frontiers | Frontiers in Physics

TYPE Original Research
PUBLISHED 20 June 2025
DOI 10.3389/fphy.2025.1562805

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OPEN ACCESS

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Adriano Tiribocchi,
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REVIEWED BY
Asad Ullah,
Karakoram International University, Pakistan
Jian Zhu,
China University of Mining and Technology, China

*CORRESPONDENCE
Chao Liu,
liu.chao@shufe.edu.cn

RECEIVED 18 January 2025
ACCEPTED 13 May 2025
PUBLISHED 20 June 2025

CITATION
Liu C (2025) First- and second-order network coherence in N-duplication weighted corona networks.
Front. Phys. 13:1562805.
doi: 10.3389/fphy.2025.1562805

First- and second-order network coherence in N-duplication weighted corona networks

Chao Liu*

School of Mathematics, Shanghai University of Finance and Economics, Shanghai, China

This paper studies first- and second-order coherence problems for N-duplication weighted corona networks subject to stochastic disturbances. Explicit expressions of the coherence for first-order (and second-order) dynamics, which are determined by the sum of the reciprocal (and square of reciprocal) of each nonzero eigenvalue of the Laplacian matrix, are derived. In particular, for both first- and second-order systems, the analytical formulas of the network coherence are presented from two different perspectives. Based on these formulas, the influence of the duplication N, the weight ω, and the factor networks G_1 and G_2 on the network coherence of the corona network $G_1 \circ G_2$ is investigated. Some noteworthy topological properties of the N-duplication weighted corona network are also revealed.

KEYWORDS

robustness, network coherence, N-duplication weighted corona network, Laplacian spectrum, join operation

1 Introduction

Over the past few years, technological development in communication networks has greatly increased interest in distributed coordination for networks of dynamical agents. As one of the fundamental problems in cooperative control, the consensus problem for multi-agent systems has been investigated from various perspectives [1–5]. In the context of networks (graphs) of agents, consensus means that agents represented by nodes (vertices) reach an agreement on a certain issue, such as pace, load, or direction and velocity.

In realistic applications, communication between agents is often degraded due to environmental uncertainty or communication uncertainty, for example, thermal, fading channel, and quantization noises during encoding and decoding. Without uncertainties, it is well known that when the graph is connected, the states of autonomous systems converge exponentially to the average of the initial state values. In the presence of stochastic disturbances, however, the state evolution becomes a stochastic process and fluctuates around the average of the current node states. Thus, it is of great interest to consider how robust distributed consensus algorithms are to external disturbances [6–16]. Network coherence [8–16] quantifies the steady-state variance of these fluctuations and can be considered a measure of the robustness of the consensus process to the additive noise. Networks with small steady-state variance have high network coherence and can be considered to be more robust to noise than networks with low coherence [13–16]. For both first- and second-order systems, network coherence can be measured by the H_2-norm of the consensus errors, which can be characterized by the spectrum of the Laplacian matrix of the underlying communication graph [15, 16].

Because massive networks often consist of small pieces, for example, communities [17] and motifs [18], graph products, by which one can build a large network out of two or more smaller ones, are widely used as an effective method for generating large-scale networks. Analysis of product networks offers critical insights into understanding the dynamics of real-world large-scale networks. Specifically, graph products have been explored to construct and reveal structural and functional relationships between factor systems and the associated composite system [19–22].

Common graph products include direct products and strong products [23], Cartesian products [24], Kronecker products [25], and corona products [26]. Among them, corona product graphs have attracted a great deal of attention due to their complex but unique structures and wide range of applications in coding theory, DNA sampling, UAV formation, and some special chemical and biological structures or communities [27–34]. The concept of the corona product of graphs was first introduced by Frucht and Harary [27]. In [28], the authors introduced the edge corona of graphs and calculated the corresponding spectrum. The properties of spectra and Laplacian spectra of corona products have been extensively studied in spectral graph theory [35–38]. Some recently widely concerned indices, such as the Sombor index and the Kirchhoff index, have been derived from corona product graphs [39, 40]. Notably, some related advancements in graph theory have been reported in [41–45]. In the literature, based on the spectral analysis of the Laplacian and normalized Laplacian matrices, Kemeny's constant, global mean-first-passage time of random walks, and the number of spanning trees were studied in various network structures. Specifically, Kemeny's constant represents the cumulative sum of relaxation time scales and has specific applications in computing a graph's Kirchhoff index. These research achievements and methods are enlightening for further studies on corona networks.

The first-order coherence of weighted corona networks was examined from the weighted Laplacian spectra perspective in [46]. It is noteworthy that, in addition to the basic corona product investigated in [46], multiple variants of corona operation have been introduced and studied, including edge corona [35], neighborhood corona [36], subdivision double corona, Q-graph double corona, R-graph double corona [37], and iterative corona [38], etc. Therefore, further research on consensus algorithms of various kinds of corona networks is necessary. Moreover, little research has clearly addressed the relationship between the coherence of the corona network and that of its factor networks, and research results on the second-order or higher-order coherence of corona networks are still rare.

The multilayer network is a frontier research branch of network science. The multilayered structure has many examples in reality, for instance, the interactions between the power grid and the Internet, friendship and family relations, or transportation and aviation networks [47, 48]. Lately, a multilayered graph based on the duplication of corona products was introduced in [49, 50]. Varghese and Susha [49] determined the Laplacian spectrum and discussed the number of spanning trees, the Kirchhoff index, and the incidence energy of the graph. The controllability of the N-duplication corona product network $G_1 \circ G_2$ was investigated in [50]. An example of this N-duplication corona network is shown in Figure 1, where G_1 and G_2 are complete graphs of order 5 and 2, respectively.

With the introduction of duplication, the classic corona graph is generalized from a single-layer structure to a multilayered structure. It is necessary and significant to extend the consensus theory to the N-duplication corona product network, which includes the basic corona network as a special case with the duplication $N = 1$. The intricate topological configurations of the N-duplication corona product network not only compounds analytical challenges in coherence studies but also raises new research questions. What influence will the duplication N or the weight factor ω have on the network coherence? What is the relationship between the coherence of the corona network $G_1 \circ G_2$ and that of its factor networks G_1 and G_2? In comparison, which factor network plays a more important role in the determination of the coherence of the composite corona network? These natural and interesting questions deserve to be considered.

