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EDITED BY

Karl Hess,
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United States

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Juergen Jakumeit,
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Omar Magana-Loaiza,
Louisiana State University, United States

*CORRESPONDENCE

Ana María Cetto,
✉ ana@fisica.unam.mx

[†]These authors have contributed equally to
this work

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Signature of matter–field coupling in quantum–mechanical statistics

Ana María Cetto^{*†} and Luis de la Peña[†]

Instituto de Física, Universidad Autónoma de México, Mexico City, Mexico

The connection between the intrinsic angular momentum (spin) of particles and quantum statistics is established by considering the response of identical particles to a common background radiation field. For this purpose, the Hamiltonian analysis previously performed in stochastic electrodynamics to derive the quantum description of a one-particle system is extended to a system of two identical bound particles subject to the same field. Depending on the relative phase of the response of the particles to a common field mode, two types of particles are distinguished by their symmetry or antisymmetry with respect to particle exchange. While any number of identical particles responding in phase can occupy the same energy state, there can only be two particles responding in antiphase. The calculation of bipartite correlations between the response functions reveals maximum entanglement as a consequence of the parallel response of the particles to the common field. The introduction of an internal rotation parameter leads to a direct link between spin and statistics and to a physical rationale for the Pauli exclusion principle.

KEYWORDS

particle-field coupling, resonant response, quantum statistics, symmetry/antisymmetry, Pauli exclusion principle

1 Introduction

The statistics of identical particles is one of the most fundamental quantum features: all quantum particles are known to obey either Fermi–Dirac or Bose–Einstein statistics. It is also well known that the intrinsic angular momentum (spin) of a particle determines its statistics and *vice versa*, with integral-spin particles being bosons and half-integral-spin particles being fermions. The symmetrization postulate and the spin statistics theorem are central to a number of key quantum applications, including all of atomic, molecular, and nuclear physics and quantum statistical physics. Nevertheless, a century after their establishment (Pauli, 1925; Heisenberg, 1926; Dirac, 1926), they continue to be taken as mathematically-justified empirical facts. All known experimental data are consistent with Pauli's exclusion principle, and experiments continue to be carried out to find possible violations of it (Kaplan, 2020). Pauli himself, who gave the first formal proof of the spin-statistics theorem in 1925, expressed his dissatisfaction with this state of affairs two decades later (Pauli, 1946; Pauli, 1950), but explanations continue to rely mainly on formal arguments based on topological properties, group-theoretical considerations, and the like.

All this leads to the conclusion that the physical underpinning of quantum statistics remains to be elucidated. What makes the state vectors of identical multipartite systems either symmetric or antisymmetric? What is the mechanism that “binds” identical particles in such a way that they obey either Fermi or Bose statistics?

The aim of this paper is to provide an answer to these questions based on general principles and previous results from stochastic electrodynamics (SED). Recent work has shown that consideration of the interaction of particles with the electromagnetic radiation field is key to understanding their quantum behavior (de la Peña et al., 2015). The ground state of the radiation field—the zero point field (ZPF)—has been identified as the source of quantum fluctuations and a key factor in driving a bound system to a stationary state. In addition, the quantum operator formalism has been obtained as the algebra describing the response of the particle's dynamical variables to the background field modes responsible for the transitions between stationary states (Cetto and de la Peña, 2024). Furthermore, bipartite entanglement was derived as a consequence of the interaction of two identical particles with the same field modes (de la Peña et al., 2015). Against this background, the theory provides us with a physically grounded explanation of the origin of the symmetry properties of identical quantum particle systems and the resulting statistics.

The paper is structured as follows. Section 2 summarizes the SED Hamiltonian derivation of the quantum operator formalism, which gives sense to this formalism as an algebraic description of the linear (dipolar) resonant response of the particle to a well-defined set of modes of the background radiation field. In Section 3, the expression of the dynamical variables of the particle in terms of linear response coefficients is applied to the analysis of a system of two identical particles in a stationary state. Section 4 identifies two types of particles according to the relative phase of their coupling to a common field mode in the bipartite case, and the multipartite case is briefly discussed. Section 5 shows that the analysis of two-particle correlations leads to entangled symmetric or antisymmetric state vectors. In Section 6, the intrinsic rotation is introduced in order to establish the connection between the spin and the quantum statistics as reflected in the symmetry of the state vector, leading to the Pauli exclusion principle for particles with half-integer spin.

2 Quantum operators as linear response functions

As shown in SED (de la Peña et al., 2015), the dynamics of an otherwise classical charged particle immersed in the zero-point radiation field of energy $\hbar\omega/2$ per mode (ZPF) and subject to a binding force and its own radiation reaction evolves irreversibly into the quantum regime, characterized by the stationary states reached as a result of the average energy balance between radiation reaction and the action of the background field. Cetto and de la Peña (2024) showed by means of a Hamiltonian analysis of the particle–field system that the *nature* of the particle dynamical variables—the kinematics—changes in the transition to the quantum regime. In this regime, $x(t)$, $p(t)$ no longer refer to trajectories but to the linear, resonant *response* of the particle to the driving force of the background field, which effects the transitions between stationary states. The radiative transitions between two states (n, k) involve precisely those field modes to which the particle responds resonantly. Thus, from the initially infinite, continuous set of canonical field variables (q, p), only those (q_{nk}, p_{nk}) so defined are relevant for the description in the quantum regime. Since the memory of the initial particle variables $x(0), p(0)$ is lost and the

dynamics are now controlled by the field, the Poisson bracket of the particle canonical variables, which initially is taken with respect to the complete set of (particle + field) variables, reduces to the Poisson bracket with respect to the (relevant) field variables. Therefore, for the particle in a stationary state n (note that Roman letters are used for the canonical field variables),

$$\{x_n(t), p_n(t)\}_{\text{qp}} = 1, \quad (1)$$

where

$$\{x_n(t), p_n(t)\}_{\text{qp}} = \sum_{k \neq n} \left(\frac{\partial x_n}{\partial q_{nk}} \frac{\partial p_n}{\partial p_{nk}} - \frac{\partial p_n}{\partial q_{nk}} \frac{\partial x_n}{\partial p_{nk}} \right).$$

