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Generalized uncertainty relations: Existing paradigms and new approaches

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Generalized uncertainty relations: Existing paradigms and new approaches

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Table of contents

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Editorial: Generalized uncertainty relations: existing paradigms and new approaches

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Editorial on the Research Topic
Generalized uncertainty relations: existing paradigms and new approaches

Introduction

This volume brings together a series of papers on the Research Topic of Generalized Uncertainty Relations (GURs). These works are chosen to provide a broad and timely overview of the current status of this important field, some 30 years after its initial development, including both phenomenological applications and foundational issues. Critical analyses of unsolved problems are also presented.

Scope and aims of the project

Our aim is to collect, in a single volume, a series of works that are representative of the current state of the art, and which, with the benefit of nearly three decades of hindsight, can contribute to the ongoing critical assessment of the field; its methods, models and techniques. We hope, therefore, that this modest collection will serve as a valuable resource and point of reference for future researchers, and, perhaps, even a small but significant mile stone in the development of the GUR research program.

After 30 years, we face the question: *quo vadis* generalized uncertainty principle? (Bosso et al., 2023) A further aim of this project is to stimulate debate about the future direction of GUR research, in particular, regarding the suitability of the current widely-used approach, of implementing modified relations via modifications of the canonical Heisenberg algebra. Despite nearly 30 years of effort, it remains unclear whether this method is conceptually and mathematically self-consistent, or whether new methods and techniques for studying GURs must be found.

For this reason, we present a variety of perspectives. Some papers in this volume explore the implications of traditional modified commutator models, while others present purely phenomenological analyses. The latter may be expected to hold, regardless of which mathematical formalism is used to describe the GURs, whereas the former are more model-specific, in general. Some papers explore well known GURs, such as the generalized uncertainty principle (GUP) and extended uncertainty principle (EUP), while others present new forms of modified relations, based on different physical assumptions. Still others explore alternative mathematical formalisms for the GUP and EUP, and investigate their phenomenological consequences. Whatever the research interests or perspectives of the individual reader, we hope that they will find something of interest to them here.

Call for papers

The non-gravitational interactions of microscopic particles are governed by the laws of quantum mechanics and so obey Heisenberg's uncertainty principle. This may be derived rigorously from the canonical quantum formalism or introduced, heuristically, via the Heisenberg microscope thought experiment. Extending the thought experiment argument to include the effects of gravitational attraction between particles and/or a background dark energy density leads to generalized uncertainty relations, which contain additional non-Heisenberg terms, but how to embed these within an extended quantum formalism remains an open problem in fundamental physics.

Two of the most widely studied relations, proposed in the phenomenological quantum gravity literature, are the generalized uncertainty principle (GUP) and extended uncertainty principle (EUP). The GUP incorporates the effects of attractive gravity and implies a minimum length scale of the order of the Planck length, whereas the EUP accounts for the effects of repulsive vacuum energy and implies a minimum momentum of the order of the de Sitter momentum. This is the momentum a canonical quantum particle would have if its de Broglie wavelength were of the order of the de Sitter radius, which is comparable to the present day radius of the Universe. Expanding the Heisenberg microscope argument to include the effects of canonical gravity and dark energy implies the extended generalized uncertainty principle (EGUP) which predicts the existence of both a minimum length and a minimum momentum scale in nature.

For almost three decades the most common method used to construct generalizations of the Heisenberg uncertainty principle has been to introduce modified commutation relations. These then lead, directly, to modified uncertainty relations, via the standard Schrödinger-Robertson relation. Unfortunately, despite its widespread use, this approach remains fraught with difficulties and modified commutator models lead to several well known pathologies, including:

(a) Violation of the equivalence principle,
(b) Reference-frame dependence of the "minimum" length,
(c) Violation of Lorentz invariance in the relativistic limit,
(d) The inability to construct sensible multi-particle states, known as the soccer ball problem.

This strongly motivates new approaches to the field, as well as critical analyses of traditional models, or their possible refinements, which aim to address these vital issues head-on.

In this Research Topic, we seek papers exploring all approaches to generalized uncertainty relations and their phenomenological implications. We aim to provide a broad overview the subject including summaries of the major approaches presented, to date, in this important field, as well as summaries of non-standard approaches based on new models.

Though many studies focus on the familiar GUP, EUP and EGUP formulae, which include position and linear momentum, we especially welcome explorations of generalized uncertainty relations for time, energy, angular momentum, quantum mechanical spin, and entropy, motivated by quantum gravitational phenomenology. Proposals for new relations, which have not yet been explored in the existing literature, are also warmly welcomed, and will be considered without prejudice.

Published papers

1. *Dark matter as an effect of a minimal length*, Bosso et al.
2. *Generalized uncertainty principle and burning stars*, Moradpour et al.
3. *Weak equivalence principle in quantum space*, Gnatenko and Tkachuk.
4. *Comments on the cosmological constant in generalized uncertainty models*, Bishop et al.
5. *The Generalized Uncertainty Principle and higher dimensions: Linking black holes and elementary particles*, Carr.
6. *Generalized uncertainty relations from finite-accuracy measurements*, Lake et al.
7. *Problems with modified commutators*, Lake and Watcharapasorn.
8. *New deformed Heisenberg algebra from the μ-deformed model of dark matter*, Gavrilik et al.
9. *Dimensionally-dependent uncertainty relations, or why we (probably) won't see black holes at the LHC, even if large extra dimensions exist*, Lake et al.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Bosso, P., Luciano, G. G., Petruziello, L., and Wagner, F. (2023). 30 years in: quo vadis generalized uncertainty principle? *Class. Quantum Gravity*. doi:10.1088/1361-6382/acf021

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Comments on the cosmological constant in generalized uncertainty models

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The existence of a small, non-zero cosmological constant is one of the major puzzles in fundamental physics. Naively, quantum field theory arguments would imply a cosmological constant which is up to $10,^{120}$ times larger than the observed one. It is believed a comprehensive theory of quantum gravity would resolve this enormous mismatch between theory and observation. In this work, we study the ability of generalized uncertainty principle (GUP) models, which are phenomenologically motivated models of quantum gravity, to address the cosmological constant problem. In particular, we focus on how these GUP models may change the phase space of QFT, and how this affects the momentum space integration of the zero-point energies of normal modes of fields. We point out several issues that make it unlikely that GUP models, in their current form, would be able to adequately address the cosmological constant problem.

KEYWORDS

generalized uncertainty principle, cosmological constant, minimal length, vacuum energy density, quantum vacuum

1 Introduction

A theory of quantum gravity, although not yet a reality, has been advertised as being able to solve many of the ills of classical general relativity, such as the singularities that occur in black hole and cosmological solutions (Penrose, 1965; Hawking and R Ellis, 1973). Quantum gravity is also supposed to resolve some of the issues surrounding the results of applying quantum field theory in a curved space-time such as what happens to a black hole at the end of evaporating via Hawking radiation (Hawking, 1975), and what happens to the information stored in a black hole due to this evaporation (Susskind and Lindesay, 2005).

The puzzle we address in this work is the apparent mismatch between the observed cosmological constant and the theoretically calculated cosmological constant—a conundrum known as the cosmological constant problem. This cosmological constant problem has been known for a long time. A nice relatively recent review of the issue is reference (Weinberg, 1989). The problem is that having a cosmological constant, Λ, is equivalent to having a constant energy density, ρ_{vac}, as a source in the Einstein field equations. The relationship is (using the units and notation of (Weinberg, 1989))

$$\rho_{vac} = \frac{\Lambda}{8\pi G} \ . \tag{1}$$

The subscript *vac* comes from quantum field theory where one obtains a constant vacuum energy density by adding up all the energy zero modes of vacuum quantum fields. The zero modes are given by $\frac{1}{2}\hbar\omega_p = \frac{1}{2}E_p = \frac{1}{2}\sqrt{\vec{p}^2 + m^2}$, and summing these up to get the vacuum energy density yields

$$\begin{aligned} \rho_{vac} &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\sqrt{\vec{p}^2 + m^2} \\ &= \frac{1}{2}\int_0^{p_c}\frac{4\pi}{(2\pi)^3}dp\, p^2\sqrt{p^2+m^2} \approx \frac{p_c^4}{16\pi^2} \ . \end{aligned} \tag{2}$$

Note that $p = |\vec{p}|$, we use these notations interchangeably throughout the rest of the paper. Since the integral is divergent, we cut off the dp integration at some scale p_c, which is usually taken to be the Planck scale, $p_c \sim (8\pi G)^{-1/2}$. Using (2) gives $\rho_{vac} \approx 2\times 10^{71}$ GeV^4. In contrast, the measured vacuum energy density (Zyla et al., 2020) is about $\rho_{vac} \approx 10^{-47}$ GeV^4. The difference between the theoretically calculated ρ_{vac} from (2) versus experimentally measured ρ_{vac} is a difference of 118 orders of magnitude. This massive discrepancy *is* the cosmological constant problem. Even if one lowers the cut off scale to the QCD scale of $\Lambda_{QCD} \sim 200$ MeV, where we think we fully understand QFT, one still gets a disagreement between theory and experiment of 41 order of magnitude. Some drastic change in our understanding of either QFT, general relativity, or both is needed to resolve this puzzle.

2 Generalized uncertainty principle and quantum gravity

One of the proposed resolutions to the cosmological constant problem is a theory of quantum gravity, a catch-all solution to all open problems in fundamental theoretical physics. In this work, we utilize the phenomenological generalized uncertainty principle (GUP) approach to quantum gravity. The GUP approach to quantum gravity is a bottom up approach [in contrast to the more top down approaches to quantum gravity such as superstring theory (Polchinski, 1998) or loop quantum gravity (Rovelli, 2008)]. There is a vast amount of literature on GUP, with a few of the important representative papers being (Veneziano, 1986; Amati et al., 1987; Gross and Mende, 1987; Gross and Mende, 1988; Amati et al., 1988; Amati et al., 1989; Amati et al., 1990; Maggiore, 1993; Garay, 1995; Kempf et al., 1995; Adler and Santiago, 1999; Scardigli, 1999; Adler et al., 2001). After this original burst of work on GUP there were various other works, a sample of where can be found in references (Myung et al., 2007; Zhu et al., 2009; Chemissany et al., 2011; Das and Mann, 2011; Sprenger et al., 2011; Ali et al., 2015; Anacleto et al., 2015; Garattini and Faizal, 2016) which further developed this area of research. There are also some very recent works (Tamburini and Licata, 2020; Fadel and Maggiore, 2022) which deal with the algebraic and physical structure of spacetime in connection with GUP.

The basic idea is that quantum gravity should modify the standard position and momentum commutator of canonical quantum mechanics from $[\hat{x}_i, \hat{p}_j] = i\delta_{ij}\hbar$ to $[\hat{X}_i, \hat{P}_j] = i\delta_{ij}\hbar f(x,p)$; with $f(x, p)$ representing the effects of quantum gravity. The capital X and P indicate that the position and momentum operators are changed from their canonical form. A common example that we will refer to often in this work is the modified commutator of (Kempf et al., 1995) of the form

$$\left[\hat{X}_i, \hat{p}_j\right] = i\delta_{ij}\hbar\left(1 + \beta|\vec{p}|^2\right) \ . \tag{3}$$

In this model the position and momentum as given by

$$\hat{X}_i = i\hbar\left(1+\beta|\vec{p}|^2\right)\frac{\partial}{\partial p_i} \quad \text{and} \quad \hat{p}_i = p_i \ , \tag{4}$$

i.e., the position operator is modified but the momentum operator is not. The constant β is a phenomenological parameter that characterizes the scale at which quantum gravity effects become important. Conventionally, it is thought β should be of the Planck scale *i.e.* $\beta \sim \frac{l_{Pl}^2}{\hbar^2}$ with l_{Pl} being the Planck length. A full analysis of the system in Equations 3, 4 is given in reference (Kempf et al., 1995), but for our purposes we recall two important results for this particular GUP model:

- Equations 3, 4 have a minimum length of $\Delta|\vec{x}| = \hbar\sqrt{\beta}$ at $\Delta|\vec{p}| = \frac{1}{\sqrt{\beta}}$
- In order for position and momentum operators to be symmetric *i.e.* $(\langle\psi|p_i\rangle|\phi\rangle = \langle\psi|(p_i|\phi\rangle)$ and $(\langle\psi|x_i\rangle|\phi\rangle = \langle\psi|(x_i|\phi\rangle)$, the scalar product of this model needs to be given by

$$\langle\psi|\phi\rangle = \int_{-\infty}^{\infty}\frac{d^3p}{1+\beta|\vec{p}|^2}\psi^*(p)\phi(p). \tag{5}$$

The modification of the scalar product as given by (5) is for three dimensions, but in n dimensions one still has the same modifying factor for the momentum integration, namely $\frac{d^np}{1+\beta|\vec{p}|^2}$. More generally, for a modified position operator of the form

$$\hat{X}_i = i\hbar f\left(|\vec{p}|^2\right)\frac{\partial}{\partial p_i} \ , \tag{6}$$

the scalar product must take the form

$$\langle\psi|\phi\rangle = \int_{-\infty}^{\infty}\frac{d^np}{f\left(|\vec{p}|^2\right)}\psi^*(p)\phi(p) \ . \tag{7}$$

These results from (5) and (7) will become important in the next section.

3 GUP and its effects on vacuum energy calculations

3.1 Vacuum energy in KMM GUP

The main issue we want to examine is how GUP affects the calculation of the vacuum energy and cosmological constant as laid out in (1), (2), and the surrounding discussion. One of the earliest and most impactful works dealing with the cosmological constant problem in the context of GUPs is the work by Chang *et al.* (Chang et al., 2002). In their work, the authors calculate how the GUP, as defined by (3) and (4), modifies Liouville's theorem and the phase space volume, *i.e.* $d^n x\, d^n p$, in n spatial dimensions. The modified phase space found in (Chang et al., 2002) for the GUP from (3) and (4) is

$$\frac{d^n x\, d^n p}{\left(1+\beta|\vec{p}|^2\right)^n}\ . \tag{8}$$

The volume in (8) is integrated out ($\int d^n x \to V$). Upon quantization, the claimed phase space volume from (Chang et al., 2002) becomes

$$\frac{V\, d^n p}{(2\pi)^n\left(1+\beta|\vec{p}|^2\right)^n}\ . \tag{9}$$

Recall we are using units with $\hbar = 1$ as consistent with reference (Weinberg, 1989). Thus, to compare 9) with the result in (Chang et al., 2002), one should replace the factor 2π by $2\pi\hbar$ in the denominator above. Using the result in (9) for three spatial dimensions, the calculation of ρ_{vac} via 2) changes to

$$\begin{aligned}\rho_{vac} &= \int \frac{d^3 p}{(2\pi)^3\left(1+\beta|\vec{p}|^2\right)^3}\,\frac{1}{2}\sqrt{\vec{p}^{\,2}+m^2}\\ &= \frac{1}{2}\int_0^\infty \frac{4\pi}{(2\pi)^3}\,dp\,\frac{p^2\sqrt{p^2+m^2}}{\left(1+\beta|\vec{p}|^2\right)^3}\ .\end{aligned} \tag{10}$$

Since the integrand of (10) is $\mathcal{O}(\frac{1}{|\vec{p}|^3})$ at large momentum, it is convergent and does not need to have the dp integration capped as in (2). One can integrate (10) exactly for any m (Chang et al., 2002); for the sake of simplicity, when $m = 0$ (10) becomes

$$\rho_{vac}(m=0) = \frac{1}{16\pi^2\beta^2}\ . \tag{11}$$

If one takes β to be of the Planck scale, then the result from (11) still leaves the GUP modified vacuum energy to be about 118 orders of magnitude larger than the measured vacuum energy of $\rho_{vac} \approx 10^{-47}\,\text{GeV}^{-4}$. In fact, by comparing 2) and 11) and using dimensional analysis, one finds that $\beta \sim p_c^{-2}$. Thus, using the GUP cutoff factor of $\frac{1}{(1+\beta|\vec{p}|^2)^3}$, while making ρ_{vac} finite, still leaves ρ_{vac} much too large which fails to resolve the cosmological constant problem. One has only replaced the "by hand" cutoff in (2) with the functional cutoff of (10). This failure of the GUP, defined by (3) and (4), to address the cosmological constant puzzle was already noted in (Chang et al., 2002).

However, there may be an additional problem with the integration over the momentum in (10): it appears to disagree with the momentum integration from (Kempf et al., 1995), as given by the definition of the scalar product in (5) or more generally in (7). In the momentum space integration in (5), there is only one factor of $(1+\beta|\vec{p}|^2)$ in the denominator, as compared to the denominator of (10), which has three factors of $(1+\beta|\vec{p}|^2)$. If one only had one factor of $(1+\beta|\vec{p}|^2)$ in the denominator of (10), as implied by (5), then the integrand would go as $\mathcal{O}(|\vec{p}|)$ and would thus diverge.

The derivation of the phase space volume carried out in (Chang et al., 2002) that gave the result in (8) is long, but straight forward, so it is hard to see any problem with this result. On the other hand, having a momentum space volume that has a factor of $(1+\beta|\vec{p}|^2)^{-n}$ for the $d^n p$ integration would then violate the symmetry of the position operator which is the requirement that led to (5); that is if the momentum integration in (10) is correct then this would imply $(\langle\psi|x_i)|\phi\rangle \neq \langle\psi|(x_i|\phi\rangle)$.

One potential solution to the difference in the integration factors between 5) and 9) could be to reconsider the spatial/volume calculation. In the transition from (8) to (9), it is assumed that the real spatial volume with GUP is the same as without GUP, that is, $\int d^n x = V$. The introduction of a minimal length may change the calculation of volumes in some way. If one could argue the $n-1$ factors of $(1+\beta|\vec{p}|^2)$ should go with the $d^n x$ integration, this would leave the correct single factor of $(1+\beta|\vec{p}|^2)$ to go with the $d^n p$ integration. This would resolve the discrepancy between 5) and (8). Ordinarily, all the factors of $(1+\beta|\vec{p}|^2)$ should fall under the $d^n p$ integration, but in the GUP given by (3) and (4) one can see that the position operator becomes dependent on the momentum. We suggest that in spherical coordinates every length r should carry with it a factor of $(1+\beta|\vec{p}|^2)^{-1}$. The n dimensional version of the GUP modified phase space given in (8) should be written as

$$\begin{aligned}&\left(\frac{d^n x}{\left(1+\beta|\vec{p}|^2\right)^{(n-1)}}\right)\left(\frac{d^n p}{\left(1+\beta|\vec{p}|^2\right)}\right)\\ &= \left(\frac{r^{n-1}dr d\Omega}{\left(1+\beta|\vec{p}|^2\right)^{(n-1)}}\right)\left(\frac{d^n p}{\left(1+\beta|\vec{p}|^2\right)}\right)\ .\end{aligned} \tag{12}$$

For low energy/momentum, where $\beta|\vec{p}|^2 \ll 1$, the length will not change much, but for high energy/momentum, where $\beta|\vec{p}|^2 \gg 1$, the length is reduced. This way, the modified momentum integration from the requirement of symmetry of the position and momentum operators as given in (5) and the GUP modified phase space of (8) now agree.

If the momentum space integration is now given by one factor of $(1+\beta|\vec{p}|^2)$, as implied by (12), rather than n factors as implied by (9) or (10), then not only does the GUP of Equations

3, 4 not solve the cosmological constant puzzle, as already acknowledged in (Chang et al., 2002), but the cosmological constant is not even finite. In the next subsection, we will investigate a different GUP, which does give a finite cosmological constant and examine to what extent this different GUP can address the cosmological constant problem.

3.2 Alternative GUP and the associated vacuum energy

From the generalized modified position operators of (6) and the associated modified momentum integration in (7), one can see that the integrand for ρ_{vac} will be of order $\mathcal{O}(\frac{|\vec{p}|^3}{f(|\vec{p}|)})$. Thus, $f(|\vec{p}|)$ must have a dependence of $|\vec{p}|^5$ or higher for the integral to be finite. One such GUP that meets this requirement is given in the recent paper (Bishop et al., 2021) which has modified operators in three spatial dimensions of the form

$$X_i = i\hbar \cosh^2\left(\frac{|\vec{p}|}{p_M}\right)\partial_{p_i} \quad ; \quad P_i = \frac{p_i p_M}{|\vec{p}|}\tanh\left(\frac{|\vec{p}|}{p_M}\right) . \tag{13}$$

From (6), we see that (13) implies $f(|\vec{p}|) = \cosh^2(\frac{|\vec{p}|}{p_M})$. Thus, the GUP in (13) implies a vacuum energy density of

$$\begin{aligned} \rho_{vac}^{tanh} &= \int \frac{d^3p}{(2\pi)^3\cosh^2\left(\frac{|\vec{p}|}{p_M}\right)}\frac{1}{2}\sqrt{\left(\vec{P}\right)^2 + m^2} \\ &\approx \frac{1}{2}\int_0^\infty \frac{4\pi}{(2\pi)^3}dp\,\frac{p^2 p_M \tanh\left(\frac{p}{p_M}\right)}{\cosh^2\left(\frac{p}{p_M}\right)} \end{aligned} \tag{14}$$

which has an integrand that exponentially decays with momentum. In (14) we set the rest mass equal to zero (m = 0), and used $|\vec{P}| = \frac{|\vec{p}|p_M}{|\vec{p}|}\tanh(\frac{|\vec{p}|}{p_M}) = p_M\tanh(\frac{p}{p_M})$. One can evaluate the last expression exactly and this yields a finite answer

$$\rho_{vac}^{tanh} = \frac{p_M^4\ln(2)}{4\pi^2} . \tag{15}$$

Thus, with this GUP model we do get a finite vacuum energy density while maintaining symmetry of the position and momentum operators. In contrast the GUP model given by Equations 3, 4, has an infinite vacuum energy density when *only one* power of $1+\beta|\vec{p}|^2$ (as argued in this work) is used in the denominator of the vacuum energy density (10). Even for the GUP models like that in (13), where the vacuum energy density is finite, the end conclusion is essentially the same as for the vacuum energy density in (2) which is obtained via a "by-hand" cutoff: both go as momentum to the fourth power. Comparing the vacuum energy densities from (2), (11), and (15), they all have essentially the same form, with different notations for the momentum scale cut-off. Thus, whether the vacuum energy is infinite and cut-off "by-hand" or is finite due to using a GUP like (13), both these models are equally ineffective at addressing the cosmological constant problem.