Inspired by these questions, this study explores the robustness of consensus algorithms for N-duplication weighted corona networks when the nodes are subject to external perturbations. The coherence in corona product networks composed of first- or second-order dynamic agents is studied, aiming to obtain exact solutions of network coherence and unveil the relationship between the network topology and network coherence. The main contributions of this work are three-fold. First, the explicit expressions for the coherence of the first- and second-order noisy consensus algorithms in N-duplication weighted corona networks are obtained. The results of [46] are now a special case of this study, and more detailed and noteworthy analysis is presented in this work. Second, for both first- and second-order consensus algorithms, the impacts of the duplication N and the weight ω on the network coherence are explored. It is found that corona networks with larger duplication N or higher weight ω usually have higher network coherence and can be considered to be more robust to noise. Note that the property of high coherence of the network with large duplication N can be regarded as a special topological characteristic of the N-duplication weighted corona network. Finally, based on the obtained formulas, the relationship between the coherence of the composite corona network $G_1 \circ G_2$ and that of the factor networks G_1 and G_2 is investigated. Little work has been done from this perspective on the study of coherence problems. It is revealed that, for both first- and second-order consensus algorithms, higher coherence of G_1 or G_2 usually also leads to higher coherence of the corona network $G_1 \circ G_2$. Especially in the situation of large duplication N or high weight ω, the network coherence of the corona network $G_1 \circ G_2$ is mainly determined by the factor network G_1. The results presented in this study not only contribute to the theoretical understanding of network coherence but also provide practical insights into how different parameters and network structures can be optimized for better coherence, which is crucial for the design and analysis of complex networks in various applications such as sensor networks, social networks, and biological networks.

Abbreviations: $\circ$ N-duplication corona product; $\vee$ join operation; $\bigcup$ union of sets; $\otimes$ Kronecker product; R real number field; I_k k-dimensional identity matrix; 1_k k-dimensional vector with all elements being 1.

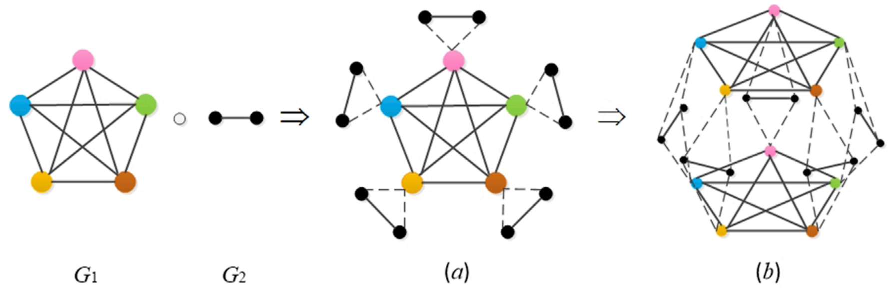

FIGURE 1
Illustration of N-duplication corona graphs $G_1 \circ G_2$ composed of G_1 and G_2, where **(a)** and **(b)** indicate 1-duplication and 2-duplication corona graphs, respectively.

The rest of this paper is organized as follows. Section 2 reviews the definition of N-duplication weighted corona network, the first- and second-order network coherence, and the relationship between the network coherence and the Laplacian spectrum of the underlying graph. Section 3 shows the explicit analytical results of first- and second-order network coherence in N-duplication weighted corona networks. The influence of the duplication N, the weight ω, and the factor networks G_1 and G_2 on the network coherence of the corona network $G_1 \circ G_2$ is investigated. Several simulation examples are presented in Section 4. Finally, Section 5 draws the conclusion.

2 Notations and preliminaries

This section briefly reviews the definition of the first- and second-order network coherence and the formation of N-duplication weighted corona networks and introduces some lemmas that will be used in the sequel.

2.1 Notations

Some available notations used in this study are given in the abbreviations.

2.2 Coherence in networks with first-order dynamics

Consider a network with underlying undirected graph $G = (V, E, A)$, where $V = \{v_1, \ldots, v_n\}$ is the set of nodes, $E \subseteq V \times V$ is the set of edges, and $A = [a_{ij}] \in R^{n\times n}$ is the weighted adjacency matrix. The Laplacian matrix L is defined as $L = D - A$, where D is the degree matrix. The network has consensus dynamics modeled by the stochastic differential equation

$$\dot{x}(t) = -Lx(t) + \chi(t), \tag{1}$$

where $x(t) \in R^n$ is the state of the system, L is the Laplacian matrix, and $\chi(t)$ is an n-vector of zero-mean and unit variance white noise. The network coherence of the first-order system in Equation 1 is denoted by the mean steady-state variance of the deviation from the average of all node values [7–9], that is,

$$H_1(G) \coloneqq \lim_{t\to\infty} \frac{1}{n}\sum_{i=1}^{n} var\left\{x_i(t) - \frac{1}{n}\sum_{j=1}^{n} x_j(t)\right\}.$$

It has been shown [6–9] that $H_1(G)$ can be completely determined by the eigenvalues of the Laplacian matrix L,

$$H_1(G) = \frac{1}{2n}\sum_{i=2}^{n}\frac{1}{\lambda_i}, \tag{2}$$

where $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n$ are the Laplacian eigenvalues.

2.3 Coherence in networks with second-order dynamics

In the second-order system, such as the vehicle formation problem, the node states consist of a position vector $x(t) \in R^n$ and a velocity vector $v(t) \in R^n$. The second-order consensus dynamics subject to noise are described by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -L & -L \end{bmatrix}\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix}\omega(t), \tag{3}$$

where $\omega(t)$ is a disturbance vector with zero-mean and unit variance.

The network coherence of the second-order system in Equation 3 is defined in terms of $x(t)$ only and denoted by the mean (over all nodes) and the steady-state variance of the deviation from the average of $x(t)$,

$$H_2(G) \coloneqq \lim_{t\to\infty} \frac{1}{n}\sum_{i=1}^{n} var\left\{x_i(t) - \frac{1}{n}\sum_{j=1}^{n} x_j(t)\right\}.$$

This value is also completely determined by the eigenvalues of the Laplacian matrix [8, 9, 15],

$$H_2(G) = \frac{1}{2n}\sum_{i=2}^{n}\frac{1}{(\lambda_i)^2}, \tag{4}$$

where $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n$ are the Laplacian eigenvalues.

Note that networks with smaller steady-state variance $H_1(G)$ or $H_2(G)$ have higher network coherence and can be considered to be more robust to noise than networks with lower coherence [11–14].

The following lemma gives the well-known Vieta's formulas, which will be leveraged in the following coherence analysis.

Lemma 1: [52] *Let $p(x)=a_nx^n+\cdots+a_1x+a_0$ be a real polynomial of degree $n\geq 2$ with $a_0\neq 0$. It shows*

$$\sum_{k=1}^{n}\frac{1}{\rho_k}=-\frac{a_1}{a_0},$$

and

$$\sum_{k=1}^{n}\frac{1}{\rho_k^2}=\left(\frac{a_1}{a_0}\right)^2-2\frac{a_2}{a_0},$$

where $\rho_k(1\leq k\leq n)$ are the roots of $p(x)$.

