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Analytical solutions for the forced KdV equation with variable coefficients

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This paper focuses on obtaining the exact solutions to the variable-coefficient forced Korteweg-de Vries (KdV) equation for modeling spatial inhomogeneity in fluids. By combining the direct similarity reduction-based CK method with the *(G′/G)* expansion method, three new similarity solutions are obtained for this variable-coefficient forced KdV equation.

KEYWORDS

forced KdV equation, direct similarity reduction-based CK method, variable coefficient, similarity solution, exact solution

1 Introduction

The forced KdV equation with variable coefficients plays a crucial role in researching the waves motion and nonlinear phenonmeno [1], and it attracts more and more research attentions [2]. In this paper, we examine the forced KdV equation with variable coefficients:

$$u_t + a(t)uu_x + b(t)u_{xxx} + c(t)u + d(t)u_x = f(x,t), \tag{1.1}$$

where $u = u(x,t)$ is the wave function and x and t are scaled spatial and temporal coordinates, respectively. $a(t)$ is a nonlinearity coefficient, $b(t)$ is a dispersive coefficient, $c(t)$ is a line-damping coefficient, $d(t)$ is a dissipative coefficient, and $f(x,t)$ represents the external force effect function. Under suitable selections of the coefficient functions, Equation 1.1 reduces to a sequence of integrable systems or describe the nonlinear waves in a fluid-filled tube [3], weakly nonlinear waves in the water of variable depth [4], and it models the spatial inhomogeneity in fluids [5]. The classical KdV equation was first derived from shallow water wave theory to describe the propagation of water waves in the long wave limit (surface gravity waves) [6]. The coefficients of variation in Equation 1.1 are due to geometric and physical inhomogeneities [3, 7].

In this context, we suppose that the external force effect function has the following form according to [5]:

$$f(x,t) = f_1(t)x + f_2(t). \tag{1.2}$$

In particular, when $a(t) = \mu_1$, $b(t) = \mu_2$, $c(t) = 0$, $d(t) = \mu_3(t) - \mu_1\mu(t)$, and $f(x,t) = 0$, H.L. Demiray [8] obtained the solitary wave solution via coordinate transformation. When $a(t) = \frac{3c^3}{2}$, $b(t) = \frac{1}{6c}$, $c(t) = -\frac{1}{2c}$, $d(t) = 0$, and $f(x,t) = 0$, H.L. Demiray [9] obtained a progressive wave-type solution through the reductive perturbation method. When $f_1(t) = 0$, Zhang et al. [10] utilized the Wronskian technique and Hirota method to obtain a bilinear form and an analytic N-soliton-like solution. On the basis of symbolic computation, Tian et al.

[11] obtained solutions for the Airy, Hermit and Jacobian elliptic functions [12]. Utilizing the Hirota bilinear method, Yu et al. [13] obtained an N-soliton solution and a type of analytic solution.

Various methods have been applied to search for exact solutions to nonlinear evolution equations (see [14–23]), including the (G/G) expansion method [22, 24], the Hirota bilinear method [18], the inverse scattering transform method [16], the CK direct method [15], and the nonclassic Lie group method [14], among others.

In general, solving ordinary differential equations is easier than directly constructing partial differential equations. However, owing to variable coefficients, it remains difficult to solve an ordinary differential equation with variable coefficients. In the present work, first, by applying the direct similarity reduction-based CK method, we transform the partial differential equation shown in (1.1) into an ordinary differential equation. Then, solutions are obtained for the above ordinary differential equation via the (G/G) expansion method. Finally, we can obtain three new similarity solutions to the variable-coefficient forced KdV equation in (1.1) and express the graphics of these similarity solutions. This equation may represent the main profile of these solutions for Equation 1.1.

2 Direct similarity reduction-based CK method

We hypothesize that the solution to Equation 1.1 takes the following form:

$$u(x,t) = U(x,t,w(z(x,t))). \tag{2.1}$$

Indeed, Equation 2.1 may represent the general form of the similarity solutions for Equation 1.1 (see [25]), and it is sufficient to consider the solutions for Equation 1.1 in the form of Equation 2.1. By substituting Equations 1.2, 2.1 into (1.1), we obtain

