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The anisotropic quantum Rabi model with diamagnetic term

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We employ a squeeze operator transformation approach to solve the anisotropic quantum Rabi model that includes a diamagnetic term. By carefully adjusting the amplitude of the diamagnetic term, we demonstrate that the anisotropic Rabi model with the A^2 term can be exactly reduced to either a Jaynes-Cummings or an anti-Jaynes-Cummings model without requiring any approximations.

KEYWORDS

Rabi model, diamagnetic term, squeeze operator, transformations, anti-JCM

1 Introduction

The interaction of atoms with cavity fields [1-3] is of great importance not only because of the fundamental questions that may be answered, but also because of the possible technological applications [4-6] as entanglement, at the core of such interaction, is the key ingredient of quantum information processing.

When analyzing this interaction several approximations are done, namely, the diamagnetic term [7-11] is dropped, the dipole and rotating wave approximations are made and the interaction with environments [12] is not considered, this is, studies are focused on high-Q cavities. However, there are intensity regimes where such approximations are not any more valid and then it is needed to consider the full interaction, *i.e.*, the quantum Rabi model [13]. Solutions for this problem have been already provided [14-17], usually in terms of infinite continued fractions [14].

It has been shown that the diamagnetic term may be of importance in the deep-strong-coupling (DSC) and ultra-strong-coupling regimes (USC) [9]. In the atom-field interaction, the diamagnetic term is usually dropped as it is a term that it is of the order of counter rotating terms [8]. However, in other regimes the impact of the diamagnetic term is non-negligible and it may become dominant in the DSC regime [9, 11].

Generalizations of the quantum Rabi model, such as the anisotropic quantum Rabi model [18-20], have been studied. In particular it has been shown the existence of entanglement [20] and antibouncing-to-bouncing transitions of photons [19].

In this contribution we show that an anisotropic Rabi model that includes the diamagnetic term may be reduced, by using a transformation that involves the squeeze operator [21], to the Jaynes-Cummings [1] and anti-Jaynes-Cummings models [22]. These kind of systems have been shown to have partner Hamiltonians in the theory of supersymmetry (SUSY) [23, 24] that allows the connection of physical models via supersymmetric operators, *i.e.*, mapping the corresponding Hilbert spaces.

2 The anisotropic quantum Rabi model

The Hamiltonian for the anisotropic quantum Rabi model, including the diamagnetic term, can be expressed as (with $\hbar = 1$):

$$\begin{aligned}\widehat{H} &= \omega\hat{a}^\dagger\hat{a} + \frac{\omega_0}{2}\hat{\sigma}_z + \left(g_1\hat{a} + g_2\hat{a}^\dagger\right)\hat{\sigma}_+ + \left(g_1\hat{a}^\dagger + g_2\hat{a}\right)\hat{\sigma}_- + D\left(\hat{a} + \hat{a}^\dagger\right)^2,\\ &= \omega_D\hat{a}^\dagger\hat{a} + \frac{\omega_0}{2}\hat{\sigma}_z + \left(g_1\hat{a} + g_2\hat{a}^\dagger\right)\hat{\sigma}_+ + \left(g_1\hat{a}^\dagger + g_2\hat{a}\right)\hat{\sigma}_- + D\left(\hat{a}^2 + \hat{a}^{\dagger 2}\right) + D,\end{aligned} \tag{1}$$

where $\omega_D = \omega + 2D$. Here, $\hat{a}$ and $\hat{a}^\dagger$ represent the annihilation and creation operators of the bosonic field, satisfying the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. The Pauli atomic operators $\hat{\sigma}_\pm$ and $\hat{\sigma}_z$ describe the two-level atomic system, obeying the commutation relations: $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$ and $[\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$. Additionally, ω and ω_0 denote the field frequency and the atomic transition frequency, respectively, while D quantifies the diamagnetic amplitude. The coupling constants g_1 and g_2 characterize the interaction strength between the atom and the field. The parameters $\{\omega, \omega_0, g_1, g_2, D\}$ are known quantities. As we will demonstrate later, depending on the parameter regime and without approximations, it is possible to recover either the Jaynes-Cummings or the anti-Jaynes-Cummings model, respectively, by adjusting judiciously one of them (except ω_0 that will not play a role on which of the models is obtained).

To eliminate the residual diamagnetic term $\left(\hat{a}^2 + \hat{a}^{\dagger 2}\right)$ from Equation 1, we apply a unitary transformation defined by the squeeze operator: $\hat{S}(r) = \exp\left[\frac{r}{2}\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right)\right]$ [2], where r is the squeezing parameter to be determined subsequently. Under this transformation, the annihilation and creation operators transform as:

$$\hat{S}^\dagger(r)\hat{a}\hat{S}(r) = \mu\hat{a} - \nu\hat{a}^\dagger, \quad \hat{S}^\dagger(r)\hat{a}^\dagger\hat{S}(r) = \mu\hat{a}^\dagger - \nu\hat{a}, \quad \text{where}$$
$$\mu = \cosh(r), \quad \nu = \sinh(r).$$