2.4 *N*-duplication weighted corona networks

As an extension of the classic corona network, an N-duplication weighted corona network was recently introduced in the literature [49, 50].

Definition 1: [49, 50] *Let G_1 and G_2 be two finite, simple, nonempty, and vertex-disjoint graphs with n_1 and n_2 vertices, respectively. The N-duplication weighted corona product $G_1\circ G_2$ is generated by taking N copies of G_1 and n_1 copies of G_2 and then joining the ith vertex of each G_1 to every vertex in the ith copy of G_2 $(i=1,2,3,\ldots,n_1)$. All the weights of the newly added edges between G_1 and G_2 are the same and positive, denoted as ω.*

Let L_1 and L_2 be the Laplacian matrices of G_1 and G_2 respectively, then the Laplacian matrix of the N-duplication weighted corona product $G_1\circ G_2$ is

$$L=\begin{bmatrix} I_N\otimes\left(L_1+n_2\omega I_{n_1}\right) & -1_N\otimes\left(\omega I_{n_1}\otimes 1_{n_2}^T\right)\\ -1_N^T\otimes\left(\omega I_{n_1}\otimes 1_{n_2}\right) & I_{n_1}\otimes\left(L_2+N\omega I_{n_2}\right)\end{bmatrix}. \tag{5}$$

The join operation of two disjoint graphs is also an effective method for generating large-scale networks [51]. As an extension, [46] presented the weighted join operation as below.

Definition 2: [46, 51] *The join of two disjoint graphs G_1 and G_2, denoted by $G_1\vee G_2$, is the graph with vertex set $V(G_1)\bigcup V(G_2)$ and the edge set $E(G_1)\bigcup E(G_2)\bigcup\{(u,v),$ for each $u\in V(G_1)$ and $v\in V(G_2)\}$. (u,v) represents the added edge joining u and v. Each link (u,v) has the same and positive weight, called the join-weight of $G_1\vee G_2$.*

In [49, 50], the characteristic polynomial of the N-duplication weighted corona graph $G_1\circ G_2$ was presented based on the eigenvalues of the factor graphs G_1 and G_2.

Lemma 2: [49, 50] *Let $\sigma(G_1)=\{\lambda_1,\lambda_2,\ldots,\lambda_{n_1}|0=\lambda_1<\lambda_2\leq\cdots\leq\lambda_{n_1}\}$ and $\sigma(G_2)=\{\mu_1,\mu_2,\ldots,\mu_{n_2}|0=\mu_1<\mu_2\leq\cdots\leq\mu_{n_2}\}$ be the Laplacian spectrum sets of G_1 and G_2, respectively. Then, the Laplacian characteristic polynomial of the N-duplication weighted corona product $G_1\circ G_2$ with the Laplacian matrix Equation 5 is*

$$\Phi(L{:}\gamma)=\gamma\left(\gamma-(N\omega+n_2\omega)\right)\prod_{j=2}^{n_2}\left(\gamma-N\omega-\mu_j\right)^{n_1}\prod_{i=1}^{n_1}\left(\gamma-n_2\omega-\lambda_i\right)^{N-1}\prod_{i=2}^{n_1}\left(\gamma^2-\left(N\omega+n_2\omega+\lambda_i\right)\gamma+N\omega\lambda_i\right). \tag{6}$$

3 Main results

This section studies the first- and second-order coherence problems for N-duplication weighted corona networks, where vertices are subject to white noise. Note that the graphs G_1, G_2 and the N-duplication weighted corona network $G_1\circ G_2$ used in the sequel are all as defined in Def. 1.

3.1 First-order coherence of the *N*-duplication weighted corona network

Theorem 1: *Let $\sigma(G_1)=\{\lambda_1,\lambda_2,\ldots,\lambda_{n_1}|0=\lambda_1<\lambda_2\leq\cdots\leq\lambda_{n_1}\}$ and $\sigma(G_2)=\{\mu_1,\mu_2,\ldots,\mu_{n_2}|0=\mu_1<\mu_2\leq\cdots\leq\mu_{n_2}\}$ be the Laplacian spectrum sets of G_1 and G_2, respectively. Then the first-order coherence of the N-duplication weighted corona network $G_1\circ G_2$ can be described as follows:*

$$H_1\left(G_1\circ G_2\right)=\frac{1}{2n_1N}\sum_{i=2}^{n_1}\frac{1}{\lambda_i}+\frac{N-1}{2n_1\left(n_2+N\right)}\sum_{i=1}^{n_1}\frac{1}{n_2\omega+\lambda_i}+\frac{1}{2\left(n_2+N\right)}\sum_{j=2}^{n_2}\frac{1}{N\omega+\mu_j}+\frac{1}{2n_1(n_2+N)^2\omega}+\frac{n_1-1}{2n_1\left(n_2+N\right)N\omega}. \tag{7}$$

Proof. By Def. 1, the N-duplication weighted corona network $G_1\circ G_2$ contains $n_1(n_2+N)$ vertices. From Lem. 2, the Laplacian eigenvalues of $G_1\circ G_2$ are the roots of $\Phi(L{:}\gamma)=0$. Then, the Laplacian spectrum of $G_1\circ G_2$ consists of

(i) 0;
(ii) $N\omega+n_2\omega$;
(iii) $N\omega+\mu_j$, repeated n_1 times for $j=2,3,\ldots,n_2$;
(iv) $n_2\omega+\lambda_i$, repeated $N-1$ times for $i=1,2,\ldots,n_1$;
(v) Two roots of the equation

$$\gamma^2-\left(N\omega+n_2\omega+\lambda_i\right)\gamma+N\omega\lambda_i=0,\ i=2,3,\ldots,n_1.$$

From Equation 2, the first-order network coherence of $G_1\circ G_2$ is determined by the sum of the inverses of nonzero Laplacian eigenvalues. Consider the last factor of the Laplacian characteristic polynomial defined in Equation 6:

$$\prod_{i=2}^{n_1}\left(\gamma^2-\left(N\omega+n_2\omega+\lambda_i\right)\gamma+N\omega\lambda_i\right). \tag{8}$$