4 Summary and conclusions

In this work, we have examined how the GUP may alter the calculation of the vacuum energy density and the related cosmological constant. In standard QFT, which was reviewed in *Introduction* section, the vacuum energy diverges and must be cut-off as in (2), which leads to a quartic dependence of the vacuum energy density on the cut-off.

GUP models with their associated minimal length scales provide a potential avenue to calculate a finite vacuum energy density. Having a minimal length implies a maximum energy-momentum which cuts off the divergence in the standard vacuum energy density given in (2). An early work (Chang et al., 2002) led to a finite vacuum energy density given by (10) and (11). However, one of our points was to argue that the calculation of the vacuum energy given in (Chang et al., 2002) by (10) is inconsistent with the requirement that the position and momentum operators are symmetric in GUP models such as (Kempf et al., 1995). This symmetry requirement leads to an integration over momentum as given in (5) for the GUP from Equations 3, 4 or for a more general modified position as in (7). Although, if one takes only a single factor of $1+\beta|\vec{p}|^2$ in the momentum integration used to calculate ρ_{vac}, then one finds that the vacuum energy density from the GUP is not finite, which conflicts with the results of (Chang et al., 2002) which has n factors of $1+\beta|\vec{p}|^2$. Alternatively, one can preserve the symmetry of the position and momentum operators, but then the vacuum energy density is infinite for some GUPs like 3) and (4). In the present work, we argued for the latter option, because when arriving at the momentum integration measure of (10), one had to integrate out the spatial volume, as is done in going from (8) to (9). However, the implication is that one is treating the spatial volume the same as in a theory with *no minimal length*. In order to take into account the minimal length of the GUP, $n-1$ of the n factors of $1+\beta|\vec{p}|^2$ should correspond to the volume integration to take into account the minimal length, leaving one factor to go with the momentum integration.

In a larger sense, GUPs may not be able to resolve the cosmological constant problem. We presented an GUP model 13) where the integrand in the vacuum energy density decayed exponentially and led to a finite integral. However, this led to the same quartic momentum behavior as the in "by-hand" cutoff of (2) which were all essentially the same up to multiplicative factors of order one. Regardless, the end result for all the models is more or less the same.

There may be a way for a GUP model to address the cosmological constant problem by requiring the function

$f(|\vec{p}|)$, which gives the modification for the position operator, take negative values for some range of $|\vec{p}|$. All the GUP functions discussed here (*i.e.* $1 + \beta p^2$ or $\cosh(p/p_M)$) were positive definite. In integrating the momentum from 0 up to the QCD scale of $\Lambda_{QCD} \sim 200$ MeV, one already had a huge disagreement between the observed and theoretically calculated vacuum energy density. To compensate for this already large disagreement, a GUP function that is negative for some range of $|\vec{p}|$ beyond the QCD scale is needed to cancel out the positive contribution from the low momentum part of the integration. This is reminiscent of the supersymmetry approach to the cosmological constant problem where the positive bosonic contribution to the vacuum energy density is canceled by the negative fermionic contribution. Note, that requiring $f(|\vec{p}|)$ to be negative is similar to a parity transformation $\vec{x} \to -\vec{x}$ but is an unusual parity transformation in that it is not discrete but rather changes continuously as momentum increases. In any case, this may provide a fruitful new avenue for addressing the cosmological constant problem with GUPs.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of interest

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Weak equivalence principle in quantum space

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Owing to the development of String Theory and Quantum Gravity, studies of quantized spaces described by deformed commutation relations for operators of coordinates and operators of momenta have received much attention. In this paper, the implementation of the weak equivalence principle is examined in the quantized spaces described by different types of deformed algebras, among them the noncommutative algebra of canonical type, Lie type, and the nonlinear deformed algebra with an arbitrary function of deformation depending on momenta. It is shown that the deformation of commutation relations leads to the mass-dependence of motion of a particle (a composite system) in a gravitational field, and, hence, to violation of the weak equivalence principle. We conclude that this principle is recovered in quantized spaces if one considers the parameters of the deformed algebras to be different for different particles (bodies) and to be determined by their masses.

KEYWORDS

minimal length, quantized space, deformed Heisenberg algebra, weak equivalence principle, parameters of deformed algebras

1 Introduction

String Theory and Quantum Gravity predict the existence of a minimal length [see, for instance, (Gross and Mende, 1988; Maggiore, 1993)]. This, one of the most important suggestions of these theories, follows from the generalized uncertainty principle (GUP)

$$\Delta X \geq \frac{\hbar}{2}\left(\frac{1}{\Delta P} + \beta \Delta P\right), \quad (1)$$

where β is a constant which is called the parameter of deformation. Notations ΔX, ΔP are used for position and momentum uncertainties. The inequality Eq. 1 leads to the existence of a minimal value of ΔX which is determined by the parameter β and reads $\Delta X_{min} = \hbar\sqrt{\beta}$.

One can obtain the generalized uncertainty principle (1) by considering a quadratic deformation of the commutation relations for the operator of coordinate and the operator of momentum

$$[X, P] = i\hbar\left(1 + \beta P^2\right). \quad (2)$$

Relation (2) can be generalized as

$$[X, P] = i\hbar F\left(\sqrt{\beta}\,|P|\right), \quad (3)$$

where $F(\sqrt{\beta}|P|)$ is a function, which is called the deformation function, β is a constant, $\beta \geq 0$, $F(0) = 1$.

For the invariance of the deformed commutation relation (3) upon reflection ($X \to -X, P \to -P$) and for preserving of the time-reversal symmetry the deformation function has to be even, so that $F = F(\sqrt{\beta}|P|)$. Also from Eq. 3 it follows that the deformation function has to be dimensionless, therefore a dependence of F on the dimensionless product $\sqrt{\beta}|P|$ is considered. In (Masłowski et al., 2012) the results of studies of the minimal length in the context of the deformed algebra (3) are presented and the answer to the question regarding what functions $F(\sqrt{\beta}|P|)$ lead to the minimal length is found.

Historically the first deformed algebra was proposed by Snyder in 1947 (Snyder, 1947). The algebra is well studied [see, for instance, (Romero and Zamora, 2008; Mignemi, 2011; Lu and Stern, 2012; Gnatenko Kh. P. and Tkachuk V. M., 2019b)]. In the nonrelativistic case the Snyder algebra reads

$$[X_i, X_j] = i\hbar\beta(X_i P_j - X_j P_i), \tag{4}$$

$$[X_i, P_j] = i\hbar(\delta_{ij} + \beta P_i P_j), \tag{5}$$

$$[P_i, P_j] = 0. \tag{6}$$

Also, a three-dimensional algebra leading to the minimal length was proposed by Kempf [see, for instance, (Kempf et al., 1995; Kempf, 1997; Sandor et al., 2002; Menculini et al., 2013; Gnatenko Kh. and Tkachuk V. M., 2019a)]

$$[X_i, X_j] = i\hbar\frac{(2\beta - \beta') + (2\beta + \beta')\beta P^2}{1 + \beta P^2}(P_i X_j - P_j X_i), \tag{7}$$

$$[X_i, P_j] = i\hbar(\delta_{ij}(1 + \beta P^2) + \beta' P_i P_j), \tag{8}$$

$$[P_i, P_j] = 0, \tag{9}$$

where β, β' are constants. Here the minimal length is determined by the parameters of deformation, it reads $\Delta X_{\min} = \hbar\sqrt{\beta + \beta'}$.

The most simple algebras leading to space quantization (i. e. the existence of a minimal length and minimal area), are noncommutative algebras of canonical type. In this algebras the commutators for coordinates and momenta are equal to constants

$$[X_i, X_j] = i\hbar\theta_{ij}, \tag{10}$$

$$[X_i, P_j] = i\hbar(\delta_{ij} + \sigma_{ij}), \tag{11}$$

$$[P_i, P_j] = i\hbar\eta_{ij}, \tag{12}$$

where θ_{ij}, σ_{ij}, η_{ij} are elements of constant antisymmetric matrixes [see, for example, (Djemai and Smail, 2004; Alavi, 2007; Bastos and Bertolami, 2008; Bertolami and Queiroz, 2011)]. Modification of the commutation relations in the form (10)-(12) leads to both a minimal length and a minimal momentum [see, for instance, (Gnatenko Kh. P. and Tkachuk V. M., 2018b)].

Another type of deformed algebra describing features of the spatial structure at the Planck scale is the noncommutative algebra of Lie type. It is characterized by the following commutation relations

$$[X_i, X_j] = i\hbar\theta_{ij}^k X_k. \tag{13}$$

Here θ_{ij}^k are the parameters of noncommutativity which are constants (see, for instance, (Lukierski and Woronowicz, 2006; Daszkiewicz and Walczyk, 2008; Lukierski et al., 2018)).

So, different deformed algebras, which describe features of the spatial structure at the Planck scale were proposed. These algebras can be divided into algebras of canonical type, noncommutative algebras of Lie type, and nonlinear deformed algebras (commutators for coordinates and momenta that are equal to a nonlinear function of coordinates and momenta). We would like to note that the status of the minimal length in the frame of all the algebras is the same. The minimal length indicates the min linear range in which a particle can be localized.

It is important to mention that a modification of the commutation relations for coordinates and momenta leads to violations of the fundamental laws and principles of physics, among them the weak equivalence principle. This principle is also known as the Galilean equivalence principle or universality of free fall, and is a restatement of the equality of gravitational and inertial mass. According to the weak equivalence principle, the kinematic characteristics, such as the velocity and position of a point mass in a gravitational field do not depend on its mass, composition and structure and are determined only by its initial position and velocity.

The equivalence principle was examined in the context of a noncommutative algebra of canonical type in (Bastos et al., 2011; Gnatenko, 2013; Saha, 2014; Bertolami and Leal, 2015; Gnatenko Kh. and Tkachuk V., 2017b, Gnatenko Kh. and Tkachuk, V. 2018a). The weak equivalence principle in noncommutative phase space was studied in (Bastos et al., 2011; Bertolami and Leal, 2015; Gnatenko Kh. and Tkachuk V., 2017b, Gnatenko Kh. and Tkachuk, V. 2018a). The authors of (Bertolami and Leal, 2015) concluded that the equivalence principle holds in the quantized space in the sense that an accelerated frame of reference is locally equivalent to a gravitational field, unless the parameters of noncommutativity are anisotropic ($\eta_{xy} \neq \eta_{xz}$). In the paper (Lake et al., 2019) generalized uncertainty relations that do not lead to the violation of the equivalence principle were presented. GUP models that do not require modified commutation relations, have also been proposed in (Bishop et al., 2021).

In this paper we study the weak equivalence principle in the context of different deformed algebras leading to space quantization. We show that the motion of a particle (a body) in a gravitational field in quantized space depends on its mass and composition. The weak equivalence principle is violated in quantized space. It is important that space quantization leads to a great violation of the weak equivalence principle if one considers the parameters of the deformed algebras to be the same for different particles (bodies). We conclude that in the context of different algebras (algebras with arbitrary deformation function depending on momentum, noncommutative algebras of canonical type, and noncommutative algebras of Lie type) the weak equivalence principle is recovered in the case when the parameters of deformation are different for different particles and are determined by their masses.

The paper is organized as follows. In Section 2 the weak equivalence principle is studied in the space with GUP. It is shown that the deformation of the commutation relations leads to a great violation of the weak equivalence principle. We find a condition on the parameter of deformation in which the weak equivalence principle is preserved. Section 3 is devoted to studies of the motion of a particle in a gravitational field in a noncommutative phase space of canonical type. The way to recover the weak equivalence principle in the space is proposed. Section 4 is devoted to studying a quantized space with Lie algebraic noncommutativity. It is shown that the weak equivalence principle is recovered due to the relation of the parameters of the noncommutative algebra with mass. Conclusions are presented in Section 5.

2 The weak equivalence principle in a quantized space with a nonlinear deformed algebra and the parameters of deformation

Let us examine the motion of a particle in a gravitational field in one-dimensional space characterized by a deformed algebra (1D) with an arbitrary function of deformation dependent on momenta (3). We study the following Hamiltonian

$$H = \frac{P^2}{2m} + mV(X), \tag{14}$$

where m is the mass of the particle, $V(X)$ corresponds to the gravitational potential. Note that in Eq. 14 we consider the inertial mass (mass in the first term) to be equal to the gravitational mass (mass in the second term).

In the classical limit $\hbar \to 0$ on the basis of Eq. 3 we find the deformed Poisson brackets

$$\{X, P\} = F\left(\sqrt{\beta}|P|\right). \tag{15}$$

The definition of the brackets reads

$$\{f, g\} = F\left(\sqrt{\beta}|P|\right)\left(\frac{\partial f}{\partial X}\frac{\partial g}{\partial P} - \frac{\partial f}{\partial P}\frac{\partial g}{\partial X}\right). \tag{16}$$

So, using Eq. 16, one can write the equations of motion of a particle in the gravitational field in the deformed space as follows

$$\dot{X} = \{X, H\} = \frac{P}{m}F\left(\sqrt{\beta}|P|\right), \tag{17}$$

$$\dot{P} = \{P, H\} = -m\frac{\partial V(X)}{\partial X}F\left(\sqrt{\beta}|P|\right). \tag{18}$$

From Eqs. 17, 18 it follows that the motion of a particle in a gravitational field in the space (3) depends on its mass. So, the deformed relation (3) leads to violation of the weak equivalence principle. Moreover, the GUP (3) causes a great violation of the weak equivalence principle (the value of the Eötvös parameter is many orders larger then that obtained experimentally). Let us show this by considering the motion of two particles in a uniform gravitational field $V(X) = -gX$, where g is the gravitational acceleration. On the basis of Eqs. 17, 18 one can write

$$\dot{X}^{(b)} = \frac{P^{(b)}}{m_b}F\left(\sqrt{\beta}|P^{(b)}|\right), \tag{19}$$

$$\dot{P} = m_b g F\left(\sqrt{\beta}|P^{(b)}|\right). \tag{20}$$

So, up to the first order in β we find

$$\ddot{X}^{(b)} = g + 3F'(0)g\sqrt{\beta}m_b|v| + \left(2F''(0) - (F'(0))^2\right)g\beta m_b^2 v^2, \tag{21}$$

where m_b is the mass of a particle labeled by index b $(b = (1, 2))$, $F'(x) = dF/dx$, $F''(x) = d^2F/dx^2$. The notation v is used for the velocity of motion in the gravitational field $V(X) = -gX$ in the ordinary space (i. e. the space with $\beta = 0$).

So, up to the first order in β the Eötvös parameter for particles with masses m_1, m_2 reads

$$\begin{aligned}\frac{\Delta a}{a} &= \frac{2\left(\ddot{X}^{(1)} - \ddot{X}^{(2)}\right)}{\ddot{X}^{(1)} + \ddot{X}^{(2)}}\\ &= 3F'(0)|v|\sqrt{\beta}(m_1 - m_2) + \left(2F''(0) - (F'(0))^2\right)v^2\beta\\ &\quad\times (m_1^2 - m_2^2).\end{aligned} \tag{22}$$

To estimate the value of Eq. 22 let us put $\hbar\sqrt{\beta} = l_P = \sqrt{\hbar G}/\sqrt{c^3}$ (l_P is the Planck length, c is the speed of light, G is the gravitational constant). So, we have

$$\begin{aligned}\frac{\Delta a}{a} &= 3F'(0)\frac{|v|}{c}\frac{(m_1 - m_2)}{m_P} + \left(2F''(0) - (F'(0))^2\right)\frac{v^2}{c^2}\\ &\quad\times\frac{(m_1^2 - m_2^2)}{m_P^2},\end{aligned} \tag{23}$$

with $m_P = \sqrt{\hbar c}/\sqrt{G}$ being the Planck mass (Gnatenko and Tkachuk, 2020).

Note that for $m_1 = 1$ kg, $m_2 = 0.1$ kg in the case of deformation function $F(\sqrt{\beta}|P|) = 1 + \beta P^2$ form Eq. 23 we obtain great violation of the weak equivalence principle $\Delta a/a \approx 0.1$ which has not been seen experimentally (Gnatenko and Tkachuk, 2020). From the tests of the weak equivalence principle it follows that this principle holds with high accuracy. For instance, on the basis of the Lunar Laser Ranging experiment it is known that the equivalence principle holds with precision $\Delta a/a = (-0.8 \pm 1.3)\cdot 10^{-13}$ (Williams et al., 2012). Similar results were obtained from the laboratory torsion-balance tests of the weak equivalence principle for Be and Ti in which $\Delta a/a = (0.3 \pm 1.8) \cdot 10^{-13}$ and $\Delta a/a = (-0.7 \pm 1.3) \cdot 10^{-13}$ for Be and Ti or Al (Wagner et al., 2012). The MICROSCOPE space mission aims to test the principle with accuracy 10^{-15} (Touboul et al., 2017).

It is important to mention that we have obtained a great violation of the weak equivalence principle in a space with GUP assuming that parameter of deformation β is the same for different particles. Let us consider a more general case in which the parameter of deformation is different for different particles. We use notation β_b for the parameter of deformation corresponding to a particle with index b. The weak equivalence principle can be recovered in a space with GUP, if we assume that β_b is determined by the mass of a particle as

$$\sqrt{\beta_b} m_b = \gamma = \text{const}, \tag{24}$$

where the constant γ is the same for different particles and does not depend on mass (Quesne and Tkachuk, 2010; Tkachuk, 2012; Gnatenko and Tkachuk, 2020).

Taking into account Eq. 24, we find that the Eötvös parameter written up to the first order in β is equal to zero

$$\frac{\Delta a}{a} = 3F'(0)|v|\left(\sqrt{\beta_1}\, m_1 - \sqrt{\beta_2}\, m_2\right) + \left(2F''(0) - (F'(0))^2\right)v^2 \times \left(\beta_1 m_1^2 - \beta_2 m_2^2\right) = 0. \tag{25}$$

Also, considering the parameter of deformation to be dependent on mass according to

$$\beta = \frac{\gamma^2}{m^2}, \tag{26}$$

(this expression follows from Eq. 24), the equations of motion of a particle in a gravitational field Eqs. 17, 18 read

$$\dot{X} = \frac{P}{m} F\left(\gamma \frac{|P|}{m}\right), \tag{27}$$

$$\frac{\dot{P}}{m} = -\frac{\partial V(X)}{\partial X} F\left(\gamma \frac{|P|}{m}\right). \tag{28}$$

On the basis of Eqs. 27, 28 we have that the equations for X and P/m do not contain mass. Therefore, the solutions $X(t)$, $P(t)/m$ of these equations also do not depend on mass. So, the weak equivalence principle is recovered due to the assumption Eq. 24 (Tkachuk, 2012; Gnatenko and Tkachuk, 2020).

Let us also study the weak equivalence principle in the more general three-dimensional case of deformed (3D) algebras. Namely, let us consider the following commutation relations

$$[X_i, X_j] = G(P^2)\left(X_i P_j - X_j P_i\right), \tag{29}$$

$$[X_i, P_j] = f(P^2)\delta_{ij} + F(P^2) P_i P_j, \tag{30}$$

$$[P_i, P_j] = 0. \tag{31}$$

The algebra Eqs. 29–31 is a generalization of the well known Snyder Eqs. 4–6 and Kempf Eqs. 7–9 algebras. The functions $G(P^2)$, $F(P^2)$ and $f(P^2)$ in Eqs. 29–31 have to satisfy the relation

$$f(F - G) - 2f'(f + FP^2) = 0, \tag{32}$$

(here $f' = \partial f/\partial P^2$) which follows from the Jacobi identity (Frydryszak and Tkachuk, 2003).