Applying the transformation $\widehat{H}_S = \hat{S}^\dagger(r)\widehat{H}\hat{S}(r)$, the Hamiltonian becomes:

$$\begin{aligned}\widehat{H}_S = &\left[\omega_D\left(\mu^2 + \nu^2\right) - 4D\mu\nu\right]\hat{a}^\dagger\hat{a} + \frac{\omega_0}{2}\hat{\sigma}_z\\ &+ \left(\mu g_1 - \nu g_2\right)\left(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-\right) + \left(\mu g_2 - \nu g_1\right)\left(\hat{a}\hat{\sigma}_- + \hat{a}^\dagger\hat{\sigma}_+\right)\\ &+ \left[D\left(\mu^2 + \nu^2\right) - \mu\nu\omega_D\right]\left(\hat{a}^2 + \hat{a}^{\dagger 2}\right) + \nu^2\omega_D - 2D\mu\nu + D.\end{aligned}$$

By imposing the condition $\frac{D}{\omega_D} = \frac{\mu\nu}{\mu^2+\nu^2}$, the Hamiltonian simplifies to:

$$\begin{aligned}\widehat{H}_S = &\frac{\omega_D}{\mu^2 + \nu^2}\hat{a}^\dagger\hat{a} + \frac{\omega_0}{2}\hat{\sigma}_z + \left(\mu g_1 - \nu g_2\right)\left(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-\right) + \left(\mu g_2 - \nu g_1\right)\\ &\times\left(\hat{a}\hat{\sigma}_- + \hat{a}^\dagger\hat{\sigma}_+\right) + \omega_D\left(\nu^2 - \frac{2\mu^2\nu^2 - \mu\nu}{\mu^2 + \nu^2}\right).\end{aligned} \tag{2}$$

Thus, the squeezing transformation eliminates the residual diamagnetic term, thereby simplifying the system to the anisotropic quantum Rabi model. In the special case where $g_1 = g_2$, the model reduces to the standard quantum Rabi model [13, 17]. However, this work focuses on the scenario where $g_1 \neq g_2$. Specifically, we investigate two distinct cases: (a) $g_2 < g_1$, which corresponds to the Jaynes-Cummings model, and (b) $g_1 < g_2$, associated with the anti-Jaynes-Cummings model [22, 23]. The Hamiltonian described by Equation 2 represents one of the key contributions of this work, providing a comprehensive framework for exploring the interplay between anisotropy, squeezing, and light-matter interactions within the anisotropic quantum Rabi model.

2.1 Jaynes-Cummings model

Once the Hamiltonian in Equation 2 is established, we fix the squeezing parameter r within the parameter regime defined by $g_2 < g_1$ to recover the Jaynes-Cummings model. This requires imposing the condition $\frac{\mu}{\nu} = \frac{g_1}{g_2}$, where $\mu = \cosh(r)$ and $\nu = \sinh(r)$ are the hyperbolic functions associated with r. Under this condition, the Hamiltonian $\widehat{H}_S$ simplifies to the Jaynes-Cummings Hamiltonian, which takes the following form:

$$\widehat{H}_{\mathrm{JCM}} = \omega_{\mathrm{eff}}\hat{a}^\dagger\hat{a} + \frac{\omega_0}{2}\hat{\sigma}_z + g_{\mathrm{eff}}\left(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-\right) + f_0,$$

where the effective frequency ω_{eff}, the effective coupling constant g_{eff}, and the zero-point energy shift f_0 are explicitly defined as:

$$\omega_{\mathrm{eff}} = \omega_D\left(\frac{g_1^2 - g_2^2}{g_1^2 + g_2^2}\right), \quad g_{\mathrm{eff}} = \sqrt{g_1^2 - g_2^2}, \quad \text{and} \quad f_0 = \omega_D\left(\frac{g_1g_2 - g_2^2}{g_1^2 + g_2^2}\right).$$

For the specific case where $g_2 < g_1$, the condition eliminates the residual diamagnetic term, which takes the following form in this parameter regime:

$$\frac{D}{\omega_D} = \frac{\mu\nu}{\mu^2 + \nu^2} = \frac{\tanh(2r)}{2} = \frac{g_1g_2}{g_1^2 + g_2^2}. \tag{3}$$

This condition, together with the inequality $g_2 < g_1$, establishes the parameter regime in which the anisotropic quantum Rabi model transitions to the Jaynes-Cummings model. Consequently, the squeezing transformation is fully described by the hyperbolic functions:

$$\cosh(r) = \frac{g_1}{\sqrt{g_1^2 - g_2^2}}, \quad \sinh(r) = \frac{g_2}{\sqrt{g_1^2 - g_2^2}}.$$

These results demonstrate how the squeezing transformation not only removes the diamagnetic term but also establishes a direct connection between the physical parameters of the system and the mathematical structure of the Jaynes-Cummings model.