Instead of the canonical field variables (the quadratures) (q_{nk}, p_{nk}), it is convenient to use the (dimensionless) normal variables $a_{nk} = \exp(i\phi_{nk})$, where ϕ_{nk} is a random phase, which are related to the former by

$$q_{nk} = \sqrt{\frac{\hbar}{2|\omega_{kn}|}} (a_{nk} + a_{nk}^*), \quad p_{nk} = -i\sqrt{\frac{\hbar|\omega_{kn}|}{2}} (a_{nk} - a_{nk}^*). \quad (2)$$

This transformation, which takes into account the energy of the field mode of frequency ω_{kn} being equal to $\hbar\omega_{kn}$, is the entry point of Planck's constant in the equations that follow.

With the transformation (2), the Poisson bracket with respect to the normal variables becomes

$$\begin{aligned} \{x(t), p(t)\}_{\text{nm}} &\equiv \sum_{k \neq n} \left(\frac{\partial x_n}{\partial a_{nk}} \frac{\partial p_n}{\partial a_{nk}^*} - \frac{\partial p_n}{\partial a_{nk}} \frac{\partial x_n}{\partial a_{nk}^*} \right) \\ &= i\hbar \sum_{k \neq n} \left(\frac{\partial x_n}{\partial q_{nk}} \frac{\partial p_n}{\partial p_{nk}} - \frac{\partial p_n}{\partial q_{nk}} \frac{\partial x_n}{\partial p_{nk}} \right), \end{aligned} \quad (3)$$

and, therefore, according to Equation 1, the transformed Poisson bracket must satisfy

$$\{x(t), p(t)\}_{\text{nm}} = i\hbar. \quad (4)$$

From this and Equation 3, it is clear that $x_n(t), p_n(t)$ must indeed be linear functions of the normal variables $\{a_{nk}\}$, $k \neq n$. Thus, $x_n(t)$ becomes expressed in the form (in one dimension, for simplicity)

$$x_n(t) = x_{\text{nm}} + \sum_{k \neq n} x_{nk} a_{nk} e^{-i\omega_{kn}t} + \text{c.c.}, \quad (5)$$

where the index k denotes any other state that can be reached by means of a transition from n (hence $k \neq n$), and ω_{kn} is the corresponding transition frequency. The coefficient x_{nk} is the response amplitude of the particle to the field mode of frequency ω_{kn} . More generally, since the field variables connecting different states n, n' are independent random variables, $(\partial a_{nk} / \partial a_{n'k}) = \delta_{nn'}$ (for equal times, one may omit the time dependence in the expression) and

$$\{x, p\}_{\text{nm}'} = i\hbar \delta_{\text{nm}'}. \quad (6)$$

Using Equation 5 for $x_n(t)$ and

$$p_n(t) = m\dot{x}_n(t) = -im \sum_{k \neq n} \omega_{kn} x_{nk} a_{nk} e^{-i\omega_{kn}t} + \text{c.c.} \quad (7)$$

to calculate the derivatives involved in Equation 3, we obtain

$$\{x(t), p(t)\}_{nm} = 2im \sum_{k \neq n} \omega_{kn} |x_{nk}|^2 = i\hbar. \quad (8)$$

For x and p real, $x_{nk}^*(\omega_{nk}) = x_{kn}(\omega_{kn})$, $p_{nk}^*(\omega_{nk}) = p_{kn}(\omega_{kn})$, $a_{nk}^*(\omega_{nk}) = a_{kn}(\omega_{kn})$. This allows us to write Equation 6 in the explicit form

$$\sum_{k \neq n} (x_{nk} p_{kn'} - p_{n'k} x_{kn}) = i\hbar \delta_{nn'}, \quad (9)$$

and to identify the response coefficients x_{nk} , $p_{n'k}$ as the elements of matrices \hat{x} , \hat{p} such that

$$[\hat{x}, \hat{p}] = i\hbar. \quad (10)$$

This central result of SED reveals the quantum commutator as *the matrix expression of the Poisson bracket* of the particle variables (x_n , p_n) in any state n with respect to the (relevant) normal field variables corresponding to the modes $\{nk\}$ to which the particle responds resonantly from that state. Furthermore, Equation 8 is identified with the Thomas–Reiche–Kuhn sum rule,

$$2im \sum_{k \neq n} \omega_{kn} |x_{nk}|^2 = i\hbar. \quad (11)$$

In summary, this is the physical essence of the quantum operators: they describe the linear, resonant response of the (bound) particle to a well-defined set of field modes. The response coefficients x_{nk} and the transition frequencies ω_{kn} contained in Equation 5 are characteristic of the mechanical system; the corresponding random normal variables a_{nk} in turn contain information about the (stationary, random) background field. By taking the derivatives of x_n and p_n given by Equations 5, 7 with respect to a_{nk} , a_{nk}^* to calculate the Poisson bracket, the latter are removed from the description; the problem seems to be reduced to be purely mechanical, although it is in essence electrodynamic. Once the operator formalism is adopted, the factor \hbar , coming from the transformation expressed in Equation 2, remains the only conspicuous imprint left by the field.

We further note that the structure of the commutator is a direct consequence of the symplectic structure of the problem; this is a feature of the Hamiltonian dynamics that remains intact in the evolution from the initial classical to the quantum regime. The correspondence between classical Poisson brackets and quantum commutators, insightfully established by Dirac on formal grounds, thus finds a physical explanation.