$$\begin{aligned}&[U_t + U_w w z_t] + a(t)U[U_x + U_w w z_x] + c(t)U + d(t)[U_x + U_w w z_x]\\&+ b(t)\{U_{xxx} + 3U_{xxw} w z_x + U_{www}(w)^3 z_x^3 + U_{xw}[3wz_x^2 + 3wz_{xx}]\\&+ U_{ww}[3wwz_x^3 + 3w^2 z_{xx} z_x] + 3U_{xww} w^2 (z_x)^2\\&+ U_w[wz_x^3 + 3wz_{xx}z_x + wz_{xxx}]\} = f_1(t)x + f_2(t),\end{aligned} \tag{2.2}$$

where $w = dw/dz$, $w = d^2w/dz^2$, and $w = d^3w/dz^3$. To reduce Equation 2.2 to an ordinary differential equation with respect to $w(z)$, the derivatives of $w(z)$ should be only functions of z and w. Moreover, when conducting normalization with a coefficient of w, namely, $U_w z_x^3 b(t)$, the coefficients of ww must satisfy

$$U_w z_x^3 b(t)\Gamma_1(w,z) = b(t)U_{ww} z_x^3, \tag{2.3}$$

where $\Gamma_1(w,z)$ is a function and can be determined as follows. From Equation 2.3, we have

$$\Gamma_1(w,z) = \frac{U_{ww}}{U_w}.$$

After an integration step, we obtain

$$\int \Gamma_1(w,z)\,dw = \ln U_w - \ln\beta_0(x,t),$$

where $\ln\beta_0(x,t)$ is the integral function. Letting $\ln\Gamma_2(w,z) = \int\Gamma_1(w,z)dw$, we have

$$\Gamma_2(w,z) = \frac{U_w}{\beta_0(x,t)}.$$

After an integration process, we obtain

$$\int \Gamma_2(w,z) = \frac{1}{\beta_0(x,t)}[U - \alpha_0(x,t)], \tag{2.4}$$

where $\alpha_0(x,t)$ is the function generated via integration. Let $\Gamma_3(w,z) = \int\Gamma_2(w,z)dw$; from Equation 2.4, we have that

$$U = \beta_0(x,t)\Gamma_3(w,z) + \alpha_0(x,t).$$

Letting $\Gamma_3(w,z) = w(z)$, $\alpha_0(x,t) = \alpha(x,t)$, and $\beta_0(x,t) = \beta(x,t)$, we have

$$u(x,t) = \alpha(x,t) + \beta(x,t)w[z(x,t)]. \tag{2.5}$$

Next, we compute $\alpha(x,t)$, $\beta(x,t)$, and $z(x,t)$. By substituting Equation 2.5 into Equation 2.2, we deduce that

$$\begin{aligned}&b(t)\beta z_x^3 w + wb(t)[3\beta_x z_x^2 + 3\beta z_x z_{xx}] + w[\beta z_t + a(t)\alpha\beta z_x + d(t)\beta z_x\\&+ b(t)(3\beta_{xx}z_x + 3\beta_x z_{xx} + \beta z_{xxx})] + wwa(t)\beta^2 z_x + w^2 a(t)\beta\beta_x\\&+ w[\beta_t + a(t)(\alpha\beta_x + \alpha_x\beta) + c(t)\beta + d(t)\beta_x + b(t)\beta_{xxx}]\\&+ \alpha_t + a(t)\alpha\alpha_x + \alpha c(t) + d(t)\alpha_x + b(t)\alpha_{xxx} - [f_1(t)x + f_2(t)] = 0.\end{aligned} \tag{2.6}$$

For the above equation to reduce to a solvable equation with respect to $w(z)$, the ratios of the coefficients $w(z)$ and the derivatives of the equation must be dependent on z alone. This condition provides the relationships for $\alpha(x,t)$, $\beta(x,t)$, and $z(x,t)$, ensuring that any solution is a similarity reduction solution. Then, we introduce the following three footnotes (see [15]).

Remark 1: We adopt the coefficient of w (i.e., $b(t)\beta z_x^3$) as the normalizing factor and thus impose the condition that the other coefficients must take the form $b(t)\beta z_x^3\Gamma(z)$, where $\Gamma(z)$ is a function of z that needs to be determined.

Remark 2: Uppercase Greek letters are reserved for undetermined functions of z, ensuring that after performing operations (such as differentiation, integration, exponentiation, and rescaling), the result can still be denoted by the same letter. For example, the derivative of $\Gamma(z)$ is denoted as $\Gamma(z)$.