From the classical limit of Eqs. 29–31 we obtain the following Poisson brackets

$$\{X_i, X_j\} = G(P^2)\left(X_i P_j - X_j P_i\right), \tag{33}$$

$$\{X_i, P_j\} = f(P^2)\delta_{ij} + F(P^2) P_i P_j, \tag{34}$$

$$\{P_i, P_j\} = 0. \tag{35}$$

Let us study the weak equivalence principle in the quantized space represented by Eqs. 33–35. Considering a particle in a gravitational field $V(\mathbf{X})$ with Hamiltonian

$$H = \sum_i \frac{P_i^2}{2m} + mV(\mathbf{X}), \tag{36}$$

and taking into account the deformation of the Poisson brackets Eqs. 33–35, we find the following equations of motion

$$\dot{X}_i = \frac{P_i}{m} f(P^2) + m\sum_j \frac{\partial V(\mathbf{X})}{\partial X_j} G(P^2)\left(X_i P_j - X_j P_i\right), \tag{37}$$

$$\dot{P}_i = -m\frac{\partial V(\mathbf{X})}{\partial X_i} \tilde{f}(\beta P^2) - m\sum_j \frac{\partial V(\mathbf{X})}{\partial X_j} F(P^2) P_i P_j. \tag{38}$$

On the basis of dimensional considerations the functions $f(P^2)$, $F(P^2)$, $G(P^2)$ can be rewritten as $f(P^2) = \tilde{f}(\beta P^2)$, $F(P^2) = \beta\tilde{F}(\beta P^2)$ and $G(P^2) = \beta\tilde{G}(\beta P^2)$, where $\tilde{f}(\beta P^2)$, $\tilde{F}(\beta P^2)$ and $\tilde{G}(\beta P^2)$ are dimensionless functions. Taking this into account, and considering the condition Eq. 26, one can rewrite the equations of motion of a particle in a gravitational field as follows

$$\dot{X}_i = P_i'\tilde{f}\left(\gamma^2 (P')^2\right) + \gamma^2 \sum_j \frac{\partial V(\mathbf{X})}{\partial X_j}\tilde{G}\left(\gamma^2 (P')^2\right) \times \left(X_i P_j' - X_j P_i'\right), \tag{39}$$

$$\dot{P}_i' = -\frac{\partial V(\mathbf{X})}{\partial X_i}\tilde{f}\left(\gamma^2 (P')^2\right) - \gamma^2 \sum_j \frac{\partial V(\mathbf{X})}{\partial X_j}\tilde{F} \times \left(\gamma^2 (P')^2\right) P_i' P_j', \tag{40}$$

where $P_i' = P_i/m$. It is important that Eqs. 39, 40 do not depend on mass. So, the weak equivalence principle is preserved in quantized space Eqs. 33–35 if the relation of the parameter of deformation with mass Eq. 26 is satisfied (Gnatenko and Tkachuk, 2020).

It is also important to mention that the relation Eq. 26 gives a possibility to preserve the additivity property of the kinetic energy in a space with GUP and to solve the problem of the significant effect of the GUP on the kinetic energy of a macroscopic body [for details see (Gnatenko and Tkachuk, 2020)].

3 Influence of noncommutativity of coordinates and noncommutativity of momenta on the motion in a gravitational field

Let us study the motion of a particle in a uniform gravitational field in the context of a noncommutative algebra of canonical type (2D)

$$[X_1, X_2] = i\hbar\theta, \quad (41)$$

$$\left[X_i, P_j\right] = i\hbar\delta_{ij}, \quad (42)$$

$$[P_1, P_2] = i\hbar\eta, \quad (43)$$

where the parameters of noncommutativity θ, η are constants and i, j = (1, 2). In the classical limit we obtain the following deformed Poisson brackets

$$\{X_1, X_2\} = \theta, \quad (44)$$

$$\left\{X_i, P_j\right\} = \delta_{ij}, \quad (45)$$

$$\{P_1, P_2\} = \eta. \quad (46)$$

Let us examine the motion of a particle in a gravitational field in the space Eqs. 44–46 and find the way to preserve the weak equivalence principle (Gnatenko, 2013; Gnatenko Kh. and Tkachuk V., 2017b, Gnatenko Kh. and Tkachuk, V. 2018a). The equations of motion of a particle with mass m in a uniform gravitational field with Hamiltonian

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} - mgX_1, \quad (47)$$

read

$$\dot{X}_1 = \{X_1, H\} = \frac{P_1}{m}, \quad (48)$$

$$\dot{X}_2 = \{X_2, H\} = \frac{P_2}{m} + mg\theta, \quad (49)$$

$$\dot{P}_1 = \{P_1, H\} = mg + \eta\frac{P_2}{m}, \quad (50)$$

$$\dot{P}_2 = \{P_2, H\} = -\eta\frac{P_1}{m}. \quad (51)$$

In Eq. 47 one considers the field directed along the X_1 axis. Note that in the two-dimensional case the noncommutative algebra of canonical type Eqs. 41–43 is rotationally invariant, so the results and conclusions we obtain, considering this particular case, can be generalized to the case of the arbitrary direction of the field.

The solution of Eqs. 48–51 with initial conditions $X_1(0) = X_{01}$, $X_2(0) = X_{02}$, $\dot{X}_1(0) = \upsilon_{01}$, $\dot{X}_2(0) = \upsilon_{02}$ is the following

$$X_1(t) = \frac{m\upsilon_{01}}{\eta}\sin\frac{\eta}{m}t + \left(\frac{m^2 g}{\eta^2} - \frac{m^2 g\theta}{\eta} + \frac{m\upsilon_{02}}{\eta}\right)\left(1 - \cos\frac{\eta}{m}t\right) + X_{01}, \quad (52)$$

$$X_2(t) = \left(\frac{m^2 g}{\eta^2} - \frac{m^2 g\theta}{\eta} + \frac{m\upsilon_{02}}{\eta}\right)\sin\frac{\eta}{m}t - \frac{m\upsilon_{01}}{\eta}\left(1 - \cos\frac{\eta}{m}t\right) - \frac{mg}{\eta}t + mg\theta t + X_{02}. \quad (53)$$

The obtained expressions Eqs. 52, 53 depend on mass, if we assume that the parameters of noncommutativity θ, η are the same for different particles. In this case the weak equivalence principle is violated in the noncommutative phase space of canonical type. The way to solve this problem is to consider, as in the previous section, that the parameters of noncommutativity are dependent upon mass (Gnatenko Kh. and Tkachuk V., 2017b).

The trajectory of a particle in the uniform gravitational field depends on $m\theta$ and η/m. So, if these values do not depend on mass then the weak equivalence principle is recovered. So, let us consider the following conditions

$$\theta m = \gamma = \text{const}, \quad (54)$$

$$\frac{\eta}{m} = \alpha = \text{const}, \quad (55)$$

where γ, α are the same for different particles. Taking into account Eqs. 54, 55, the trajectory Eqs. 52, 53 transforms to

$$X_1(t) = \frac{\upsilon_{01}}{\alpha}\sin\alpha t + \left(\frac{g}{\alpha^2} - \frac{g\gamma}{\alpha} + \frac{\upsilon_{02}}{\alpha}\right)(1 - \cos\alpha t) + X_{01}, \quad (56)$$

$$X_2(t) = \left(\frac{g}{\alpha^2} - \frac{g\gamma}{\alpha} + \frac{\upsilon_{02}}{\alpha}\right)\sin\alpha t - \frac{\upsilon_{01}}{\alpha}(1 - \cos\alpha t) - \frac{g}{\alpha}t + \gamma g t + X_{02}. \quad (57)$$

The trajectory of a particle in the gravitational field Eqs. 56, 57 is determined by its initial coordinates and velocities and does not depend on its mass. So, the weak equivalence principle is recovered in the noncommutative phase space of canonical type due to the relations Eqs. 54, 55 (Gnatenko Kh. and Tkachuk V., 2017b).

It is worth also mentioning that for $\eta \to 0$ the expressions Eqs. 52, 53 reduce to the well known result for the trajectory of a particle in a uniform gravitational field in ordinary space, $X_1(t) = gt^2/2 + \upsilon_{01}t + X_{01}$, $X_2(t) = \upsilon_{02}t + X_{02}$. At the same time, the noncommutativity of the coordinates affects the relation between the momenta and velocities, such that

$$P_1 = m\dot{X}_1, \quad P_2 = m\left(\dot{X}_2 + mg\theta\right). \quad (58)$$

In the case when the parameter of coordinate noncommutativity is inversely proportional to the mass on the basis of Eq. 58 we obtain that the momentum P_2 is proportional to mass, as it is in ordinary space $P_2 = m(\dot{X}_2 + \gamma g)$.

In the more general case of a particle in a nonuniform gravitational field $V(X_1, X_2)$ with Hamiltonian

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + mV(X_1, X_2), \quad (59)$$

The equations of motion read

$$\dot{X}_1 = \{X_1, H\} = \frac{P_1}{m} + m\theta\frac{\partial V(X_1, X_2)}{\partial X_2}, \tag{60}$$

$$\dot{X}_2 = \{X_2, H\} = \frac{P_2}{m} - m\theta\frac{\partial V(X_1, X_2)}{\partial X_1}, \tag{61}$$

$$\dot{P}_1 = \{P_1, H\} = -m\frac{\partial V(X_1, X_2)}{\partial X_1} + \eta\frac{P_2}{m}, \tag{62}$$

$$\dot{P}_2 = \{P_2, H\} = -m\frac{\partial V(X_1, X_2)}{\partial X_2} - \eta\frac{P_1}{m}. \tag{63}$$

To obtain Eqs. 60–63 we take into account Eqs. 44–46. From Eqs. 60–63 we can conclude that the weak equivalence principle is violated, if the parameters of noncommutativity are the same for different particles. In the case when the conditions on the parameters of noncommutativity Eqs. 54, 55 hold, introducing the notation $P_i' = P_i/m$, we can write

$$\dot{X}_1 = P_1' + \gamma\frac{\partial V(X_1, X_2)}{\partial X_2}, \quad \dot{X}_2 = P_2' - \gamma\frac{\partial V(X_1, X_2)}{\partial X_1}, \tag{64}$$

$$\dot{P}_1' = -\frac{\partial V(X_1, X_2)}{\partial X_1} + \alpha P_2', \quad \dot{P}_2' = -\frac{\partial V(X_1, X_2)}{\partial X_2} - \alpha P_1'. \tag{65}$$

Eqs. 64, 65 depend on the parameters γ and α and do not depend on mass. As a result, $X_i = X_i(t)$, $P_i' = P_i'(t)$ also do not depend on mass. So, conditions Eqs. 54, 55 give a possibility to preserve the weak equivalence principle also in the case of motion in a nonuniform gravitational field (Gnatenko Kh. and Tkachuk V., 2017b).

It is worth noting that in this section we consider the two-dimensional case of the noncommutative algebra of canonical type Eqs. 41–43, because it is rotationally-invariant. In the three-dimensional noncommutative phase space of canonical type one faces the problem of rotational symmetry breaking. A 3D algebra which is rotationally invariant and equivalent to the noncommutative algebra of canonical type was proposed in (Gnatenko K. P. and Tkachuk V. M., 2017a). It is important to mention that to recover the weak equivalence principle in the context of this algebra the idea to relate the parameters of noncommutativity with mass has to be considered [for details see (Gnatenko, 2018)].

4 Weak equivalence principle in noncommutative space of Lie type

Let us also study the motion of a composite system in a gravitational field in a space with a noncommutative algebra of Lie type and examine the weak equivalence principle. We consider the following algebra

$$\{X_i, X_j\} = \theta^0_{ij}t + \theta^k_{ij}X_k, \tag{66}$$

$$\{X_i, P_j\} = \delta_{ij} + \bar{\theta}^k_{ij}X_k + \tilde{\theta}^k_{ij}P_k, \quad \{P_i, P_j\} = 0, \tag{67}$$

where $i, j, k = (1, 2, 3)$, θ^0_{ij}, θ^k_{ij}, $\bar{\theta}^k_{ij}$, $\tilde{\theta}^k_{ij}$ are the parameters of noncommutativity, that are antisymmetric in their lower indexes $\theta^0_{ij} = -\theta^0_{ji}$, $\bar{\theta}^k_{ij} = -\bar{\theta}^k_{ji}$, $\tilde{\theta}^k_{ij} = -\tilde{\theta}^k_{ij}$ (Miao et al., 2011).

From the Jacobi identity it follows that the parameters θ^0_{ij}, θ^k_{ij}, $\bar{\theta}^k_{ij}$, $\tilde{\theta}^k_{ij}$ can not be arbitrary. In the particular case when

$$\theta^0_{kl} = -\theta^0_{ky} = \frac{1}{\kappa}, \quad \theta^0_{ly} = \frac{1}{\kappa}, \tag{68}$$

$$\theta^l_{ky} = -\theta^k_{ly} = \bar{\theta}^l_{ky} = -\tilde{\theta}^k_{ly} = \frac{1}{\tilde{\kappa}}, \tag{69}$$

the noncommutative algebra of Lie type reads

$$\{X_k, X_y\} = -\frac{t}{\kappa} + \frac{X_l}{\tilde{\kappa}}, \quad \{X_l, X_y\} = \frac{t}{\kappa} - \frac{X_k}{\tilde{\kappa}}, \tag{70}$$

$$\{X_k, X_l\} = \frac{t}{\kappa}, \quad \{P_k, X_y\} = \frac{P_l}{\tilde{\kappa}}, \tag{71}$$

$$\{P_l, X_y\} = -\frac{P_k}{\tilde{\kappa}}, \quad \{X_i, P_j\} = \delta_{ij}, \tag{72}$$

$$\{X_y, P_y\} = 1 \quad \{P_m, P_n\} = 0, \tag{73}$$

[see (Miao et al., 2011)]. Choosing

$$\theta^0_{ly} = -\theta^0_{ky} = \frac{1}{\kappa}, \quad \theta^l_{ky} = -\theta^k_{ly} = \frac{1}{\tilde{\kappa}}, \tag{74}$$

$$\tilde{\theta}^l_{ky} = -\tilde{\theta}^k_{ly} = \frac{1}{\tilde{\kappa}}, \tag{75}$$

$$\bar{\theta}^l_{ky} = -\bar{\theta}^k_{ly} = \frac{1}{\tilde{\kappa}}, \tag{76}$$

We obtain the following noncommutative algebra

$$\{X_k, X_y\} = -\frac{t}{\kappa} + \frac{X_l}{\tilde{\kappa}}, \quad \{X_l, X_y\} = \frac{t}{\kappa} - \frac{X_k}{\tilde{\kappa}}, \tag{77}$$

$$\{X_k, X_l\} = 0, \quad \{P_k, X_y\} = \frac{X_l}{\tilde{\kappa}} + \frac{P_l}{\tilde{\kappa}}, \tag{78}$$

$$\{P_l, X_y\} = \frac{X_k}{\tilde{\kappa}} - \frac{P_k}{\tilde{\kappa}}, \quad \{X_i, P_j\} = \delta_{ij}, \tag{79}$$

$$\{X_y, P_y\} = 1, \quad \{P_m, P_n\} = 0, \tag{80}$$

[see (Miao et al., 2011)].

The equations of motion of a particle with mass m in a gravitational field $V = V(X_1, X_2, X_3)$ with Hamiltonian $H = \frac{\mathbf{P}^2}{2m} + mV(X_1, X_2, X_3)$ in a space with a noncommutative algebra of Lie type read

$$\dot{X}_i = \frac{P_i}{m} + \bar{\theta}^k_{ij}\frac{P_jX_k}{m} + \tilde{\theta}^k_{ij}\frac{P_jP_k}{m} + m\left(\theta^0_{ij}t + \theta^k_{ij}X_k\right)\frac{\partial V}{\partial X_j}, \tag{81}$$

$$\dot{P}_i = -m\frac{\partial V}{\partial X_i} - m\left(\bar{\theta}^k_{ij}X_k + \tilde{\theta}^k_{ij}P_k\right)\frac{\partial V}{\partial X_j}. \tag{82}$$

The equivalence principle is recovered if the following conditions are satisfied (Gnatenko, 2019)

$$\theta^{0(b)}_{ij}m_b = \gamma^0_{ij} = \text{const}, \quad \theta^{k(b)}_{ij}m_b = \gamma^k_{ij} = \text{const}, \tag{83}$$

$$\tilde{\theta}^{k(b)}_{ij}m_b = \tilde{\gamma}^k_{ij} = \text{const}, \quad \bar{\theta}^{k(b)}_{ij} = \bar{\theta}^k_{ij}. \tag{84}$$

The constants γ^0_{ij}, γ^k_{ij}, $\tilde{\gamma}^k_{ij}$ are the same for different particles, $\gamma^0_{ij} = -\gamma^0_{ji}$, $\gamma^k_{ij} = -\gamma^k_{ji}$, $\tilde{\gamma}^k_{ij} = -\tilde{\gamma}^k_{ji}$. Taking into account (83), (84) and using the notation $P'_i = P_i/m$ the equations of motion of a particle in an arbitrary gravitational field can be rewritten as

$$\dot{X}_i = P'_i + \bar{\theta}^k_{ij} P'_j X_k + \tilde{\gamma}^k_{ij} P'_j P'_k + \left(\gamma^0_{ij} t + \gamma^k_{ij} X_k\right) \frac{\partial V}{\partial X_j}, \tag{85}$$

$$\dot{P}'_i = -\frac{\partial V}{\partial X_i} - \left(\bar{\theta}^k_{ij} X_k + \tilde{\gamma}^k_{ij} P'_k\right) \frac{\partial V}{\partial X_j}. \tag{86}$$

The obtained Eqs. 85, 86 do not contain mass. So, conditions Eqs. 83, 84 give a possibility to recover the weak equivalence principle in a space characterized by a noncommutative algebra of Lie type Eqs. 66, 67.

5 Discussion

The idea to describe features of the spatial structure at the Planck scale (the existence of a minimal length) with the help of deformed algebras has been considered. Deformed algebras of different types have been studied. Among them are deformed algebras with arbitrary functions of deformation that depends on momenta (these algebras are generalizations of the nonrelativistic Snyder and Kempf algebras), algebras with noncommutativity of the coordinates and noncommutativity of the momenta of canonical type, and noncommutative algebras of Lie type. The implementation of the weak equivalence principle has been examined in the quantized spaces described by these deformed algebras.

We have shown that, considering the parameters of the deformed algebras to be the same for different particles (different bodies), one faces the problem of violation of the weak equivalence principle. In this case the motion of a particle in a gravitational field in quantized space depends on its mass and composition. Even in the case of equality between the gravitational and the inertial masses of a body the Eötvös parameter is not equal to zero. Besides space quantization leads to great violation of this principle which should have been seen experimentally (see Eq. 23). To solve this problem the dependence of the parameters of the deformed algebras on mass has been considered. We have shown that if the parameters of the deformed algebras for coordinates and momenta are related to the particle mass the weak equivalence principle is preserved in noncommutative phase spaces of canonical type, in spaces with Lie algebraic noncommutativity, and in spaces with an arbitrary function of deformation dependent on momenta. In addition, the same relations for the parameters of deformation (parameters of noncommutativity) on mass give a possibility to recover the properties of the kinetic energy (its additivity and independence of compositions) and to solve the problem of the great effect of the minimal length on the motion of macroscopic bodies which is well known in the literature as the soccer-ball problem (Gnatenko and Tkachuk, 2020, Gnatenko Kh. and Tkachuk, V. 2017b; Gnatenko, 2019).

Data availability statement

The datasets analyzed during the current study are available from the corresponding author on reasonable request.

Author contributions

KG and VT contributed in equal part to the manuscript.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Dark matter as an effect of a minimal length

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In this work, we consider the implications of a phenomenological model of quantum gravitational effects related to a minimal length, implemented *via* the generalized uncertainty principle. Such effects are applied to the Bekenstein–Hawking entropy to derive a modified law of gravity through Verlinde's conjecture. Implications on galactic scales, and in particular on the shape of rotational curves, are investigated, exploring the possibility to mimic dark matter-like effects via a minimal length.

KEYWORDS

minimal length, dark matter, rotational curves, entropic force, gravity

1 Introduction

Dark matter (DM) is one of the dominant components in the energy budget of the Universe. Evidence for its existence ranges from galaxy clusters, rotational curves of galaxies, and gravitational lensing all to the cosmic microwave background (CMB) (Freese, 2009; Drees and Gerbier, 2012; Arbey and Mahmoudi, 2021). However, effects related to DM have not been observed on the scale of the solar system, whereas they become significant on galactic and intergalactic scales. Nonetheless, the nature of DM remains one of the most debated problems in physics up to this day. Several proposals and speculations concerning DM have been put forward, among which are MACHOs, WIMPs, axions, sterile neutrinos, and modified Newtonian dynamics (MOND) (Freese, 2009; Drees and Gerbier, 2012; Arbey and Mahmoudi, 2021; Oks, 2021). In this work, we propose an alternative explanation for DM. Specifically, we argue that the phenomenology related to DM can be partially described in terms of quantum gravitational effects.