To connect with quantum formalism in the Heisenberg representation, we consider an appropriate Hilbert space on which the operators act. In the present case, the natural choice is the Hilbert space spanned by the set of orthonormal vectors $\{|n\rangle\}$ representing the stationary states with energy \mathcal{E}_n . With the components of $\hat{x}(t)$ given by $x_{nk}e^{-i\omega_{kn}t}$ (see Equation 5), we have

$$\hat{x}(t) = \sum_{n,k} x_{nk} e^{-i\omega_{kn}t} |n\rangle \langle k|. \quad (12)$$

The matrix elements of $\hat{x}(t)$ are

$$x_{nk}(t) = \langle n | \hat{x}(t) | k \rangle \quad (13)$$

in the Heisenberg picture, or

$$x_{nk}(t) = \langle n(t) | \hat{x} | k(t) \rangle \quad (14)$$

in the Schrödinger picture, where the time dependence has been transferred to the state vector,

$$|n(t)\rangle = e^{-i\mathcal{E}_n t/\hbar} |n\rangle. \quad (15)$$

Finally, with the evolution of x , p into operators, the initial Hamiltonian equations evolve in the quantum regime into the Heisenberg equations

$$\frac{1}{i\hbar} [\hat{x}, \hat{H}] = \hat{\dot{x}}, \quad \frac{1}{i\hbar} [\hat{p}, \hat{H}] = \hat{\dot{p}}, \quad (16)$$

with $\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}$, $\hat{x} = \hat{p}/m$ and $\hat{p} = -(\hbar d\widehat{V}/dx)$. By taking the matrix element (nk) of the first of these equations, we confirm that $\omega_{kn} = (\mathcal{E}_n - \mathcal{E}_k)/\hbar$ —that is, that the energy $\hbar\omega_{kn}$ transferred to (or from) the field to the particle in a transition is equal to the energy difference between the two stationary states.

3 Response of a bipartite system to the background field

Now consider a system consisting of two identical particles. When the particles are isolated from each other, they are subject to different realizations of the background field, in which case their behavior can be studied separately for each particle using the procedure above. However, if they are part of one and the same system, they are subject to the same realization of the field and, being identical, they respond to the same set of relevant field modes, whether or not they interact with each other. In the following, we assume that the particles do not interact directly with each other.

Our purpose is to describe the response of the composite system to the background field when in a stationary state characterized by the total energy $\mathcal{E}_{(nm)} = \mathcal{E}_n + \mathcal{E}_m$ with $\mathcal{E}_n \neq \mathcal{E}_m$, the subindices n and m referring to single-particle states. If particle 1 is in state n , it responds to the set of modes $\{nk\}$, and similarly particle 2 in state m responds to the set $\{ml\}$,

$$\begin{aligned} x_{1n}(t) &= \sum_k e^{i\theta_{nk}^1} x_{1nk} a_{nk} e^{-i\omega_{kn}t} + \text{c.c.}, \quad x_{2m}(t) \\ &= \sum_l e^{i\theta_{ml}^2} x_{2ml} a_{ml} e^{-i\omega_{lm}t} + \text{c.c.} \end{aligned} \quad (17)$$

where we have added the factor $\exp(i\theta)$ to each term to allow for the (random) phase of the response of the particle to the field modes.

When $n \neq m$, the sums in Equation 17 involve the different, mutually independent normal variables a_{nk} and a_{ml} , except when $k = m$ and $l = n$, since $a_{nm} = a_{nm}^*$. Therefore, the Poisson bracket of $x_1(t)$ and $x_2(t)$, calculated in the state of the composite system (nm) , reduces to a single term:

$$[x_1, x_2]_{(nm)} = \left(\frac{\partial x_{1n}}{\partial a_{nm}} \frac{\partial x_{2m}}{\partial a_{nm}^*} - \frac{\partial x_{2m}}{\partial a_{nm}} \frac{\partial x_{1n}}{\partial a_{nm}^*} \right) = 2i |x_{nm}|^2 \sin \theta_{nm}^{12}. \quad (18)$$

Since the particles are identical, the interchange of labels 1 and 2 should not alter the value of the Poisson bracket, and therefore this equation must be equal to 0. This sets an important restriction on the possible values of the phase difference. With

$$|\theta_{nm}^1 - \theta_{nm}^2| = |\theta_{nm}^{12}| \equiv \pi \zeta_{nm}^{12}, \quad (19)$$

we see that ζ_{nm}^{12} must be an integer so that

$$[x_1, x_2]_{(nm)} = 0 \quad (n \neq m). \quad (20)$$

Furthermore, with $p_2(t)$ obtained from the second Equation 17,

$$p_{2m}(t) = -im \sum_l e^{i\theta_{ml}^{12}} \omega_{lm} x_{2ml} a_{ml} e^{-i\omega_{lm}t} + \text{c.c.},$$

the Poisson bracket of $x_1(t)$ and $p_2(t)$ calculated for the same state (nm) gives

$$[x_1, p_2]_{(nm)} = \left(\frac{\partial x_{1n}}{\partial a_{nm}} \frac{\partial p_{2m}}{\partial a_{nm}^*} - \frac{\partial p_{2m}}{\partial a_{nm}} \frac{\partial x_{1n}}{\partial a_{nm}^*} \right) = 2im\omega_{nm}|x_{nm}|^2 \cos \theta_{nm}^{12}. \quad (21)$$

In terms of the parameter ζ_{nm}^{12} defined in Equation 19, we have

$$\cos \theta_{nm}^{12} = (-1)^{\zeta_{nm}^{12}}, \quad \zeta_{nm}^{12} = 0, 1, 2, \dots \quad (22)$$

and therefore, from Equation 21,

$$[x_1, p_2]_{(nm)} = (-1)^{\zeta_{nm}^{12}} 2im\omega_{nm}|x_{nm}|^2. \quad (23)$$

This result shows that a correlation is established between the response variables of the two particles to the shared field mode (nm) for $n \neq m$; in other words, the field mode serves as a bridge between the particles and correlates their responses. It is important to note that Equation 23 involves only the field mode connecting the two states with $\mathcal{E}_n \neq \mathcal{E}_m$, and it is different from 0 only when these states are connected by a dipolar transition element, $x_{nm} \neq 0$.