Remark 3: Three degrees of freedoms exist in the determination of α, β, z and w and can be exploited (without loss of generality) to keep the proposed method manageable:

(i) if $\alpha(x,t)$ is expressed as $\alpha = \alpha_0(x,t) + \beta(x,t)\Omega(z)$, we can simplify it by setting $\Omega(z) = 0$, which is equivalent to performing the substitution $w(z) \to w(z) - \Omega(z)$;
(ii) if $\beta(x,t)$ is expressed as $\beta = \beta_0(x,t)\Omega(z)$, we can simplify it by setting $\Omega(z) = 1$, which is equivalent to performing the substitution $w(z) \to w(z)/\Omega(z)$];
(iii) if $z(x,t)$ is determined by an equation with the form $\Omega(z) = z_0(x,t)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z) = z$ [by substituting $z \to \Omega^{-1}(z)$].

The general similarity reduction of Equation 1.1 can be determined via this method.

For Equation 2.2 to be reduced to an ordinary differential equation in terms of $w(z)$, the ratios of the various derivatives of $w(z)$ must depend on the functions of w and z. When implementing normalization with the coefficient of w, namely, $b(t)\beta z_x^3$, the coefficients of ww must satisfy certain conditions.

$$b(t)\beta z_x^3\Gamma_4(z) = a(t)\beta^2 z_x,$$

where $\Gamma_4(z)$ is a function to be determined. From Remark 3 (ii), we have

$$\beta = \frac{b(t)}{a(t)} z_x^2. \tag{2.7}$$

The coefficient of w^2 requires that

$$b(t)\beta z_x^3\Gamma_5(z) = a(t)\beta\beta_x, \tag{2.8}$$

where $\Gamma_5(z)$ is a function to be determined. By substituting Equation 2.8 into Equation 2.7, we obtain

$$z_x^2\Gamma_5(z) = 2z_{xx}.$$

From Remark 2, after applying a scale transformation, we have

$$z_x\Gamma_5(z) + \frac{z_{xx}}{z_x} = 0.$$

An integration step yields the following:

$$\Gamma_5(z) + lnz_x = \Theta(t),$$

where $\Theta(t)$ is the integration function. From Remark 2, this can be exponentiated:

$$z_x\Gamma_5 = \Theta(t).$$

Integrating again gives

$$\Gamma_5 = x\Theta(t) + \Sigma(t),$$

where $\Sigma(t)$ is another integration function. We obtain the following equation through Remark 3 (iii).

$$z = x\theta(t) + \sigma(t), \tag{2.9}$$

where $\theta(t)$ and $\sigma(t)$ are functions to be determined.

By substituting Equation 2.9 into Equation 2.7, we have that

$$\beta = \frac{b(t)}{a(t)}\theta(t)^2. \tag{2.10}$$

With the coefficient of w, we have

$$b(t)\beta z_x^3\Gamma_6(z) = \beta z_t + a(t)\alpha\beta z_x + d(t)\beta z_x + b(t)\left(3\beta_{xx}z_x + 3\beta_x z_{xx} + \beta z_{xxx}\right).$$

where $\Gamma_6(z)$ is a function to be determined. From Equations 2.9, 2.10, we have

$$b(t)\theta^3\Gamma_6(z) = x\frac{d\theta}{dt} + \frac{d\sigma}{dt} + a(t)\alpha\theta + d(t)\theta,$$

From Remark 3(i), we obtain

$$\alpha = -\frac{1}{a(t)\theta}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right) - \frac{d(t)}{a(t)}. \tag{2.11}$$

Substituting Equations 2.9–2.11 into (Equation 2.2), we have

$$\begin{aligned}
&\theta^5\frac{b^2(t)}{a(t)}(w + ww) + w\left\{\frac{a(t)\left[b(t)\theta^2 + 2b(t)\theta\frac{d\theta}{dt}\right] - a(t)b(t)\theta^2}{a(t)^2}\right.\\
&\left.\quad - \frac{b(t)\theta}{a(t)}\frac{d\theta}{dt} + \frac{b(t)c(t)\theta^2}{a(t)}\right\}\\
&+ \left[\frac{1}{\theta}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right) + d(t)\right]\frac{1}{a(t)\theta}\frac{d\theta}{dt}\\
&- c(t)\left[\frac{1}{a(t)\theta}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right) + \frac{d(t)}{a(t)}\right] - \frac{x\frac{d^2\theta}{dt^2} + \frac{d^2\sigma}{dt^2}}{a(t)\theta}\\
&+ \frac{\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right)\left[\frac{d\theta}{dt}a(t) + a(t)\theta\right]}{\theta^2 a^2(t)} - \frac{d(t)a(t) - d(t)a(t)}{a^2(t)}\\
&- \left[f_1(t)x + f_2(t)\right] = 0.
\end{aligned} \tag{2.12}$$

