, p_4) = 0.3.$$

Thus, we get the edge set of $\mathcal{SFDG}$ as follows:

$$\breve{B} = \{((\breve{u}_1, \breve{u}_4), 0.1, 0.2, 0.4), ((\breve{u}_1, \breve{u}_6), 0.1, 0.2, 0.3), ((\breve{u}_2, \breve{u}_4), 0.2, 0.3, 0.3), ((\breve{u}_2, \breve{u}_5), 0.3, 0.3, 0.2), ((\breve{u}_3, \breve{u}_5), 0.2, 0.1, 0.3), ((\breve{u}_3, \breve{u}_6), 0.3, 0.1, 0.4), ((\breve{u}_6, \breve{u}_5), 0.5, 0.1, 0.4), ((\breve{u}_4, \breve{u}_6), 0.1, 0.2, 0.3)\}.$$

In Figure 13, the $\mathcal{SFDG}$ $\breve{G} = (\breve{V}, \breve{C}, \breve{D})$ of G is plotted by dotted lines.

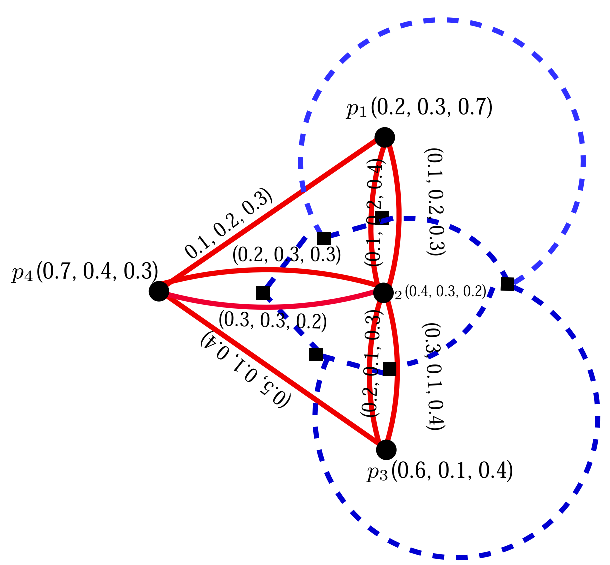

FIGURE 13
Dual of $\mathcal{SFPG}$.

6 Application

In real-world power distribution systems, maintaining efficient and reliable connections between transformers and households is crucial. Using the spherical fuzzy multigraph model allows us to better visualize and optimize these complex networks. By representing transformers as vertices and the wires connecting them to households as edges, we can account for uncertainties or varying degrees of connectivity in the system. This approach provides insights into how power is distributed, identifies potential vulnerabilities, and suggests improvements in efficiency and reliability. In the example of a transformer system represented as a spherical fuzzy planar graph, the transformer is considered a node and the electrical connections (wires) between components are treated as edges. The number of intersecting points (where wires cross) directly impacts the damage rate. The more intersections there are, the higher the likelihood of damage, as each crossing increases the risk of overheating or failure. Using the spherical fuzzy graph, we can model the uncertainty and degree of risk associated with each connection, helping minimize the damage rate by optimizing the placement of wires and reducing intersections. A vertex denotes each transformer $L_1, L_2, L_3, L_4, L_5, L_6, L_7$, while the edge denotes each electric connection made between the transformer via a small wire as given in Figure 14.

The rate of decomposition grows with the number of crossings. Z_1, Z_2, Z_3, Z_4, Z_5, and Z_6 are crossings among the pairs of wire (L_1L_6, L_2L_7), (L_1L_3, L_2L_7), (L_1L_3, L_2L_6), (L_1L_3, L_2L_5), (L_2L_5, L_3L_6), (L_2L_5, L_4L_6), respectively. The strength of the wire $L_1L_3 = (1, 0.666, 0.833)$, $L_2L_6 = (0.75, 0.8, 0.8)$, $L_1L_6 = (0.5, 0.75, 0.83)$, $L_2L_7 = (0.666, 0.6, 0.8)$, $L_2L_5 = (0.5, 0.75, 0.8)$, $L_3L_6 = (0.4, 0.33, 0.75)$, and $L_4L_6 = (1, 0.8, 0.714)$. For crossings, the points of intersection are