We now consider two equal particles in the same energy state: $n = m$. In this case, the particles share all field modes, so that the Poisson brackets become, by virtue of Equation 22,

$$[x_1, x_2]_{(nm)} = \sum_k \left(\frac{\partial x_{1n}}{\partial a_{nk}} \frac{\partial x_{2n}}{\partial a_{nk}^*} - \frac{\partial x_{2n}}{\partial a_{nk}} \frac{\partial x_{1n}}{\partial a_{nk}^*} \right) = 2i \sum_k \sin \theta_{nk}^{12} |x_{nk}|^2 = 0, \quad (24)$$

$$[x_1, p_2]_{(nm)} = \sum_k \left(\frac{\partial x_{1n}}{\partial a_{nk}} \frac{\partial p_{2n}}{\partial a_{nk}^*} - \frac{\partial p_{2n}}{\partial a_{nk}} \frac{\partial x_{1n}}{\partial a_{nk}^*} \right) = 2im \sum_k \omega_{kn} \cos \theta_{nk}^{12} |x_{nk}|^2 = 2im \sum_k (-1)^{\zeta_{nk}^{12}} \omega_{kn} |x_{nk}|^2. \quad (25)$$

4 Two families of particles

Equation 23 indicates that there are two distinct types of identical particles, depending on whether the phase parameter ζ_{nm}^{12} given by Equation 19 is an even or odd number. Since this condition applies to all modes that are shared by the two particles, we can write, using Equation 19:

$$\zeta_{nm}^{12} = \zeta^{12} = |\zeta^1 - \zeta^2|, \quad (26)$$

so that the two types of particles are characterized by

$$\text{Type B: } \zeta_B^{12} = 0, 2, 4, \dots, \quad (27a)$$

$$\text{Type F: } \zeta_F^{12} = 1, 3, 5, \dots \quad (27b)$$

In Appendix A, it is shown that for all ζ_B^{12} to be even, the individual ζ_B^i must be integers, and that for all ζ_F^{12} to be odd, the individual ζ_F^i must be half-integers:

$$\text{Type B: } |\zeta_B^i| = 0, 1, 2, \dots \Upsilon_B, \quad (28a)$$

$$\text{Type F: } |\zeta_F^i| = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots \Upsilon_F, \quad (28b)$$

where Υ_B and Υ_F are the maximum values of the individual ζ_B^i, ζ_F^i . This means that B and F actually stand for two distinct families of particles, the members of which are characterized by the respective value of Υ . Identical particles of family B can have any value of ζ_B^i integer such that $|\zeta_B^i| \leq \Upsilon_B$, but when combined they must satisfy Equation 27a; similarly, those of family F must satisfy Equation 27b. In other words, according to Equations 27a, b, only pairwise combinations of ζ_B^i that are even and only pairwise combinations of ζ_F^i that are odd are allowed. Since, in both cases, ζ^i can be positive or negative, this gives a total of $g = 2\Upsilon + 1$ possible different states of the bipartite system.

With these results, Equation 17 take the form (except for a remaining common phase factor $e^{i\theta}$ that can be neglected)

$$x_{1n}(t) = e^{i\pi\zeta^1} \sum_k x_{1nk} a_{nk} e^{-i\omega_{kn}t} + \text{c.c.},$$

$$x_{2m}(t) = e^{i\pi\zeta^2} \sum_l x_{2ml} a_{ml} e^{-i\omega_{lm}t} + \text{c.c.}, \quad (29)$$

and Equation 25 is reduced to

$$[x_1, p_2]_{(nm)} = (-1)^{\zeta^{12}} i\hbar. \quad (30)$$

Therefore, in comparison with the one-particle commutator $[x_1, p_1]_{(nm)} = i\hbar$, we note that in the B case—when Equation 27a holds—particle 2 responds in the same way as particle 1. Indeed, according to Equation 19, the response of the two particles to the shared field modes is *in phase*, and a correlation is established between the particles. By contrast, according to Equation 27b, ζ_F^{12} is an odd number; hence, the response of the two identical type F particles to the shared field modes is *in antiphase*.

4.1 Extension to three or more particles

In light of the above results, we now briefly analyze the possible correlations for a system composed of three or more identical particles.

In the first case of three type- B particles, when total energy $\mathcal{E}_{(nml)} = \mathcal{E}_n + \mathcal{E}_m + \mathcal{E}_l$ with $\mathcal{E}_n \neq \mathcal{E}_m \neq \mathcal{E}_l$, Equation 27a applies, and the three particles are pairwise correlated. According to Equation 30, correlation also exists when $\mathcal{E}_n \neq \mathcal{E}_m = \mathcal{E}_l$ or $\mathcal{E}_n = \mathcal{E}_m = \mathcal{E}_l$ because the responses of the three particles to common field modes are always in phase. Therefore, all three particles may in principle occupy the same state n and respond coherently. The argument can of course be extended to four or more particles; consequently, there may in principle be an arbitrary number N of type- B particles in the same state and responding coherently to the field modes, like a well-disciplined troop.

In the type- F case, we have already concluded that particles 1 and 2 respond in antiphase to a common mode, and the same applies of course to any pair of identical particles. When total energy $\mathcal{E}_{(nml)} = \mathcal{E}_n + \mathcal{E}_m + \mathcal{E}_l$ with $\mathcal{E}_n \neq \mathcal{E}_m \neq \mathcal{E}_l$, the three particles are pairwise correlated according to Equation 27b. However, when at least two energy levels coincide, two particles respond in antiphase

to the shared modes, thus preventing a third one from responding in antiphase to the same modes and therefore from being correlated to the other two. Therefore, contrary to the type-*B* case, there can be no coherent response of more than two type-*F* particles in this case.