Finally, to establish the complete relationship between the Jaynes-Cummings Hamiltonian and the Hamiltonian of the anisotropic quantum Rabi model with the diamagnetic term, Equation 1, for the case $g_2 < g_1$, it is essential to recall the relation $\widehat{H}_{\mathrm{JCM}} = \hat{S}^\dagger(r)\widehat{H}\hat{S}(r)$. This relation is fundamental for finding the eigenvalues of $\widehat{H}$ in this parameter regime. Therefore, by multiplying the relation $\hat{S}^\dagger(r)\widehat{H}\hat{S}(r)$ by $\hat{S}(r)$, we obtain:

$$\widehat{H}\hat{S}(r)|\psi_n^{\mathrm{JCM}}, \pm\rangle = E_{n,\pm}^{\mathrm{JCM}}\hat{S}(r)|\psi_n^{\mathrm{JCM}}, \pm\rangle,$$

where the eigenvalues $E_{n,\pm}^{\rm JCM}$ and the eigenvectors $|\psi_n^{\rm JCM},\pm\rangle$ are those of the Jaynes-Cummings Hamiltonian, determined by:

$$E_{n,\pm}^{\rm JCM} = \omega_{\rm eff}\left(n+\frac{1}{2}\right) \pm \frac{1}{2}\sqrt{(\omega_0-\omega_{\rm eff})^2+4g_{\rm eff}^2(n+1)}+f_0,$$
$$\hat{S}(r)|\psi_n^{\rm JCM},+\rangle = \hat{S}(r)\left(\cos(\theta_n)|n,\uparrow\rangle+\sin(\theta_n)|n+1,\downarrow\rangle\right),$$
$$\hat{S}(r)|\psi_n^{\rm JCM},-\rangle = \hat{S}(r)\left(-\sin(\theta_n)|n,\uparrow\rangle+\cos(\theta_n)|n+1,\downarrow\rangle\right),$$

with $\tan(2\theta_n)=\frac{2g_{\rm eff}\sqrt{n+1}}{\omega_0-\omega_{\rm eff}}$. Here, $|n,\uparrow\rangle=|n\rangle\otimes|\uparrow\rangle$ and $|n+1,\downarrow\rangle=|n+1\rangle\otimes|\downarrow\rangle$ are the basis vectors in the Fock space and the atomic subspace, respectively. Clearly, the eigenvalues of $\widehat{H}$, given by Equation 1, are the same as those of the Jaynes-Cummings Hamiltonian. Moreover, the eigenstates of $\widehat{H}$ are connected to those of the Jaynes-Cummings model through the action of the squeeze operator $\hat{S}(r)$ on the eigenstates $|\psi_n^{\rm JCM},\pm\rangle$.

2.2 Anti- Jaynes-Cummings model

On the other hand, starting from Equation 2 and considering the parameter region defined by $g_1<g_2$, the squeezing parameter r necessary to recover the anti-Jaynes-Cummings model is derived by imposing the condition $\frac{\mu}{\nu}=\frac{g_2}{g_1}$. Under these conditions, the Hamiltonian $\widehat{H}_S$ adopts the following form:

$$\widehat{H}_{\rm AJCM} = \tilde{\omega}_{\rm eff}\hat{a}^\dagger\hat{a}+\frac{\omega_0}{2}\hat{\sigma}_z+\tilde{g}_{\rm eff}\left(\hat{a}\hat{\sigma}_-+\hat{a}^\dagger\hat{\sigma}_+\right)+\tilde{f}_0,$$

where

$$\tilde{\omega}_{\rm eff}=\omega_D\left(\frac{g_2^2-g_1^2}{g_1^2+g_2^2}\right),\quad \tilde{g}_{\rm eff}=\sqrt{g_2^2-g_1^2},\quad \text{and}\quad \tilde{f}_0=\omega_D\left(\frac{g_1g_2-g_1^2}{g_1^2+g_2^2}\right).$$

Consequently, in a manner analogous to the previous case, and in addition to Equation 3, which eliminates the diamagnetic term, the anti-Jaynes-Cummings model can be recovered within the parameter region defined by $g_1<g_2$. Therefore, in this parameter regime, it follows that:

$$\cosh(r)=\frac{g_2}{\sqrt{g_2^2-g_1^2}},\qquad \sinh(r)=\frac{g_1}{\sqrt{g_2^2-g_1^2}}. \tag{4}$$

Finally, to establish the complete relationship between the anti-Jaynes-Cummings Hamiltonian and the Hamiltonian of the anisotropic quantum Rabi model with the diamagnetic term, Equation 1, for the case $g_1<g_2$, it is essential to recall the relation $\widehat{H}_{\rm AJCM}=\hat{S}^\dagger(r)\widehat{H}\hat{S}(r)$. Therefore, by multiplying the relation $\hat{S}^\dagger(r)\widehat{H}\hat{S}(r)$ by $\hat{S}(r)$, we obtain:

$$E_{n,\pm}^{\rm AJCM} = \tilde{\omega}_{\rm eff}\left(n+\frac{1}{2}\right) \pm \frac{1}{2}\sqrt{(\omega_0+\tilde{\omega}_{\rm eff})^2+4\tilde{g}_{\rm eff}^2(n+1)}+\tilde{f}_0,$$
$$\hat{S}(r)|\psi_{n,+}^{\rm AJCM}\rangle = \hat{S}(r)\left(\cos(\theta_n)|n+1,\uparrow\rangle+\sin(\theta_n)|n,\downarrow\rangle\right),$$
$$\hat{S}(r)|\psi_{n,-}^{\rm AJCM}\rangle = \hat{S}(r)\left(-\sin(\theta_n)|n+1,\uparrow\rangle+\cos(\theta_n)|n,\downarrow\rangle\right),$$