We proceed to transform Equation 2.12 into an ordinary differential equation for $w(z)$. By using the coefficients of w and ww (i.e., $\theta^5\frac{b^2(t)}{a(t)}$), we deduce that the coefficients of w and w^0 must satisfy

$$\begin{aligned}
\theta^5\frac{b^2(t)}{a(t)}\gamma_1(z) &= \frac{a(t)\left[b(t)\theta^2 + 2b(t)\theta\frac{d\theta}{dt}\right] - a(t)b(t)\theta^2}{a^2(t)}\\
&\quad - \frac{b(t)\theta}{a(t)}\frac{d\theta}{dt} + \frac{b(t)c(t)\theta^2}{a(t)},
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
\theta^5\frac{b^2(t)}{a(t)}\gamma_2(z) &= \left[\frac{1}{\theta}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right) + d(t)\right]\frac{1}{a(t)\theta}\frac{d\theta}{dt}\\
&\quad - c(t)\left[\frac{1}{a(t)\theta}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right) + \frac{d(t)}{a(t)}\right] - \frac{x\frac{d^2\theta}{dt^2} + \frac{d^2\sigma}{dt^2}}{a(t)\theta}\\
&\quad + \frac{\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right)\left[\frac{d\theta}{dt}a(t) + a(t)\theta\right]}{\theta^2 a^2(t)}\\
&\quad - \frac{d(t)a(t) - d(t)a(t)}{a^2(t)} - \left[f_1(t)x + f_2(t)\right],
\end{aligned} \tag{2.14}$$

where $\gamma_1(z)$ and $\gamma_2(z)$ are functions to be determined. Since $z = x\theta(t) + \sigma(t)$ and the right-hand side of Equation 2.14 is linear in terms of x, we can assume that $\gamma_2(z) = Az + B$, where A and B are constants. From Equation 2.14, we have that

$$\begin{aligned}
\theta^5\frac{b^2(t)}{a(t)}\left[A(x\theta + \sigma) + B\right] &= \left[\frac{1}{\theta}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right) + d(t)\right]\frac{1}{a(t)\theta}\frac{d\theta}{dt}\\
&\quad - c(t)\left[\frac{1}{a(t)\theta}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right) + \frac{d(t)}{a(t)}\right]\\
&\quad - \frac{x\frac{d^2\theta}{dt^2} + \frac{d^2\sigma}{dt^2}}{a(t)\theta} + \frac{\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right)\left[\frac{d\theta}{dt}a(t) + a(t)\theta\right]}{\theta^2 a^2(t)}\\
&\quad - \frac{d(t)a(t) - d(t)a(t)}{a^2(t)}\\
&\quad - \left[f_1(t)x + f_2(t)\right].
\end{aligned} \tag{2.15}$$

By comparing the coefficients of x, we obtain

$$\begin{aligned} A\theta^6 \frac{b^2(t)}{a(t)} &= -\frac{1}{a(t)\theta}\frac{d^2\theta}{dt^2} + \frac{1}{a^2(t)\theta^2}\frac{d\theta}{dt}\left[\frac{d\theta}{dt}a(t) + a(t)\theta\right] \\ &\quad + \frac{1}{a(t)\theta^2}\left[\frac{d\theta}{dt}\right]^2 - \frac{c(t)}{a(t)\theta}\frac{d\theta}{dt} - f_1(t), \end{aligned} \tag{2.16}$$

$$\begin{aligned} \theta^5 \frac{b^2(t)}{a(t)}(A\sigma + B) &= -\frac{1}{a(t)\theta}\frac{d^2\sigma}{dt^2} + \frac{1}{a^2(t)\theta^2}\frac{d\sigma}{dt}\left[\frac{d\theta}{dt}a(t) + a(t)\theta\right] \\ &\quad - \frac{1}{a^2(t)\theta^2}\left[d(t)a(t) - d(t)a(t)\right] \\ &\quad + \left[\frac{1}{\theta}\frac{d\theta}{dt} + d(t)\right]\frac{1}{a(t)\theta}\frac{d\theta}{dt} \\ &\quad + c(t)\left[-\frac{1}{a(t)\theta}\frac{d\sigma}{dt} - \frac{d(t)}{a(t)}\right] - f_2(t). \end{aligned} \tag{2.17}$$