The development of a theory of quantum gravity (QG) is an open problem in physics. Several candidate theories have been proposed, and numerous thought experiments have shaped the expected features of such a theory. However, none could have been directly tested due to current experimental and technological limitations. For this reason, phenomenological approaches have become some of the main tools to tackle the problem of QG (Magueijo, 2006; Ali et al., 2011; Hamma and Markopoulou, 2011; Dos Santos et al., 2013; Feller and Livine, 2016; Danshita et al., 2017; Haine, 2018; van de Kamp et al., 2020) (see Addazi et al. (2022) for a recent review). Such approaches usually consider the implications of features of a full QG theory on lower energy scales, possibly accessible to current experiments and observations. Among such features, a common one is the existence of a fundamental minimal length. Such a minimal length strongly opposes the traditional Heisenberg uncertainty principle of quantum mechanics, which should be properly modified approaching the QG scale. The set of models corresponding to a modified uncertainty relation is collectively referred to as the generalized uncertainty principle (GUP) (Maggiore, 1993a; Maggiore, 1993b; Maggiore, 1994; Kempf et al., 1996; Kempf, 1997; Scardigli, 1999; Scardigli and Casadio, 2003; Ali et al., 2011; Pikovski et al., 2012; Bosso et al., 2017; Scardigli et al., 2017; Bosso, 2018; Kumar and Plenio, 2018; Gnatenko and Tkachuk, 2019; Luciano, 2021a; Bosso, 2021; Bosso and Luciano, 2021; Das and Fridman, 2021; Giardino and Salzano, 2021; Gomes, 2022). Such models are inspired from candidate theories of quantum gravity, such as string theory and loop quantum gravity, in which an effective minimal observable length appears in scattering experiments or as a structural feature of space-time. The phenomenological implications are then accounted for in terms of a minimal uncertainty in position or non-commutativity of space-time. The commutator corresponding to one of the most common GUP models can be cast as (Kempf et al., 1996; Bosso, 2021):

$$[x_i, p_j] = i\hbar\left[\delta_{ij} + \beta\left(p^2\delta_{ij} + 2p_ip_j\right)\right], \tag{1}$$

where x_i and p_j are the position and momentum operators, respectively, $\beta \equiv \beta_0/(M_P c)^2$, where β_0 is a dimensionless parameter, and $M_P = \sqrt{\hbar c/G}$ is the Planck mass. Such a model implies a modification of the uncertainty relation between the position and momentum, as found using the Schrödinger–Robertson relation and thus leads to a minimal uncertainty in the position. For example, in the one-dimensional case, the modified uncertainty relation reads (Kempf et al., 1996):

$$\Delta x \Delta p \gtrsim \hbar\left[1 + 3\beta\Delta p^2\right]. \tag{2}$$

In this study, we propose how minimal length phenomenology can give rise to features similar to DM on galactic scales. Specifically, we deduce effects from GUP that contribute to the flatness of rotational curves. Such effects are obtained as a consequence of the modifications to the Bekenstein–Hawking entropy through the holographic principle, induced by GUP (Cai et al., 2008; Zhu et al., 2009; Awad and Ali, 2014; Giardino and Salzano, 2021; Buoninfante et al., 2022; Das et al., 2022). Therefore, we obtain corrections to the corresponding entropic force due to the presence of a minimal length. Based on Verlinde's conjecture (Verlinde, 2011), such a modified entropic force turns out as a modified Newton's law of gravity, thus providing a basis to study the implications of a minimal length on gravitational systems. Specifically, we require the holographic principle to hold, that is, we consider spherically symmetric surfaces of area $A = 4\pi R^2$ separating points in space. Such surfaces behave as the natural place to store information about particles that are inside the surfaces and that can move from one side to another. In this way, the information about the location of particles is stored in discrete bits on the surfaces. This naturally leads to the assumption that the number N_b of bits on the screens can be approximated with the number of particles N_p enclosed by the surfaces, that is, $N_b \sim N_p = N$. Then, the total number N of bits of the system, which is measured by its entropy S, can be naturally assumed to be proportional to the surface area A, that is, $N \sim S \sim A$. The total energy E of the system inside the surfaces is distributed on such bits and is related to the surface temperature by the equipartition theorem (or the GUP-modified version thereof). Such energy can be written in terms of the uniformly distributed mass M inside the surface as $E = Mc^2$. It should be noticed that the aforementioned reasoning can in principle be applied to any mass distribution, as long as one defines a proper holographic screen of radius R containing the whole distribution.

It turns out that a distance dependence for the GUP parameter β_0 must be assumed to provide a reasonable mechanism to study minimal length effects on rotational curves of galaxies. Such dependence has been proposed in other works as well (see, for e.g., Ref. Ong (2018)) and suggested by the different estimations of the GUP parameter in tabletop experiments, where $\beta_0 > 0$ (Pikovski et al., 2012; Bosso et al., 2017; Scardigli et al., 2017; Kumar and Plenio, 2018; Das and Fridman, 2021), and in astrophysical/cosmological observations, where $\beta_0 < 0$ (Jizba et al., 2010; Ong, 2018; Buoninfante et al., 2019; Nenmeli et al., 2021; Das et al., 2022; Jizba et al., 2022) (see also Luciano (2021b) for a recent review).

The article is structured as follows: in Section 2, a modification to the equipartition theorem due to GUP is introduced; in Section 3, a modified Newton's law of gravity is derived from the GUP-modified equipartition theorem and the Bekenstein–Hawking entropy; in Section 4, we summarize our results and include some final remarks.

2 GUP-modified equipartition theorem

One of the reasons to introduce DM is the flatness of galactic rotational curves, which deviate from the behavior predicted based on Kepler's model considering only luminous matter. In particular, Kepler's laws predict that the orbital velocities for stars outside the bulge decrease as the square root of the distance from the center, $v(R) \propto 1/\sqrt{R}$. The observation that the orbital velocities are approximately independent of the distance from the center, $v(R) \propto const.$, even at distances comparable with the galactic radius and beyond, suggests that either Newton's law of gravity does not hold at such scales, or that non-visible matter, present in galaxies, affects stellar dynamics. As mentioned in Introduction, here we will consider the former, with the intent of studying the implications of a minimal length on galactic rotational curves. Specifically, following Verlinde (2011), we introduce the gravitational force F as an entropic force, which is given as:

$$F\Delta x = T\Delta S\ , \tag{3}$$

where $\Delta x = \eta\lambda_C = \eta\frac{\hbar}{mc}$ is the displacement from the source of a test particle in terms of its Compton wavelength λ_C, m is the mass of the test particle, η is the modification factor ($\eta = 1$ if no modification), T is the effective temperature at a given radius, and $\Delta S = 2\pi k_B$ is the minimal change in entropy, as stated by the information theory (Adami, 2004), with k_B being the Boltzmann constant. The effective temperature T can be expressed in terms of energy E of a thermodynamical system via the equipartition theorem. In our case, we assume that the mass of the system is contained within a sphere of radius R. However, when a minimal length is introduced considering the effects of GUP on statistical mechanics, the equipartition theorem presents some corrections. To see this, we first notice that the GUP-modified density of states in three spatial dimensions reads as (Das and Fridman, 2021),

$$g(\varepsilon) = \frac{V(2m)^{3/2}\varepsilon^{1/2}}{4\pi^2\hbar^3}\left(1 - \frac{75}{4}\beta m\varepsilon\right)\ , \tag{4}$$

where V is the volume of the system and ε is the single particle energy. Since the results in Das and Fridman, (2021) are valid up to order β, the density of states presented previously, and its consequences are understood to hold up to the same order. We anticipate that the volume V will cancel out in our considerations and does not affect the results. Furthermore, it is worth mentioning that GUP is not expected to modify the value of geometrical quantities such as volumes or areas. Since GUP is a phenomenological model of quantum mechanics including a minimal uncertainty in position, GUP only affects the precision with which particles are localized, and therefore the precision with which geometrical quantities are determined, not their actual value. Returning to the expression mentioned previously, it is worth noticing that it reduces to the standard density of states for $\beta \to 0$, and that such a result is quantum in nature since it is based on the quantum energy spectrum of a particle in a box. In the classical limit $\varepsilon - \mu \gg T$, where μ is the chemical potential, and assuming no particles are added or removed from the system, $\mu = 0$, the Bose–Einstein and Fermi–Dirac distributions reduce to

$$f_{\text{BE,FD}}(\varepsilon) = \frac{1}{\exp\left(\frac{\varepsilon-\mu}{k_BT}\right) \mp 1} \approx f(\varepsilon) = \exp\left(-\frac{\varepsilon}{k_BT}\right)\ , \tag{5}$$

where the − and the + signs refer to the Bose–Einstein and Fermi–Dirac distributions, respectively. The limit on the right hand side of Eq. 5 is the Maxwell–Boltzmann distribution.

To proceed further, we compute the number of particles in the system by considering the following ensemble average using the GUP-modified density of states from Eq. 4 and the classical limit for the particle distribution in Eq. 5. We then obtain

$$\begin{aligned} N = \int_0^\infty g(\varepsilon)\, f_{\text{BE,FD}}(\varepsilon)\, d\varepsilon \simeq \frac{V(2m)^{3/2}}{8\pi^{3/2}\hbar^3}(k_BT)^{3/2} \\ \times\left[1 - \frac{225}{8}\beta m(k_BT)\right]\ , \end{aligned} \tag{6}$$

where the additional term with β represents the GUP correction to the number of particles in the system, given the temperature T of the system and the mass of the constituent particles m. The energy of the system is obtained in a similar manner as the number of particles in the system from Eq. 6. In this case, we find

$$\begin{aligned} E = \int_0^\infty \varepsilon\, g(\varepsilon)\, f_{\text{BE,FD}}(\varepsilon)\, d\varepsilon \simeq \frac{3V(2m)^{3/2}}{16\pi^{3/2}\hbar^3}(k_BT)^{5/2} \\ \times\left[1 - \frac{375}{8}\beta m(k_BT)\right]\ . \end{aligned} \tag{7}$$

The aforementioned formula represents the thermal energy of a system in three spatial dimensions. We can recast the expression for the thermal energy in a more familiar form by combining Eqs 6, 7 to obtain the GUP-modified equipartition theorem, which reads as

$$E = \frac{n_s}{2}Nk_BT\left[1 - \frac{75}{4}\beta m(k_BT)\right]\ , \tag{8}$$

in n_s-spatial dimensions. We are going to use this expression with $n_s = 1$, since the only relevant spatial degree of freedom in the system contributing to the entropic force is the radial one. We can then find an expression of the temperature T as a function of the energy E up to the first order in β, that is,

$$T \sim \frac{2E}{k_BN} + \beta\frac{75mE^2}{k_BN^2}\ . \tag{9}$$

Since this expression is derived from a quadratic equation, two solutions in principle are allowed. However, only the solution with the minus sign is considered since it is the only one which returns the standard case as $\beta \to 0$. For the case of the entropic force, the energy in Eq. 9 is not simply the thermal energy of particles in a given volume but the total energy of the system in that volume.

3 GUP-modified law of gravity

As shown in Verlinde (2011), one can derive Newton's law of gravity as an entropic force. The same procedure is applied here, with the difference that one includes GUP corrections everywhere they apply. A similar consideration has been discussed in Sheykhi (2020), where a Tsallis entropy modification to the Bekenstein–Hawking (BH) entropy has been used to derive the modified law of gravity. Such an entropy can be modified by GUP through the holographic principle as well (Awad and Ali, 2014; Das et al., 2022). It can then be written as

$$S = \frac{c^3 k_B}{8\hbar G}\left[A + \sqrt{A^2 - \beta^* A} - \frac{\beta^*}{2} \times \ln\left(1 - \frac{2}{\beta^*}\left(A + \sqrt{A^2 - \beta^* A}\right)\right)\right], \quad (10)$$

where A is the area of the horizon and $\beta* = 12\pi\hbar^2\beta$. Such a modification implies a deformation of the temperature of the system. To see this, first, we notice that the total number N of bits of information on the surface of the holographic horizon can be expressed in terms of the horizon entropy as follows (Verlinde, 2011):

$$N = \frac{4S}{k_B}\,. \quad (11)$$

Thus, substituting Eq. 11 and Eq. 10 in Eq. 9, we get

$$T = \frac{2\hbar GE}{k_B c^3 A} + \beta_0\left[\frac{6\pi\hbar^3 GE}{k_B M_p^2 c^5 A^2}\left(1 + \ln\left(2 - \frac{M_p^2 c^2 A}{3\pi\hbar^2\beta_0}\right)\right) + \frac{75\hbar^2 G^2 mE^2}{k_B M_p^2 c^8 A^2}\right]. \quad (12)$$

It is worth noticing that the temperature T depends on the area A of the holographic horizon and the total energy E of the system. These quantities are related to the radius R of the horizon and the mass M contained within that radius through $A = 4\pi R^2$ and $E = Mc^2$, respectively. The temperature from Eq. 12, expressed in terms of R and M, and the aforementioned Δx and ΔS are substituted in Eq. 3, where the GUP-modified Compton wavelength with $\eta = 1 + \beta_0 \frac{m^2}{M_p^2}$ (Carr et al., 2020) is considered. Up to leading orders in β_0, we obtain the GUP-modified law of gravity, which is

$$F = \frac{GmM}{R^2} + \beta_0\left[\frac{3\hbar^2 GmM}{4M_p^2 c^2 R^4}\left(1 + \ln\left(2 - \frac{4M_p^2 c^2 R^2}{3\hbar^2\beta_0}\right)\right) + \frac{75\hbar G^2 m^2 M^2}{8M_p^2 c^3 R^4} - \frac{Gm^3 M}{M_p^2 R^2}\right]. \quad (13)$$

Using the expression for the centripetal force $F = m\frac{v^2}{R}$ at radius R, we can then find the GUP-modified rotational velocity, which is

$$v = \sqrt{\frac{GM}{R} + \beta_0\left[\frac{3\hbar^2 GM}{4M_p^2 c^2 R^3}\left(1 + \ln\left(2 - \frac{4M_p^2 c^2 R^2}{3\hbar^2\beta_0}\right)\right) + \frac{75\hbar G^2 mM^2}{8M_p^2 c^3 R^3} - \frac{Gm^2 M}{M_p^2 R}\right]}\,. \quad (14)$$

The test particle, of mass m, can in principle be anything, from a subatomic particle to a large star. However, when a composite system is considered, GUP effects are reduced by the number n of constituents (Amelino-Camelia, 2013). In the present case, this amounts to replacing the modification parameter β_0 by the reduced parameter scaling with the square of the inverse number of constituents, that is, $\beta_0 \to \beta_0/n^2$. It should be noted that n corresponds to the number of constituent particles of the test mass which is different from N, introduced in Eq. 6, which corresponds to the number of bits of information on the holographic screen.

The last term in Eq. 13 that dominates at galactic distances, compared to other correction terms, can be easily verified. Therefore, Eq. 13 can be rewritten as

$$F \simeq \frac{Gm_{eff}M}{R^2}\,, \quad (15)$$

where we have defined an effective gravitational mass $m_{eff} = m[1 - \frac{\beta_0}{n^2}\frac{m^2}{M_p^2}]$. We notice that this implies a potential GUP-induced violation of the weak equivalence principle, since $m_{eff} \equiv m_g \neq m \equiv m_i$, where m_g is the gravitational mass and m_i is the inertial mass (see also Casadio and Scardigli, (2020)). Since in our framework $\beta_0 < 0$, we have $m_g > m_i$, which might partially justify the higher galaxy rotation velocities with respect to standard cosmological predictions.

From Eq. 14, the GUP-modified velocity for large distances is given as

$$v \simeq \sqrt{\frac{GM}{R}\left[1 - \frac{\beta_0}{n^2}\frac{m^2}{M_p^2}\right]} \quad \text{with} \quad \beta_0 < 0\,. \quad (16)$$

We point out here that other terms in Eqs 13, 14 dominate at smaller scales, where the GUP effects are significantly smaller. Furthermore, it is worth noticing that, due to the scaling of the modification parameter in Eq. 16, the mass of a star orbiting with velocity v is scaled by the number of fundamental constituents. Assuming that such constituents correspond to the elements in the plasma (mostly electrons and protons for a typical main sequence star), the quantity m/n is of the order of the proton mass regardless of the actual values of m and n. Such assumption will be considered in the following estimations.

It turns out that a distance dependence of the GUP parameter β_0 must be assumed to properly describe the flatness of rotational curves of galaxies. In fact, as can be noticed from Eq. 16, a constant GUP parameter simply shifts the orbital velocity at any given radial position by a constant factor. The assumption that the GUP parameter β_0 takes a distance dependence is compatible with the fact that similar effects are not observed at the solar system scale, at which Kepler's laws hold, while effects usually associated with DM tend to become relevant approaching galactic and intergalactic scales. This suggests that the GUP parameter β_0 must be distance-dependent since DM effects appear to be distance-dependent. Such an assumption is also supported by Ong (2018) and the difference between estimations of the quadratic GUP parameter in tabletop experiments, where $\beta_0 > 0$ (Pikovski et al., 2012; Bosso et al., 2017; Scardigli et al., 2017; Kumar and Plenio, 2018; Das and Fridman, 2021), and in astrophysical/cosmological observations, where $\beta_0 < 0$ (Jizba et al., 2010; Ong, 2018; Buoninfante et al., 2019; Nenmeli

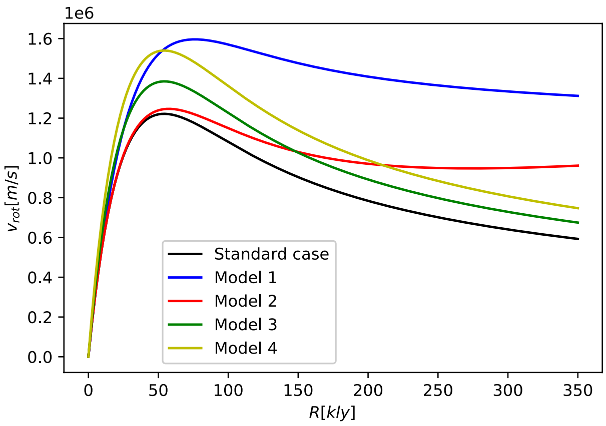

FIGURE 1
Galactic rotational curves for different models of R-dependence of β_0.

et al., 2021; Das et al., 2022; Jizba et al., 2022). We propose different models of a distance-dependent β_0:

- Model 1: $\beta_0(R) = \gamma \frac{R}{R*}$,
- Model 2: $\beta_0(R) = \gamma \frac{R^2}{R^{*2}}$,
- Model 3: $\beta_0(R) = \gamma \ln(1 + \frac{R}{R*})$, and
- Model 4: $\beta_0(R) = \gamma \frac{2}{\pi} \arctan(\frac{R}{R*})$,

where $\gamma < 0$ is a constant parameter and $R* \sim 1$ly is the scale at which effects, associated with DM, become significant. We consider a toy model galaxy with the following matter distribution (Freeman, 1970):

$$\rho(R) = \rho_0 e^{-\frac{R}{R_d}}, \tag{17}$$

where we chose $\rho_0 = 2 \times 10^{-19}$ kg/m^3 for the central density and $R_d = 16{,}000$ ly for the galaxy scale parameter. We use the matter distribution from Eq. 17 to obtain the mass of the galaxy within a certain radius $M(R)$, used in Eq. 16, to obtain Figure 1.

From Figure 1, we can see that the model which best describes the flatness of rotational curves is model 1, since the Compton correction term dominates at large distances, and the linear model 1 renders it constant. As for the natural logarithm and arc tangent models, they require a much higher peak for the rotational velocities in order to explain the flatness of the curves. Since observations of rotational curves of galaxies show no significant discrepancy from standard Newton's theory up to about the peak of the curve, such models are not able to fit the observations. The quadratic model 2 can potentially constitute a good description for a different choice of the parameters γ and $R*$. Since models 2 and 4 do not describe the DM effects satisfactorily, we are left with models 1 and 2. For these models, we can consider the two different parameters γ and $R*$ as only one parameter $\gamma/R*$ and γ/R^{*2} for models 1 and 2, respectively. The values for these parameters, which were used to obtain the aforementioned graphs, are $\gamma/R* = -1.9, \times, 10^{33}$ ly^{-1} and $\gamma/R^{*2} = -3.6 \times 10^{27}$ ly^{-2} for models 1 and 2, respectively. These values constitute upper bounds for such parameters, namely, $|\gamma/R*| \leq 1.9 \times 10^{33}$ ly^{-1} and $|\gamma/R^{*2}| \leq 3.6 \times 10^{27}$ ly^{-2}, respectively.

4 Conclusion

Newton's law of gravity can be derived as an entropic force through the holographic principle (Verlinde, 2011). In the present work, we have revised the derivation considering the influence of GUP. Specifically, we have considered the influence of GUP on the temperature T in the equipartition theorem, the Bekenstein–Hawking entropy, and the Compton wavelength. The GUP-corrected law of gravity has then been used to provide an explanation for the flatness of the rotational curves of galaxies. Specifically, alongside the proposed DM content in galaxies, we proposed that GUP effects can contribute to the observed shape of rotational curves. In the case that the GUP parameter β_0 remains constant, the rotational curves of galaxies only get magnified by a constant factor. Therefore, for GUP to effectively influence the rotational curves, we argued that the GUP parameter must be distance-dependent. This claim is directly supported by the work from Ong (2018) and indirectly by a comparison of positive bounds of quadratic GUP parameters estimated from laboratory experiments (Pikovski et al., 2012; Bosso et al., 2017; Scardigli et al., 2017; Kumar and Plenio, 2018; Das and Fridman, 2021) and negative bounds estimated from astrophysical/cosmological observations (Jizba et al., 2010; Ong, 2018; Buoninfante et al., 2019; Nenmeli et al., 2021; Das et al., 2022; Jizba et al., 2022).

