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Bifurcation, chaotic behavior, and traveling wave solutions of the space−time fractional Zakharov−Kuznetsov−Benjamin−Bona−Mahony equation

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The space−time fractional Zakharov−Kuznetsov−Benjamin−Bona−Mahony (ZKBBM) equation is a significant nonlinear model used to illustrate numerous physical phenomena, such as water wave mechanics, fluid flow, marine and coastal science, and control systems. In this article, the dynamical behavior of the space−time fractional ZKBBM equation is analyzed, and its traveling wave solutions are investigated based on the theory of the cubic polynomial complete discriminant system. First, the equation is transformed into a nonlinear ordinary differential equation through a complex wave transformation. Then, the dynamical behavior analysis of the equation is using the bifurcation theory from planar dynamical systems. Subsequently, by utilizing the polynomial complete discriminant system and root formulas, several new exact traveling wave solutions of the equation are obtained. Finally, the plots of some solutions are shown using MATLAB software in order to demonstrate their structure.

KEYWORDS

nonlinear fractional partial differential equation, bifurcation theory, dynamic analysis, planar dynamical system, polynomial complete discriminant system

1 Introduction

A nonlinear fractional partial differential equation (NLFPDE) was first proposed by Zabusky and Kruskal in 1965 [1]. Fractional calculus is a natural extension of traditional integral calculus and plays a key role in describing some non-local and non-Markov processes. By studying and applying the solutions of NLFPDEs, we can better predict the behavior in complex systems [2] and provide more in-depth analysis and solutions for phenomena such as electromagnetic field propagation in nonlinear media [3], the growth and diffusion of biological tissues [4], seismic wave propagation [5], and groundwater flow [6]. Many literature studies have used various strategies to study the explicit solutions of nonlinear evolution equations, and an abundance of remarkable results has been obtained.

These approaches, like the Jacobi elliptic function [7, 8], the sine–cosine method [9, 10], the (G', G)-expansion approach [11, 12], the two variable $(G'/G, 1/G)$-expansion method [13, 14], the $e^{\phi(\eta)}$-expansion method [15], the F-expansion method [16], the Riccati–Bernoulli sub-ODE [17], the modified extended tanh-function method [18], the improved tanh method [19], directed extended Riccati method [20, 21], the analysis method

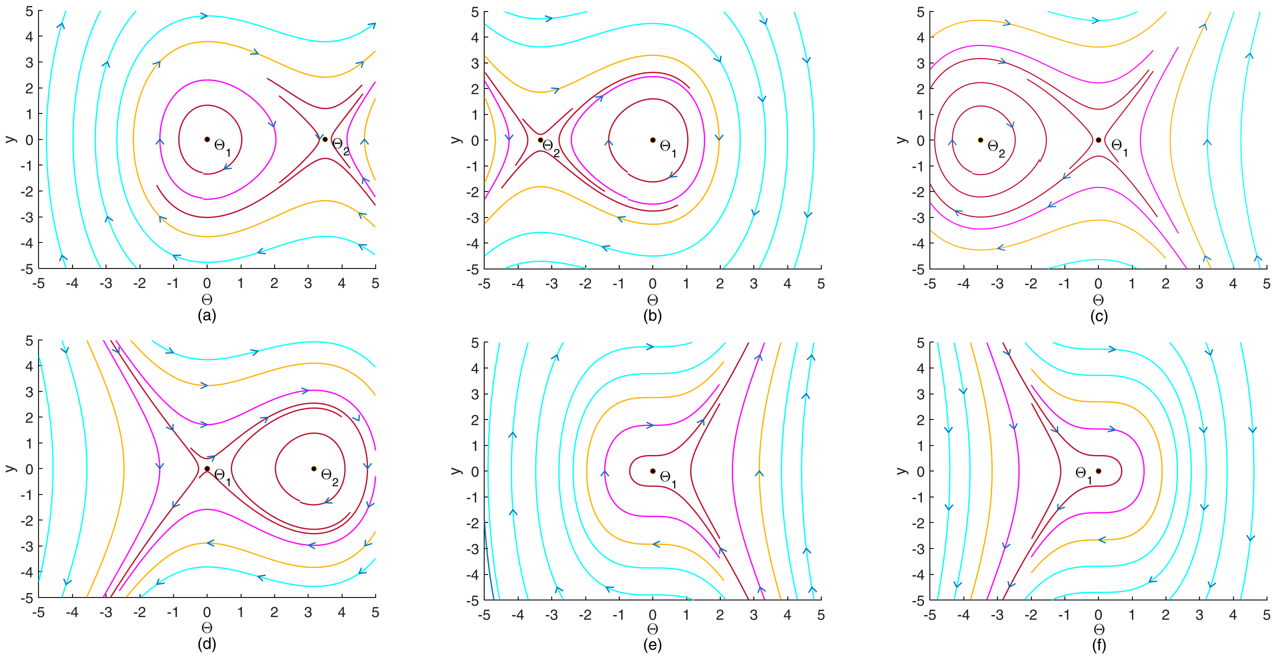

FIGURE 1
Phase portraits of Equation 2.4. **(a)** $\lambda_1 = -2.1$ and $\lambda_2 = 0.6$. **(b)** $\lambda_1 = -2$ and $\lambda_2=0.6$. **(c)** $\lambda_1 = 2.1$ and $\lambda_2 = 0.6$. **(d)** Phase portrait with $\lambda_1 = 1.9$ and $\lambda_2 = -0.6$. **(e)** Phase portrait with $\lambda_1 = 0$ and $\lambda_2 = 2.1$. **(f)** Phase portrait with $\lambda_1 = 0$ and $\lambda_2=-1.6$.

of planar dynamical system [22, 23], Riemann–Hilbert approach [24–26], and the finite difference methods [27], have been thoroughly investigated for the solutions of NLFPDEs.