Let

$$f_1(t) = \frac{1}{a(t)\theta^2}A^2 + \frac{Aa(t)\theta + 2A^2 a(t)}{a^2(t)\theta^2} - A\frac{c(t)}{a(t)\theta} - A\theta^6\frac{b^2(t)}{a(t)}, \tag{2.18}$$

and

$$\begin{aligned} f_2(t) &= -\theta^5\frac{b^2(t)}{a(t)}(A\sigma + B) + \frac{1}{a^2(t)\theta^2}B\left[Aa(t) + a(t)\theta\right] \\ &\quad - \frac{1}{a^2(t)\theta^2}\left[d(t)a(t) - d(t)a(t)\right] \\ &\quad + \left[\frac{1}{\theta}A + d(t)\right]\frac{1}{a(t)\theta}A + c(t)\left[-\frac{1}{a(t)\theta}B - \frac{d(t)}{a(t)}\right]. \end{aligned} \tag{2.19}$$

It follows from Equations 2.18, 2.19, 2.16, 2.17 that

$$\begin{cases} \theta = At + A_0, \\ \sigma = Bt + B_0, \end{cases} \tag{2.20}$$

where A_0 and B_0 are integral constants. For the convenience of the calculation process, we assume that

$$\frac{a(t)}{a(t)} - \frac{b(t)}{b(t)}\theta = A + c(t)\theta - A\theta^4 b(t). \tag{2.21}$$

By substituting Equations 2.20–2.21 into Equation 2.13, we obtain $\gamma_1(z) = A$. Then, substituting $\gamma_1(z) = A$ and $\gamma_2(z) = Az + B$ into Equation 2.12 yields

$$w + ww + Aw + Az + B = 0, \tag{2.22}$$

where A and B are arbitrary constants. Upon substituting Equations 2.9–2.11 and (Equation 2.20) into Equation 2.5, we can deduce that

$$\begin{aligned} u &= \frac{b(t)}{a(t)}(At + A_0)^2 w(z) - \frac{1}{a(t)(At + A_0)}(xA + B) - \frac{d(t)}{a(t)}, \\ z &= x(At + A_0) + Bt + B_0, \end{aligned} \tag{2.23}$$

where $w(z)$ satisfies Equation 2.22.

Next, we try to compute the exact solutions for Equation 2.22 based on the (G/G) expansion method. More precisely, we suppose that Equation 2.22 has solutions in the following form:

$$w(z) = \sum_{i=1}^{m} a_i(z)(G/G),$$

where $m \in N$. To balance w and ww, we can take $m = 2$, and then we have

$$w(z) = a_0(Z) + a_1(z)(G/G) + a_2(z)(G/G)^2, \tag{2.24}$$

where G satisfies the following ordinary differential equation:

$$G + \lambda G + \mu G = 0, \tag{2.25}$$

where λ and μ are constants. By substituting Equation 2.24 and Equation 2.25 into Equation 2.22 and comparing their coefficients, we deduce that

$$\begin{cases} -24a_2 - 2a_2^2 = 0, \\ -3(-4a_2 + a_1 + 10a_2\lambda) + 6a_2 - 24a_2\lambda \\ \quad + \left[a_2(a_1 - a_1\lambda - 2\mu a_2) - 2a_1a_2\right] = 0, \\ -2(a_2 - 2a_1 - 4\lambda a_2 + 3\lambda a_1 + 8a_2\mu + 4a_2\lambda^2) - 4a_2 + 2a_1 + 10a_2\lambda \\ \quad - 3\lambda(-4a_2 + 2a_1 + 10a_2\lambda) \\ -24a_2\mu - 2a_0a_2 + a_1(a_1 - a_1\lambda - 2a_2\mu) + a_2(a_1 - \lambda a_1 - 2a_2\mu) = 0, \\ -(a_1 - 2a_1\lambda - 4a_2\mu + a_1\lambda^2 + 6a_2\lambda\mu + 2a_1\mu) + a_2 - 2a_1 - 4a_2\lambda \\ \quad + 3a_1\lambda + 8a_2\mu + 4a_2\lambda^2 \\ -2\lambda(a_2 - 2a_1 - 4a_2\lambda + 8a_2\mu + 4a_2\lambda^2) - 3\mu(-4a_2 + 2a_1 + 10a_2)a_0 \\ \quad (a_1 - a_1\lambda - 2a_2\mu) \\ + a_1(a_1 - a_1\lambda - 2a_2\mu) + a_2(a_0 - a_1\mu) + Aa_2 = 0, \\ a_1 - 2a_1\lambda - 4a_2\mu + a_1\lambda^2 + 6a_2\lambda\mu + 2a_1\mu - \lambda \\ \quad (a_1 - 2a_1\lambda - 4a_2\mu + a_1\lambda + 6a_2\lambda\mu \\ +2a_1\mu) - 2\mu(a_2 - 2a_1 - 4a_2\lambda + 3a_1\lambda + 8a_2\mu + 4a_2\lambda^2) \\ \quad + a_1(a_0 - a_1\mu) \\ a_0(a_1 - a_1\lambda - 2a_2\mu) + Aa_1 = 0, \\ a_0 - a_1\mu - \mu(a_1 - a_1\lambda - 2a_2\mu) - \mu(a_1 - 2a_1\lambda - 4a_2\mu + a_1\lambda^2 + 6a_2\lambda\mu + 2a_1\mu) \\ \quad + Aa_0 + Az + B = 0. \end{cases}$$