Let a_0 and a_1 denote the constant term and the coefficient of the linear term of Equation 8, respectively. Then,

$$\begin{aligned} a_0&=\prod_{i=2}^{n_1}\left(N\omega\lambda_i\right),\\ a_1&=-\prod_{i=2}^{n_1}\left(N\omega\lambda_i\right)\sum_{i=2}^{n_1}\left[\left(N\omega+n_2\omega+\lambda_i\right)/\left(N\omega\lambda_i\right)\right],\end{aligned} \tag{9}$$

and

$$-\frac{a_1}{a_0}=\sum_{i=2}^{n_1}\frac{N\omega+n_2\omega+\lambda_i}{N\omega\lambda_i}=\frac{n_1-1}{N\omega}+\frac{N+n_2}{N}\sum_{i=2}^{n_1}\frac{1}{\lambda_i}. \tag{10}$$

Combining the Laplacian spectrum of $G_1\circ G_2$ as shown in (i)-(v), Lemma 1, and Equations 2, 10, one obtains

$$H_1(G_1 \circ G_2) = \frac{1}{2n_1(n_2+N)}\left(\frac{1}{(n_2+N)\omega} + \frac{n_1-1}{N\omega} + \frac{N+n_2}{N}\sum_{i=2}^{n_1}\frac{1}{\lambda_i} + \sum_{i=1}^{n_1}\frac{N-1}{n_2\omega+\lambda_i} + \sum_{j=2}^{n_2}\frac{n_1}{N\omega+\mu_j}\right)$$
$$= \frac{1}{2n_1N}\sum_{i=2}^{n_1}\frac{1}{\lambda_i} + \frac{N-1}{2n_1(n_2+N)}\sum_{i=1}^{n_1}\frac{1}{n_2\omega+\lambda_i} + \frac{1}{2(n_2+N)}\sum_{j=2}^{n_2}\frac{1}{N\omega+\mu_j}$$
$$+ \frac{1}{2n_1(n_2+N)^2\omega} + \frac{n_1-1}{2n_1(n_2+N)N\omega}.$$

The proof is completed.

Remark 1: *From Equation 7, Theorem 1 implies that $H_1(G_1 \circ G_2)$ decreases as the duplication N or the weight factor ω increases. Accordingly, corona networks $G_1 \circ G_2$ with larger duplication N or weight factor ω have higher first-order network coherence and can be considered to be more robust to noise than networks with smaller duplication or weight factors.*

It is worth noting that the phenomenon of high coherence of the corona network $G_1 \circ G_2$ with a large duplication N is interesting because it differs from the results reported in prior literature, such as [14, 15]. [14] considered the first-order network coherence in a kind of 5-rose graph and found that 5-rose networks with small network sizes have high network coherence. In [15], the authors investigated the coherence problem of the Koch network and revealed that enhancing the iteration or the network size of the Koch network will reduce the network coherence. Thus, high coherence or strong robustness of the corona network $G_1 \circ G_2$ with a large duplication N (thereby large network size) can be regarded as a distinctive topological characteristic that may lead to significant application value. For example, in [8], the vehicular formation control problem was studied based on the analysis of performance measures in large-scale networks. It is found that the network coherence, which varies with network size and dimension, plays an important role in the performance limitation of the vehicle formation. From this point of view, the N-duplication weighted corona network $G_1 \circ G_2$ can be considered a graph with good robustness to external disturbances, which provides new insights into its practical applications.

In Equation 7, $H_1(G_1 \circ G_2)$ is characterized by the Laplacian eigenvalues of the associated matrices G_1 and G_2, the weight ω, and the duplication N. To further explore the relationship between the first-order coherence of the composite network $G_1 \circ G_2$ and that of the factor networks G_1 and G_2, we derive another analytical formula for $H_1(G_1 \circ G_2)$.

Lemma 3: *Let G be a simple graph with n vertices and Laplacian eigenvalues $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n$. There is an orthogonal matrix $P = (p_{ij})_{n\times n}$, such that $P^T L(G) P = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Moreover, $p_1 = \sqrt{n}, p_2 = \cdots = p_n = 0$, where $p_i (i = 1, 2, \ldots, n)$ is the sum of the ith column of the matrix P.*

Proof. For the Laplacian matrix $L(G)$, it is obvious that $\xi_1 = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$ is the unit eigenvector associated with eigenvalue $\lambda_1 = 0$. Let $\xi_i \quad (1 < i \le n)$ be an eigenvector of $L(G)$ associated with eigenvalue λ_i. Then, $\xi_1^T \xi_i = 0$; that is to say, the sum of all the entries of $\xi_i \quad (1 < i \le n)$ is 0. The conclusion of the lemma follows from the orthogonal decomposition theorem.

Theorem 2: *Let k_1 denote the complete graph of order 1 (i.e., the trivial graph), $G_1 \vee k_1$ be the join graph of G_1 and k_1 with the join-wight $n_2\omega$, and $G_2 \vee k_1$ be the join graph of G_2 and k_1 with the join-wight $N\omega$. Then, the first-order coherence of the N-duplication weighted corona network $G_1 \circ G_2$ can be described as follows:*

$$H_1(G_1 \circ G_2) = \frac{1}{N}H_1(G_1) + \frac{(N-1)(n_1+1)}{n_1(n_2+N)}H_1(G_1 \vee k_1) + \frac{n_2+1}{n_2+N}H_1(G_2 \vee k_1)$$
$$+ \frac{1}{2(n_2+N)\omega}\left[\frac{n_1-1}{Nn_1} + \frac{1}{n_1(n_2+N)} + \frac{N-1}{n_2(n_1+1)} - \frac{1}{N(n_2+1)}\right]. \tag{11}$$

Proof. Let A_1 and L_1 denote the adjacency and Laplacian matrices of G_1, respectively. The block form of the adjacency matrix of $G_1 \vee k_1$ is

$$A(G_1 \vee k_1) = \begin{bmatrix} A_1 & n_2\omega 1_{n_1} \\ n_2\omega 1_{n_1}^T & 0 \end{bmatrix},$$

where 1_{n_1} represents the all-ones column vector of dimension n_1. The Laplacian matrix of $G_1 \vee k_1$ is

$$L(G_1 \vee k_1) = \begin{bmatrix} L_1 + n_2\omega I_{n_1} & -n_2\omega 1_{n_1} \\ -n_2\omega 1_{n_1}^T & n_2\omega \end{bmatrix}.$$

Suppose that $\sigma(G_1) = \{\lambda_1, \lambda_2, \ldots, \lambda_{n_1} | 0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_{n_1}\}$ is the spectrum set of G_1. Then, there is an orthogonal matrix P such that $P^T L_1 P = diag(\lambda_1, \lambda_2, \ldots, \lambda_{n_1})$. In addition,

$$\begin{bmatrix} P^T & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} L_1 + n_2\omega I_{n_1} & -n_2\omega 1_{n_1} \\ -n_2\omega 1_{n_1}^T & n_2\omega \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} P^T\left(L_1 + n_2\omega I_{n_1}\right)P & -n_2\omega P^T 1_{n_1} \\ -n_2\omega 1_{n_1}^T P & n_2\omega \end{bmatrix}.$$