We have proposed different models concerning the distance dependence for the GUP parameter β_0 and introduced a new scale parameter $R*$ at which GUP effects start to contribute to the shape of the rotational curves of galaxies. Here, we note that the GUP length scale need not be of the order of the Planck length ℓ_p but can be any intermediate length scale $\sqrt{\beta_0}\ell_p$ between the electroweak and Planck scales, determined by the GUP parameter β_0. For the cases of models 1 and 2, we introduce parameters $\gamma/R*$ and γ/R^{*2}, respectively, since we cannot obtain bounds for γ and $R*$ separately. Models 3 and 4 were found to be inadequate to explain the observed DM effects, and there would also be no possibility to combine parameters γ and $R*$ in a similar fashion as for models 1 and 2.

Models 1 and 2 constrain the newly defined parameters to $|\gamma/R*| \leq 1.9 \times 10^{33}$ ly^{-1} and $|\gamma/R^{*2}| \leq 3.6 \times 10^{27}$ ly^{-2}, respectively. However, these models can explain the flatness of rotational curves of galaxies only up to an extent. For example, we notice that for model 1 the rotational velocities would remain constant for $R \to \infty$, while for model 2 they would diverge for $R \to \infty$ for any values of the parameters. Furthermore, the approximations used to obtain Eqs 12, 13 break down at sufficiently large *R*.

The contribution of GUP effects to the shape of rotational curves of galaxies should be determined, once more information on the exact nature of particle DM and its abundance in galaxies is known. Furthermore, the feature of a distance-dependent GUP parameter, leading toward a partial explanation of galactic rotational curves, can be considered a possible additional structure of models of quantum mechanics with a minimal length having astrophysical and cosmological consequences. Finally, it is worth exploring the correspondence between our results and those presented by Ong (2018), Gnatenko and Tkachuk, (2019), and Gnatenko and Tkachuk, (2020), which still exhibit the possibility of a mass-dependent GUP parameter.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Generalized uncertainty principle and burning stars

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Gamow's theory of the implications of quantum tunneling for star burning has two cornerstones: quantum mechanics and the equipartition theorem. It has been proposed that both of these foundations are affected by the existence of a non-zero minimum length, which usually appears in quantum gravity scenarios and leads to the generalized uncertainty principle (GUP). Mathematically, in the framework of quantum mechanics, the effects of the GUP are considered as perturbation terms. Here, generalizing the de Broglie wavelength relation in the presence of a minimal length, GUP corrections to the Gamow temperature are calculated, and in parallel, an upper bound for the GUP parameter is estimated.

KEYWORDS
quantum gravity, minimal length, generalized uncertainty principle, Gamow theory, stellar formation

Introduction

In the first step of star burning, its constituents must overcome the Coulomb barrier to participate in nuclear fusion (NF). This means that when the primary gas ingredients have mass m and velocity v, then using the equipartition theorem, one gets

$$\frac{1}{2}mv^2 = \frac{3}{2}K_B T \geq U_c(r_0), \tag{1}$$

where K_B denotes the Boltzmann constant, the subscript c in $U_c(r_0)$ indicates the Coulomb potential, and correspondingly, $U_c(r_0) = \frac{Z_i Z_j e^2}{r_0}$ denotes the maximum of the Coulomb potential between the ith and jth particles located at a distance r_0 from each other (Prialnik, 2000). In this article, Kelvin (K) is the temperature unit. Finally, we reach

$$T \geq \frac{2Z_i Z_j e^2}{3K_B r_0} \simeq 1\cdot 1\times 10^{10}\frac{Z_i Z_j}{r_0}, \tag{2}$$

for the temperature required to overcome the Coulomb barrier. Therefore, NF happens whenever the temperature of the primary gas is comparable to Eq. 2, which clearly shows that, for the heavier nuclei, NF happens at higher temperatures. On the contrary, for the temperature of gas with mass M and radius R, we have (Prialnik, 2000)

$$\mathcal{T} \approx 4\times 10^6\left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right), \tag{3}$$

TABLE 1 Some bounds on the GUP parameter β_0.

Measurement/experiment	β_0	Refs.
Modified mass-temperature relation	10^{78}	Scardigli and Casadio (2015)
Light deflection	10^{78}	Scardigli and Casadio (2015)
Pulsar PRS B 1913 + 16 data	10^{71}	Scardigli and Casadio (2015)
Solar system data	10^{69}	Scardigli and Casadio (2015)
GW150914	10^{60}	Feng et al. (2017)
Dresselhaus interaction	10^{51}	Aghababaei et al. (2021)
Landau levels	10^{50}	Das and Vagenas (2008)
Sagnac effect	10^{49}	Feleppa et al. (2021)
Rashba effect	10^{46}	Aghababaei et al. (2021)

where $M_\odot$ and $R_\odot$ are the Sun mass and radius, respectively. Clearly, $\mathcal{T}$ and T are far from each other, meaning that NF cannot cause star burning (Prialnik, 2000). Therefore, NF occurs if a process reduces the required temperature (2). In fact, we need a process that decreases Eq. 2 to the values comparable to Eq. 3. Quantum tunneling lets particles pass through the Coulomb barrier, which finally triggers star burning, meaning that quantum tunneling allows NF to occur at temperatures lower than T (Prialnik, 2000). Indeed, if the distance between particles (r_0) becomes of the order of their de Broglie wavelength ($r_0 \simeq \frac{\hbar}{p} \equiv \lambda_Q$ where Q implies that we are in the purely quantum mechanical regime), then quantum tunneling happens and simple calculations lead to (Prialnik, 2000)

$$T \geq \frac{2Z_iZ_je^2}{3K_B\lambda_Q} \simeq 9 \cdot 6 \times 10^6 Z_i^2 Z_j^2 \left(\frac{m}{\frac{1}{2}}\right) \equiv \mathrm{T}, \quad (4)$$

instead of Eq. 2 for the temperature required to launch star burning. λ_Q can also be obtained by solving $\frac{p^2}{2m} = U_c(r_0)|_{r_0=\lambda_Q}$ which gives (Prialnik, 2000)

$$\lambda_Q = \frac{\hbar^2}{2mZ_iZ_je^2}, \quad (5)$$

meaning that quantum tunneling provides a platform for NF in stars (Prialnik, 2000). As an example, for hydrogen atoms, one can see that quantum tunneling leads to $\mathrm{T} \simeq 9 \cdot 6 \times 10^6\ K$ (comparable to (3)) as the Gamow temperature at which NF is underway. Based on the above argument, it is expected that any change in p affects λ_Q and, thus, these results.

It is also useful to mention here that the quantum tunneling theory allows the above process because the tunneling probability is not zero. Indeed, quantum tunneling is also the backbone of Gamow's theory of the α decay process (Gamow, 1928). Relying on the inversion of the Gamow formula for α decay, which gives the transmission coefficient, a method has also been proposed for studying the inverse problem of Hawking radiation (Völkel et al., 2019).

The backbone of quantum mechanics is the Heisenberg uncertainty principle (HUP),

$$\Delta x \Delta p \geq \frac{\hbar}{2}, \quad (6)$$

where x and p are ordinary canonical coordinates satisfying $[x_i, p_j] = i\hbar\delta_{ij}$. It has been proposed that, in quantum gravity scenarios, the HUP is modified such that (Kempf et al., 1995; Kempf, 1996)

$$\Delta X \Delta P \geq \frac{\hbar}{2}\left(1 + \frac{\beta_0 l_p^2}{\hbar^2}(\Delta P)^2\right), \quad (7)$$

called the GUP, where l_p denotes the Planck length and β_0 is the dimensionless GUP parameter. X and P are called generalized coordinates, and we work in a framework in which $X_i = x_i$, and up to the first order, we have $P_i = p_i(1 + \frac{\beta_0 l_p^2}{3\hbar^2}p^2)$ and $[X_i, P_j] = i\hbar(1 + \frac{\beta_0 l_p^2}{\hbar^2}P^2)\delta_{ij}$ (Das and Vagenas, 2008; Motlaq and Pedram, 2014). Moreover, the GUP implies that there is a non-zero minimum length $(\Delta X)_{\min} = \sqrt{\beta_0}\ l_p$. Indeed, the existence of a non-zero minimum length also emerges even when the gravitational regime is Newtonian (Mead, 1964), a common result with quantum gravity scenarios (Hossenfelder, 2013). More studies on quantum gravity can be traced to earlier studies (Lake et al., 2019; Lake et al., 2020; Lake, 2021; Lake, 2022). There have been various attempts to estimate the maximum possible upper bound on β_0 (Zhu et al., 2009; Chemissany et al., 2011; Das and Mann, 2011; Sprenger et al., 2011; Pikovski et al., 2012; Husain et al., 2013; Ghosh, 2014; Jalalzadeh et al., 2014; Scardigli and Casadio, 2015; Bosso et al., 2017; Feng et al., 2017; Gecim and Sucu, 2017; Bushev et al., 2019; Luciano and Petruzziello, 2019; Park, 2020; Aghababaei et al., 2021; Feleppa et al., 2021; Mohammadi Sabet et al., 2021), and among them, it seems that the maximum estimation for the upper bound is of the order of 10^{78} (Scardigli and Casadio, 2015). The implications of GUP on stellar evolution (Moradpour et al., 2019; Shababi and Ourabah, 2020) and the thermodynamics of various gases (Chang et al., 2002; Fityo, 2008; Wang et al., 2010; Hossenfelder, 2013; Motlaq and Pedram, 2014; Moradpour et al., 2021) have also been studied.

Indeed, the existence of a minimal length leads to the emergence of the GUP (Hossenfelder, 2013), and it affects thermodynamics (Chang et al., 2002; Fityo, 2008; Wang et al., 2010; Hossenfelder, 2013; Motlaq and Pedram, 2014; Moradpour et al., 2021) and quantum mechanics (Kempf et al., 1995; Kempf, 1996), as P can be expanded as a function of p. This letter deals with the GUP effects on star burning facilitated by quantum tunneling. Loosely speaking, we investigate the effects of a minimal length on T (the Gamow temperature).

GUP corrections to the tunneling temperature

To proceed further and in the presence of the quantum features of gravity, we introduce the generalized de Broglie wavelength as

$$\lambda_{\text{GUP}} \equiv \frac{\hbar}{P}. \quad (8)$$

It is obvious that, as $\beta_0 \to 0$, one obtains $P \to p$ and thus $\lambda_{\text{GUP}} \to \lambda_Q$, which is the quantum mechanical result. Indeed, up to first order in β_0, we have $\lambda_{\text{GUP}} = \lambda_Q (1 - \frac{\beta_0 l_p^2}{3\lambda_Q^2})$, and the thermal energy per particle with temperature T is (Motlaq and Pedram, 2014)

$$\langle K \rangle = \langle \frac{P^2}{2m} \rangle = \frac{3}{2} K_B T - 3 \frac{\beta_0 l_p^2}{\hbar^2} m K_B^2 T^2. \quad (9)$$

Mathematically, one should find the corresponding de Broglie wavelength by solving the following equation:

$$\frac{P^2}{2m} = U_c(r_0)\Big|_{r_0=\lambda_{\text{GUP}}}. \quad (10)$$

Inserting the result into

$$\frac{3}{2} K_B T - 3 \frac{\beta_0 l_p^2}{\hbar^2} m K_B^2 T^2 \geq U_c(r_0)\Big|_{r_0=\lambda_{\text{GUP}}}, \quad (11)$$

one can finally find the GUP corrected version of Eq. 4.

Now, inserting λ_{GUP} into Eq. 10 and then combining the results with Eq. 11, we find

$$T^{\pm}_{\text{GUP}} = \frac{\hbar^2 \left(1 \pm \sqrt{1 - 8\beta_0 l_p^2 m K_B \mathrm{T}/\hbar^2}\right)}{4\beta_0 K_B l_p^2 m}. \quad (12)$$

in which Eq. 4 has been used for simplification. To estimate the magnitude of $l_p^2 m K_B \mathrm{T}/\hbar^2$, we consider the hydrogen atom for which $m \sim 10^{-27}\, kg$. Now, since $l_p \sim 10^{-35}\, m$, $K_B \sim 10^{-23}\, \frac{m^2 kg}{s^2 K}$, $\hbar \sim 10^{-34}\, \frac{m^2 kg}{s}$, and $\mathrm{T} \sim 10^6\, K$, one easily finds $l_p^2 m K_B \mathrm{T}/\hbar^2 \sim 10^{-46}$. Moreover, because the effects of the GUP in the quantum mechanical regime are small (Hossenfelder, 2013), a reasonable basic assumption could be that $\beta_0 l_p^2 m K_B \mathrm{T}/\hbar^2 \ll 1$. Indeed, if $\beta_0 \ll 10^{46}$, then we always have $\beta_0 l_p^2 m K_B \mathrm{T}/\hbar^2 \ll 1$ meaning that *i)* we can Taylor expand our results and *ii)* 10^{46} is an upper bound for β_0, which is comparable to those found in previous works (Das and Vagenas, 2008; Scardigli and Casadio, 2015; Feng et al., 2017; Aghababaei et al., 2021; Feleppa et al., 2021) summarized in Table 1.

Expanding the above solutions (12) and bearing in mind that the true solution should recover T at $\beta = 0$, one can easily find that T^-_{GUP} is the proper solution leading to

$$T^-_{\text{GUP}} = \mathrm{T}\left(1 + 2\beta_0 \left(\frac{l_p^2 m K_B \mathrm{T}}{\hbar^2}\right)\right). \quad (13)$$

up to first order in β_0. Hence, because it seems that β_0 is positive (Das and Vagenas, 2008; Scardigli and Casadio, 2015; Feng et al., 2017; Aghababaei et al., 2021; Feleppa et al., 2021), one can conclude that $\mathrm{T} < T^-_{\text{GUP}}$.

Conclusion

Motivated by the GUP proposal and the vital role of the HUP in quantum mechanics and, thus, the quantum tunneling process that facilitates star burning, we studied the effects of the GUP on the Gamow temperature. In order to determine this, the GUP modification to the de Broglie wavelength was addressed, which finally helped us to find the GUP correction to the Gamow temperature and also estimate an upper bound for β_0 (10^{46}), which agrees well with those found in previous works (Das and Vagenas, 2008; Scardigli and Casadio, 2015; Feng et al., 2017; Aghababaei et al., 2021; Feleppa et al., 2021).

Finally, based on the obtained results, it may be expected that the GUP also affects the transmission coefficients (Gamow's formula) (Gamow, 1928; Hossenfelder, 2013; Völkel et al., 2019), meaning that the method of Völkel et al. (2019) will also be affected. This is an interesting topic for future study because Hawking radiation is a fascinating issue in black hole physics (Wald, 2001).

Data availability statement

The original contributions presented in the study are included in the article/Supplementary material. Further inquiries can be directed to the corresponding author.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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The Generalized Uncertainty Principle and higher dimensions: Linking black holes and elementary particles

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Black holes play an important role in linking microphysics with macrophysics, with those of the Planck mass ($M_P \sim 10^{-5}$ g) featuring in any theory of quantum gravity. In particular, the Compton-Schwarzschild correspondence posits a smooth transition between the Compton wavelength ($R_C \propto 1/M$) below the Planck mass and the Schwarzschild radius ($R_S \propto M$) above it. The duality between R_C and R_S implies a form of the Generalized Uncertainty Principle (GUP) and suggests that elementary particles may be sub-Planckian black holes. The simplest possibility is that the ADM mass has the form $M + \beta M_P^2/M$ for some constant β and this model can be extended to charged and rotating black holes, clearly relevant to elementary particles. Another possibility is that sub-Planckian black holes may arise in loop quantum gravity and this explicitly links black holes and elementary particles. Higher dimensions may modify both proposals. If there are n extra dimensions, all with the same compactification scale, one expects $R_S \propto M^{1/(1+n)}$ below this scale but R_C depends on the form of the higher-dimensional wave-function. If it is spherically symmetric, then $R_C \propto M^{-1}$, so duality is broken and the Planck mass is reduced, allowing the possibility of TeV quantum gravity. If the wave-function is pancaked in the extra dimensions, $R_C \propto M^{-1/(1+n)}$ and so duality is preserved but the Planck mass is unchanged.

KEYWORDS

Generalized Uncertainty Principle, Compton-Schwarzschild correspondence, black holes, higher dimensions, elementary particles

1 Introduction

Whatever final theory amalgamates relativity theory and quantum mechanics, it is likely to involve two features: 1) what is termed the Black Hole Uncertainty Principle (BHUP) correspondence; and 2) the existence of extra dimensions on sufficiently small scales. Both features are expected to become important at the Planck length, $R_P \sim 10^{-33}$ cm, and possibly on much larger scales. It is striking that black holes are involved in both these features and indeed there are many other ways in which these objects provide a link between macrophysics and microphysics (Carr, 2018).

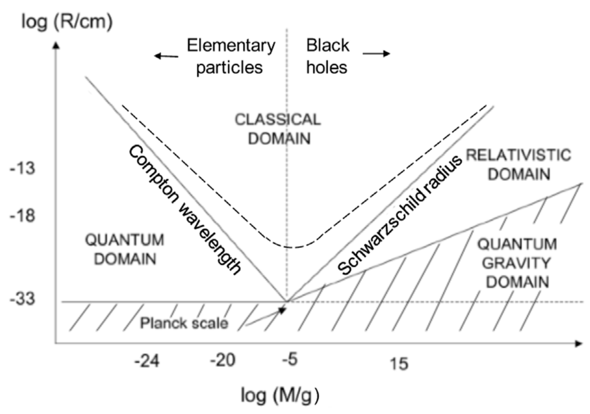

FIGURE 1
Division of (*M*, *R*) diagram into classical, quantum, relativistic and quantum gravity domains. The boundaries are specified by the Planck density, Compton wavelength and Schwarzschild radius. From Carr (2018).

As regards feature (1), the duality under the transformation $M \to M_P^2/M$ between the Compton wavelength for a particle of mass M, $R_C = \hbar/(Mc)$, and the Schwarzschild radius for a black hole of mass M, $R_S = 2GM/c^2$, suggests a unified Compton-Schwarzschild expression with a smooth minimum in the (M, R) plane. This implies that elementary particles may in some sense be sub-Planckian black holes. This proposal goes back to the 1970s, when it was motivated in the context of strong gravity theories.

As regards feature (2), if there are n extra spatial dimensions compactified on some scale R_E, then R_S scales as $R^{1/(1+n)}$ for $R < R_E$, leading to the possibility of TeV quantum gravity and black hole production at accelerators if R_C scales as M^{-1} for $R < R_E$. However, the higher-dimensional Compton wavelength depends on the form of the $(3 + n)$-dimensional wavefunction and in some circumstances one might expect $R_C \propto M^{-1/(1+n)}$ for $R < R_E$. This preserves the duality between R_C and R_S but TeV quantum gravity is precluded. Nevertheless, the extra dimensions could still have consequences for the detectability of black hole evaporations and the enhancement of pair-production at accelerators on scales below R_E.

The plan of this paper is as follows. Section 2 discusses the BHUP correspondence in general terms. Section 3 applies the correspondence to black holes in Loop Quantum Gravity (LQG), this being the first historical study of this kind. Section 4 then considers the simplest application of the BHUP correspondence: the '$M + 1/M$' Schwarzschild model and its extension to charged and rotating black holes. Higher-dimensional black holes are discussed in Section 5 and some concluding remarks about the connection between particles and black holes are made in Section 6.

2 The Black Hole Uncertainty Principle correspondence

A key feature of the microscopic domain is the (reduced) Compton wavelength for a particle of rest mass M, with the region $R < R_C$ in the (M, R) diagram of Figure 1 being regarded as the "quantum domain". A key feature of the macroscopic domain is the Schwarzschild radius for a body of mass M, with the region $R < R_S$ being regarded as the "relativistic domain". The Compton and Schwarzschild lines intersect at around the Planck scales,

$$R_P = \sqrt{\hbar G/c^3} \sim 10^{-33}\text{cm}, \quad M_P = \sqrt{\hbar c/G} \sim 10^{-5}\text{g}, \tag{1}$$

and divide the (M, R) diagram into three regimes, which we label quantum, relativistic and classical. There are several other interesting lines in the figure. The vertical line $M = M_P$ marks the division between elementary particles ($M < M_P$) and black holes ($M > M_P$), since the size of a black hole is usually required to be larger than the Compton wavelength associated with its mass. The horizontal line $R = R_P$ is significant because quantum fluctuations in the metric should become important below this (Wheeler, 1955). Quantum gravity effects should also be important whenever the density exceeds the Planck value, $\rho_P = c^5/(G^2\hbar) \sim 10^{94}\,\text{g cm}^{-3}$, corresponding to the sorts of curvature singularities associated with the big bang or the centres of black holes. This implies $R < R_P (M/M_P)^{1/3}$, which is well above the $R = R_P$ line for $M \gg M_P$, so the shaded region specifies the 'quantum gravity' domain.

The Compton and Schwarzschild lines transform into one another under the transformation $M \to M_P^2/M$, which suggests some connection between elementary particles and black holes. This relates to what is termed "T-duality" in string theory and maps momentum-carrying states to winding states (Zwiebach, 2009). Although the Compton and Schwarzschild boundaries correspond to straight lines in the logarithmic plot of Figure 1, this form presumably breaks down near the Planck point due to quantum gravity effects. One might envisage two possibilities: either there is some form of critical point at the Planck scale, so that the separation between particles and black holes is maintained (Isi et al., 2013), or there is a smooth minimum, as indicated by the broken line in Figure 1, so that the Compton and Schwarzschild lines merge (Carr et al., 2016). Which alternative applies has important implications for the relationship between elementary particles and black holes.