Among the most well-known model equations, the ZKBBM equation is a vital type of NLFPDEs, which describes the phenomenon of gravitational water waves occurring when long waves propagate bidirectionally in a nonlinear dispersive system [28]. Many scholars studied the solution of this equation and its fractional form. To date, many types of traveling wave solutions of the ZKBBM evolution equation have been obtained utilizing the new (G'/G)-expansion method [29], the $\exp(-\phi(\eta))$-function method [30–32], the differential transformation method (DTM) [33], the generalized exponential rational function method [34], the extended tanh-function approach [35], and the Lie symmetry method [36].

Stability is a critical factor in the design and control of a nonlinear system. By analyzing the equilibrium points and phase trajectories of the space–time fractional ZKBBM equation and observing its chaotic behavior, one can establish an important basis for the practical application of the system. Therefore, this paper investigates the dynamics of the space–time fractional ZKBBM equation according to the plane dynamics theory [37]. Furthermore, different forms of traveling wave solutions describe the same complex physical phenomenon in different ways and provide important initial and boundary conditions for numerical simulation. This allows for a more in-depth and comprehensive understanding of the properties of the equation and the structure of its solutions. In addition to the methods mentioned above, this paper uses the method of polynomial complete discriminant system proposed by Liu C [38] to derive a new traveling wave solution of the equation. This method has been widely used in solving a variety of NLFPDEs [39, 40], enabling the discovery of multiple types of solutions.

In this article, we adopt the following α–order conformable derivative of the function f proposed by Khalil et al [41]:

$$D_t^{\alpha}(f)(t) = \lim_{\epsilon\to 0}\frac{f\left(t+\epsilon t^{1-\alpha}\right)-f(t)}{\epsilon}, \tag{1.1}$$

For all $t>0, \alpha\in(0,1)$. If f is α-differentiable for an interval $(0,a), a>0$ and $\lim_{t\to 0^+} D_t^{\alpha}(f)(t)$ exists, then $D_t^{\alpha}(f)(0) = \lim_{t\to 0^+} D_t^{\alpha}(f)(t)$. It has been verified that the fractional-order definition (1.1) satisfies the following properties [41]:

- $D_t^{\alpha}(t^k) = kt^{k-\alpha}$;
- $D_t^{\alpha}(af_1+bf_2) = aD_t^{\alpha}(f_1)+bD_t^{\alpha}(f_2)$;
- $D_t^{\alpha}(\frac{f_1}{f_2}) = \frac{f_2D_t^{\alpha}(f_1)-f_1D_t^{\alpha}(f_2)}{f_2^2}$;
- $D_t^{\alpha}(f_1f_2) = f_2D_t^{\alpha}(f_1)+f_1D_t^{\alpha}(f_2)$;
- If f is a constant, then $D_t^{\alpha}(f)\equiv 0$;
- If f becomes differentiable, then $D_t^{\alpha}(f) = (t^{1-\alpha})\frac{df(t)}{dt}$.

The space–time fractional ZKBBM equation ([28] and [35]) is given as follows:

$$D_t^{\alpha}\Theta + D_x^{\beta}\Theta - 2a\Theta D_x^{\beta}\Theta - bD_t^{\alpha}\left(D_x^{\beta}D_x^{\beta}\Theta\right) = 0, \tag{1.2}$$

where $t>0$, $a,b\neq 0$, and α,β are the fractional order derivatives and $0<\alpha,\beta\le 1$. The objective of this article is to analyze the dynamics of Equation 1.2 and apply the polynomial complete discriminant system method to construct some new exact traveling wave solutions of Equation 1.2.

The remainder of this work is structured as follows: Section 2 discusses the dynamical behavior and presents the wave equation of Equation 1.2. Section 3 provides several new traveling wave

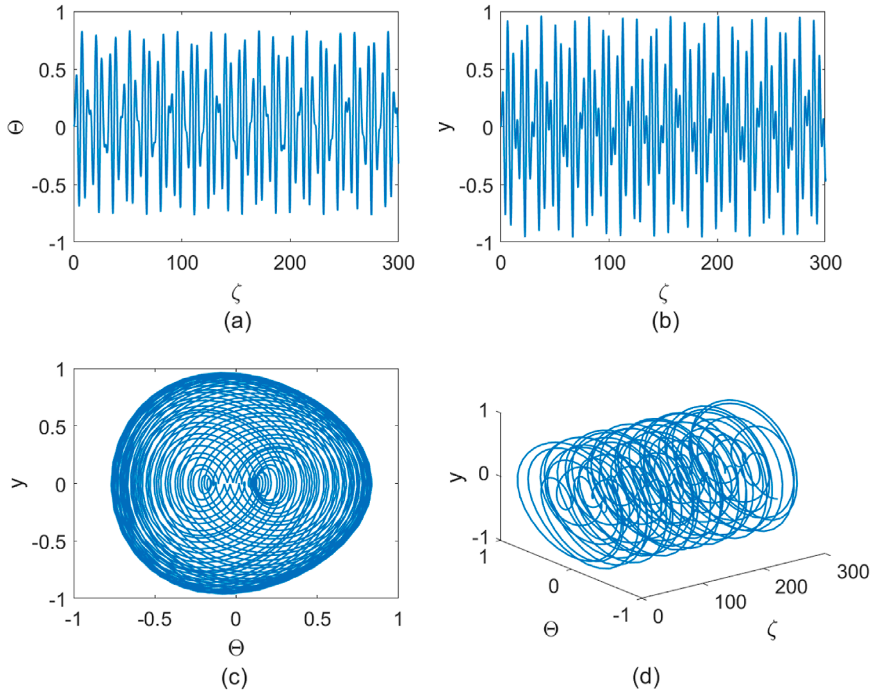

FIGURE 2
Chaotic behavior of Equation 2.8 at $\lambda_1 = -2.1, \lambda_2 = 0.6, A = 0.5, \varpi = 1$. **(a)** Times series about (θ). **(b)** Times series about y. **(c)** 2D phase portrait. **(d)** 3D trajectory.

solutions of Equation 1.2 by utilizing the theory of cubic polynomial complete discriminant system and root formulas. In addition, some plots of the new solutions are showed using MATLAB software. Section 4 concludes the paper.