5 Field-induced covariance and entanglement

To calculate the effect of the background field on the correlation of the responses, we consider two generic dynamical variables associated with particles 1 and 2. These can be the variables $x(t)$ and $p(t)$ considered so far, a linear combination of them, or any other variable of the form given by Equation 29, where n, m are, as before, two stationary states of the system, with energies $\mathcal{E}_n, \mathcal{E}_m$,

$$f_{1n}(t) = f_{1nm} + e^{i\pi\zeta^1} \sum_{k \neq n} f_{1nk} a_{nk} e^{-i\omega_{kn}t} + \text{c.c.}, \quad (31)$$

$$g_{2m}(t) = g_{2mm} + e^{i\pi\zeta^2} \sum_{l \neq m} g_{2ml} a_{ml} e^{-i\omega_{lm}t} + \text{c.c.}, \quad (32)$$

The time-independent terms in these equations represent in each case the average value of the function, taken over the distribution of the normal variables $a_{nk} = \exp(i\phi_{nk})$ where ϕ_{nk} is a random phase, as mentioned in Section 2,

$$\overline{f_{1n}(t)} = f_{1nm}, \quad \overline{g_{2m}(t)} = g_{2mm}. \quad (33)$$

To calculate the correlation, we take the average of the product of $f_1(t)$ and $g_2(t)$. When particles 1 and 2 do not form part of the same system, they respond to independent realizations of the field modes, and therefore the covariance is given by

$$\Gamma(f_{1n}g_{2m}) = (\overline{f_{1n}(t)} - f_{1nm})(\overline{g_{2m}(t)} - g_{2mm}) = 0, \quad (34)$$

which simply confirms that the variables are not correlated.

However, when the particles form a bipartite system, they respond to the same realization of the field modes. To calculate the covariance in this case, we must take into account the double degeneracy of the combined state, $\mathcal{E} = \mathcal{E}_{1n} + \mathcal{E}_{2m} = \mathcal{E}_{1m} + \mathcal{E}_{2n}$. In order to distinguish between the two configurations, we define

$$\mathcal{E}_C = \mathcal{E}_{1n} + \mathcal{E}_{2m}, \quad \mathcal{E}_D = \mathcal{E}_{1m} + \mathcal{E}_{2n}. \quad (35)$$

Let us consider the first case, $\mathcal{E}_C = \mathcal{E}_{1n} + \mathcal{E}_{2m}$, and use Equations 31, 32 to calculate the average product of $f_1(t)$ and $g_2(t)$, which we call \overline{fg}^C (the left factor always refers to particle 1 and the right to particle 2, so that we omit the indices 1 and 2 in the following). Taking into account that, for random independent normal variables, $\overline{a_{ij}a_{jk}} = \overline{a_{ij}a_{kj}} = \delta_{ik}$ and hence

$$\overline{a_{nk}a_{ml}} = \delta_{nk}\delta_{ml} + \delta_{nl}\delta_{km}, \quad (36)$$

we obtain

$$\overline{fg}^C = f_{nm}g_{nm} + (-1)^\zeta f_{nm}g_{mm}. \quad (37)$$

Similarly, for the *D* configuration, we obtain

$$\overline{fg}^D = f_{nm}g_{nm} + (-1)^\zeta f_{mm}g_{nm}. \quad (38)$$

Since the two configurations have the same weight, the averages of $f_1(t)$ and $g_2(t)$ are

$$\bar{f} = \frac{1}{2}(f_{nm} + f_{mm}), \quad \bar{g} = \frac{1}{2}(g_{nm} + g_{mm}),$$

and the average of the product of $f_1(t)$ and $g_2(t)$ is given by

$$\begin{aligned} \overline{fg} &= \frac{1}{2}(\overline{fg}^C + \overline{fg}^D) \\ &= \frac{1}{2}[f_{nm}g_{nm} + (-1)^\zeta f_{nm}g_{mm} + f_{mm}g_{nm} + (-1)^\zeta f_{mm}g_{mm}]. \end{aligned} \quad (39)$$

The covariance is therefore given by

$$\begin{aligned} \Gamma(fg) &= \overline{fg} - \bar{f}\bar{g} \\ &= -\frac{1}{4}(f_{nm} - f_{mm})(g_{nm} - g_{mm}) + \frac{1}{2}(-1)^\zeta [f_{nm}g_{mm} + f_{mm}g_{nm}]. \end{aligned} \quad (40)$$

In this equation, the two contributions to the covariance are of a very different nature: the first is a classical covariance of f_1 and g_2 due to the different average values of these functions in states n, m under the condition of degeneracy, $\mathcal{E}_{1n} + \mathcal{E}_{2m} = \mathcal{E}_{1m} + \mathcal{E}_{2n}$. The second term, though, has no classical counterpart: it is entirely due to the joint response of particles 1 and 2 to the shared mode (nm) and is therefore a signature of the matter-field interaction. Evidently, both particles must respond to the mode (nm) for this term not to be zero; if any of the two matrices \hat{f}, \hat{g} is diagonal, there is no quantum contribution to $\Gamma(fg)$.

5.1 Emergence of entanglement

In quantum formalism, entanglement is reflected in the non-factorizability of the bipartite state vector. Therefore, in order to show the emergence of entanglement in the present context, we will translate Equation 40 into the language of the product Hilbert space $H_1 \otimes H_2$, where H_1, H_2 are respectively spanned by the sets of orthonormal state vectors $\{|n\rangle\}$ of particles 1 and 2 (see Section 2 for the one-particle case). In the shorthand notation introduced above, configurations *C, D* are represented by the product state vectors

$$|C\rangle = |n\rangle_1 |m\rangle_2, \quad |D\rangle = |m\rangle_1 |n\rangle_2. \quad (41)$$

In this notation, Equation 40 reads

$$\begin{aligned} \Gamma(fg) &= -\frac{1}{4}(f_{nm} + f_{mm})(g_{nm} + g_{mm}) \\ &\quad + \frac{1}{2}[C + (-1)^\zeta D] \hat{f} \hat{g} [C + (-1)^\zeta D]. \end{aligned} \quad (42)$$