where $\tan(2\theta_n)=\frac{2\tilde{g}_{\rm eff}\sqrt{n+1}}{\omega_0+\tilde{\omega}_{\rm eff}}$. The eigenvectors are given by linear combinations of the states $|n+1,\uparrow\rangle$ and $|n,\downarrow\rangle$. This structure aligns with the anti-Jaynes-Cummings model, where the anti-resonant coupling links the states $|n+1,\uparrow\rangle$ and $|n,\downarrow\rangle$, in contrast to the Jaynes-Cummings model, which connects $|n,\uparrow\rangle$ and $|n+1,\downarrow\rangle$. Once again, the eigenvalues are identical to those of the anti-Jaynes-Cummings Hamiltonian, and the eigenvectors are related via the squeeze operator, $\hat{S}(r)|\psi_n^{\rm AJCM},\pm\rangle$.

3 Results and discussion

In this section, we analyze the eigenvalues and atomic inversion for the anisotropic quantum Rabi model with diamagnetic term in the two distinct parameter regimes: (a) $g_2<g_1$, corresponding to the Jaynes-Cummings model, and (b) $g_1<g_2$, associated with the anti-Jaynes-Cummings model. We first discuss the eigenvalues in both regimes and then examine the behavior of the atomic inversion.

The eigenvalues $E_{n,\pm}$ of the Hamiltonian $\widehat{H}$ are determined by the effective parameters obtained from the squeezing transformation $\hat{S}(r)$ in each parameter regime: (a) $g_2<g_1$ and (b) $g_1<g_2$ (with $g_1\neq g_2$). As established in the previous section, these eigenvalues are expressed as:

$$E_{n,\pm}=\begin{cases}\omega_{\rm eff}\left(n+\frac{1}{2}\right)\pm\frac{1}{2}\sqrt{(\omega_0-\omega_{\rm eff})^2+4g_{\rm eff}^2(n+1)}+f_0, & \text{if}\quad g_2<g_1,\\ \tilde{\omega}_{\rm eff}\left(n+\frac{1}{2}\right)\pm\frac{1}{2}\sqrt{(\omega_0+\tilde{\omega}_{\rm eff})^2+4\tilde{g}_{\rm eff}^2(n+1)}+\tilde{f}_0, & \text{if}\quad g_1<g_2.\end{cases} \tag{5}$$

Figure 1 displays the first energy levels for both the (a) Jaynes-Cummings and (b) anti-Jaynes-Cummings models as a function of the coupling parameter g_2, without loss of generality, taking $g_1=1$. (a) In the regime $g_2<g_1$, corresponding to the Jaynes-Cummings model, the energy levels $E_{n,\pm}^{\rm JCM}$ depend on g_2, with their structure dictated by the effective coupling strength $g_{\rm eff}=\sqrt{g_1^2-g_2^2}$. As g_2 approaches g_1 from the left, the energy levels gradually converge due to the vanishing effective coupling, $g_{\rm eff}\to 0$, and effective frequency, $\omega_{\rm eff}\to 0$. (b) For $g_2>g_1$, the system transitions to the anti-Jaynes-Cummings regime, where the energy levels $E_{n,\pm}^{\rm AJCM}$ are now governed by the effective coupling $\tilde{g}{\rm eff}=\sqrt{g_2^2-g_1^2}$. As g_2 gradually increases beyond g_1, the effective coupling constant and effective frequency, $\tilde{g}{\rm eff}$ and $\tilde{\omega}_{\rm eff}$, respectively, start from zero in the limiting case and increase with g_2. Although Figure 1 focuses on the first energy levels, a clear pattern emerges for higher-order eigenvalues: they either converge toward g_1 or diverge from it as g_2 varies. This behavior underscores the role of the coupling parameters in shaping the energy spectrum and provides insight into the system's response to variations in g_2.

To conclude this section, we present an analysis of the atomic inversion for the anisotropic quantum Rabi model with diamagnetic term in the two distinct coupling regimes. The atomic inversion, denoted as $W(t)$, is a fundamental quantity that characterizes the dynamics of the system. It is defined as the difference in population between the atomic states $|\uparrow\rangle$ and $|\downarrow\rangle$, and is mathematically expressed as: $W(t)=\langle\hat{\sigma}_z(t)\rangle$. This quantity provides insight into the temporal evolution of the atomic populations and serves as a key indicator of the system's behavior under different coupling conditions. In the two parameter regimes under consideration, the atomic inversion is given by: $W(t)=\langle\psi(0)|\widehat{U}^\dagger(t)\hat{\sigma}_z\widehat{U}(t)|\psi(0)\rangle$, where $\widehat{U}(t)$ represents the time evolution operator. This operator is defined as: $\widehat{U}(t)=\exp[-i\widehat{H}t]$. Here, $|\psi(0)\rangle$ denotes the initial state of the system, which is the tensor product of the field state and the initial atomic state. The atomic inversion captures the interplay between the atomic and field degrees of freedom, reflecting the influence of the coupling parameters g_1 and g_2. For each parameter regime, the atomic inversion takes the following form:

$$W(t)=\begin{cases}\langle\psi(0)|\hat{S}(r)\widehat{U}_{\rm JCM}^\dagger(t)\hat{S}^\dagger(r)\quad\hat{\sigma}_z\quad\hat{S}(r)\widehat{U}_{\rm JCM}(t)\hat{S}^\dagger(r)|\psi(0)\rangle, & \text{if}\quad g_2<g_1,\\ \langle\psi(0)|\hat{S}(r)\widehat{U}_{\rm AJCM}^\dagger(t)\hat{S}^\dagger(r)\quad\hat{\sigma}_z\quad\hat{S}(r)\widehat{U}_{\rm AJCM}(t)\hat{S}^\dagger(r)|\psi(0)\rangle, & \text{if}\quad g_1<g_2.\end{cases}$$