Then, we obtain

$$\begin{cases} a_0(z) = 11\lambda^2 - 8\mu + \left[3\lambda - 12\mu - \frac{2\mu(6\lambda - 24\mu)}{2\lambda + 3}\right]e^{(2\lambda+3)z} \\ \quad + \frac{1 + 2\lambda^2 + 3\lambda}{24(2\lambda + 3)}(12\lambda - 48\mu)^2 e^{2(2\lambda+3)z}, \\ a_1(z) = (6\lambda - 24\mu)\frac{2}{2\lambda + 3}e^{(2\lambda+3)z}, \\ a_2(z) = -12. \end{cases} \tag{2.26}$$

The following exact solution forms exist for Equation 2.22, and they can be acquired by substituting Equation 2.26 into Equation 2.24.

Case 1. If $\lambda^2 - 4\mu > 0$, then

$$w_1(z) = 11\lambda^2 - 8\mu + \left[3\lambda - 12\mu - \frac{2\mu(6\lambda - 24\mu)}{2\lambda + 3}\right]e^{(2\lambda+3)z} + \frac{1 + 2\lambda^2 + 3\lambda}{24(2\lambda + 3)}(12\lambda - 48\mu)^2 e^{2(2\lambda+3)z} + (6\lambda - 24\mu)\frac{2}{2\lambda + 3}e^{(2\lambda+3)z} \left[\frac{\sqrt{\lambda^2 - 4\mu}}{2}\frac{A_1 cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right) + A_2 sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right)}{A_1 sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right) + A_2 cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right)} - \frac{\lambda}{2}\right] - 12\left[\frac{\sqrt{\lambda^2 - 4\mu}}{2}\frac{A_1 cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right) + A_2 sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right)}{A_1 sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right) + A_2 cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right)} - \frac{\lambda}{2}\right]^2. \tag{2.27}$$

Case 2. If $\lambda^2 - 4\mu < 0$, then

$$w_2(z) = 11\lambda^2 - 8\mu + \left[3\lambda - 12\mu - \frac{2\mu(6\lambda - 24\mu)}{2\lambda + 3}\right]e^{(2\lambda+3)z} + \frac{1 + 2\lambda^2 + 3\lambda}{24(2\lambda + 3)}(12\lambda - 48\mu)^2 e^{2(2\lambda+3)z} + (6\lambda - 24\mu)\frac{2}{2\lambda + 3}e^{(2\lambda+3)z} \left[\frac{\sqrt{4\mu - \lambda^2}}{2}\frac{A_1 sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right) + A_2 cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right)}{A_1 cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right) + A_2 sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right)} - \frac{\lambda}{2}\right] - 12\left[\frac{\sqrt{4\mu - \lambda^2}}{2}\frac{A_1 sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right) + A_2 cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right)}{A_1 cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right) + A_2 sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right)} - \frac{\lambda}{2}\right]^2. \tag{2.28}$$

Case 3. If $\lambda^2 - 4\mu = 0$, then

$$w_3(z) = 11\lambda^2 - 8\mu + \left[3\lambda - 12\mu - \frac{2\mu(6\lambda - 24\mu)}{2\lambda + 3}\right]e^{(2\lambda+3)z} + \frac{1 + 2\lambda^2 + 3\lambda}{24(2\lambda + 3)}(12\lambda - 48\mu)^2 e^{2(2\lambda+3)z} + (6\lambda - 24\mu)\frac{2}{2\lambda + 3}e^{(2\lambda+3)z}\left[\frac{A_2}{A_1 + A_2 z} - \frac{\lambda}{2}\right] - 12\left[\frac{A_2}{A_1 + A_2 z} - \frac{\lambda}{2}\right]^2. \tag{2.29}$$

3 Conclusion

Substitute Equation 2.27–2.29 into Equation 2.23 and the following forms of similarity solutions exist for Equation 1.1.