In writing the second term, we have used the fact that $(-1)^\zeta = \pm 1$ according to Equations 27a and b. Note that the average of fg is now taken over the (normalized) state vector

$$|\Psi\rangle \equiv \frac{1}{\sqrt{2}}[C + (-1)^\zeta D], \quad (43)$$

or in terms of the individual state vectors,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}[|n\rangle_1 |m\rangle_2 + (-1)^\zeta |m\rangle_1 |n\rangle_2]. \quad (44)$$

As a result, we obtain

$$\Gamma(fg) = \langle \Psi | \hat{f} \hat{g} | \Psi \rangle - \langle \Psi | \hat{f} | \Psi \rangle \langle \Psi | \hat{g} | \Psi \rangle, \quad (45)$$

which is exactly the quantum covariance of $\hat{f}\hat{g}$ calculated in the entangled state given by Equation 44. The covariance coincides with the correlation of f and g since the state vector $|\Psi\rangle$ is normalized to unity.

We stress that the above calculation is restricted to the case $n \neq m$; when $n = m$, there is no field mode correlating the responses of the two particles, so there is no entanglement. On the other hand, if there is degeneracy—that is, $\mathcal{E}_C = \mathcal{E}_D$ —the two-particle system is necessarily in an entangled state if f_{mn}, g_{mn} are different from zero—that is, if the response variables f, g connect the single-particle states n, m . The origin of the entanglement is thus traced back to the action of the common relevant field mode (mn), and the responses of the two particles to this mode are maximally correlated (anticorrelated) according to Equation 40 with $(-1)^\zeta = +1$ (-1). More generally, entanglement occurs whenever there is degeneracy, be it in energy or any other variable that defines the state of the bipartite system, as discussed in the next section.

Equations 43–45 were previously obtained in the context of SED by a somewhat laborious procedure using the Hilbert-space formalism. In contrast to such an abstract procedure, the present derivation has the advantage of keeping track at every moment of the physical quantities involved: the field mode variables, the particles' response variables, and the phase difference of the responses.

It is clear from Equation 44 that the two families of identical particles identified in Section 4 are distinguished by their entangled state vectors. The symmetry or antisymmetry of the state vector is uniquely linked to the phase difference of the responses of the two particles to the shared field mode. When the coupling is in phase (type B particles), the state vector is symmetric with respect to the exchange of particles; when the relative coupling is out of phase (type F particles), the state vector is antisymmetric.

It should be stressed that no direct interaction between the components of the system is involved in the derivation leading to entangled states; entanglement arises as a result of their indirect interaction via the shared field modes and, therefore, does not entail a non-local action.

6 The Pauli exclusion principle

6.1 Introduction of spin

Among the various proposals that have been made to justify the spin-statistics theorem, some that are relevant to this work involve the inclusion of the internal (spin) coordinates among the parameters affected by the exchange operation (e.g. Hunter et al., 2005 and Jabs, 2010, and additional references cited the latter). In particular, in Jabs (2010), the spin-statistics connection is derived under the postulates that the original and the exchange wave functions are simply added and the azimuthal phase angle, which defines the orientation of the spin part of each single-particle spin component in the plane normal to the spin-quantization axis, is exchanged along with the other parameters.

In dipolar transitions, atomic electrons interact with field modes of circular polarization, as expressed in the selection rule $\Delta l = \pm 1$, and is increasingly exploited for practical applications in spin-resolved

spectroscopy and magneto-optics (e.g. Okuda et al., 2011; De et al., 2021). Furthermore, the interaction of the particle with circular polarized modes of the ZPF, which are known to have an intrinsic angular momentum equal to $\hbar/2$ (Sobelman, 1979; Mandel and Wolf, 1995), was indeed shown in Cetto et al. (2014) to be responsible for the origin of the electron spin itself. It is reasonable to assume that a similar mechanism is responsible for the neutron spin, since the neutron has a magnetic moment that couples to the radiation field.

Therefore, following Jabs (2010) and Cetto and de la Peña (2015), in order to include the spin in the present analysis, we add an (internal) rotation angle ϕ to the expression for the dynamic variables. Strictly speaking, the problem becomes a three-dimensional one. However, for simplicity, we can still use our one-dimensional expressions for the dynamic variables if we decompose the radiation field into (statistically independent) modes of circular polarization. So instead of (31) and (32), we write

$$f_{1n}(t, \phi) = e^{i\pi\zeta^1} \sum_k f_{1nk} a_{nk} e^{i\gamma_{nk}\phi - i\omega_{kn}t} + \text{c.c.}, \quad (46)$$

$$g_{2m}(t) = e^{i\pi\zeta^2} \sum_l g_{2ml} a_{ml} e^{i\gamma_{ml}\phi - i\omega_{lm}t} + \text{c.c.}, \quad (47)$$

where $\gamma_{nk}\phi$ is the difference of two rotation angles,

$$\gamma_{nk}\phi = (\gamma_n - \gamma_k)\phi, \quad (48)$$

and γ_n, γ_k stand for counterclockwise (clockwise) rotation. If n, m are two stationary states of a system of identical particles, as before, we obtain for the partial covariances in configurations C and D (see Equations 37 and 38)

$$\overline{fg}^C = f_{mn}g_{mm} + (-1)^\zeta f_{mn}e^{i\gamma_{mn}\phi}g_{mn}e^{i\gamma_{mn}\phi}, \quad (49)$$

$$\overline{fg}^D = f_{mn}g_{mn} + (-1)^\zeta f_{mn}e^{i\gamma_{mn}\phi}g_{mn}e^{i\gamma_{mn}\phi}, \quad (50)$$

and, therefore,

$$\begin{aligned} \overline{fg} &= \frac{1}{2}(\overline{fg}^C + \overline{fg}^D) = \frac{1}{2}[f_{mn}g_{mm} + f_{mn}g_{mn}] \\ &+ \frac{1}{2}(-1)^\zeta [f_{mn}e^{i\gamma_{mn}\phi}g_{mn}e^{i\gamma_{mn}\phi} + f_{mn}e^{i\gamma_{mn}\phi}g_{mn}e^{i\gamma_{mn}\phi}]. \end{aligned} \quad (51)$$