$$T_{Z_1} = (0.58, 0.675, 0.816), T_{Z_2} = (0.833, 0.63, 0.8165), T_{Z_3} = (0.875, 0.733, 0.8165),$$

$$T_{Z_4} = (0.75, 0.708, 0.8165), T_{Z_5} = (0.45, 0.54, 0.775), T_{Z_6} = (0.75, 0.775, 0.755).$$

Thus, the spherical fuzzy planarity value $R = (0.1909, 0.1976, 0.1721)$. When the planarity value is at minimum, it indicates that graph has a high number of edge crossings, which can make it harder to interpret and manipulate, as shown in Figure 15. The spherical fuzzy planarity value indicates a high number of edge crossings at the Z_5 intersection, which increases complexity and the risk of damage. Reducing the number of these crossings can improve the system's planarity and lower the destruction rate. To mitigate the risks of electric hazards, high-quality wires are recommended for critical connections, specifically between L_3 and L_6, and L_2 and L_5. These high-quality wires help minimize the potential for overheating and failures at the intersection. This model aids in monitoring and detecting destruction rates in real-time, enabling proactive management of the system and identification of high-risk areas, such as the Z_5 intersection, to reduce potential hazards. By applying these strategies, the Z_5 intersection's risk can be minimized, improving both the safety and efficiency of the system. Through meticulous examination and the implementation of enhanced security measures, it is feasible

INPUT: 1. Let L represent a collection of electric connections, where L = L_1, L_2, …, L_n
2. The set of edges that connect the units L = $\{L_1, L_2, \ldots, L_n\}$is denoted by $E = \{E_1, E_2, E_3, \ldots, E_m\}$, and the set of intersecting points is denoted by $C = \{b_1, b_2, \ldots, b_r\}$.
3. Compute the strength of the edge E_i= $(L_j, L_k) \in L$by the equation $\left(\frac{\mathfrak{f}(L_j, L_k)}{\min(\mathfrak{f}(L_j), \mathfrak{f}(L_k))}, \frac{\mathfrak{g}(L_j, L_k)}{\min(\mathfrak{g}(L_j), \mathfrak{g}(L_k))}, \frac{\mathfrak{h}(L_j, L_k)}{\max(\mathfrak{h}(L_j), (\mathfrak{L}_k))}\right)$.
4. Calculate the value of intersecting points.
5. Determine the value of planarity for $\mathcal{SFPG}$.

Algorithm 1. Method to calculate the value of planarity for electric connections.

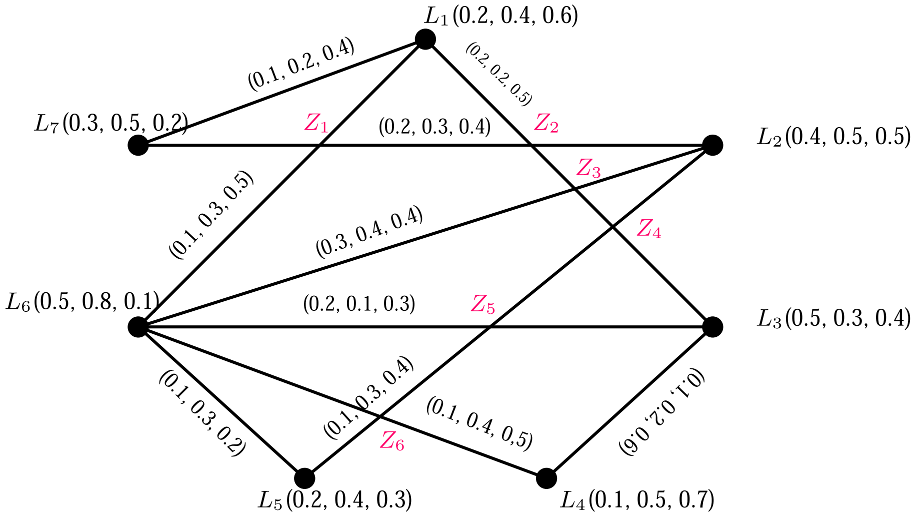

FIGURE 14
Transformer unit connections.