From Lemma 3, the characteristic polynomial of $L(G_1 \vee k_1)$ is

$$|\lambda I_{n_1+1} - L(G_1 \vee k_1)| = \begin{vmatrix} P^T\left(\lambda I_{n_1} - L_1 - n_2\omega I_{n_1}\right)P & n_2\omega P^T 1_{n_1} \\ n_2\omega 1_{n_1}^T P & \lambda - n_2\omega \end{vmatrix}$$
$$= \begin{vmatrix} \lambda - n_2\omega & 0 & \cdots & n_2\omega\sqrt{n_1} \\ 0 & \lambda - n_2\omega - \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ n_2\omega\sqrt{n_1} & 0 & \cdots & \lambda - n_2\omega \end{vmatrix}.$$

Therefore, the Laplacian eigenvalues of $G_1 \vee k_1$ with the join-weight $n_2\omega$ are

$$0, \lambda_i + n_2\omega \, (i = 2, \ldots, n_1), (n_1+1)n_2\omega. \tag{12}$$

Similarly, the Laplacian eigenvalues of $G_2 \vee k_1$ with the join-weight $N\omega$ are

$$0, \mu_i + N\omega \, (i = 2, \ldots, n_2), (n_2+1)N\omega. \tag{13}$$

Moreover, we have

$$H_1(G_1 \vee k_1) = \frac{1}{2(n_1+1)}\left(\sum_{i=2}^{n_1}\frac{1}{n_2\omega+\lambda_i} + \frac{1}{(n_1+1)n_2\omega}\right) \tag{14}$$

and

$$H_1(G_2 \vee k_1) = \frac{1}{2(n_2+1)}\left(\sum_{j=2}^{n_2}\frac{1}{N\omega+\mu_j} + \frac{1}{(n_2+1)N\omega}\right). \tag{15}$$

Equation 11 is then obtained by combining Equations 2, 7, 14, 15.

Remark 2: *Setting $N = 1$ in Equation 11, we have*

$$H_1(G_1 \circ G_2) = H_1(G_1) + H_1(G_2 \vee k_1) + \frac{(n_1-1)n_2}{2n_1(n_2+1)^2\omega}, \tag{16}$$

which is consistent with the result of [46] (see Theorem 3 of [46] for details).

In the proof of Theorem 2, the first-order coherence of the join graphs $G_1 \vee k_1$ and $G_2 \vee k_1$ is also derived, as presented in Equations 14, 15, respectively. From Equation 14, $H_1(G_1 \vee k_1)$ generally increases with the increase of $H_1(G_1)$. The assertion holds true also for $H_1(G_2 \vee k_1)$ with $H_1(G_2)$. Therefore, from Equation 11, lower $H_1(G_1)$ or $H_1(G_2)$ generally indicates lower $H_1(G_1 \circ G_2)$. Furthermore, for a fixed ω, we have $H_1(G_1 \circ G_2) \to (1+\frac{1}{n_1})H_1(G_1 \vee k_1) + \frac{1}{2n_2(n_1+1)\omega}$ as $N \to \infty$. On the other hand, given a constant N, $H_1(G_1 \circ G_2) \to \frac{1}{N} H_1(G_1)$ as $\omega \to \infty$. The above analysis leads to the following remark.

Remark 3: *From* Equation 11, Theorem 2 *shows, for fixed values of n_1, n_2, N and ω, lower $H_1(G_1)$ or $H_1(G_2)$ generally leads to lower $H_1(G_1 \circ G_2)$. In other words, higher first-order coherence of the factor network G_1 or G_2 usually implies higher first-order coherence of the N-duplication weighted corona network $G_1 \circ G_2$. Especially, in the situation of large duplication N or weight ω, the first-order coherence of $G_1 \circ G_2$ is mainly determined by the factor network G_1.*

3.2 Second-order coherence of the *N*-duplication weighted corona network

This subsection investigates the second-order coherence of the N-duplication weighted corona networks.

Theorem 3: *Let $\sigma(G_1) = \{\lambda_1, \lambda_2, \ldots, \lambda_{n_1} | 0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_{n_1}\}$ and $\sigma(G_2) = \{\mu_1, \mu_2, \ldots, \mu_{n_2} | 0 = \mu_1 < \mu_2 \le \cdots \le \mu_{n_2}\}$ be the Laplacian spectrum sets of G_1 and G_2, respectively. Then, the second-order coherence of the N-duplication weighted corona network $G_1 \circ G_2$ can be described as follows:*

$$\begin{aligned} H_2(G_1 \circ G_2) = {} & \frac{n_2}{n_1(n_2+N)N^2\omega} \sum_{i=2}^{n_1} \frac{1}{\lambda_i} + \frac{n_2+N}{2n_1 N^2} \sum_{i=2}^{n_1} \frac{1}{\lambda_i^2} \\ & + \frac{N-1}{2n_1(n_2+N)} \sum_{i=1}^{n_1} \frac{1}{(n_2\omega+\lambda_i)^2} \\ & + \frac{1}{2(n_2+N)} \sum_{j=2}^{n_2} \frac{1}{\left(N\omega+\mu_j\right)^2} + \frac{1}{2n_1(n_2+N)^3\omega^2} \\ & + \frac{n_1-1}{2n_1(n_2+N)N^2\omega^2}. \end{aligned} \tag{17}$$

Proof. Based on Equation 4, to evaluate the second-order network coherence of $G_1 \circ G_2$, we need to obtain the sum of squared reciprocals of all nonzero Laplacian eigenvalues. The analysis of the Laplacian spectrum of $G_1 \circ G_2$ is presented in Theorem 1. For Equation 8, the last factor of the Laplacian characteristic polynomial Equation 6, the constant term a_0, and the coefficient of the linear term a_1 are given in Equation 9. Let a_2 denote the coefficient of the quadratic term of Equation 8. We have

$$a_2 = \prod_{i=2}^{n_1} (N\omega\lambda_i) \left[\sum_{i=2}^{n_1} \frac{1}{N\omega\lambda_i} + \sum_{i=2}^{n_1-1} \sum_{j=i+1}^{n_1} \frac{(N\omega+n_2\omega+\lambda_i)\left(N\omega+n_2\omega+\lambda_j\right)}{N^2\omega^2\lambda_i\lambda_j} \right]$$

and

$$\left(\frac{a_1}{a_0}\right)^2 - 2\frac{a_2}{a_0} = \frac{2n_2}{N^2\omega} \sum_{i=2}^{n_1} \frac{1}{\lambda_i} + \frac{(N+n_2)^2}{N^2} \sum_{i=2}^{n_1} \frac{1}{\lambda_i^2} + \frac{n_1-1}{N^2\omega^2}. \tag{18}$$

The result of the theorem is then deduced by combining the analysis of the Laplacian spectrum of $G_1 \circ G_2$, Lemma 1, and Equations 4, 18.