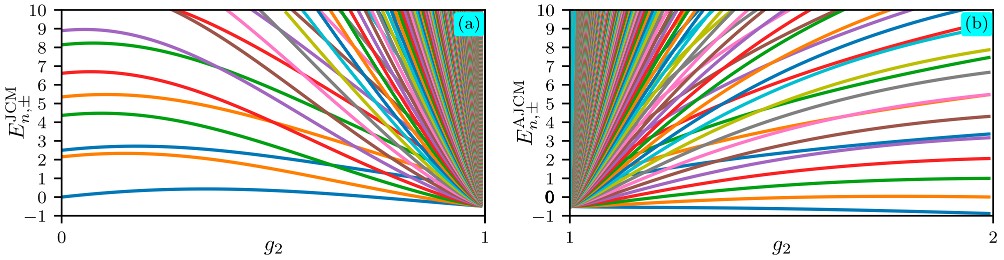

FIGURE 1
Energy levels of the anisotropic quantum Rabi model for the first ten states ($n, \pm = 10$) are plotted as a function of the coupling parameter g_2, with fixed parameters $\omega_D = 2.5$, $\omega_0 = 1.0$, and $g_1 = 1.0$. In Panel **(a)**, the eigenvalues correspond to the Jaynes-Cummings regime ($g_2 < g_1$), where the energy levels $E_{n,\pm}^{\mathrm{JCM}}$ are derived from Equation 5. Panel **(b)** illustrates the eigenvalues for the anti-Jaynes-Cummings regime ($g_2 > 1$), with the energy levels $E_{n,\pm}^{\mathrm{AJCM}}$ also determined by Equation 5. This separation highlights the distinct behaviors of the system in the two coupling regimes.

The time evolution operator for the anisotropic quantum Rabi model with diamagnetic term is expressed as:

$$\widehat{U}(t) = \begin{cases} \hat{S}(r)\,\widehat{U}_{\mathrm{JCM}}(t)\,\hat{S}^\dagger(r), & \text{for} \quad g_2 < g_1, \\ \hat{S}(r)\,\widehat{U}_{\mathrm{AJCM}}(t)\,\hat{S}^\dagger(r), & \text{for} \quad g_1 < g_2. \end{cases} \tag{6}$$

The evolution operators corresponding to the Jaynes-Cummings and anti-Jaynes-Cummings Hamiltonians are given by:

$$\widehat{U}_{\mathrm{JCM}} = e^{-it\omega_{\mathrm{eff}}\left(\hat{a}^\dagger\hat{a}+\frac{\hat{\sigma}_z}{2}\right)}\begin{bmatrix} \widehat{U}_{11}^{\mathrm{JCM}}(t) & \widehat{U}_{12}^{\mathrm{JCM}}(t) \\ \widehat{U}_{21}^{\mathrm{JCM}}(t) & \widehat{U}_{22}^{\mathrm{JCM}}(t) \end{bmatrix},$$

$$\widehat{U}_{\mathrm{AJCM}} = e^{-it\tilde{\omega}_{\mathrm{eff}}\left(\hat{a}^\dagger\hat{a}-\frac{\hat{\sigma}_z}{2}\right)}\begin{bmatrix} \widehat{U}_{11}^{\mathrm{AJCM}}(t) & \widehat{U}_{12}^{\mathrm{AJCM}}(t) \\ \widehat{U}_{21}^{\mathrm{AJCM}}(t) & \widehat{U}_{22}^{\mathrm{AJCM}}(t) \end{bmatrix},$$

respectively. The matrix elements of the evolution operators are explicitly given by:

$$\widehat{U}_{11}^{\mathrm{JCM}}(t) = \cos\left(\frac{\Omega_{\hat{n}+1}t}{2}\right) - i\frac{\Delta_{\mathrm{eff}}}{\Omega_{\hat{n}+1}}\sin\left(\frac{\Omega_{\hat{n}+1}t}{2}\right),$$

$$\widehat{U}_{12}^{\mathrm{JCM}}(t) = -i2g_{\mathrm{eff}}\frac{\sin\left(\frac{\Omega_{\hat{n}+1}t}{2}\right)}{\Omega_{\hat{n}+1}}\hat{a},$$

$$\widehat{U}_{21}^{\mathrm{JCM}}(t) = -i2g_{\mathrm{eff}}\hat{a}^\dagger\frac{\sin\left(\frac{\Omega_{\hat{n}+1}t}{2}\right)}{\Omega_{\hat{n}+1}},$$

$$\widehat{U}_{22}^{\mathrm{JCM}}(t) = \cos\left(\frac{\Omega_{\hat{n}}t}{2}\right) + i\frac{\Delta_{\mathrm{eff}}}{\Omega_{\hat{n}}}\sin\left(\frac{\Omega_{\hat{n}}t}{2}\right),$$