By translating this result into the language of the product Hilbert space and using Equation 48, we obtain after some algebra

$$\Gamma(fg) = \langle \Psi | \hat{f}\hat{g} | \Psi \rangle - \langle \Psi | \hat{f} | \Psi \rangle \langle \Psi | \hat{g} | \Psi \rangle, \quad (52)$$

where $|\Psi\rangle$ now stands for the complete bipartite state vector, including the internal rotation components,

$$\begin{aligned} |\Psi\rangle &\equiv \frac{1}{\sqrt{2}} |e^{-i\gamma_n\phi} e^{-i\gamma_m\phi} C + (-1)^\zeta e^{-i\gamma_m\phi} e^{-i\gamma_n\phi} D\rangle \\ &= \frac{1}{\sqrt{2}} |e^{-i\gamma_n\phi} |n\rangle_1 e^{-i\gamma_m\phi} |m\rangle_2 + (-1)^\zeta e^{-i\gamma_m\phi} |m\rangle_1 e^{-i\gamma_n\phi} |n\rangle_2\rangle. \end{aligned} \quad (53)$$

In Equation 53, the first angular factor is always associated with particle 1 and the second with particle 2. This suggests writing each individual state vector in the form $e^{-i\gamma\phi} |n\rangle$. In quantum language, this implies the introduction of two orthonormal vectors $|y\rangle = |+\rangle, |-\rangle$ spanning the two-dimensional Hilbert space, $|n\rangle|y\rangle \equiv |ny\rangle$; Equation 53 thus takes the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}} [|n\gamma_n\rangle_1 |m\gamma_m\rangle_2 + (-1)^\zeta |m\gamma_m\rangle_1 |n\gamma_n\rangle_2]. \quad (54)$$

Since the parameter γ is associated with the internal rotation, we identify it with the spin of the electron, which means that

$$\gamma_{n,m} = \pm \frac{1}{2}. \quad (55)$$

6.2 The connection between spin and symmetry

We now examine the symmetry properties of the complete entangled state function (53) under particle exchange. When particles 1 and 2 are exchanged, in addition to switching their positions in three-dimensional space, their internal angles change: particle 1 rotates to the azimuthal position of particle 2 and *vice versa*, with both rotations occurring in the same direction (clockwise or counterclockwise). Consider a clockwise rotation. As shown in Jabs (2010) and Cetto and de la Peña (2015), when $\phi_2 > \phi_1$ ϕ_1 transforms into ϕ_2 and ϕ_2 transforms into $\phi_1 + 2\pi$,

$$\phi_2 - \phi_1 \rightarrow \phi_1 - \phi_2 + 2\pi, \quad (56)$$

and $|\Psi\rangle$ given by Equation 53 transform into

$$|\Psi\rangle_{1 \leftrightarrow 2} = \frac{1}{\sqrt{2}} [e^{-i\gamma_m(\phi_1+2\pi)} |m\rangle_1 e^{-i\gamma_n\phi} |n\rangle_2 + (-1)^\zeta e^{-i\gamma_n(\phi_1+2\pi)} |n\rangle_1 e^{-i\gamma_m\phi} |m\rangle_2].$$

Since γ_n, γ_m are half-integers, the overall effect of the particle exchange is to multiply the original state vector by a factor of

$$|\Psi\rangle_{1 \leftrightarrow 2} = (-1)^\zeta (-1)^{2\gamma_n} |\Psi\rangle. \quad (57)$$

If instead $\phi_2 < \phi_1$, ϕ_2 transforms into ϕ_1 and ϕ_1 transforms into $\phi_2 + 2\pi$, so that

$$\phi_2 - \phi_1 \rightarrow \phi_1 - \phi_2 - 2\pi, \quad (58)$$

and the transformation of the state vector is again given by Equation 57. Of course, the same result is obtained if the rotation is anticlockwise. Since particles 1 and 2 are identical, their exchange should have no effect on the state vector, which implies that

$$(-1)^\zeta (-1)^{2\gamma_n} = 1. \quad (59)$$

Therefore, taking into account Equation 55, we conclude that $(-1)^\zeta = -1$. Thus, symmetry of the total state vector under particle exchange, obtained from Equation 54 with $(-1)^\zeta = -1$,

$$|\Psi\rangle = \frac{1}{\sqrt{2}} [|n\gamma_n\rangle_1 |m\gamma_m\rangle_2 - |m\gamma_m\rangle_1 |n\gamma_n\rangle_2]. \quad (60)$$

implies antisymmetry of the (energy) state vector (44),

$$|\Psi\rangle = \frac{1}{\sqrt{2}} [|n\rangle_1 |m\rangle_2 - |m\rangle_1 |n\rangle_2]. \quad (61)$$

6.3 The Pauli principle

The above procedure is of course applicable to particles with higher spin; thus, for any half-integer value of γ , $(-1)^{2\gamma} = -1$ and according to Equation 59, the bipartite (energy) state vector will be antisymmetric with respect to particle exchange, as in Equation 61.