Case 1. If $\lambda^2 - 4\mu > 0$, then

$$u_1(x,t) = \left\{11\lambda^2 - 8\mu + \left[3\lambda - 12\mu - \frac{2\mu(6\lambda - 24\mu)}{2\lambda + 3}\right]e^{(2\lambda+3)z} + \frac{1 + 2\lambda^2 + 3\lambda}{24(2\lambda + 3)}(12\lambda - 48\mu)^2 e^{2(2\lambda+3)z} + (6\lambda - 24\mu)\frac{2}{2\lambda + 3}e^{(2\lambda+3)z} \left[\frac{\sqrt{\lambda^2 - 4\mu}}{2}\frac{A_1 cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right) + A_2 sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right)}{A_1 sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right) + A_2 cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right)} - \frac{\lambda}{2}\right] - 12\left[\frac{\sqrt{\lambda^2 - 4\mu}}{2}\frac{A_1 cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right) + A_2 sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right)}{A_1 sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right) + A_2 cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}z\right)} - \frac{\lambda}{2}\right]^2\right\} \frac{b(t)}{a(t)}(At + A_0)^2 - \frac{1}{a(t)(At + A_0)}(xA + B) - \frac{d(t)}{a(t)}. \tag{3.1}$$

The similar solution corresponding to Case 1 is expressed in Figure 1.

The values determined for $u_1(x,t)$ when $\lambda = 1$, $\mu = -1$, $A_1 = 100$, $A_2 = A = B = A_0 = B_0 = a(t) = b(t) = d(t) = 1$.

Case 2. If $\lambda^2 - 4\mu < 0$, then

$$u_2(x,t) = \left\{11\lambda^2 - 8\mu + \left[3\lambda - 12\mu - \frac{2\mu(6\lambda - 24\mu)}{2\lambda + 3}\right]e^{(2\lambda+3)z} + \frac{1 + 2\lambda^2 + 3\lambda}{24(2\lambda + 3)}(12\lambda - 48\mu)^2 e^{2(2\lambda+3)z} + (6\lambda - 24\mu)\frac{2}{2\lambda + 3}e^{(2\lambda+3)z} \left[\frac{\sqrt{4\mu - \lambda^2}}{2}\frac{A_1 sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right) + A_2 cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right)}{A_1 cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right) + A_2 sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right)} - \frac{\lambda}{2}\right] - 12\left[\frac{\sqrt{4\mu - \lambda^2}}{2}\frac{A_1 sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right) + A_2 cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right)}{A_1 cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right) + A_2 sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}z\right)} - \frac{\lambda}{2}\right]^2\right\} \frac{b(t)}{a(t)}(At + A_0)^2 - \frac{1}{a(t)(At + A_0)}(xA + B) - \frac{d(t)}{a(t)}. \tag{3.2}$$

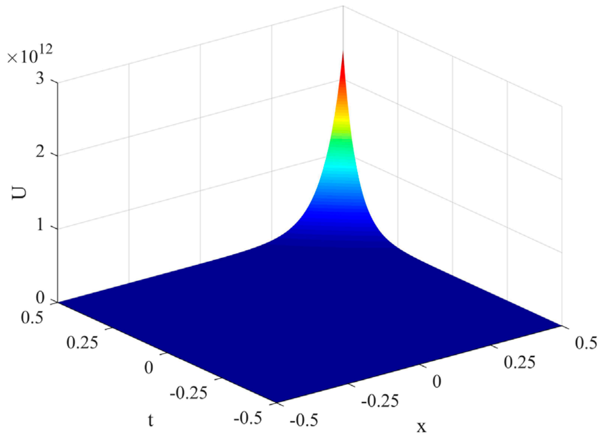

FIGURE 1
The values determined for $u_1(x, t)$ when $\lambda = 1$, $\mu = -1$, $A_1 = 100$, $A_2 = A = B = A_0 = B_0 = a(t) = b(t) = d(t) = 1$.

The similar solution corresponding to Case 2 is expressed in Figure 2.

The values determined for $u_2(x,t)$ when $\lambda = 1$, $\mu = 1$, $A_1 = 100$, $A_2 = A = B = A_0 = B_0 = a(t) = b(t) = d(t) = 1$.