where $\Delta_{\mathrm{eff}} = \omega_0 - \omega_{\mathrm{eff}}$, and $\Omega_{\hat{n}} = \sqrt{\Delta_{\mathrm{eff}}^2 + 4g_{\mathrm{eff}}^2\hat{n}}$ (with $\hat{n} = \hat{a}^\dagger\hat{a}$). Similarly, for the anti-Jaynes-Cummings model:

$$\widehat{U}_{11}^{\mathrm{AJCM}}(t) = \cos\left(\frac{\tilde{\Omega}_{\hat{n}}t}{2}\right) - i\frac{\tilde{\Delta}_{\mathrm{eff}}}{\tilde{\Omega}_{\hat{n}}}\sin\left(\frac{\tilde{\Omega}_{\hat{n}}t}{2}\right),$$

$$\widehat{U}_{12}^{\mathrm{AJCM}}(t) = -i2\tilde{g}_{\mathrm{eff}}\hat{a}^\dagger\frac{\sin\left(\frac{\tilde{\Omega}_{\hat{n}+1}t}{2}\right)}{\tilde{\Omega}_{\hat{n}+1}},$$

$$\widehat{U}_{21}^{\mathrm{AJCM}}(t) = -i2\tilde{g}_{\mathrm{eff}}\frac{\sin\left(\frac{\tilde{\Omega}_{\hat{n}+1}t}{2}\right)}{\tilde{\Omega}_{\hat{n}+1}}\hat{a},$$

$$\widehat{U}_{22}^{\mathrm{AJCM}}(t) = \cos\left(\frac{\tilde{\Omega}_{\hat{n}+1}t}{2}\right) + i\frac{\Delta_{\mathrm{eff}}}{\tilde{\Omega}_{\hat{n}+1}}\sin\left(\frac{\tilde{\Omega}_{\hat{n}+1}t}{2}\right),$$

with $\tilde{\Delta}_{eff} = \omega_0 + \tilde{\omega}_{eff}$, and $\tilde{\Omega}_{\hat{n}} = \sqrt{\tilde{\Delta}_{eff}^2 + 4\tilde{g}_{eff}^2\hat{n}}$.

Figure 2 illustrates the time evolution of the atomic inversion $W(t)$ for the anisotropic quantum Rabi model, including the diamagnetic term, in two distinct coupling regimes: (a) $g_2 < g_1$ and (b) $g_1 < g_2$. The initial state of the system is chosen as $|\psi(0)\rangle = |\alpha\rangle \otimes |\uparrow\rangle = |\alpha, \uparrow\rangle$, where $|\alpha\rangle$ represents a coherent state of the field. Therefore, considering the corresponding evolution operator, Equation 6, the atomic inversion is given by

$$W(t) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} |A_n|^2 \left(\frac{\Delta_{\mathrm{eff}}^2 + 4g_{\mathrm{eff}}^2(n+1)\cos(\Omega_{n+1}t)}{\Delta_{\mathrm{eff}}^2 + 4g_{\mathrm{eff}}^2(n+1)}\right) & \text{if} \quad g_2 < g_1, \\ \displaystyle\sum_{n=0}^{\infty} |\tilde{A}_n|^2 \left(\frac{\tilde{\Delta}_{\mathrm{eff}}^2 + 4\tilde{g}_{\mathrm{eff}}^2 n\cos(\tilde{\Omega}_n t)}{\tilde{\Delta}_{\mathrm{eff}}^2 + 4\tilde{g}_{\mathrm{eff}}^2 n}\right) & \text{if} \quad g_1 < g_2, \end{cases}$$

where

$$A_n = \frac{1}{\sqrt{\cosh(r)}}\exp\left(-\frac{|\alpha_r|^2}{2} - \frac{\tanh(-r)}{2}\alpha_r^{*2}\right)\frac{\tanh^{n/2}(-r)}{2^{n/2}\sqrt{n!}}H_n \times \left[\alpha_r\frac{1+\tanh(-r)}{\sqrt{2\tanh(-r)}}\right],$$

with $\alpha_r = \alpha\cosh(r) + \alpha^*\sinh(r)$, and $H_n(x)$ being the n-th Hermite polynomial. Here, A_n determines the photon probability distribution $|A_n|^2$ for $g_2 < g_1$, while $|\tilde{A}_n|^2$, obtained from Equation 4, describes the distribution for $g_1 < g_2$, corresponding to the anti-Jaynes-Cummings model. These coefficients arise from the initial coherent state $|\alpha\rangle$ and the squeeze operator $\hat{S}^\dagger(r)$, derived from the evolution operator in Equation 6, such that $\hat{S}^\dagger(r)\widehat{D}(\alpha)|0\rangle = \widehat{D}(\alpha_r)\hat{S}^\dagger(r)|0\rangle$ [25]. The typical revivals of the Jaynes-Cummings model depend on the effective coupling $g_{\mathrm{eff}} = \sqrt{g_1^2 - g_2^2}$, reflecting coherent energy exchange between the atom and the field. However, $\hat{S}^\dagger(r)$ modifies these dynamics by altering the photon distribution and enhancing or suppressing transitions based on the squeezing parameter r. In the anti-Jaynes-Cummings regime ($g_2 \gg g_1$), revivals stabilize, characterized by $\tilde{g}_{\mathrm{eff}} = \sqrt{g_2^2 - g_1^2}$. As shown in Figure 3, the average photon number and Fock state populations are influenced by g_2: higher Fock states populate as g_2 approaches g_1, while they depopulate for $g_2 \gg g_1$. This behavior underscores the role of $\hat{S}^\dagger(r)$ in modulating quantum dynamics across both regimes.