2 Dynamic analysis of Equation 1.2

In this section, we transform Equation 1.2 into a nonlinear ordinary differential equation using a complex traveling wave. Then, the dynamical behavior of Equation 1.2 is studied, which is exhibited through some phase portraits.

Implementing the following nonlinear complex wave transformation on Equation 1.2, we obtain

$$\zeta = \frac{pt^{\alpha}}{\alpha} + \frac{qx^{\beta}}{\beta}, \Theta(x,t) = \Theta(\zeta), \tag{2.1}$$

where p, q are both arbitrary constants and q is the velocity of the traveling wave. Substituting Equation 2.1 into Equation 1.2, we obtain

$$(p+q)\Theta' - 2aq\Theta\Theta' - bpq^2\Theta''' = 0, \tag{2.2}$$

where Θ' denotes the derivative of Θ with respect to ζ. Integrating Equation 2.2 with regard to ζ once and considering 0 as the integrating constant, we obtain

$$(p+q)\Theta - aq\Theta^2 - bpq^2\Theta'' = 0. \tag{2.3}$$

If $q = 0$ or $p = 0$, then $\Theta = 0$. Assuming that $p, q \neq 0$, we denote $\Theta' = y$. Therefore, Equation 2.3 can be transformed into the following planar dynamical system:

$$\begin{cases} \dfrac{d\Theta}{d\zeta} = y, \\ \dfrac{dy}{d\zeta} = \lambda_2\Theta^2 + \lambda_1\Theta \end{cases}, \tag{2.4}$$

using the following Hamiltonian function

$$H(\Theta, y) = \frac{1}{2}y^2 - \frac{1}{3}\lambda_2\Theta^3 - \frac{1}{2}\lambda_1\Theta^2 = \lambda_0, \tag{2.5}$$

where $\lambda_2 = -\frac{a}{bpq}, \lambda_1 = \frac{p+q}{bpq^2}$ and λ_0 are arbitrary constants. It is known that $\lambda_2 \neq 0$ due to $a \neq 0$.

Let $G(\Theta) = \lambda_2\Theta^2 + \lambda_1\Theta$. The equation $G(\Theta) = 0$ has two roots, i.e., $\Theta_1 = 0$ and $\Theta_2 = -\frac{\lambda_1}{\lambda_2}$. We denote

$$M = \begin{bmatrix} 0 & 1 \\ 2\lambda_2\Theta + \lambda_1 & 0 \end{bmatrix}. \tag{2.6}$$

The Jacobian determinant of Equation 2.4 is

$$J(\Theta) = |M| = -2\lambda_2\Theta - \lambda_1. \tag{2.7}$$

Next, the two equilibrium points of Equation 2.4 are discussed based on the bifurcation theory of planar dynamical systems [37]. The correlative conclusions are provided as follows:

(I) If $\lambda_1 < 0$, then $J(\Theta_1) = -\lambda_1 > 0, J(\Theta_2) = \lambda_1 < 0$, and $Trace(M) = 0$. So, there will be a center point at $(\Theta_1, 0)$ and a saddle point at $(\Theta_2, 0)$, as shown in Figures 1a,b.