From Equation 17, a conclusion similar to Remark 1 can be drawn for the second-order network coherence in N-duplication weighted corona networks.

Remark 4: Theorem 3 *implies that $H_2(G_1 \circ G_2)$ decreases as the duplication N or the weight factor ω increases. Therefore, similar to the case of first-order coherence, second-order noisy corona networks $G_1 \circ G_2$ with larger duplication N or weight factor ω can be considered to be more robust to noise than networks with smaller duplication or weight factor values.*

From Remark 4, the notable topological property of high coherence of corona networks $G_1 \circ G_2$ with large duplication N remains valid for N-duplication weighted corona networks with second-order dynamics. As in the case of the first-order coherence, the relationship between the second-order coherence of the corona network $G_1 \circ G_2$ and that of the factor networks G_1 and G_2 is also explored.

Theorem 4: *Let k_1 denote the trivial graph, $G_1 \vee k_1$ be the join graph of G_1 and k_1 with the join-weight $n_2\omega$, and $G_2 \vee k_1$ be the join graph of G_2 and k_1 with the join-weight $N\omega$. Then, the second-order coherence of the N-duplication weighted corona network $G_1 \circ G_2$ can be described as follows:*

$$\begin{aligned} H_2(G_1 \circ G_2) = {} & \frac{2n_2}{(n_2+N)N^2\omega} H_1(G_1) + \frac{n_2+N}{N^2} H_2(G_1) \\ & + \frac{(n_1+1)(N-1)}{n_1(n_2+N)} H_2(G_1 \vee k_1) + \frac{n_2+1}{n_2+N} H_2(G_2 \vee k_1) \\ & + \frac{1}{2(n_2+N)\omega^2} \left[\frac{n_1-1}{n_1 N^2} + \frac{1}{n_1(n_2+N)^2} + \frac{(n_1+2)(N-1)}{n_2^2(n_1+1)^2} \right. \\ & \left. - \frac{1}{(n_2+1)^2 N^2} \right]. \end{aligned} \tag{19}$$

Proof. The Laplacian eigenvalues of $G_1 \vee k_1$ and $G_2 \vee k_1$ are given in Equations 12, 13, respectively. From Equation 4, one obtains

$$H_2(G_1 \vee k_1) = \frac{1}{2(n_1+1)} \left(\sum_{i=2}^{n_1} \frac{1}{(n_2\omega+\lambda_i)^2} + \frac{1}{(n_1+1)^2 n_2^2\omega^2} \right) \tag{20}$$

and

$$H_2(G_2 \vee k_1) = \frac{1}{2(n_2+1)} \left(\sum_{j=2}^{n_2} \frac{1}{\left(N\omega+\mu_j\right)^2} + \frac{1}{(n_2+1)^2 N^2\omega^2} \right). \tag{21}$$

The theorem is then proved by combining the analysis of the Laplacian spectrum of $G_1 \circ G_2$, Lemma 1, and Equations 17, 20, 21.

Remark 5: *Setting $N = 1$ in* Equation 19, *the second-order network coherence of the 1-duplication corona network (or simply the corona network) can be expressed as follows:*

$$\begin{aligned} H_2(G_1 \circ G_2) = {} & \frac{2n_2}{(n_2+1)\omega} H_1(G_1) + (n_2+1) H_2(G_1) + H_2(G_2 \vee k_1) \\ & + \frac{n_2(n_1-1)(n_2+2)}{2n_1(n_2+1)^3\omega^2}. \end{aligned} \tag{22}$$

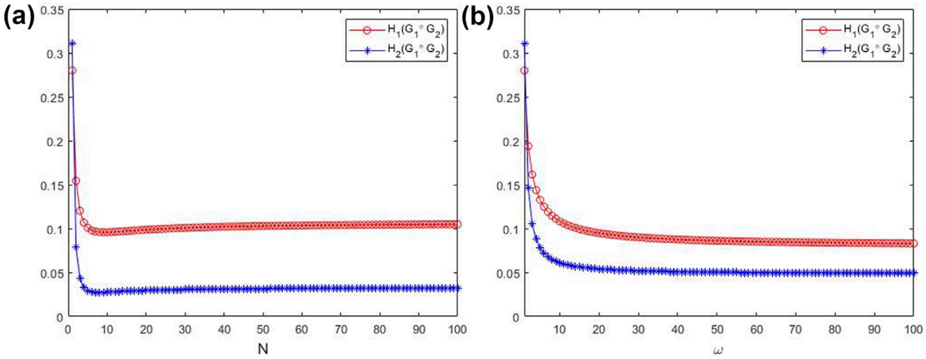

FIGURE 2
(a) $H_1(G_1 \circ G_2)$ and $H_2(G_1 \circ G_2)$ versus duplication N; **(b)** $H_1(G_1 \circ G_2)$ and $H_2(G_1 \circ G_2)$ versus weight factor ω.

In the proof of Theorem 4, the second-order coherence for the join graphs $G_1 \vee k_1$ and $G_2 \vee k_1$ is derived, as shown in Equations 20, 21, respectively. From Equation 20, it can be seen that $H_2(G_1 \vee k_1)$ generally increases with the increase of $H_2(G_1)$. The assertion holds also true for the relationship between $H_2(G_2 \vee k_1)$ and $H_2(G_2)$. Therefore, from Equation 19, lower $H_1(G_1)$, $H_2(G_1)$, or $H_2(G_2)$ generally leads to lower $H_2(G_1 \circ G_2)$. Furthermore, for a fixed ω, $H_2(G_1 \circ G_2) \to (1 + \frac{1}{n_1})H_2(G_1 \vee k_1) + \frac{n_1+2}{2n_2^2(n_1+1)^2\omega^2}$ as $N \to \infty$. On the other hand, given a constant N, $H_2(G_1 \circ G_2) \to \frac{n_2+N}{N^2}H_2(G_1)$ as $\omega \to \infty$. The above analysis leads to the following remark.

Remark 6: *Similar to the first-order noisy consensus algorithms, for fixed values of n_1, n_2, N, and ω, the higher second-order coherence of the factor network G_1 or G_2 generally implies higher second-order coherence of the corona network $G_1 \circ G_2$. Especially, in the situation of large duplication N or weight ω values, the second-order coherence of $G_1 \circ G_2$ is mainly determined by the factor network G_1.*

4 Examples and simulations

This section verifies the theoretical results of Section 3 with numerical examples.