Finally, in Figure 3, we show the probability distribution (a_1), the average photon number $\langle n(t)\rangle$ (a_2), and the field quadrature

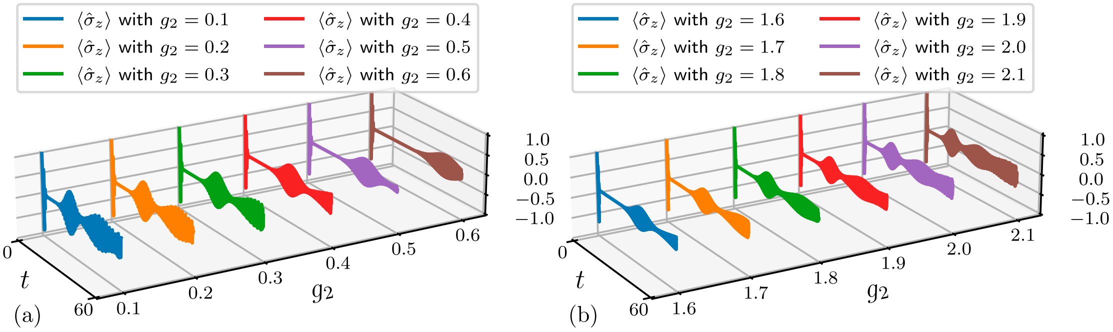

FIGURE 2
Dynamics of the atomic inversion in the anisotropic quantum Rabi model. The figure illustrates the temporal evolution of the atomic inversion $W(t)$ for two distinct coupling regimes: **(a)** $g_2 < g_1$ and **(b)** $g_1 < g_2$. The initial state of the system is chosen as $|\psi(0)\rangle = |\alpha\rangle \otimes |\uparrow\rangle = |\alpha, \uparrow\rangle$, where $|\alpha\rangle$ represents a coherent state of the field with $\alpha = 3$. Additionally, we have considered different values of g_2 for the two distinct parameter regimes with $g_1 = 1$, and we set $\omega = \omega_0 = 1$.

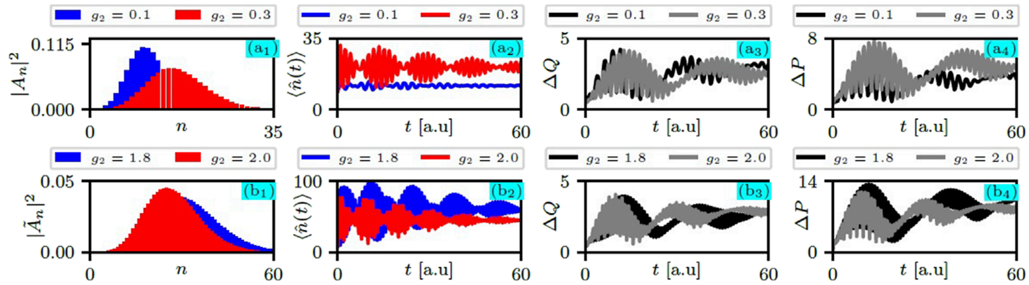

FIGURE 3
Dynamics of the anisotropic quantum Rabi model in two distinct coupling regimes: **(a)** Jaynes-Cummings regime ($g_2 < g_1$) and **(b)** anti-Jaynes-Cummings regime ($g_2 > g_1$), with $g_1 = 1$ and $\omega = \omega_0 = 1$. The panels show a_1, b_1 the photon probability distribution $|A_n|^2$ or $|\tilde{A}_n|^2$, a_2, b_2 the average photon number $\langle\hat{n}(t)\rangle$, a_3, b_3 the quadrature dispersion ΔQ, and a_4, b_4 the quadrature dispersion ΔP, as functions of time t.

dispersions $\Delta Q^2 = \langle x^2(t)\rangle - \langle x(t)\rangle^2$ and $\Delta P^2 = \langle p^2(t)\rangle - \langle p(t)\rangle^2$ (a_3 and a_4, respectively; the same is shown for b_j): Jaynes-Cummings $g_2 = [0.1, 0.3]$ and anti-Jaynes-Cummings $g_2 = [1.8, 2.0]$. The average photon number in the Jaynes-Cummings regime is given by

$$\langle\hat{n}(t)\rangle = \frac{1}{g_1^2 - g_2^2}\Bigg[g_2^2 + (g_1^2 + g_2^2)\sum_{n=0}^{\infty}|A_n|^2\left(n\left|\tau_{n+1}^{(11)}\right|^2 + (n+1)^2\left|\tau_{n+1}^{(21)}\right|^2\right) - 2g_1g_2 \times \sum_{n=0}^{\infty}\sqrt{(n+1)(n+2)}\,\mathrm{Re}\left\{A_n^* A_{n+2}e^{-i2\omega_{eff}t}\left[\left(\tau_{n+1}^{(11)*}\tau_{n+3}^{(11)}\right) + (n+3)\,\tau_{n+1}^{(21)*}\tau_{n+3}^{(21)}\right]\right\}\Bigg]$$