We recall that Equation 61 is valid for $|n\rangle \neq |m\rangle$. If $|n\rangle = |m\rangle$ and the spin is not taken into account, the state vector is simply the product of the individual energy eigenvectors, $|\Psi\rangle = |n\rangle_1 |n\rangle_2$; according to Equation 40 the particle variables are not correlated and the bipartite system is obviously not entangled. However, with the introduction of spin, the complete state function is different from zero for $|n\rangle = |m\rangle$, under the condition that $|\gamma_n\rangle \neq |\gamma_m\rangle$. If this is the case, Equation 60 is reduced to

$$|\Psi\rangle = \frac{|n\rangle_1 |n\rangle_2}{\sqrt{2}} [|\gamma_1\rangle |\gamma_2\rangle - |\gamma_2\rangle |\gamma_1\rangle]. \quad (62)$$

In other words, entanglement can arise from energy degeneracy, if $\mathcal{E} = \mathcal{E}_n + \mathcal{E}_m$ with $\mathcal{E}_n \neq \mathcal{E}_m$, or from spin degeneracy, if $\gamma = \gamma_1 + \gamma_2$ with $\gamma_1 \neq \gamma_2$. Since for the electron (and other spin-1/2 particles) $\gamma_i = \pm \frac{1}{2}$, Equation 62 takes the form (except for an irrelevant overall sign)

$$|\Psi\rangle = \frac{|n\rangle_1 |n\rangle_2}{\sqrt{2}} \left[\left| \frac{1}{2} \right\rangle \left| -\frac{1}{2} \right\rangle - \left| -\frac{1}{2} \right\rangle \left| \frac{1}{2} \right\rangle \right]. \quad (63)$$

In Section 5, it was shown that the correlation between particle variables results from the antiphase response to the single common field mode of frequency ω_{mn} with $\mathcal{E}_n \neq \mathcal{E}_m$. On the other hand, when $|n\rangle = |m\rangle$, we note from Equation 25 that the two particles respond in antiphase to all (common) field modes; in this case, correlation is established as a result of the response of both particles to a common field mode of circular polarization. In other words, the entanglement results not from the response to a single mode connecting two states separated by their energies, $\Delta\mathcal{E}_{nm} = |\mathcal{E}_n - \mathcal{E}_m|$ but from a mode connecting two states separated by their spins, $\Delta\gamma_{12} = |\gamma_1 - \gamma_2|$. Just as in the first case $\Delta\mathcal{E} = \hbar\omega_{mn}$ is the energy exchanged with the field in a transition, in the second case $\hbar\Delta\gamma_{12} = \hbar$ is the angular momentum exchanged with the field in a transition.

Equation 63 leaves no room for a third electron in the same energy state $|n\rangle$ because its spin parameter would be either equal to γ_1 or γ_2 . The conclusion holds for any pair of identical half-integer spins because the condition $\Delta\gamma_{ij} = |\gamma_i - \gamma_j| = 1$ cannot be satisfied simultaneously for $i, j = 1, 2, 3$: if two half-integer values of γ satisfy $\Delta\gamma_{ij} = 1$, the third value of γ differs from the first two by an even number. To illustrate, consider $\Gamma_F = \frac{3}{2}$. Possible pairs (γ_1, γ_2) are $(\frac{3}{2}, \frac{1}{2})$, $(\frac{3}{2}, -\frac{1}{2})$, and $(-\frac{3}{2}, -\frac{1}{2})$; there is no γ_3 that simultaneously satisfies $\Delta\gamma_{31} = |\gamma_3 - \gamma_1| = 1$ and $\Delta\gamma_{32} = |\gamma_3 - \gamma_2| = 1$.

This is a clear example of Pauli's exclusion principle. The present discussion reveals the physical basis of the phenomenon: two particles in the same energy state respond in antiphase to a single (circularly polarized) mode of the field and a third particle cannot respond in antiphase to the first two.

7 Discussion

In this work, the symmetrization postulate and the spin-statistics theorem were shown to follow from the in-phase or antiphase response of identical particles to specific modes of the common background radiation field. The inclusion of spin in the

analysis allowed the identification of the type B and F families introduced in Section 4 as bosons and fermions and led to the Pauli exclusion principle in the case of fermions.

Key quantum phenomena that were introduced as postulates in the foundational phase of quantum mechanics and that have been repeatedly confirmed both formally and experimentally thus find a physical justification. The picture provided by the present approach is very suggestive. In particular, it shows that the collective behavior of identical particles, which leads to the respective quantum statistics, is a consequence of the mediation of specific field modes that “connect” the particles and correlate their dynamics, producing entanglement. A mysterious, apparently non-local connection between particles, as described by quantum formalism, is thus shown to be an entirely causal and local effect of the bridging role of the common background field. Given the increasing attention paid to entanglement phenomena and their applications, particularly in the fields of quantum information, computing, and communication, the insight gained from this perspective should prove highly fruitful. In particular, since entanglement and other quantum phenomena discussed here are shown to depend critically on the correlations established between identical particles by their coherent binding to certain common field modes, the cancellation or significant modification of these modes by Casimir cavity techniques (e.g. Kleppner, 1986; Walther et al., 2006) could be an interesting way to analyze the effect on such correlations.

The results reported here suggest further investigation. In particular, extending the one-dimensional analysis carried out here to three dimensions would allow an adequate treatment of more general problems involving additional dynamical variables, including orbital angular momentum.

Data availability statement

The original contributions presented in the study are included in the article; further inquiries can be directed to the corresponding author.

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Author contributions

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Appendix A

Equations 24, 25 must be satisfied for any pair of identical particles—that is, ζ^{ij} is either even or odd for identical particles $i, j = 1, 2, \dots$. This means that ζ^{12} expresses a distinctive property of the particles themselves, which manifests when the particles form part of the same system and couple either in phase or antiphase to the shared modes. This property is identified in Equation 26 and the following with the parameters $\zeta_B^i, \zeta_F^i, i = 1, 2$, which must satisfy either Equation 27a or 27b, respectively.

If we take the smallest possible value of ζ^1 in the F case, which is $|\zeta_m^1| = 1/2$, any integer value of another type- F particle would violate both Equations 27a, b; hence, type- F particles can only have half-integer values of the parameter ζ^i . Similarly, taking the smallest possible value of ζ^1 in the B case, which is $|\zeta_m^1| = 0$, any half-integer value for another type- B particle would violate both Equations 27a, b, so type- B particles can only have integer values of the parameter ζ^i .

This confirms the correctness of Equations 28a, b.