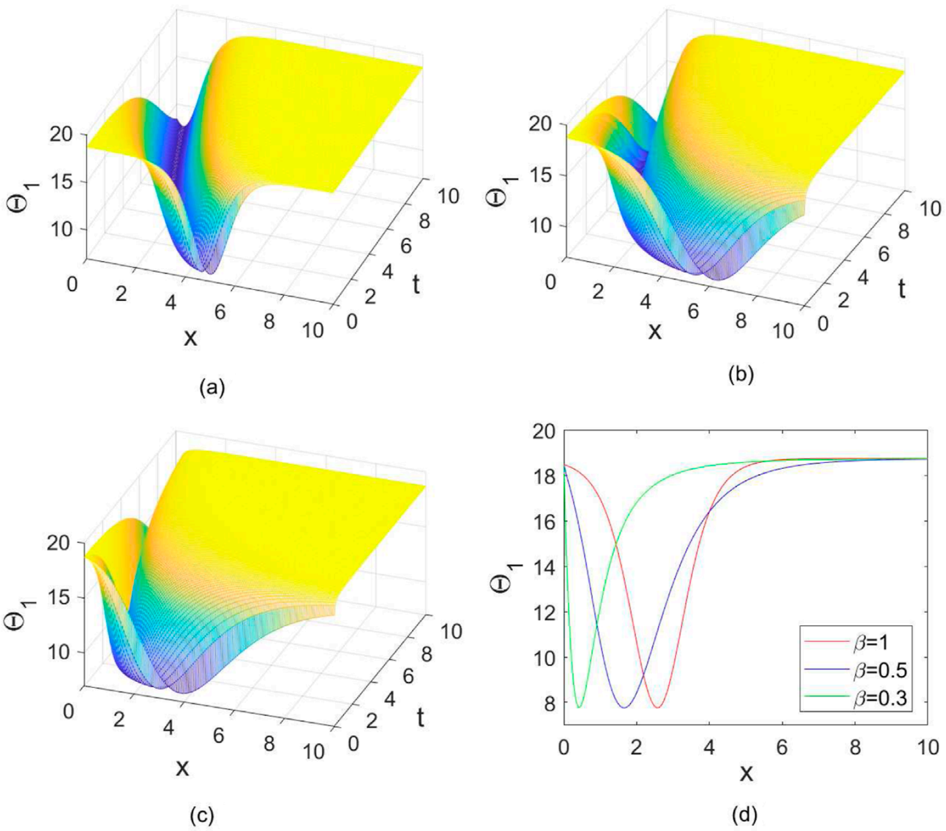

FIGURE 3
Figures of the solutions Θ_1 with $a = 2$, $b = -5$, $p = 3$, $q = 4$, $\alpha = 0.5$, $\eta_1 = 10$, $\eta_0 = 20$, $\zeta_0 = 20$. **(a)** $\beta = 1$. **(b)** $\beta = 0.5$. **(c)** $\beta = 0.3$. **(d)** Change in θ_1 with x when t = 2.6174.

(II) If $\lambda_1 > 0$, then $J(\Theta_1) = -\lambda_1 < 0, J(\Theta_2) = \lambda_1 > 0$, and $Trace(M) = 0$. Therefore, a saddle point will be at $(\Theta_1, 0)$ and a center point will be at $(\Theta_2, 0)$, as shown in Figures 1c,d.

(III) If $\lambda_1 = 0$, then $J(\Theta_1) = J(\Theta_2) = 0$. Thereby, $(\Theta_1, 0)$ and $(\Theta_2, 0)$ both are degraded points, as shown in Figures 1e,f.

Multiple attractors and bifurcation phenomena often occur in some nonlinear dynamical systems. Small perturbations can cause the system to shift from one attractor to another, causing the orbit of the system state to become irregular and chaotic. Therefore, we will explore whether Equation 2.4 has chaotic behavior in the case of small external perturbations. In order to simulate the chaotic phenomena in the system, a bounded periodic function can be added to Equation 2.4 as an uncertain perturbation factor. The new equations under the perturbation are described as

$$\begin{cases} \dfrac{d\Theta}{d\zeta} = y, \\ \dfrac{dy}{d\zeta} = \lambda_2\Theta^2 + \lambda_1\Theta + A\sin\varpi\zeta. \end{cases} \tag{2.8}$$

It can be found that when periodic perturbations are added, some systems become divergent, even though they were previously bounded, such as when $\lambda_1 = 1.1$ and $\lambda_2 = -0.6$. Some the phase portraits of Equation 2.8 with bounded phenomenon under the reasonable parameters are shown in Figures 2a–d.

3 Traveling wave solutions of Equation 1.2

In this section, some new exact traveling wave solutions of Equation 1.2 are studied based on the theory of the polynomial complete discriminant system [38] and root formula for a cubic polynomial equation. Finally, we demonstrate the solution structure using some two- or three-dimensional pictures.

3.1 Solving procedure

Equation 2.3 is integrated with regard to ζ once again to obtain

$$(\Theta')^2 = \eta_3\Theta^3 + \eta_2\Theta^2 + \eta_1\Theta + \eta_0, \tag{3.1}$$

where $\eta_3 = -\frac{2a}{3bpq}, \eta_2 = \frac{p+q}{bpq^2}$ and η_1 and η_0 are arbitrary constants. Let $\omega = (\eta_3)^{\frac{1}{3}}\Theta$, $m_2 = \eta_2(\eta_3)^{-\frac{2}{3}}$, $m_1 = \eta_1(\eta_3)^{-\frac{1}{3}}$, and $m_0 = \eta_0$. Then, Equation 3.1 can be transformed as

$$(\omega')^2 = \omega^3 + m_2\omega^2 + m_1\omega + m_0. \tag{3.2}$$

We can obtain the integral form of Equation 3.2 as follows

$$\pm(\eta_3)^{\frac{1}{3}}(\zeta - \zeta_0) = \int \frac{1}{\sqrt{\omega^3 + m_2\omega^2 + m_1\omega + m_0}} d\omega, \tag{3.3}$$