Example 1: Consider the network coherence of the N-duplication weighted corona network $G_1 \circ G_2$, where G_1 and G_2 are complete graphs of orders 5 and 2, respectively. Examples of 1-duplication and 2-duplication corona networks are shown in Figure 1.

The Laplacian eigenvalues of G_1 and G_2 are $0, \underbrace{5, \ldots, 5}_{4}$ and $0, 2$, respectively. The first- and second-order coherence of the N-duplication weighted corona network $G_1 \circ G_2$ can be derived from Equations 7, 17, respectively. Especially, setting $\omega = 1$, $H_1(G_1 \circ G_2) \to \frac{1}{2n_1}\sum_{i=1}^{n_1}\frac{1}{n_2+\lambda_i} \approx 0.1071$, and $H_2(G_1 \circ G_2) \to \frac{1}{2n_1}\sum_{i=1}^{n_1}\frac{1}{(n_2+\lambda_i)^2} \approx 0.0332$, as $N \to \infty$; setting $N = 1$, $H_1(G_1 \circ G_2) \to \frac{1}{2n_1}\sum_{i=2}^{n_1}\frac{1}{\lambda_i} = H_1(G_1) = 0.08$ and $H_2(G_1 \circ G_2) \to \frac{n_2+1}{2n_1}\sum_{i=2}^{n_1}\frac{1}{(\lambda_i)^2} = (n_2+1)H_2(G_1) = 0.048$ as $\omega \to \infty$.

Figure 2 shows the asymptotic trend of $H_1(G_1 \circ G_2)$ and $H_2(G_1 \circ G_2)$ with the increasing duplication N and weight factor ω, respectively. Furthermore, we can see the steep decline of $H_1(G_1 \circ G_2)$ and $H_2(G_1 \circ G_2)$ occurring at the small values of N and ω.

The dependence of $H_1(G_1 \circ G_2)$ and $H_2(G_1 \circ G_2)$ on duplication N and weight ω is depicted in Figure 3. It can be seen from the hook face that both $H_1(G_1 \circ G_2)$ and $H_2(G_1 \circ G_2)$ generally decrease as ω or N increases. Accordingly, a corona network with large duplication N and weight factor ω can be considered to be more robust to noise than networks with small N and ω.

Example 2: In this example, the relationship between the coherence of the N-duplication weighted corona network $G_1 \circ G_2$ and that of the factor networks G_1 and G_2 is explored. To this aim, two different cases are considered.

In case (I), the N-duplication weighted corona networks $G_1 \circ G_2$ are composed of the same factor graph G_2 (complete graph of order 5) but different G_1 (complete, cycle, and star graphs, all with eight vertices). The first- and second-order coherence of the three different G_1 is $H_1(G_c) = 0.0547$, $H_1(G_{cy}) = 0.2613$, $H_1(G_s) = 0.3828$, $H_2(G_c) = 0.0068$, $H_2(G_{cy}) = 0.2604$, and $H_2(G_s) = 0.3760$, where the subscripts c, cy, and s stand for the complete, cycle, and star graph, respectively. The results of case (I) are shown in Figure 4.

In case (II), the N-duplication weighted corona networks $G_1 \circ G_2$ are composed of the same factor graph G_1 (complete graph of order 8) but different G_2 (complete, cycle, and star graphs, all with five vertices). The first- and second-order coherence of the three different G_2 is $H_1(G_c) = 0.0800$, $H_1(G_{cy}) = 0.1632$, $H_1(G_s) = 0.3200$, $H_2(G_c) = 0.0160$, $H_2(G_{cy}) = 0.0808$, and $H_2(G_s) = 0.3040$, respectively. The results of case (II) are shown in Figure 5.

The influence of the factor networks G_1 and G_2 on the coherence of the corona network $G_1 \circ G_2$ is illustrated in Figures 4, 5, respectively. As shown in Figure 4, in the case of the same factor graph G_2, a smaller value of $H_1(G_1)$ (or $H_2(G_1)$) also generally leads to a smaller value of $H_1(G_1 \circ G_2)$ (or $H_2(G_1 \circ G_2)$). In other words, for the same G_2, the N-duplication weighted corona network $G_1 \circ G_2$ will generally have higher network coherence when the factor network G_1 has higher network coherence. From Figure 5, it can be seen that the above assertion also holds true for the influence of G_2 on the network coherence of $G_1 \circ G_2$. Moreover, compared with the results of Figure 5, the values of $H_1(G_1 \circ G_2)$ (or $H_2(G_1 \circ G_2)$) in Figure 4 show a more notable difference, which indicates that the factor network G_1

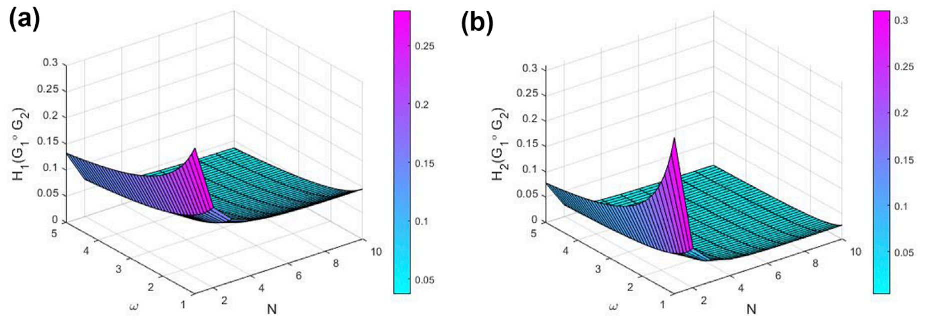

FIGURE 3
(a) $H_1(G_1 \circ G_2)$ with various duplication N and weight factor ω values; **(b)** $H_2(G_1 \circ G_2)$ with various duplication N and weight factor ω values.