One way of smoothing the transition between the Compton and Schwarzschild lines is to invoke some form of unified expression which asymptotes to the Compton wavelength and Schwarzschild radius in the appropriate regimes (Carr, 2015). The simplest such expression would be

$$R_{CS} = \frac{\beta\hbar}{Mc} + \frac{2GM}{c^2}, \tag{2}$$

where β is a dimensionless constant. In the super-Planckian regime, this becomes

$$R'_S = \frac{2GM}{c^2}\left[1 + \frac{\beta}{2}\left(\frac{M_P}{M}\right)^2\right] \quad (M \gg M_P), \tag{3}$$

with the second term corresponding to a small correction to the usual Schwarzschild expression. In the sub-Planckian regime, it becomes

$$R'_C = \frac{\beta\hbar}{Mc}\left[1 + \frac{2}{\beta}\left(\frac{M}{M_P}\right)^2\right] \quad (M \ll M_P), \tag{4}$$

with the second term corresponding to a small correction to the usual expression for the Compton wavelength. More generally, one might consider any unified expression $R'_C(M) \equiv R'_S(M)$ which has the asymptotic behaviour $\beta\hbar/(Mc)$ for $M \ll M_P$ and $2GM/c^2$ for $M \gg M_P$.

An expression of the form (3) arises in the quantum N-portrait model of Dvali *et al.* (Dvali et al., 2011), which regards a black hole as a weakly-coupled Bose-Einstein condensate of gravitons. From holographic considerations, the number of gravitons in the black hole is $N \approx M^2/M_P^2$ and one can then argue that the black hole radius is (Frassino et al., 2016)

$$R_{CS} \approx \frac{2GM}{c^2}\left(1 + \frac{\beta}{2N}\right) \quad (M > M_P), \tag{5}$$

which is equivalent to Eq. 3. An expression of the form (4) also arises in the context of the Generalized Uncertainty Principle (GUP). This is because it can be argued that the Uncertainty Principle should be modified to the form (Adler, 2010)

$$\Delta x = \frac{\hbar}{\Delta p} + \alpha\frac{R_P^2\,\Delta p}{\hbar}, \tag{6}$$

where α is a dimensionless constant. The first term represents the uncertainty in the position due to the momentum of the probing photon and leads to the usual expression for the Compton wavelength if one substitutes $\Delta x \to R$ and $\Delta p \to cM$. The second term represents the gravitational effect of the probing photon and is much smaller than the first term for $\Delta p \ll cM_P$. Variants of Eq. 6 are also motivated by string theory (Veneziano, 1986; Witten, 1996), non-commutative quantum mechanics (Gross and Mende, 1988; Amati et al., 1989; Yoneya, 1989; Konishi et al., 1990; Scardigli, 1999), general minimum length considerations (Maggiore, 1993a; Maggiore, 1993b; Maggiore, 1994), polymer corrections in the structure of spacetime in LQG (Ashtekar et al., 2003a; Hossain et al., 2010) and some approaches to quantum decoherence (Kay, 1998).

The GUP is usually restricted to the sub-Planckian domain ($M < M_P$). However, if we rewrite Eq. 6 using $\Delta x \to R$ and $\Delta p \to cM$ even in the super-Planckian regime, we obtain a revised Compton wavelength which applies for all M:

$$R_{CS} = \frac{\hbar}{Mc} + \alpha\frac{GM}{c^2}. \tag{7}$$

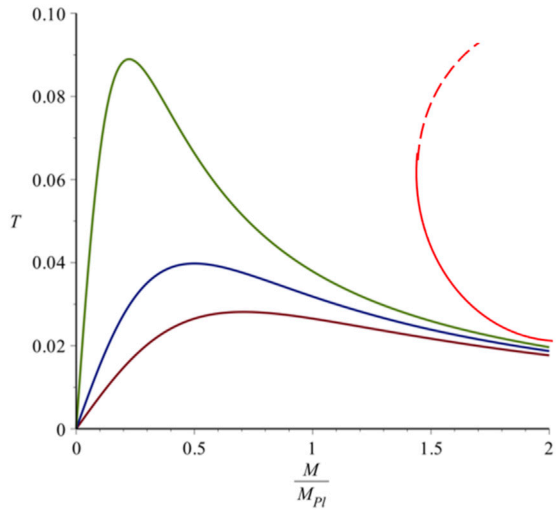

FIGURE 2
Hawking temperature (in Planck units) from Eq. 3 and surface gravity argument as a function of M/M_P for $\beta = 1$ (bottom), $\beta = 0.5$ (middle) and $\beta = 0.1$ (top). Also shown on the right is the Adler prediction. From Carr et al. (2016).

This resembles Eq. 2 except that the constant is associated with the second term. This suggests that there is a different kind of positional uncertainty for an object larger than the Planck mass, related to the size of a black hole. This is not unreasonable since the usual Compton wavelength is below the Planck length here and also an outside observer cannot localize an object on a scale smaller than its Schwarzschild radius. This is termed the Black Hole Uncertainty Principle (BHUP) correspondence (Carr, 2015) or the Compton-Schwarzschild correspondence when discussing an interpretation in terms of extended de Broglie relations (Lake and Carr, 2015).

Strictly speaking, Eqs 2, 7 are consistent only if $\alpha = 2$ and $\beta = 1$ but that would leave no free parameter at all. Therefore an interesting issue is whether one should associate the free constant in R_{CS} with the $1/M$ term, as in Eq. 2, or the M term, as in Eq. 7. Here we adopt the former approach, on the grounds that the expression for the Schwarschild radius is exact, whereas there is some ambiguity in the meaning of the Compton scale. However, for comparison with the GUP literature, we still need to identify an effective value of α and a simple rescaling of the relationship between Δx and R suggests $\alpha = 2/\beta$. Another approach is to identify Δp with $1/M$ rather than M for $M > M_P$ and Eq. 6 then equates α and β directly. One might even argue that Δp has the form $(M + 1/M)^{-1}$, in which case $\Delta x \sim 1/M$ and $\Delta p \sim M$ in the particle case ($M < M_P$) and $\Delta x \sim M$ and $\Delta p \sim 1/M$ in the black hole case ($M > M_P$). This would be consistent with the extended de Broglie relations (Hawking, 1974; Lake and Carr, 2015).

In the standard picture, one can calculate the black hole temperature from the Uncertainty Principle by identifying it with a multiple η of Δp. This gives (Hawking, 1974)

$$kT = \eta c\Delta p = \frac{\eta\hbar c}{\Delta x} = \frac{\eta c^2 M_P^2}{2M}, \tag{8}$$

which is precisely the Hawking temperature if we take $\eta = 1/(4\pi)$. If one adopts the GUP but assumes the usual black hole size, one obtains the Adler form (Chen and Adler, 2003)

$$kT = \frac{\eta Mc^2}{\alpha}\left(1 \pm \sqrt{1 - \frac{\alpha M_P^2}{M^2}}\right). \tag{9}$$

The negative sign gives a small perturbation to the standard Hawking temperature

$$kT \approx \frac{\eta M_P^2 c^2}{2M}\left[1 - \frac{\alpha M_P^2}{4M^2}\right] \quad (M \gg M_P) \tag{10}$$

at large M. However, the solution becomes complex when M falls below $\sqrt{\alpha}\, M_P$, corresponding to a minimum mass, and it then connects to the positive branch of Eq. 9. This form is indicated by the curve on the right of Figure 2.

Eq. 9 is inconsistent with the BHUP correspondence since this also modifies the relationship between the black hole radius Δx and M. If we adopt Eq. 3 instead, then the surface gravity argument gives a temperature

$$kT = \frac{M_P^2 c^2}{4\pi M\left(2 + \beta M_P^2/M^2\right)} \approx \begin{cases} \dfrac{M_P^2 c^2}{8\pi M}\left[1 - \dfrac{\beta}{2}\left(\dfrac{M_P}{M}\right)^2\right] & (M \gg M_P) \\ \dfrac{Mc^2}{4\pi\beta}\left[1 - \dfrac{2}{\beta}\left(\dfrac{M}{M_P}\right)^2\right] & (M \ll M_P). \end{cases} \tag{11}$$

This is plotted in Figure 2 and is very different from the Adler form. As M decreases, the temperature reaches a maximum of around T_P and then goes to zero as $M \to 0$.

An important caveat is that Eq. 6 assumes that the two uncertainties add linearly. On the other hand, since they are independent, it might be more natural to assume that they add quadratically (Carr et al., 2011):

$$\Delta x = \sqrt{(\hbar/\Delta p)^2 + (\alpha R_P^2 \Delta p/\hbar)^2}. \tag{12}$$

We refer to Eqs 6, 12 as the *linear* and *quadratic* forms of the GEP, respectively. The latter corresponds to a unified expression

$$R_{CS} = \sqrt{(\beta\hbar/Mc)^2 + (2GM/c^2)^2}, \tag{13}$$

where we have we have again introduced β. This leads to the approximations

$$R_S' \approx \frac{2GM}{c^2}\left[1 + \frac{\beta^2}{8}\left(\frac{M_P}{M}\right)^4\right] \quad (M \gg M_P) \tag{14}$$

and

$$R_C' \approx \frac{\beta\hbar}{Mc}\left[1 + \frac{2}{\beta^2}\left(\frac{M}{M_P}\right)^4\right] \quad (M \ll M_P). \tag{15}$$

These might be compared to the *exact* expressions in the linear case, given by Eqs 3, 4. As we now show, a model inspired by LQG permits the existence of a black hole whose horizon size has precisely the form (13).

3 Loop black holes

Loop Quantum Gravity is based on a canonical quantization of the Einstein equations, written in terms of the Ashtekar variables. One feature of this is that area is quantized, with its smallest possible value being

$$A_{\min} = 4\pi\sqrt{3}\,\gamma\, R_P^2, \tag{16}$$

where γ is the Immirzi parameter and of order 1. The quantity $a_o \equiv A_{\min}/8\pi$, together with the dimensionless polymeric parameter δ, determines the deviation from classical theory.

One version of LQG, using the mini-superspace approximation, gives rise to cosmological solutions which resolve the initial singularity problem (Bojowald, 2001; Ashtekar et al., 2003b; Bojowald, 2005). Another version gives the loop black hole (LBH) solution (Modesto, 2010) and this replaces the singularity in the Schwarzschild solution with another asymptotically flat region. The metric depends only on the combined dimensionless parameter $\varepsilon \equiv \delta\gamma$, which must be small, and can be expressed as

$$ds^2 = -G(r)c^2dt^2 + \frac{dr^2}{F(r)} + H(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{17}$$

with

$$G(r) = \frac{(r - r_+)(r - r_-)(r + r_*)^2}{r^4 + a_o^2}, \quad F(r) = \frac{(r - r_+)(r - r_-)r^4}{(r + r_*)^2\left(r^4 + a_o^2\right)}. \tag{18}$$

Here $r_+ = 2GM/c^2$ and $r_- = 2GMP^2/c^2$ are the outer and inner horizons, respectively, and $r_* \equiv \sqrt{r_+ r_-} = 2GMP/c^2$, where M is the black hole mass and

$$P \equiv \frac{\sqrt{1 + \varepsilon^2} - 1}{\sqrt{1 + \varepsilon^2} + 1} \approx \varepsilon^2/4 \ll 1 \tag{19}$$

is called the polymeric function. In the limit $r \to \infty$ one has

$$G(r) \to 1 - \frac{2G\mathcal{M}}{c^2 r}\left(1 - \varepsilon^2\right), \quad F(r) \to 1 - \frac{2G\mathcal{M}}{c^2 r}, \tag{20}$$

where $\mathcal{M} \equiv M(1 + P)^2$ is the ADM mass (i.e. the mass measured as $r \to \infty$). The function $H(r)$ in Eq. 17 is not r^2 (as in the Schwarzschild case) but

$$H(r) = r^2 + \frac{a_o^2}{r^2} \quad \Rightarrow \quad R \equiv \sqrt{r^2 + \frac{a_o^2}{r^2}}. \tag{21}$$

Here R is the *physical* radial coordinate, in the sense that the circumference function is $2\pi R$. As r decreases from infinity to zero, R first decreases from infinity to a minimum value of $\sqrt{2a_0}$ at $r = \sqrt{a_0}$ and then increases again to infinity. In particular, the value of R associated with the outer event horizon is

$$R_S' = \sqrt{\left(\frac{2GM}{c^2}\right)^2 + \left(\frac{a_o c^2}{2GM}\right)^2}. \tag{22}$$

This corresponds to Eq. 13 if $\beta = a_o c^2/G$. The important physical implication of Eq. 21 is that central singularity of the Schwarzschild solution is replaced with another asymptotic region, so the collapsing matter bounces and the black hole becomes part of a wormhole. The fact that a purely geometrical condition in LQG implies the quadratic version of the GUP suggests some deep connection between general relativity and quantum theory. The duality between the two asympotic spaces also suggests a link between elementary particles with $M \ll M_P$ and black holes with $M \gg M_P$ (Modesto and Premont-Schwarz, 2009), which is clearly relevant to the theme of this paper.

The temperature implied by the black hole's surface gravity is

$$T \propto \frac{GM}{R_S'^2} \propto \begin{cases} M^{-1} & (M \gg M_P) \\ M^3 & (M \ll M_P). \end{cases} \tag{23}$$

However, if one calculates the temperature using the GUP expression for Δp, one obtains

$$kT \approx \begin{cases} \dfrac{\eta \hbar c^3}{2GM}\left[1 - \dfrac{\beta^2}{8}\left(\dfrac{M_P}{M}\right)^4\right] & (M \gg M_P) \\ \dfrac{\eta M c^2}{\beta}\left[1 - \dfrac{2}{\beta^2}\left(\dfrac{M}{M_P}\right)^4\right] & (M \ll M_P). \end{cases} \tag{24}$$

This is similar to Eq. 11 but inconsistent with Eq. 23 in the sub-Planckian regime. The source of the discrepancy is that there are two asymptotic spaces—one on each side of the wormhole throat—and the temperature is different in these. Observer only detect radiation from the horizon on their side of the throat, so the inner horizon with respect to $r = \infty$ corresponds to the outer horizon with respect to $r = 0$ (Carr et al., 2011). For $M < \sqrt{\beta/2}\, M_P$, $T \propto M^3$ in our space and $T \propto M$ in the other space, which explains the predictions of Eqs 23, 24. For $\sqrt{\beta/2}\, M_P < M < P^{-2}\sqrt{\beta/2}\, M_P$, $T \propto M^{-1}$ in our space and $T \propto M$ in the other space. For $M > P^{-2}\sqrt{\beta/2}\, M_P$, $T \propto M^{-1}$ in our space and $T \propto M^{-3}$ in the other space.

4 Carr-Mureika-Nicolini approach

This section describes a particular interpretation of the linear version of the BHUP correspondence, described in my work with Mureika and Nicolini (Carr et al., 2016), in which the Arnowitt-Deser-Misner (ADM) mass is taken to be

$$M_{\text{ADM}} = M\left(1 + \frac{\beta}{2}\frac{M_P^2}{M^2}\right). \tag{25}$$

This is equivalent to Eq. 3 and we noted a possible connection with the energy-dependent metric in the "gravity's rainbow" proposal (Magueijo and Smolin, 2004) and with the QFT renormalization of mass in the presence of stochastic metric fluctuations (Camacho, 2003). Putting $\hbar = c = 1$, the Schwarzschild radius for the modified metric is

$$R_S' = \frac{2M_{\text{ADM}}}{M_P^2} \approx \begin{cases} 2M/M_P^2 & (M \gg M_P) \\ \beta/M & (M \ll M_P) \end{cases} \tag{26}$$

and the temperature is $kT = M_P^2/(8\pi M_{\text{ADM}})$, corresponding to Eq. 11. This is not the only black hole metric allowed by the GUP. This is illustrated by the discussion of LQG in Sec. 3 but there could be other relevant solutions in GR itself. However, Eq. 25 gives the simplest such solution.

The black hole luminosity in this model is $L = \xi^{-1} M_{\text{ADM}}^{-2}$ where $\xi \sim t_P/M_P^3$, so the mass loss rate decreases when M falls below M_P and the black hole never evaporates completely. There are two values of M for which the evaporation time ($\tau \sim M/L$) is comparable to the age of the Universe ($t_0 \sim 10^{17}$s). One is super-Planckian, $M_* \sim (t_0/\xi)^{1/3} \sim 10^{15}$g, this being the standard expression for the mass of a PBH evaporating at the present epoch. The other is sub-Planckian, $M_{**} \sim \beta^2\,(t_P/t_0)M_P \sim 10^{-65}\beta^2$ g, although the mass cannot actually reach this value at the present epoch because the black hole is cooler than the CMB temperature for $M < M_{\text{CMB}} \sim 10^{-36}\beta$ g. This leads to *effectively* stable relics of this mass.

It is interesting to consider observational constraints on the parameter β and these are discussed in Carr et al. (2022). Within the GUP context, these only arise in the microscopic domain and a variety of mechanical oscillator experiments imply $\alpha < 4 \times 10^4$ (Pikovski et al., 2012; Bushev et al., 2019). A similar bound arises from the AURIGA gravitational bar detector (Marin et al., 2014). Since $\beta = 2/\alpha$, both bounds corresponding to a *lower* limit $\beta > 10^{-4}$. Within the context of the BHUP correspondence, there are also constraints in the macroscopic domain from measuring the gravitational force between 100 mg masses with mm separation (Westphal et al., 2021) and these imply $\beta < 10^6$. Clearly these limits still allow a wide range of values for β. One might also constrain β by observations on astrophysical scales but in this domain the effects of the Extended Uncertainty Principle, in which $\Delta x\ \Delta p \sim 1 + (\delta x)^2$ rather than $1 + (\delta p)^2$, becomes more relevant (Mureika, 2019).

Recently we have extended this work, together with Heather Mentzer, to charged and rotating black holes (Carr et al., 2020), since this is clearly relevant to elementary particles. The standard Reissner-Nordström (RN) already exhibits features of the GUP-

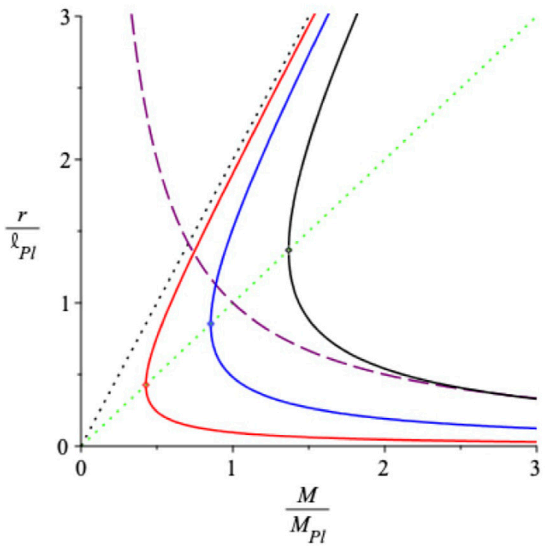

FIGURE 3
The solid curves show the outer and inner horizons for a standard RN black hole with n = 5, 11, 16 (left to right). For each n, the horizons meet at the extremal mass on the line $r = M/M_P^2$ (green dotted) and are bounded from above by the Schwarzschild radius $r_S = 2M/M_P^2$ (black dotted line). The Compton curve is shown by the dashed line and the inner horizon asymptotes to this for n = 16. Solutions with $11 < n < 16$ penetrate the sub-Planckian RN regime. From Carr et al. (2020).

modified Schwarzschild solution. This is because the RN metric has an outer (+) and inner (-) horizon at

$$r_\pm = \frac{M}{M_P^2}\left(1 \pm \sqrt{1 - \frac{\alpha_e M_P^2 n^2}{M^2}}\right) \approx \begin{cases} \frac{2M}{M_P^2}\left(1 - \frac{\gamma M_P^2}{M^2}\right) & (+) \\ \frac{2\gamma}{M}\left(1 + \frac{\gamma M_P^2}{M^2}\right) & (-) \end{cases} \tag{27}$$

where ne is the black hole charge, $\gamma \equiv \alpha_e n^2/4$ with $\alpha_e \approx 1/137$ being the electric fine structure constant, and the last expression applies for a black hole which is far from extremal ($M \gg \sqrt{\alpha_e}\, nM_P$). The form of the outer and inner horizons for different values of n are shown by the upper and lower parts of the solid curves in Figure 3, respectively. There are two asymptotic behaviors: the outer horizon correponds to Eq. 2 but with a negative value of β; the inner horizon resembles the Compton expression and it asymptotes to the Compton wavelength itself for n = 16, this being the integer part of $\sqrt{2/\alpha_e}$.