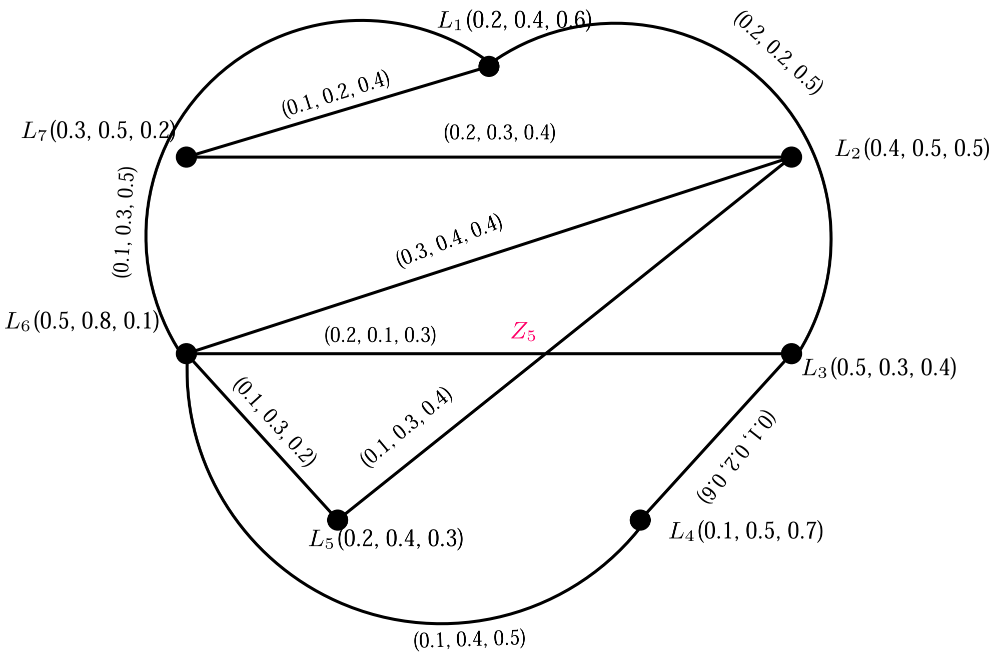

FIGURE 15
Spherical fuzzy electric connection model.

to diminish the percentage of destruction, thereby safeguarding numerous lives.

6.1 Comparison with already existing methods

There are already existing methods, such as fuzzy planar graph and interval-valued planar graph, which can be used to determine if a discrete process is planar or if an interval representing a continuous process is planar, respectively. However, spherical planar graphs can be understood by combining principles from spherical fuzzy sets and planar graphs.

In planar graphs, interval-valued planar graphs are considered simultaneously [40]. However, we use a more versatile technique that involves three types of degree of membership such as truthfulness, falseness, and abstinence memberships, which gives more clear information about uncertain data.

7 Limitation and advantages

The proposed technique of spherical planar graph is restricted to undirected graphs only.

The spherical fuzzy planar graph extends traditional fuzzy graph theory by incorporating membership, non-membership, and hesitancy degrees, offering a more comprehensive representation of uncertainty. Compared to standard fuzzy graphs, it provides higher precision in modeling ambiguous or imprecise relationships. Its planar structure simplifies visualization and analysis of complex networks, reducing computational complexity. The approach supports dynamic adaptability, making it suitable for systems with evolving uncertainties. It also enhances decision-making capabilities by capturing partial truths more effectively than traditional fuzzy models. Furthermore, its multi-dimensional representation is particularly useful for handling conflicting and incomplete information in engineering and optimization problems. The proposed study, applying spherical fuzzy planar graphs to transformer systems, effectively models uncertain and dynamic connections between components like wires and nodes. It can address network reliability analysis, fault detection, and load balancing in power systems by capturing imprecise data and ambiguous relationships. This approach also aids in optimizing energy distribution and minimizing losses through flexible modeling of uncertainty. Additionally, it supports scalability and adaptability, making it suitable for evolving smart grid systems and renewable energy integration challenges.

8 Conclusion

Graph theory has enormous applications to problems in transportation, operations research, computer science, image capture, data mining, etc. Sometimes, to deal with uncertainty and vagueness in various network problems, various graph theoretical concepts are used based on Zadeh's fuzzy relations. $\mathcal{SFG}$ as the generalization of fuzzy graphs, intuitionistic fuzzy graphs, Pythagorean fuzzy graphs, and picture fuzzy graphs can be used to tackle various models based on real-world problems more effectively due to the enlargement of the space of uncertainty. In this article, the idea of spherical fuzzy graphs is utilized, as are the notions of $\mathcal{SFG}$, $\mathcal{SFMG}$, and $\mathcal{SFPG}$. The concept of spherical fuzzy planarity value provides a new method to evaluate the edge intersection. Furthermore, the spherical fuzzy planarity value has been defined based on strong, weak, and considerable edges. In addition, we present the idea of a $\mathcal{SFPG}$. Planar graphs are very useful in designing circuits as well as various network models. In this article, we delve into potential synergies between $\mathcal{SFPG}$ and two distinct yet interconnected fields: neutral networks and geographical information system(*GIS*). By harnessing this strength of $\mathcal{SFPG}$, we aim to enhance the capabilities of neutral networks in modeling complex relationships, while also extending the functionality of *GIS* for spatial analysis and decision-making. We can deal with such problems using a planar graph. Planar graphs can be used to build circuits and road networks. We gave an example of transformer connections to check crossing between edges, so the planarity value can also be calculated. Our future plans regarding our research work are as follows:

Soft fuzzy planar graphs, vague planar graphs, hesitant $\mathcal{SFG}$, spherical fuzzy hypergraphs, and single-valued $\mathcal{SFG}$.

Data availability statement

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

Author contributions

HG: conceptualization and writing–original draft. SH: project administration, supervision, and writing–original draft. Sadaf: investigation, methodology, and writing–original draft. AK: project administration and writing–review and editing. JS: data curation, supervision, and writing–review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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