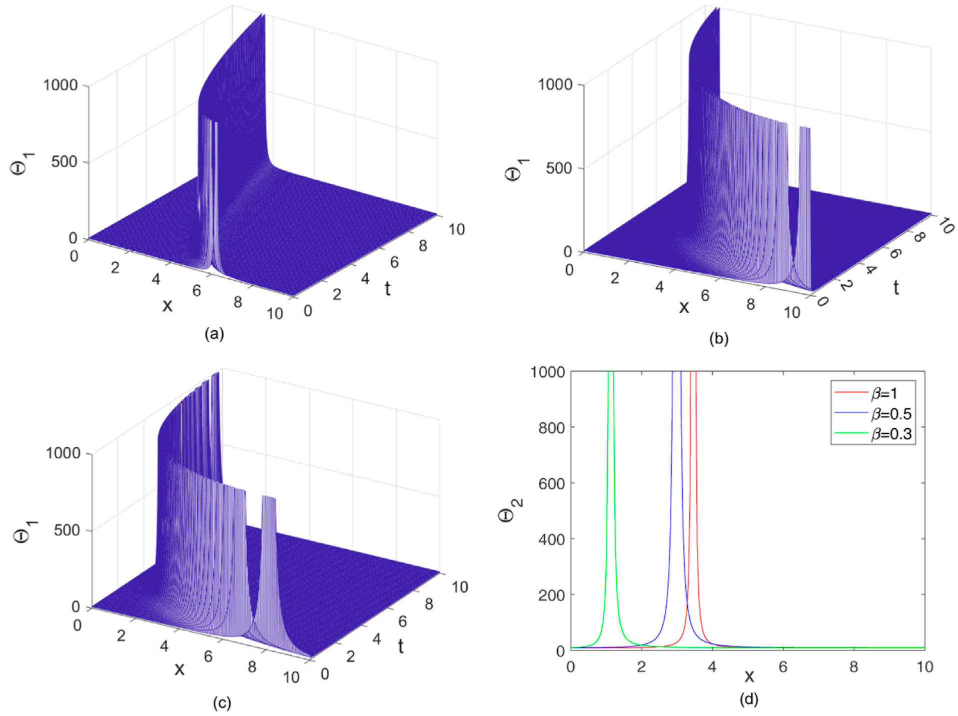

FIGURE 4
Figures of the solutions Θ_2 with $a = 5$, $b = -5$, $p = 3$, $q = 4$, $\alpha = 0.5$, $\eta_1 = 10$, $\eta_0 = 11$, $\zeta_0 = 25$. **(a)** $\beta = 1$. **(b)** $\beta = 0.5$. **(c)** $\beta = 0.3$. **(d)** Change in θ_2 with x when t = 3.4673.

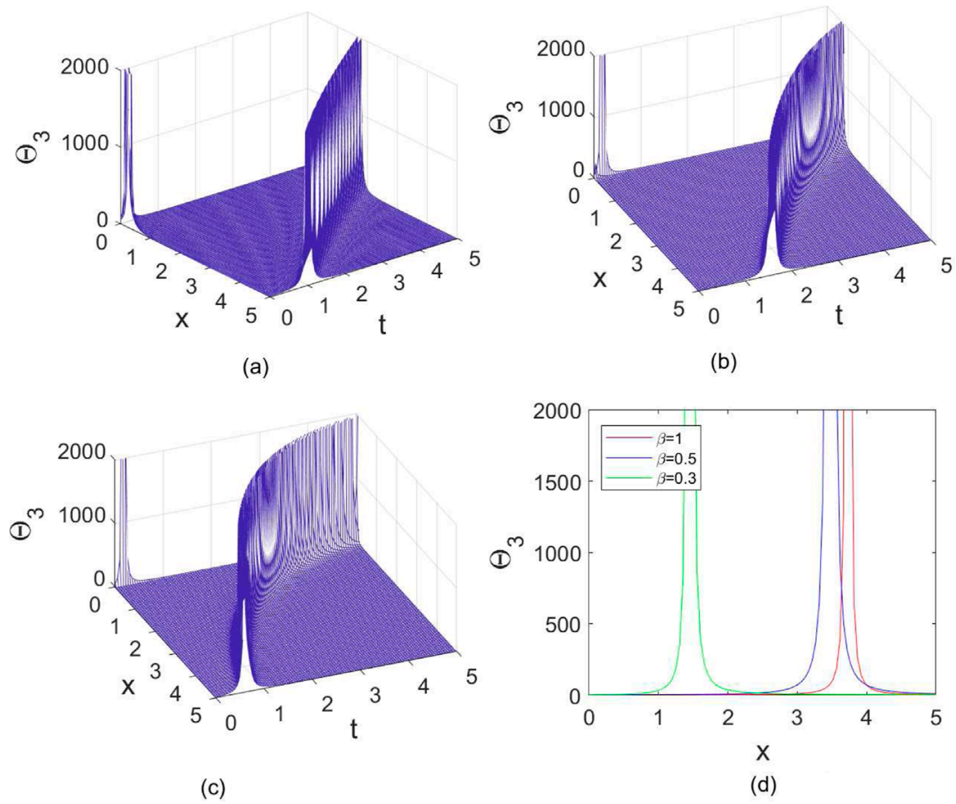

FIGURE 5
Figures of the solutions Θ_3 with $a = 6$, $b = -5$, $p = 3$, $q = 3$, $\alpha = 0.8$, $\eta_1 = 7$, $\eta_0 = 1$, $\zeta_0 = -8$. **(a)** $\beta = 1$. **(b)** $\beta = 0.5$. **(c)** $\beta = 0.3$. **(d)** Change in θ_3 with x when t = 2.4121.

where ζ_0 is the integration constant. Let $F(\omega) = \omega^3 + m_2\omega^2 + m_1\omega + m_0$. We can obtain the complete discrimination system of $F(\omega)$ as follows:

$$\begin{cases} \Delta = -27\left(\dfrac{2m_2^3}{27} + m_0 - \dfrac{m_1 m_0}{3}\right)^2 - 4\left(m_1 - \dfrac{m_2^2}{3}\right)^3 \\ H_1 = m_1 - \dfrac{m_2^2}{3} \end{cases}. \tag{3.4}$$

According to the complete discrimination system (3.4), the solution of Equation 1.2 has the following four situations.

Case 1. $\Delta = 0, H_1 < 0$. There is $-27\left(\frac{2m_2^3}{27} + m_0 - \frac{m_1 m_0}{3}\right)^2 = 4\left(m_1 - \frac{m_2^2}{3}\right)^3 < 0$. Then, $F(\omega)$ has two real roots and a single real root. Using the cubic derivation formula, we obtain

$$F(\omega) = (\omega - \upsilon)^2 (\omega - \mu),$$

where $\upsilon = -\frac{m_2 m_1 - 9m_0}{2(m_2^2 - 3m_1)}, \mu = -m_2 + \frac{m_2 m_1 - 9m_0}{m_2^2 - 3m_1}$. Then, the solutions of Equation 3.2 are