Case 3. If $\lambda^2 - 4\mu = 0$, then

$$
\begin{aligned}
u_3(x,t) = &\Bigg\{11\lambda^2 - 8\mu + \left[3\lambda - 12\mu - \frac{2\mu(6\lambda - 24\mu)}{2\lambda + 3}\right] e^{(2\lambda+3)z} \\
&+ \frac{1 + 2\lambda^2 + 3\lambda}{24(2\lambda + 3)}(12\lambda - 48\mu)^2 e^{2(2\lambda+3)z} \\
&+ (6\lambda - 24\mu)\frac{2}{2\lambda + 3} e^{(2\lambda+3)z}\left[\frac{A_2}{A_1 + A_2 z} - \frac{\lambda}{2}\right] \\
&-12\left[\frac{A_2}{A_1 + A_2 z} - \frac{\lambda}{2}\right]^2\Bigg\}\frac{b(t)}{a(t)}(At + A_0)^2 \\
&- \frac{1}{a(t)(At + A_0)}(xA + B) - \frac{d(t)}{a(t)}.
\end{aligned} \tag{3.3}
$$

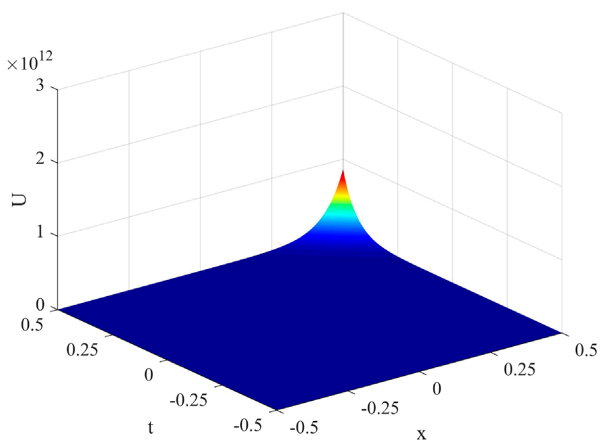

FIGURE 2
The values determined for $u_2(x, t)$ when $\lambda = 1$, $\mu = 1$, $A_1 = 100$, $A_2 = A = B = A_0 = B_0 = a(t) = b(t) = d(t) = 1$.

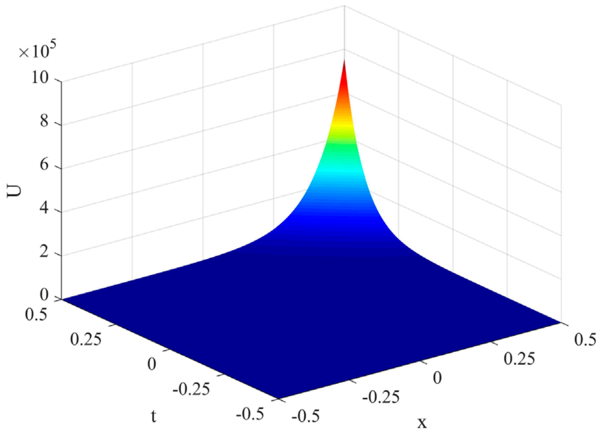

FIGURE 3
The values determined for $u_3(x, t)$ when $\lambda = 2$, $A_1 = 100$, $\mu = A_2 = A = B = A_0 = B_0 = a(t) = b(t) = d(t) = 1$.

The similar solution corresponding to Case 3 is expressed in Figure 3.

The values determined for $u_3(x,t)$ when $\lambda = 2$, $A_1 = 100$, $\mu = A_2 = A = B = A_0 = B_0 = a(t) = b(t) = d(t) = 1$.

According to the above discussion, the shape of the graphic is sensitive to the values of x and t. Indeed, we can clearly see the change trends exhibited by these three graphs, which are in exponential form. The constraints on the solution of the variable coefficient equation obtained by Method 1 are simple and less categorical than those of the previous article [26].

In the present work, we investigate the variable-coefficient forced KdV equation. As a result, this paper not only reduces the equation to an ordinary differential equation via the direct similarity reduction-based CK method but also obtains similarity solutions from the solutions of the above ordinary differential equation. This provides a simpler method for studying the variable-coefficient forced KdV equation. It simplifies the mathematical solution process and facilitates fluctuation control and application design in engineering practice. Many other variable-coefficient nonlinear partial differential equations can also be investigated via this method.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

JW: Conceptualization, Writing – original draft, Writing – review and editing. JF: Writing – review and editing, Visualization. JD: Visualization, Writing – original draft, Writing – review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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