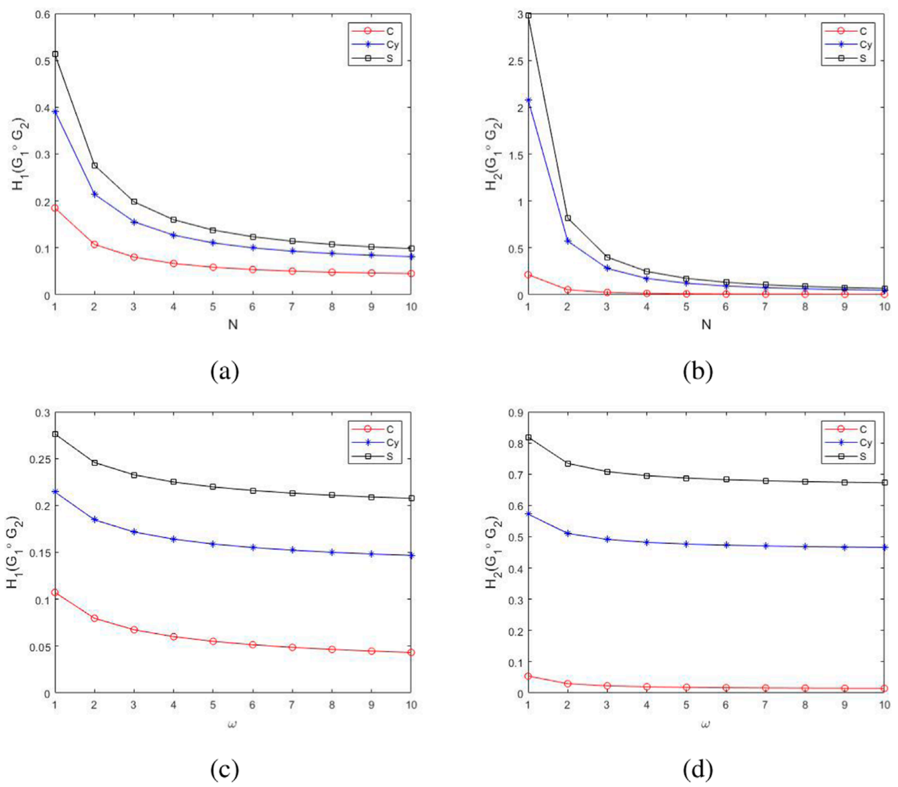

FIGURE 4
$H_1(G_1 \circ G_2)$ and $H_2(G_1 \circ G_2)$ of the N-duplication weighted corona networks $G_1 \circ G_2$ composed of the same factor graph G_2 but different G_1, where **(a, b)** are versus duplication N and **(c, d)** are versus weight ω. The different factor graphs G_1 are a complete graph (C), a cycle graph (Cy), and a star graph (S), respectively.

plays a more important role than G_2 in the network coherence of the N-duplication weighted corona network $G_1 \circ G_2$.

5 Conclusion

In this paper, coherence problems in N-duplication weighted corona networks with first- or second-order dynamics are addressed. As a special case with N =1, the network coherence in the classic corona network is also investigated (see Equations 16, 22). For both first- and second-order consensus problems, explicit expressions of the network coherence are derived and presented in two different ways. In one way, the network coherence is expressed in terms of the Laplacian spectra of the factor networks G_1 and G_2, the weight factor ω of edges connecting G_1 and G_2, and the duplication N. Based on this kind of expression, it is found that corona networks with large duplication N or weight ω usually have high network coherence and can be considered to be more robust to noise. High

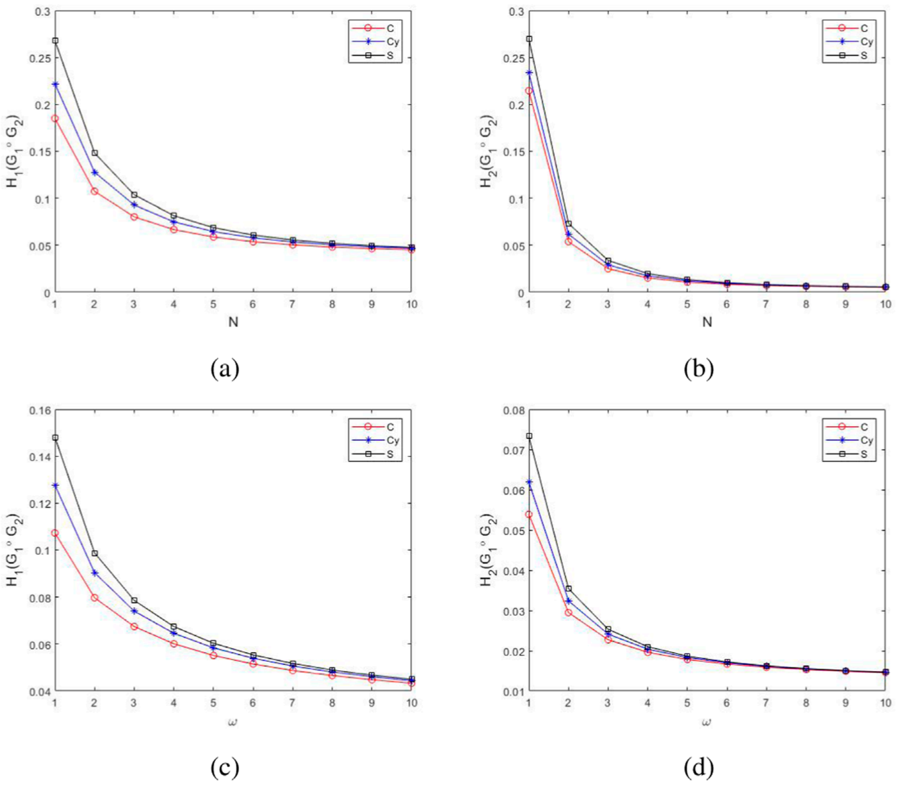

FIGURE 5
$H_1(G_1{\circ}G_2)$ and $H_2(G_1{\circ}G_2)$ of the N-duplication weighted corona networks $G_1{\circ}G_2$ composed of the same factor graph G_1 but different G_2, where **(a, b)** are versus duplication N and **(c, d)** are versus weight ω. The different factor graphs G_2 are a complete graph (C), a cycle graph (Cy), and a star graph (S), respectively.

coherence or strong robustness of the corona network $G_1{\circ}G_2$ with large duplication N (thereby large network size) can be regarded as a special and notable topological property of the N-duplication weighted corona network. In another way, the coherence of the corona network $G_1{\circ}G_2$ is expressed in terms of that of the factor networks G_1 and G_2. Little work has been done from this perspective on the study of the consensus problems in product networks, and it deserves further research. Based on this kind of expression, the influence of the factor networks G_1 and G_2 on the network coherence of the corona network $G_1{\circ}G_2$ is investigated. The results show that higher coherence of G_1 or G_2 usually also leads to higher coherence of the corona network $G_1{\circ}G_2$. Especially, in the situation of large duplication N or weight ω, G_1 plays a more important role than G_2 in the network coherence of the N-duplication weighted corona network $G_1{\circ}G_2$.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material; further inquiries can be directed to the corresponding author.

Author contributions

CL: Writing – original draft and Writing – review and editing.

Funding

The author(s) declare that no financial support was received for the research and/or publication of this article.

Conflict of interest

The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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