$$\Theta_1(\zeta) = (\eta_3)^{-\frac{1}{3}} \left\{ \left(m_2 - \frac{3(m_2 m_1 - 9m_0)}{2(m_2^2 - 3m_1)} \right) \cdot \tanh^2\left[\frac{1}{2}\left(m_2 - \frac{3(m_2 m_1 - 9m_0)}{2(m_2^2 - 3m_1)} \right)^{\frac{1}{2}} (\eta_3)^{\frac{1}{3}} (\zeta - \zeta_0) \right] - m_2 + \frac{m_2 m_1 - 9m_0}{m_2^2 - 3m_1} \right\}, \tag{3.5}$$

$$\Theta_2(\zeta) = (\eta_3)^{-\frac{1}{3}} \left\{ \left(m_2 - \frac{3(m_2 m_1 - 9m_0)}{2(m_2^2 - 3m_1)} \right) \cdot \coth^2\left[\frac{1}{2}\left(m_2 - \frac{3(m_2 m_1 - 9m_0)}{2(m_2^2 - 3m_1)} \right)^{\frac{1}{2}} (\eta_3)^{\frac{1}{3}} (\zeta - \zeta_0) \right] - m_2 + \frac{m_2 m_1 - 9m_0}{m_2^2 - 3m_1} \right\}, \tag{3.6}$$

$$\Theta_3(\zeta) = (\eta_3)^{-\frac{1}{3}} \left\{ \left(-m_2 + \frac{3(m_2 m_1 - 9m_0)}{2(m_2^2 - 3m_1)} \right) \cdot \tan^2\left[\frac{1}{2}\left(-m_2 + \frac{3(m_2 m_1 - 9m_0)}{2(m_2^2 - 3m_1)} \right)^{\frac{1}{2}} (\eta_3)^{\frac{1}{3}} (\zeta - \zeta_0) \right] - m_2 + \frac{m_2 m_1 - 9m_0}{m_2^2 - 3m_1} \right\}. \tag{3.7}$$

We substitute $\zeta = \frac{pt^\alpha}{\alpha} + \frac{qx^\beta}{\beta}$, $m_2 = \eta_2(\eta_3)^{-\frac{2}{3}}$, $m_1 = \eta_1(\eta_3)^{-\frac{1}{3}}$, $\eta_3 = -\frac{2a}{3bpq}$, and $\eta_2 = \frac{p+q}{bpq^2}$ into Equations 3.5–3.7. Then, the traveling wave solutions of Equation 1.2 are obtained as follows

$$\Theta_1(x,t) = -\left(\frac{3bpq}{2a}\right)^{\frac{1}{3}} \left\{ R_1 \tanh^2\left[\frac{R_1^{\frac{1}{2}}}{2} \left(-\frac{2a}{3bpq}\right)^{\frac{1}{3}} \left(\frac{pt^\alpha}{\alpha} + \frac{qx^\beta}{\beta} - \zeta_0 \right) \right] + R_2 \right\}, \tag{3.8}$$

$$\Theta_2(x,t) = -\left(\frac{3bpq}{2a}\right)^{\frac{1}{3}} \left\{ R_1 \coth^2\left[\frac{R_1^{\frac{1}{2}}}{2} \left(\frac{2a}{3bpq}\right)^{\frac{1}{3}} \left(\frac{pt^\alpha}{\alpha} + \frac{qx^\beta}{\beta} - \zeta_0 \right) \right] + R_2 \right\}, \tag{3.9}$$

$$\Theta_3(x,t) = -\left(\frac{3bpq}{2a}\right)^{\frac{1}{3}} \left\{ -R_1 \tanh^2\left[\frac{(-R_1)^{\frac{1}{2}}}{2} \left(-\frac{2a}{3bpq}\right)^{\frac{1}{3}} \left(\frac{pt^\alpha}{\alpha} + \frac{qx^\beta}{\beta} - \zeta_0 \right) \right] + R_2 \right\}, \tag{3.10}$$

where $R_1 = \frac{p+q}{bpq^2}\left(\frac{3bpq}{2a}\right)^{\frac{2}{3}} - \frac{9(\frac{\eta_1(p+q)}{2aq} - 3\eta_0)}{(\frac{3bpq}{2a})^{\frac{1}{3}} \frac{(p+q)^2 - 3\eta_1}{abpq^3}}, R_2 = -\frac{p+q}{bpq^2}\left(\frac{3bpq}{2a}\right)^{\frac{2}{3}} + \frac{3(\frac{\eta_1(p+q)}{2aq} - 3\eta_0)}{(\frac{3bpq}{2a})^{\frac{1}{3}} \frac{(p+q)^2 - 3\eta_1}{2abpq^3}}$.

Case 2. $\Delta = 0, H_1 = 0$. There is $-27\left(\frac{2m_2^3}{27} + m_0 - \frac{m_1 m_0}{3}\right)^2 = 4\left(m_1 - \frac{m_2^2}{3}\right)^3$ and $m_1 = \frac{m_2^2}{3}$. Therefore, $F(\omega)$ has three same real roots. Due to the cubic derivation formula, there is $F(\omega) = (\omega - \varrho)^3$, where $\varrho = -\frac{m_2}{3}$. Then, the solutions of Equation 3.2 are

$$\Theta_4(\zeta) = 4(\eta_3)^{-\frac{2}{3}} (\zeta - \zeta_0)^{-2} - \frac{m_2}{3}. \tag{3.11}$$

We substitute $\zeta = \frac{pt^\alpha}{\alpha} + \frac{qx^\beta}{\beta}$, $m_2 = \eta_2(\eta_3)^{-\frac{2}{3}}$, and $\eta_3 = -\frac{2a}{3bpq}$ into Equation 3.11. Then, the traveling wave solution of Equation 1.2 is obtained as follows:

$$\Theta_4(x,t) = 4\left(\frac{3bpq}{2a}\right)^{\frac{2}{3}} \left(\frac{pt^\alpha}{\alpha} + \frac{qx^\beta}{\beta} - \zeta_0 \right)^{-2} - \frac{p+q}{(12a^2 bpq^4)^{\frac{1}{3}}}. \tag{3.12}$$