For each n, the two horizons merge on the line $r = GM$ (lower dotted curve) at the minimum value of M. This corresponds to a sequence of "extremal" solutions (shown by the dots in Figure 3) with a spectrum of masses $\sqrt{\alpha_e}\, nM_P$. For given n, there are no solutions with M less than this since these would correspond to naked singularities. In particular, n could be at most the integer part of $1/\sqrt{\alpha_e}$ (i.e. 11) for a Planck-mass black hole. As in the GUP case, the temperature of the RN solution reaches a maximum and then goes to zero as M tends to the limiting value $\sqrt{\alpha_e}\, nM_P$. One might want to associate elementary particles only with extremal solutions (since they are stable) but these states all have masses in the range $(0.1–1)M_P$, which is too large. Also even extremal black holes may discharge through the Schwinger mechanism (Schwinger, 1951).

The (standard) Compton line intersects the outer black hole horizon, as required if one wants a smooth connection between particles and black holes, at the mass

$$M = \frac{M_P}{\sqrt{2 - \alpha_e n^2}} \approx \frac{M_P}{\sqrt{2 - n^2/137}}. \tag{28}$$

(This is also termed the self-completeness condition (Isi et al., 2013).) For $n = 0$, the intersect is $M_P/\sqrt{2}$ but it increases with n and tends to M_P as $n \to \sqrt{137}$ (middle curve). This implies a constraint $n \le 11$ on the charge of a self-complete RN black hole. The Compton line still intersects the *inner* horizon for $\sqrt{137} < n < \sqrt{274}$, but these solutions penetrate the $r < R_P$ region.

One can extend this model to the GUP-modified RN solutions by replacing M with M_{ADM} given by Eq. 25. Providing $n < \sqrt{2\beta/\alpha_e}$ for fixed β, the outer horizon behaves as in the GUP-Schwarzschild case, with a continuous transition between the gravitational ($r_{CS} \propto M$) and Compton ($r_{CS} \propto M^{-1}$) scaling. Also, r_+ has a minimum and r_- has a maximum at

$$M = M_{crit} \equiv \sqrt{\beta/2}\, M_P, \quad r_\pm = \left[\sqrt{2\beta} \pm \sqrt{2\beta - n^2\alpha_e}\,\right] R_P. \tag{29}$$

This is indicated by the curves on the left of Figure 4. In principle, the particle-like black holes can have arbitrarily low mass in this case. However, it is unclear that these solutions are candidates for *stable* particles since none of them are extremal, this possibility arising only in the limit $n_{max} = \sqrt{2\beta/\alpha_e}$. For larger values of $\boldsymbol{n}$, the form of the solutions changes, as indicated by the curves on the right of Figure 4. These represent super-Planckian black holes on the right (similar to the standard RN case with an extremal solution at the smallest value of M) and sub-Planckian particles on the left, with a mass gap in between. Equivalently, for a given value of n, there is a critical value of $\beta = n^2\alpha_e/2$ below which the solutions bifurcate and become separated by a mass gap.

The Kerr metric exhibits similar behaviour but there is a critical spin ($n\hbar$) rather than a critical charge. The extremal case corresponds to the spectrum of masses $\sqrt{n}\, M_P$, while the self-completeness condition corresponds to

$$M = M_P\sqrt{\frac{1 + n^2}{2}}. \tag{30}$$

This allows all values of n, whereas n could not exceed $[1/\sqrt{\alpha_e}\,] = 11$ in the RN case. In the GUP Kerr case, an expression similar to Eq. 29 still applies and there is a change in the form of the solutions for $n > 2\beta$.

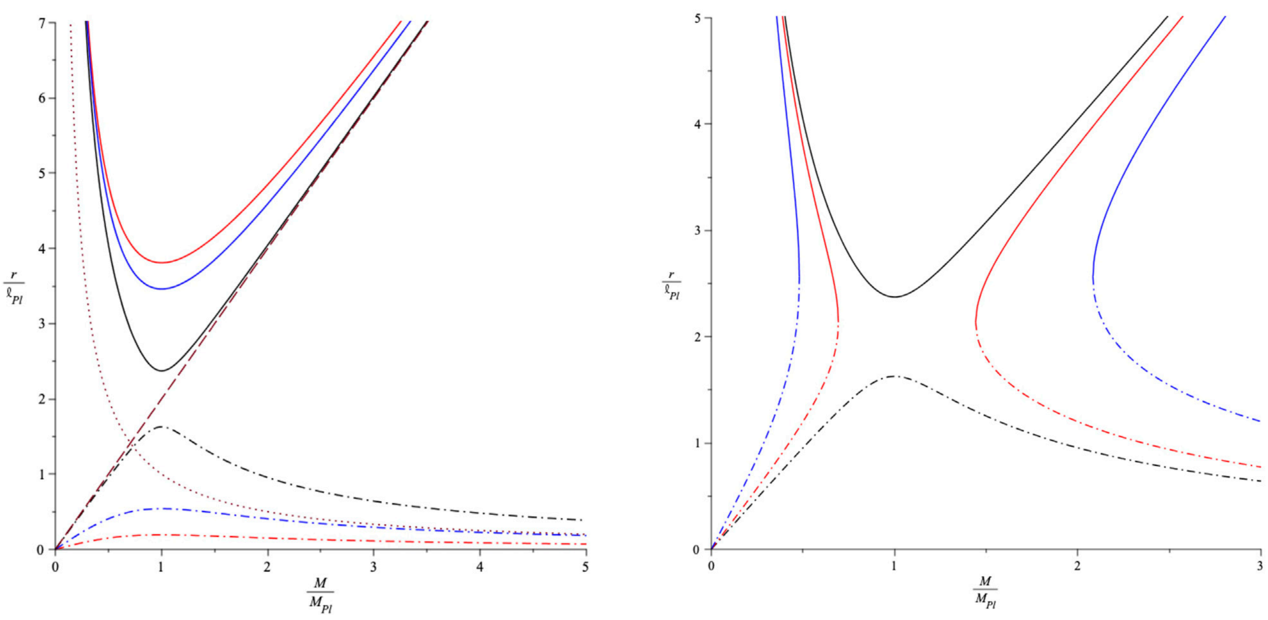

FIGURE 4
Outer (solid) and inner (dash-dot) horizon size for GUP-RN black hole with $\beta = 2$. Left: Outer (top) and inner (bottom) horizons for $n = 10$ (red), $n = 16$ (blue) and $n = 23$ (black). The dashed/dotted lines show the usual Schwarzschild/Compton scales. The inner horizon is nearly asymptotic to the Compton wavelength at large M for $n = 16$. There is a discontinuity when n reaches 23, this being close to an extremal solution. Right: Outer (top) and inner (bottom) horizons for $n = 23$ (black), $n = 25$ (blue) and $n = 30$ (black). The horizons in this case have a maximum value of M on the left and a minimum value on the right. There are no black holes between these values. From Carr et al. (2020).

5 Higher-dimensional black holes

The black hole boundary in Figure 1 assumes there are three spatial dimensions but many theories suggest that the dimensionality could increase on small scales. Although the extra dimensions are often assumed to be compactified on the Planck length, there are also models in which they are much larger. For example, the model of Arkani-Hamed et al. (1998) has n extra spatial dimensions, all compactified on the same scale R_E. If we assume that the standard expression for the Compton wavelength ($R_C \propto M^{-1}$) still applies, then the masses with Compton and Schwarzschild scales R_E are

$$M_E \equiv \frac{\hbar}{cR_E} \simeq M_P \frac{R_P}{R_E}, \quad M_E' \equiv \frac{c^2 R_E}{G} \simeq M_P \frac{R_E}{R_P}. \tag{31}$$

For $R < R_E$, the gravitational potential generated by a mass M is

$$V_{\rm grav} = \frac{G_D M}{R^{1+n}}, \tag{32}$$

where G_D is the higher-dimensional gravitational constant and $D = 4 + n$ is the total number of spacetime dimensions. For $R > R_E$, one recovers the usual form, $V_{\rm grav} = GM/R$ with $G = G_D/R_E^n$. Thus the effective gravitational constants at large and small scales are different.

Eq. 32 implies that the usual expression for the Schwarzschild radius no longer applies for masses below M_E'. If the black hole is assumed to be spherically symmetric in the higher-dimensional space, one has (Kanti, 2016)

$$R_S \simeq R_E \left(\frac{M}{M_E'}\right)^{1/(n+1)}. \tag{33}$$

Therefore the slope of the black hole boundary in Figure 1 becomes shallower for $M < M_E'$, as indicated in Figure 5A. The intersect with the Compton line then becomes

$$R_P' \simeq \left(R_P^2 R_E^n\right)^{1/(2+n)}, \quad M_P' \simeq \left(M_P^2 M_E^n\right)^{1/(2+n)}, \tag{34}$$

so $M_P' \ll M_P$ and $R_P' \gg R_P$ for $R_E \gg R_P$.

In principle, the lowering of the Planck mass could permit the possibility of TeV quantum gravity and the production of small black holes at the Large Hadron Collider (LHC), with their evaporation leaving a distinctive signature (Dimopoulos and Landsberg, 2001; Anchordoqui et al., 2002; Giddings and Thomas, 2002). If the accessible energy is $E_{\rm max} \approx 10$ TeV, then the extra dimensions can be probed for

$$R_E > 10^{-18+30/n}\ {\rm cm} \simeq \begin{cases} 10^{12}\ {\rm cm} & (n=1) \\ 10^{-3}\ {\rm cm} & (n=2) \\ 10^{-14}\ {\rm cm} & (n=7). \end{cases} \tag{35}$$

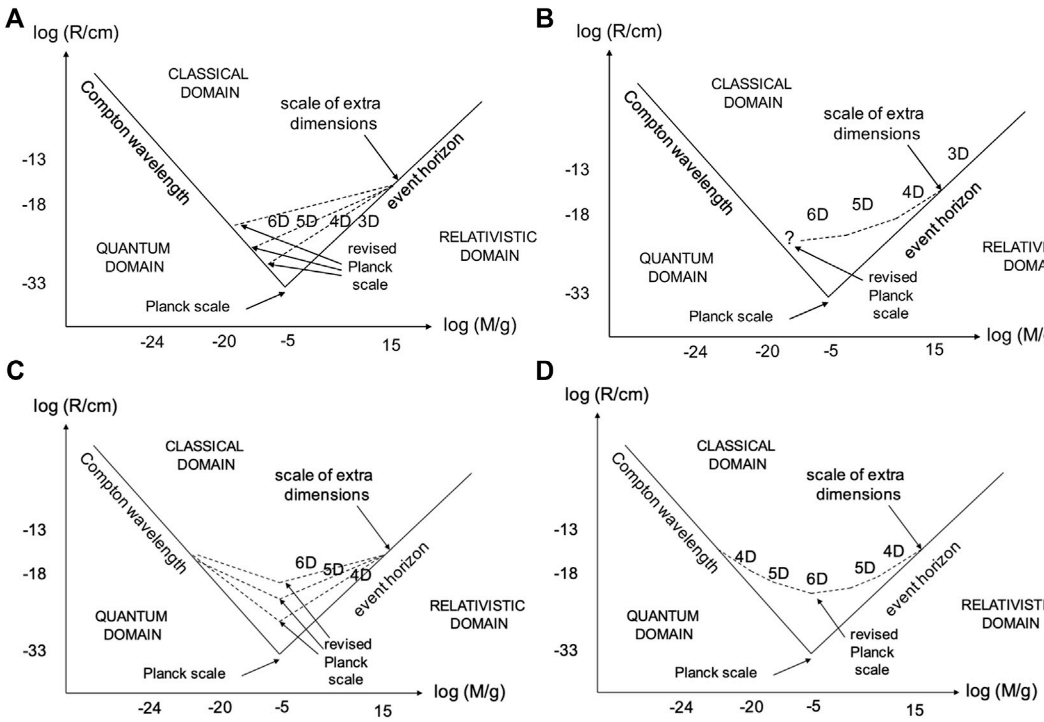

FIGURE 5
Modification of the Schwarzschild line and Planck scales in the (*M*, *R*) diagram for extra compact dimensions associated with **(A)** a single length scale or **(B)** a hierarchy of length scales **(B)** if the Compton scale preserves its usual form. **(C)** and **(D)** are the corresponding diagrams if the duality between the Compton and Schwarzschild expressions is preservd. From Lake and Carr (2019).

Clearly, $n = 1$ is excluded on empirical grounds but $n = 2$ is possible. One expects $n = 7$ in M-theory, so it is interesting that R_E must be of order a Fermi in this case. One could also consider a scenario with a hierarchy of compactification scales, $R_i = \alpha_i R_P$ with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 1$, such that the dimensionality progressively increases as one goes to smaller scales (Carr, 2013). This situation is represented in Figure 5B. There is still no evidence for the extra dimensions (ATLAS collaboration, 2016), which suggests that either they do not exist or they have a compactification scale R_E which is so small that M_P' exceeds the energy attainable by the LHC

Another possible reason for the non-detection of accelerator black holes is that the M dependence of R_C is also affected by the extra dimensions. Lake and myself have argued that the *effective* Compton wavelength depends on the form of the $(3 + n)$-dimensional wavefunction (Lake and Carr, 2019). If this is spherically symmetric in all the dimensions, then one has $R_C \propto M^{-1}$ (as usually assumed). However, if the wave function is pancaked in the extra dimensions and maximally asymmetric, then we find $R_C \propto M^{-1/(1+n)}$. This implies that the duality between the Compton wavelength and the Schwarzschild radius persists in the higher dimensional case but that there is no accelerator production of black holes. Thus the constraint on R_E given by Eq. 35 no longer applies. This scenario is illustrated in Figure 5C for extra dimensions compactified on a single length scale R_E and in Figure 5D for a hierarchy of length scales, when the extra dimensions help to smooth the minimum. The latter case resembles the smooth minimum in Figure 1, which suggests that higher dimensions might themselves underlie the BHUP correspondence.

The above discussion of higher-dimensional black holes has assumed that the simple power-law forms for R_S and R_C apply all the way to their intersect at the (modified) Planck scale. However, the BHUP correspondence suggests that they should be unified in some way, which would smooth the minima in Figure 5. This raises the issue of the form of the GUP and BHUP correspondence in the higher-dimensional case. If the Compton wavelength preserves its 3-dimensional form, one might expect the generalized Compton wavelength to become

$$R_C' = \frac{\hbar}{Mc}\left[1 + \left(\frac{M}{M_P'}\right)^{(n+2)/(n+1)}\right] \quad (R < R_E), \tag{36}$$

so that R_C' becomes R_S' at large M. If duality between R_S and R_C is preserved in the higher-dimensional case, one might expect

$$R_C' = R_*\left(\frac{M_P}{M}\right)^{1/(1+n)}\left[1 + \left(\frac{M}{M_P'}\right)^{2/(n+1)}\right] \quad (R < R_E) \tag{37}$$

However, the literature on this gives different results (Koppel et al., 2017; Knipfer et al., 2019).

Finally, if we interpret the Compton wavelength as marking the boundary in the (M, R) diagram below which pair-production rates becomes significant, we might expect the presence of compact extra dimensions to affect pair-production rates at high energies. Specifically, pair-production above the energy scale $M_E c^2 \equiv \hbar c/R_E$, should be enhanced relative to the 3-dimensional case. Indeed, there is tentative evidence that this is a generic feature of higher-dimensional theories (He, 1999; Eboli et al., 2000).

6 Linking black holes and elementary particles

The suggestion that there could be a fundamental link between elementary particles and black holes goes back to the 1970s, when it was motivated in the context of strong gravity theories (Sivaram and Sinha, 1977). Various arguments supported this suggestion: 1) both hadrons and Kerr-Newman black holes are characterised by three parameters (M,J,Q); 2) both have a magnetic dipole moment and a gyromagnetic ratio of two but no electric dipole moment; 3) Regge trajectories and extreme Kerr solutions have the same relationship between angular momentum and mass ($J \sim M^2$); 4) when classical black holes interact, their surface area can never decrease, which is analogous to the increase in cross-sections found in hadron collisions.

Of course, elementary particles cannot be black holes with normal gravity, since their Compton wavelength is much larger than their Schwarschild radius, as illustrated in Figure 1. The early models therefore assumed that gravity becomes stronger by a factor of $G_F/G \sim (M_P/m_p)^2 \sim 10^{38}$ on the hadronic scale. This requires the existence of a massive spin-2 meson and corresponds to a short-range force. If the hadronic resonances are extremal black holes, their mass and spin should satisfy $G_F m_h^2 = J$, corresponding to a Regge slope of $(1\text{Gev})^{-2}$, and they should have a spectrum of masses $M_n \sim n^{1/2}$ GeV (Oldershaw, 2010).

The current proposal—explored in more detail in work with Mureika and Nicolini (Carr et al., 2022)—is prompted by the Generalized Uncertainty Principle and the duality between the Compton and Schwarzschild expressions under the transformation $M \to M_P^2/M$, so the context is somewhat different. Also elementary particles are regarded as sub-Planckian black holes under normal gravity rather conventional black holes under strong gravity However, there is a link with strong gravity because the force between two masses, while still obeying the inverse-square law, is much enhanced for $M < \sqrt{\beta} M_P$.

Extending the BHUP correspondence to charged black holes adds important insights. Although the Reissner-Nordstrom itself has a nearly Planckian mass and therefore cannot represent an elementary particle, adding a GUP term introduces sub-Planckian solutions. This explains why the charge cannot exceed $\sqrt{2\beta/\alpha_e} \approx 1$ for $\beta \sim 10^{-2}$, as observed for elementary particles. Similar considerations apply for spinning black holes. However, these solutions only correspond to extremal black holes if the mass is $\sqrt{\beta/2}\, M_P$, which is too large for an elementary particle (given the allowed range of β).

These considerations must be modified if there are extra dimensions on small scales. Although there is some uncertainty in the modifications to the GUP in this case, Figure 5 shows that the extra dimensions themselves smooth the Compton-Schwarzschild transition. Furthermore, the black hole mass may be shifted down towards the hadron scale, the effective strength of gravity being increased by the extra dimensions. However, the higher-dimensional analysis has not yet been extended to the charged and rotating black holes. Extra dimensions may also play an important role in amalgamating general relativity and quantum theory, with higher-dimensional relativity permitting a classical-type interpretation of some quantum anomalies.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

This paper is the work of the author BC but also includes some review of earlier work done in collaboration with other authors.

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Conflict of interest

The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Generalised uncertainty relations from finite-accuracy measurements

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In this short note we show how the Generalised Uncertainty Principle (GUP) and the Extended Uncertainty Principle (EUP), two of the most common generalised uncertainty relations proposed in the quantum gravity literature, can be derived within the context of canonical quantum theory, without the need for modified commutation relations. A generalised uncertainty principle-type relation naturally emerges when the standard position operator is replaced by an appropriate Positive Operator Valued Measure (POVM), representing a finite-accuracy measurement that localises the quantum wave packet to within a spatial region $\sigma_g > 0$. This length scale is the standard deviation of the envelope function, g, that defines the positive operator valued measure elements. Similarly, an extended uncertainty principle-type relation emerges when the standard momentum operator is replaced by a positive operator valued measure that localises the wave packet to within a region $\tilde{\sigma}_g > 0$ in momentum space. The usual generalised uncertainty principle and extended uncertainty principle are recovered by setting $\sigma_g \simeq \sqrt{\hbar G/c^3}$, the Planck length, and $\tilde{\sigma}_g \simeq \hbar\sqrt{\Lambda/3}$, where Λ is the cosmological constant. Crucially, the canonical Hamiltonian and commutation relations, and, hence, the canonical Schrödinger and Heisenberg equations, remain unchanged. This demonstrates that generalised uncertainty principle and extended uncertainty principle phenomenology can be obtained without modified commutators, which are known to lead to various pathologies, including violation of the equivalence principle, violation of Lorentz invariance in the relativistic limit, the reference frame-dependence of the "minimum" length, and the so-called soccer ball problem for multi-particle states.