where $\tau_n^{(11)} = \cos(\Omega_n t/2) - i\Delta_{\mathrm{eff}}\sin(\Omega_n t/2)/\Omega_n$, and $\tau_n^{(21)} = -i2g_{\mathrm{eff}}\sin(\Omega_n t/2)/\Omega_n$. On the other hand, the average photon number in the anti-Jaynes-Cummings regime is

$$\langle\hat{n}(t)\rangle = \frac{1}{g_2^2 - g_1^2}\Bigg[g_1^2 + (g_1^2 + g_2^2)\sum_{n=0}^{\infty}|\tilde{A}_{n+1}|^2\left((n+1)\left|\tilde{\tau}_{n+1}^{(11)}\right|^2 + n(n+1)\left|\tilde{\tau}_{n+1}^{(21)}\right|^2\right)\Bigg] - 2g_1g_2\sum_{n=0}^{\infty}\mathrm{Re}\Big\{\tilde{A}_n^*\tilde{A}_{n+2}\sqrt{(n+1)(n+2)}e^{-i2\tilde{\omega}_{\mathrm{eff}}t}\tilde{\tau}_n^{(11)*}\tilde{\tau}_{n+2}^{(11)} + \tilde{A}_{n+1}^*\tilde{A}_{n+3}(n+1)\sqrt{(n+2)(n+3)}e^{-i2\tilde{\omega}_{\mathrm{eff}}t}\tilde{\tau}_{n+1}^{(21)*}\tilde{\tau}_{n+3}^{(21)}\Big\}$$

where $\tilde{\tau}_n^{(11)} = \cos(\tilde{\Omega}_n t/2) - i\tilde{\Delta}_{\mathrm{eff}}\sin(\tilde{\Omega}_n t/2)\tilde{\Omega}_n$, and $\tilde{\tau}_n^{(21)} = -i2\tilde{g}_{\mathrm{eff}}\sin(\tilde{\Omega}_n t/2)\tilde{\Omega}_n$. The squeeze operator $\hat{S}^\dagger(r)$ fundamentally modifies both the average photon number $\langle\hat{n}(t)\rangle$ and quadrature dispersions ΔQ, ΔP in Jaynes-Cummings ($g_2 < g_1$) and anti-Jaynes-Cummings ($g_1 < g_2$) regimes. Its action transforms photon operators as $\hat{S}^\dagger(r)\hat{a}^\dagger\hat{a}\hat{S}(r) = \mu^2\hat{a}^\dagger\hat{a} - \mu\nu(\hat{a}^2 + \hat{a}^{\dagger 2}) + \nu^2\hat{a}\hat{a}^\dagger$, demonstrating how it enhances/suppresses fluctuations via parameter r. When applied to a coherent state initial condition $|\alpha\rangle$, it generates a squeezed coherent state $|\alpha_r, -r\rangle = \hat{S}^\dagger(r)|\alpha\rangle$, whose photon statistics and quadrature properties are distinctly modified compared to the unsqueezed case, with $\alpha_r = \alpha\cosh r + \alpha^*\sinh r$ characterizing the displaced squeezed state. These results demonstrate the operator's role in controlling quantum dynamics in each regime.

4 Conclusion

It has been demonstrated that, by judiciously tuning the diamagnetic amplitude, the anisotropic quantum Rabi model can be reduced to either the Jaynes-Cummings model or the anti-Jaynes-Cummings model through the application of a squeezing transformation. Specifically, when the condition $\frac{\mu}{\nu} = \frac{g_1}{g_2}$ or $\frac{\mu}{\nu} = \frac{g_2}{g_1}$ is satisfied, where $\mu = \cosh(r)$ and $\nu = \sinh(r)$ are the hyperbolic functions associated with the squeezing parameter r, the Hamiltonian of the anisotropic quantum Rabi model transforms into the Jaynes-Cummings Hamiltonian for $g_2 < g_1$ or the anti-Jaynes-Cummings Hamiltonian for $g_1 < g_2$. In the case of the standard quantum Rabi model (*i.e.*, when $g_1 = g_2$), the system cannot be reduced to either the Jaynes-Cummings or anti-Jaynes-Cummings models. However, the squeezing transformation still allows us to eliminate the diamagnetic term, thereby removing the A^2 interaction from the Hamiltonian. This result highlights the versatility of the squeezing transformation in simplifying the anisotropic quantum Rabi model and its connection to well-known models in quantum optics.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

JA-C: Investigation, Methodology, Validation, Writing – original draft, Writing – review and editing. IR-P: Conceptualization, Investigation, Methodology, Software, Writing – original draft, Writing – review and editing. AZ-S: Investigation, Methodology, Supervision, Validation, Writing – original draft, Writing – review and editing. HM-C: Conceptualization, Investigation, Supervision, Writing – original draft, Writing – review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Generative AI statement

The author(s) declare that no Generative AI was used in the creation of this manuscript.

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