Case 3. $\Delta > 0, H_1 < 0$. There is $-27\left(\frac{2m_2^3}{27} + m_0 - \frac{m_1 m_0}{3}\right)^2 - 4\left(m_1 - \frac{m_2^2}{3}\right)^3 > 0$ and $m_1 - \frac{m_2^2}{3} < 0$. Then, $F(\omega)$ has three real different roots. Because of the cubic derivation formula, we obtain

$$F(\omega) = (\omega - \upsilon)(\omega - \mu)(\omega - \varrho),$$

where

$$\upsilon = \frac{-m_2 - 2(m_2^2 - 3m_1)^{\frac{1}{2}} \cos\frac{\theta}{3}}{3},$$

$$\mu = \frac{-m_2 + (m_2^2 - 3m_1)^{\frac{1}{2}} \left(\cos\frac{\theta}{3} - 3^{\frac{1}{2}} \sin\frac{\theta}{3}\right)}{3},$$

$$\varrho = \frac{-m_2 + (m_2^2 - 3m_1)^{\frac{1}{2}} \left(\cos\frac{\theta}{3} + 3^{\frac{1}{2}} \sin\frac{\theta}{3}\right)}{3},$$

$$\theta = \arccos\frac{2(m_2^2 - 3m_1) - 3m_2 m_1 + 27m_0}{2(m_2^2 - 3m_1)^{\frac{2}{3}}}.$$

Furthermore, $\upsilon < \mu < \varrho$ as $\theta \in (0, \pi)$.

If $\upsilon < \omega < \varrho$, then the solution of Equation 3.2 is

$$\Theta_5(\zeta) = (\eta_3)^{-\frac{1}{3}} \left[\upsilon + (\mu - \upsilon)\,\mathrm{sn}^2\left(\frac{1}{2}(\varrho - \upsilon)^{\frac{1}{2}} (\eta_3)^{\frac{1}{3}} (\zeta - \zeta_0), m \right) \right], \tag{3.13}$$

where $m = \left(\frac{\cos(\frac{\pi}{6} + \frac{\theta}{3})}{\cos(\frac{\pi}{6} - \frac{\theta}{3})} \right)^{\frac{1}{2}}$ and sn is the Jacobi elliptic sine function. Replacing these variables ζ, m_2, m_1, η_3, and η_2 by their specific expressions for Equation 3.13, the traveling wave solutions of Equation 1.2 can be obtained as follows:

$$\Theta_5(x,t) = -\left(\frac{3bpq}{2a}\right)^{\frac{1}{3}} \left[\frac{-p-q}{(12a^2bpq^4)^{\frac{1}{3}}} - \frac{2}{3}R_1 \cos\frac{\theta}{3} + \frac{\sqrt[3]{3}R_1}{3}\cos\left(\frac{\pi + 2\theta}{6}\right) \times \mathrm{sn}^2\left(-\frac{a^{\frac{1}{3}} \cos^{\frac{1}{2}}\left(\frac{\pi - 2\theta}{6}\right)}{(12bpq)^{\frac{1}{3}}} \left(\frac{pt^\alpha}{\alpha} + \frac{qx^\beta}{\beta} - \zeta_0 \right), m \right) \right]. \tag{3.14}$$

If $\omega > \varrho$, then the solution of Equation 3.2 is

$$\Theta_6(\zeta) = (\eta_3)^{-\frac{1}{3}} \left[\frac{\varrho - \mu\,\mathrm{sn}^2\left(\frac{1}{2}(\varrho - \upsilon)^{\frac{1}{2}} (\eta_3)^{\frac{1}{3}} (\zeta - \zeta_0), m \right)}{\mathrm{cn}^2\left(\frac{1}{2}(\varrho - \upsilon)^{\frac{1}{2}} (\eta_3)^{\frac{1}{3}} (\zeta - \zeta_0), m \right)} \right], \tag{3.15}$$

where cn is the Jacobi elliptic cosine function. Substituting the specific expression of these variables ζ, m_2, m_1, η_3, and η_2 into

Equation 3.15, the traveling wave solutions of Equation 1.2 can be given as follows

$$\Theta_6(x,t) = -\left(\frac{3bpq}{2a}\right)^{\frac{1}{3}} \left[\frac{\varrho - \mu \mathrm{sn}^2\left(\frac{-a^{\frac{1}{3}}\cos^{\frac{1}{2}}\left(\frac{\pi-2\theta}{6}\right)}{(12bpq)^{\frac{1}{3}}}\left(\frac{pt^\alpha}{\alpha}+\frac{qx^\beta}{\beta}-\zeta_0\right), m\right)}{\mathrm{cn}^2\left(\frac{-a^{\frac{1}{3}}\cos^{\frac{1}{2}}\left(\frac{\pi-2\theta}{6}\right)}{(12bpq)^{\frac{1}{3}}}\left(\frac{pt^\alpha}{\alpha}+\frac{qx^\beta}{\beta}-\zeta_0\right), m\right)}\right], \quad (3.16)$$

where

$$\varrho = \frac{-p-q}{(12a^2bpq^4)^{\frac{1}{3}}} + \frac{1}{3}R_1\cos\left(\frac{\pi-2\theta}{6}\right),$$
$$\mu = \left(\frac{-p-q}{(12a^2bpq^4)^{\frac{1}{3}}} + \frac{1}{3}R_1\cos\left(\frac{\pi+2\theta}{6}\right)\right),$$
$$\theta = \arccos\frac{2R_1 + \frac{243^{\frac{1}{3}}\eta_1(p+q)}{2aq} + 27m_0}{2R_1^{\frac{2}{3}}}.$$

Case 4. $\Delta < 0, H_1 < 0$. There is $-27\left(\frac{2m_2^3}{27} + m_0 - \frac{m_1 m_0}{3}\right)^2 - 4\left(m_1 - \frac{m_2^2}{3}\right)^3 < 0$. Then, $F(\omega)$ has only one real root. By applying the cubic derivation formula, we obtain

$$F(\omega) = (\omega - \upsilon)(\omega^2 + \mu\omega + \varrho),$$

where

$$\upsilon = -\frac{1}{3}(m_2 + Y),\ \mu = \frac{1}{3}(-2m_2 + Y),\ \varrho = \frac{1}{9}m_2^2 + \frac{1}{36}\left(\sqrt{3} - 4m_2\right)Y^2,$$
$$Y = \sqrt[3]{m_2^3 - 9m_2m_1 + \frac{27}{2}m_0 + \frac{3}{2}(-3m_2^2m_1^2 - 54m_2m_1m_0 + 81m_0^2 + 12m_2^3m_0 + 12m_1^3)^{\frac{1}{2}}}$$
$$+\sqrt[3]{m_2^3 - 9m_2m_1 + \frac{27}{2}m_0 - \frac{3}{2}(-3m_2^2m_1^2 - 54m_2m_1m_0 + 81m_0^2 + 12m_2^3m_0 + 12m_1^3)^{\frac{1}{2}}}.$$

If $\omega > \upsilon$, then the solution of Equation 3.2 is given as

$$\Theta_7(\zeta) = (\eta_3)^{-\frac{1}{3}} \cdot \left[\upsilon + \frac{2\sqrt{\upsilon^2+\upsilon\mu+\varrho}}{1+\mathrm{cn}\left(\left(\upsilon^2+\upsilon\mu+\varrho\right)^{\frac{1}{4}}(\eta_3)^{-\frac{1}{3}}(\zeta-\zeta_0), \frac{1}{2}\left(1-\frac{\alpha+\frac{1}{2}\beta}{\upsilon^2+\upsilon\mu+\varrho}\right)\right)} - \sqrt{\upsilon^2+\upsilon\mu+\varrho}\right]. \quad (3.17)$$

Substituting the specific expression of the following variables ζ, m_2, m_1, η_3, and η_2 into Equation 3.17, the traveling wave solutions of Equation 1.2 can be obtained as follows

$$\Theta_7(x,t) = -\left(\frac{3bpq}{2a}\right)^{\frac{1}{3}} \cdot \left[\frac{-p-q}{\left(36a^2bpq^4\right)^{\frac{1}{3}}} - \frac{1}{3}R_3 + \frac{2\sqrt{R_4}}{1+\mathrm{cn}\left((R_4)^{\frac{1}{4}} - \left(\frac{3bpq}{2a}\right)^{\frac{1}{3}}\left(\frac{pt^\alpha}{\alpha}+\frac{qx^\beta}{\beta}-\zeta_0\right), n\right)} - \sqrt{R_4}\right], \quad (3.18)$$

where

$$R_3 = \sqrt[3]{\frac{9(p+q)^3}{4a^2bpq^4} + \frac{9\sqrt[3]{9}\eta_1(p+q)}{2aq} + \frac{27m_0}{2} + \frac{3}{2}W}$$
$$+\sqrt[3]{\frac{9(p+q)^3}{4a^2bpq^4} + \frac{9\sqrt[3]{9}\eta_1(p+q)}{2aq} + \frac{27m_0}{2} - \frac{3}{2}W},$$
$$R_4 = \frac{4^{\frac{1}{3}}(p+q)^2}{(9a^4b^2p^6q^2)^{\frac{1}{3}}} + \frac{R_3(p+q)}{(48a^2bpq^4)^{\frac{1}{3}}} - \frac{(p+q)R_3^2}{(324a^2bpq^4)^{\frac{1}{3}}} + \frac{\sqrt{3}}{36}R_3^2,$$
$$W = \left(-\frac{27\eta_1(p+q)^2}{4a^2q^2} + \frac{162m_0\eta_1(p+q)}{2aq} + \frac{27m_0(p+q)^3}{a^2bpq^4} - \frac{18\eta_1^3bpq}{2a} + 81m_0^2\right)^{\frac{1}{2}},$$
$$n = \frac{1}{2} + \frac{(p+q)}{8(6a^2bpq^4)^{\frac{1}{3}}R_4} + \frac{R_3}{12R_4}.$$

3.2 Graphical description

To visualize the structure of these new solutions, the solutions Θ_1, Θ_2 and Θ_3 are described in the form of two- or three-dimensional pictures (see Figures 3–5). According to the derivation conditions, the appropriate parameters are taken to produce the graphs of the solutions. In each traveling wave solution, there are two fractional derivative parameters, α and β. We fix $\alpha = 0.5$ and observe the effect of β on the shapes of the solutions. As observed from these comparison graphs, the smaller the value of β, the more curved the shape of the solutions.

4 Conclusion

This paper analyzes the dynamical behavior of the space–time fractional ZKBBM equation and presents seven types of new exact traveling wave solutions by utilizing the theory of the cubic polynomial complete discriminant system and root formulas. These new solutions, including rational, trigonometric, hyperbolic, and Jacobi elliptic function solutions, can be directly applied to simulation, prediction, and control in practical scenarios. Finally, the phase portraits and some of the solutions are plotted using MATLAB software. From these figures, we can clearly and intuitively understand the properties of the equation and the shapes of its solutions under different conditions.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material; further inquiries can be directed to the corresponding author.

Author contributions

SZ: Writing – original draft. ZL: Writing – review and editing.

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