KEYWORDS

generalised uncertainty relations, generalised uncertainty principle, extended uncertainty principle, finite-accuracy measurements, POVM

1 Introduction

In canonical quantum mechanics the Heisenberg uncertainty principle (HUP) implies a fundamental trade-off between the precisions of position and momentum measurements. [1] It can be introduced heuristically, *via* the famous Heisenberg microscope thought experiment, giving (Heisenberg, 1927; Heisenberg, 1930)

$$\Delta x^i \, \Delta p_j \gtrsim \frac{\hbar}{2}\delta^i{}_j, \tag{1.1}$$

or derived rigorously from the canonical quantum formalism, yielding (Isham, 1995; Rae, 2002)

$$\Delta_\psi x^i \, \Delta_\psi p_j \geq \frac{\hbar}{2}\delta^i{}_j. \tag{1.2}$$

The inequality in Eq. 1.2 is exact and, unlike the heuristic uncertainties Δx^i and Δp_j in Eq. 1.1, $\Delta_\psi x^i$ and $\Delta_\psi p_j$ represent well-defined standard deviations of the probability distributions $|\psi(\mathbf{x})|^2$ and $|\tilde{\psi}_\hbar(\mathbf{p})|^2$, respectively, where the momentum space representation of the particle wave function is given by the $\hbar$-scaled Fourier transform of its position space representation:

$$\tilde{\psi}_\hbar(\mathbf{p}) = \left(\frac{1}{\sqrt{2\pi\hbar}}\right)^3 \int \psi(\mathbf{x})\, e^{-\frac{i}{\hbar}\mathbf{p}.\mathbf{x}} \mathrm{d}^3\mathrm{x}. \tag{1.3}$$

We emphasise the scale-dependence of the canonical quantum Fourier transform, which is often neglected in standard treatments, by introducing the subscript $\hbar$. Eq. 1.2 is obtained by combining the Schrödinger-Robertson relation for arbitrary Hermitian operators, $\hat{O}_1$ and $\hat{O}_2$ (Robertson, 1929; Schrödinger, 1930),

$$\Delta_\psi O_1 \, \Delta_\psi O_2 \geq \frac{1}{2}|\langle\psi|\left[\hat{O}_1, \hat{O}_2\right]|\psi\rangle|, \tag{1.4}$$

with the canonical position-momentum commutator,

$$\left[\hat{x}^i, \hat{p}_j\right] = i\hbar\delta^i{}_j \, \hat{\mathbb{I}}. \tag{1.5}$$

In recent years, thought experiments in quantum gravity research have suggested the existence of generalised uncertainty relations (GURs). By reconsidering Heisenberg's 1927 gedanken experiment, and accounting for the gravitational interaction between the massive particle and the probing photon, we obtain the generalised uncertainty principle (GUP),

$$\Delta x^i \gtrsim \frac{\hbar}{2\Delta p_j}\delta^i{}_j\left[1 + \alpha_0 \frac{2G}{\hbar c^3}\left(\Delta p_j\right)^2\right], \tag{1.6}$$

where α_0 is a numerical constant of order unity (Maggiore, 1993; Adler and Santiago, 1999; Scardigli, 1999). By minimising the right-hand side with respect to Δp_j, the GUP implies the existence of a minimum position uncertainty of the order of the Planck length, $l_{\rm Pl} = \sqrt{\hbar G/c^3} \simeq 10^{-33}$ cm.

Reconsidering Heisenberg's arguments in the presence of a constant dark energy density $\rho_\Lambda = \Lambda c^2/(8\pi G) \simeq 10^{-30}$ g.cm^{-3} (Riess et al., 1998; Perlmutter et al., 1999), or, equivalently, an asymptotically de Sitter background with minimum scalar curvature of the order of the cosmological constant, $\Lambda \simeq 10^{-56}$ cm^{-2} (Ade et al., 2014; Betoule et al., 2014), gives the extended uncertainty principle (EUP),

$$\Delta p_j \gtrsim \frac{\hbar}{2\Delta x^i}\delta^i{}_j\left[1 + 2\eta_0\Lambda(\Delta x^i)^2\right], \tag{1.7}$$

where η_0 is of order one (Bolen and Cavaglia, 2005; Park, 2008; Bambi and Urban, 2008). The EUP implies the existence of a minimum momentum uncertainty of the order of the de Sitter momentum, $m_{\rm dS}c = \hbar\sqrt{\Lambda/3} \simeq 10^{-56}$ g . cm s^{-1}. This is physically reasonable since it is the minimum momentum that a canonical quantum particle can possess, when its wave function is localised within the asymptotic de Sitter horizon, which is comparable to the present day radius of the Universe $r_{\rm U}(t_0) \simeq l_{\rm dS} = \sqrt{3/\Lambda} \simeq 10^{28}$ cm.

Combining both effects yields the extended generalised uncertainty principle (EGUP),

$$\Delta x^i \Delta p_j \gtrsim \frac{\hbar}{2}\delta^i{}_j\left[1 + \alpha_0\frac{2G}{\hbar c^3}\left(\Delta p_j\right)^2 + 2\eta_0\Lambda(\Delta x^i)^2\right], \tag{1.8}$$

which implies the existence of both minimum length and momentum scales in nature (Bolen and Cavaglia, 2005; Park, 2008; Bambi and Urban, 2008). Like their forebearer Eq. 1.1 all three relations Eqs 1.6–1.8 are heuristic in nature and it remains an open problem how to rigorously derive GURs from within a modified quantum formalism.

Perhaps the simplest way to obtain the GUP, EUP or EGUP, given Eq. 1.4, is to modify the canonical position-momentum commutator Eq. 1.5 and it is clear that a modification of the form

$$\begin{aligned}\left[\hat{x}^i, \hat{p}_j\right] = i\hbar\delta^i{}_j \, \hat{\mathbb{I}} &\mapsto \left[\hat{X}^i, \hat{P}_j\right] \\ &= i\hbar\delta^i{}_j\left(\hat{\mathbb{I}} + \alpha_0\frac{2G}{\hbar c^3}\left(\hat{P}_j\right)^2 + 2\eta_0\Lambda\left(\hat{X}^i\right)^2\right)\end{aligned} \tag{1.9}$$

gives rise to an EGUP-type uncertainty relation, at least when both $\langle\hat{P}_j\rangle_\psi = 0$ and $\langle\hat{X}^i\rangle_\psi = 0$ (Kempf et al., 1995). Here, we use capital letters to denote modified operators, which generate modified commutators, and lower case letters to denote their canonical quantum counterparts. However, the assumption above is problematic since, even if both $\langle\hat{P}_j\rangle_\psi = 0$ and $\langle\hat{X}^i\rangle_\psi = 0$ in a given frame of reference, a simple shift of coordinate origin or a Galilean velocity boost of the observer alters the numerical value of the associated Schrödinger-Robertson bound:

$$\begin{aligned}\Delta_\psi X^i \Delta_\psi P_j \geq \frac{\hbar}{2}\delta^i{}_j\Big\{1 + \alpha_0\frac{2G}{\hbar c^3}\left[\left(\Delta_\psi P_j\right)^2 + \langle\hat{P}_j\rangle^2_\psi\right] \\ + 2\eta_0\Lambda\left[\left(\Delta_\psi X^i\right)^2 + \langle\hat{X}^i\rangle^2_\psi\right]\Big\}.\end{aligned} \tag{1.10}$$

This leads immediately to the reference frame-dependence of the (supposedly invariant) minimum length. In fact, the situation is even worse since even a redefinition of the position-coordinate origin alters the value of the bound on the right-hand side. This gives rise to a coordinate-dependent "minimum" length, which is clearly unphysical, and which strongly suggests that GUR models based on modified commutation relations are not mathematically self-consistent (Lake, 2020; Lake et al., 2023).

In addition, the modified position-momentum commutator Eq. 1.9 implies a modification of the canonical Heisenberg equation, which immediately gives rise to mass-dependent accelerations for quantum particles, violating the equivalence principle (Tawfik and Diab, 2014; Tawfik and Diab, 2015). Such models also violate Lorentz invariance in the relativistic limit and suffer from the so-called soccer ball problem, so that sensible GUP-compatible multi-particle

1 In classical error analysis the term "precision" is used to refer to the statistical spread of the results whereas the term "accuracy" refers to the discrepancy between the measured value of a quantity and its true value. In keeping with this general usage, we use the term precision to refer to the quantum mechanical uncertainty and accuracy to refer to the width of the error bars associated with each individual measurement.

states cannot be defined (Hossenfelder, 2013; Amelino-Camelia, 2017) [2].

The heuristic, model-independent nature of the gedanken experiments that lead to the relations Eqs. 1.6–1.8, together with the pathologies displayed by modified commutator models, motivate us to consider alternative ways to generate GUP, EUP, and EGUP phenomenology, without modifying the canonical Heisenberg algebra. In this paper, we consider one way in which such a scheme can be implemented from within the canonical quantum formalism. The physical basis of the model is the notion of a finite-accuracy measurement and these are represented mathematically by the construction of appropriate POVM. Roughly speaking, since errors add in quadrature for independent random variables, finite-accuracy measurements of position and momentum with detection "sweet spots" of width $\sigma_g \simeq l_{\rm Pl}$ and $\tilde{\sigma}_g \simeq m_{\rm dS}c$, respectively, give rise to the GUP and EUP, to first order in the relevant Taylor expansion. These individual relations may then be combined to give the EGUP.

2 GUR from finite-accuracy measurements described by POVM

In this section, we show that GUP, EUP and EGUP-type uncertainty relations can be derived in an effective model, where position and momentum measurements in canonical quantum theory are not perfectly accurate, and are described by POVM, rather than perfect projective measurements.

Let us begin by replacing the usual position-measurement operator, $\hat{\mathbf{x}}$, with POVM elements corresponding to the result $\mathbf{x}$:

$$\hat{E}_{\mathbf{x}} := \int g(\mathbf{x}' - \mathbf{x})\,|\mathbf{x}'\rangle\langle\mathbf{x}'|{\rm d}^3x', \tag{2.1}$$

where $g(\mathbf{x}' - \mathbf{x})$ is any normalised function, $\int|g(\mathbf{x}' - \mathbf{x})|^2{\rm d}^3x' = 1$. These elements satisfy the relations $\hat{E}^\dagger_{\mathbf{x}}\hat{E}_{\mathbf{x}} \geq 0$ and $\int \hat{E}^\dagger_{\mathbf{x}}\hat{E}_{\mathbf{x}}{\rm d}^3x = \hat{\mathbb{I}}$, as required, so that Eq. 2.1 defines a standard POVM in canonical quantum mechanics (Nielsen and Chuang, 2000). From here on, we refer to g as the "envelope function" of the measure. For spherically symmetric functions the envelope is centred on the value $\mathbf{x}$, and, for the sake of concreteness, we may imagine $|g(\mathbf{x}' - \mathbf{x})|^2$ as a three-dimensional Gaussian distribution with mean $\mathbf{x}$ and standard deviation σ_g.

Finite-accuracy position measurements, conducted on an arbitrary state $|\psi\rangle$, then give rise to the first and second order moments

$$\langle E_{\mathbf{x}}\rangle_\psi = \int \mathbf{x}\langle\psi|\hat{E}^\dagger_{\mathbf{x}}\hat{E}_{\mathbf{x}}|\psi\rangle{\rm d}^3x = \langle\mathbf{x}\rangle_g + \langle\mathbf{x}\rangle_\psi,$$
$$\langle E^2_{\mathbf{x}}\rangle_\psi = \int \mathbf{x}^2\langle\psi|\hat{E}^\dagger_{\mathbf{x}}\hat{E}_{\mathbf{x}}|\psi\rangle{\rm d}^3x = \langle\mathbf{x}^2\rangle_g + \langle\mathbf{x}^2\rangle_\psi, \tag{2.2}$$

where $\langle\mathbf{x}^n\rangle_f := \int \mathbf{x}^n\,|f(\mathbf{x})|^2\,{\rm d}^3x$ with $f(\mathbf{x}) = g(\mathbf{x})$ or $\psi(\mathbf{x})$. Since $|g(\mathbf{x}' - \mathbf{x})|^2$ is a normalised function centred on $\mathbf{x}' = \mathbf{x}$, $\langle\mathbf{x}\rangle_g = 0$, and the corresponding variance is given by

$$\left(\Delta_\psi E_{\mathbf{x}}\right)^2 = \left(\Delta_\psi\mathbf{x}\right)^2 + \boldsymbol{\sigma}^2_g, \tag{2.3}$$

where $\boldsymbol{\sigma}_g := \sigma^i_g\mathbf{e}_i$ and σ^i_g denotes the width of $|g|^2$ in each coordinate direction x^i. By spherical symmetry, $\sigma^i_g = \sigma_g$ for all i, and we may rewrite Eq. 2.3 in terms of the individual components as

$$\left(\Delta_\psi E_i\right)^2 = \left(\Delta_\psi x^i\right)^2 + \sigma^2_g, \tag{2.4}$$

where we have used the shorthand notation $\Delta_\psi E_i \equiv \Delta_\psi E_{x^i}$.

In like manner, finite-accuracy momentum measurements may be introduced *via* the operators

$$\hat{\mathbb{E}}_{\mathbf{p}} := \int \tilde{g}(\mathbf{p}' - \mathbf{p})\,|\mathbf{p}'\rangle\langle\mathbf{p}'|{\rm d}^3p', \tag{2.5}$$

where $\int|\tilde{g}(\mathbf{p}' - \mathbf{p})|^2{\rm d}p' = 1$, but it is important to note that there is no *intrinsic* relation between the functions g and $\tilde{g}$, which may be chosen independently for a given POVM model. Nevertheless, if both $|g|^2$ and $|\tilde{g}|^2$ represent Gaussian distributions, which is perhaps the most natural choice for an envelope function, then g and $\tilde{g}$ *are* related *via* a Fourier transform,

$$\tilde{g}(\mathbf{p}' - \mathbf{p}) = \int g(\mathbf{x}' - \mathbf{x})\,e^{\frac{i}{\beta}(\mathbf{x}'-\mathbf{x}).(\mathbf{p}'-\mathbf{p})}{\rm d}^3x', \tag{2.6}$$

where the new action scale $\beta \neq \hbar$ is given by

$$\beta := 2\sigma_g\tilde{\sigma}_g, \tag{2.7}$$

and $\tilde{\sigma}_g$ is the standard deviation of $|\tilde{g}|^2$. However, it is equally important to note that there is nothing fundamental about the relation Eq. 2.6. Unlike the $\hbar$-scaled Fourier transform relating the position and momentum space representations of the quantum wave function, Eq. 1.3, the β-scaled transform relates the "envelope functions" of the model.

Finite-accuracy momentum measurements, conducted on an arbitrary state $|\psi\rangle$, then give rise to the first and second order moments

$$\langle\mathbb{E}_{\mathbf{p}}\rangle_\psi = \int \mathbf{p}\langle\psi|\hat{\mathbb{E}}^\dagger_{\mathbf{p}}\hat{\mathbb{E}}_{\mathbf{p}}|\psi\rangle{\rm d}^3p = \langle\mathbf{p}\rangle_g + \langle\mathbf{p}\rangle_\psi,$$
$$\langle\mathbb{E}^2_{\mathbf{p}}\rangle_\psi = \int \mathbf{p}^2\langle\psi|\hat{\mathbb{E}}^\dagger_{\mathbf{p}}\hat{\mathbb{E}}_{\mathbf{p}}|\psi\rangle{\rm d}^3p = \langle\mathbf{p}^2\rangle_g + \langle\mathbf{p}^2\rangle_\psi, \tag{2.8}$$

2 In Amelino-Camelia (2017) an ingenious solution to the soccer ball problem was proposed. In this approach, the generalised momentum operators of a given modified commutator model are defined to be the generators of "generalised spatial translations." The unitary transformation $\hat{\mathcal{U}}(\mathbf{X}) := exp[(i/\hbar)\mathbf{X}.\hat{\mathbf{P}}]$, which acts non-trivially only on the $\hat{X}^i$ operators, is required to leave the modified $[\hat{X}^i, \hat{P}_j]$, $[\hat{X}^i, \hat{X}^j]$ and $[\hat{P}_i, \hat{P}_j]$ algebras, as well as the multi-particle Hamiltonian of the model, $\hat{\mathcal{H}}$, invariant. This defines the "generalised translation symmetries" of the system and, when these symmetries hold, the corresponding Noether charge for an N-particle state is represented by the operator $\hat{\mathbf{P}}_{\rm Total} := \sum_{I=1}^N \hat{\mathbf{P}}_I$, where $[\hat{\mathbf{P}}_{\rm Total}, \hat{\mathcal{H}}] = 0$. The usual law of linear momentum addition therefore holds for multi-particle states but a different non-linear addition law, derived ultimately from the notion of spatial locality, holds for transfers of momentum between individual particles, due to the interactions specified by $\hat{\mathcal{H}}$. Unfortunately for GUP models, in the example system considered in Amelino-Camelia (2017), the definition of the generalised spatial translations required to maintain the linear addition law also requires one of the position-momentum commutators to equal zero, i.e., $[\hat{X}^i, \hat{P}_i] = 0$, for some i. In this case there is no Heisenberg uncertainty principle, let alone a GUP, even though a minimum length scale l still appears in the model *via* the position-position commutator, e.g., $[\hat{X}_1, \hat{X}_2] = il\hat{X}_1$. This illustrates a general point, that it is by no means certain whether a particular modified momentum operator, corresponding to a particular modification of the canonical Heisenberg algebra, and, hence, a particular form of the GUP, is compatible with a linear addition law derived *via* Amelino-Camelia's procedure. Therefore, although this procedure represents a useful criterion for defining physically viable GUP models, it is clear that arbitrary deformations of the canonical Heisenberg algebra are not consistent with the existence of a linear momentum addition law and that further work is required to determine which models truly suffer from a soccer ball problem and which ones do not. Though some GUP models may be free from this pathology, a great many could still be afflicted by it.

where $\langle \mathbf{p}^n \rangle_{f} := \int \mathbf{p}^n |\tilde{f}(\mathbf{p})|^2 \, \mathrm{d}^3\mathrm{p}$ with $\tilde{f}(\mathbf{p}) = \tilde{g}(\mathbf{p})$ or $\tilde{\psi}_\hbar(\mathbf{p})$. Since $|\tilde{g}(\mathbf{p}' - \mathbf{p})|^2$ is normalised and centred at $\mathbf{p}' = \mathbf{p}$, $\langle \mathbf{p} \rangle_g = 0$, and

$$\left(\Delta_\psi \mathbb{E}_{\mathbf{p}}\right)^2 = \left(\Delta_\psi \mathbf{p}\right)^2 + \tilde{\boldsymbol{\sigma}}_g^2, \tag{2.9}$$

where $\tilde{\boldsymbol{\sigma}}_g := \tilde{\sigma}_{gj}\mathbf{e}^j$ and $\tilde{\sigma}_{gj}$ denotes the width of $|\tilde{g}|^2$ in each momentum space direction p_j. Again employing spherical symmetry, $\tilde{\sigma}_{gj} = \tilde{\sigma}_g$ for all j, Eq. 2.9 may be rewritten in terms of the individual components as

$$\left(\Delta_\psi \mathbb{E}_j\right)^2 = \left(\Delta_\psi p_j\right)^2 + \tilde{\sigma}_g^2, \tag{2.10}$$

where we have again used the shorthand $\Delta_\psi \mathbb{E}_j \equiv \Delta_\psi \mathbb{E}_{p_j}$.

To obtain a GUP-type relation from Eq. 2.4 we simply take the square root, Taylor expand the right-hand side to first order, and substitute for $\Delta_\psi x^i$ from the HUP Eq. 1.2. Likewise, an EUP-type relation is obtained from Eq. 2.10 by taking the square root, Taylor expanding to first order, and substituting for $\Delta_\psi p_j$. Next, using the substitutions

$$\sigma_g := \sqrt{2\alpha_0}\, l_{\mathrm{Pl}}, \quad \tilde{\sigma}_g := \sqrt{6\eta_0}\, m_{\mathrm{dS}} c, \tag{2.11}$$

where

$$l_{\mathrm{Pl}} := \sqrt{\hbar G/c^3}, \quad m_{\mathrm{dS}} c := \hbar\sqrt{\Lambda/3}, \tag{2.12}$$

immediately gives

$$\Delta_\psi X^i \gtrsim \frac{\hbar}{2\Delta_\psi P_j}\delta^i{}_j\left[1 + \alpha_0\frac{2G}{\hbar c^3}\left(\Delta_\psi P_j\right)^2\right], \tag{2.13}$$

$$\Delta_\psi P_j \gtrsim \frac{\hbar}{2\Delta_\psi x^i}\delta^i{}_j\left[1 + 2\eta_0\Lambda\left(\Delta_\psi x^i\right)^2\right], \tag{2.14}$$

where we have relabelled $\Delta_\psi E_i \equiv \Delta_\psi X^i$ and $\Delta_\psi \mathbb{E}_j \equiv \Delta_\psi P_j$, for convenience. These expressions are formally analogous to the heuristic relations, Eqs. 1.6, 1.7, respectively, but with Δp_j and Δx^i on the right replaced by the well-defined standard deviations $\Delta_\psi p_j$ and $\Delta_\psi x^i$.

This proves that GUP- and EUP-type relations can be derived rigorously, from within the canonical quantum formalism, but a remaining criticism of the formulae above is that the uncertainties on the right-hand sides of Eqs 2.13, 2.14 are not equivalent to the uncertainties on the left. Indeed, according to the POVM model, $\Delta_\psi p_j$ and $\Delta_\psi x^i$ are not operationally *observable* quantities. They arise only in the limits $\sigma_g \to 0$ and $\tilde{\sigma}_g \to 0$, respectively, in which both Eqs 2.13, 2.14 reduce to the standard HUP Eq. 1.2. This objection can be overcome, however, by first substituting for $\Delta_\psi x^i$ from Eq. 1.2 in Eq. 2.4 and then again for $\Delta_\psi p_i$ from Eq. 2.10. This gives rise to an uncertainty relation between the observable standard deviations, $\Delta_\psi E_i \equiv \Delta_\psi X^i$ and $\Delta_\psi \mathbb{E}_j \equiv \Delta_\psi P_j$. It is straightforward to show that, taking the square root, Taylor expanding to first order, and neglecting the final term of order $\sigma_g \tilde{\sigma}_g \simeq l_{\mathrm{Pl}}.\, m_{\mathrm{dS}} c$, this relation reduces to

$$\Delta_\psi X^i \Delta_\psi P_j \gtrsim \frac{\hbar}{2}\delta^i{}_j\left[1 + \alpha_0\frac{2G}{\hbar c^3}\left(\Delta_\psi P_j\right)^2 + 2\eta_0\Lambda\left(\Delta_\psi X^i\right)^2\right]. \tag{2.15}$$

Therefore, the EGUP can be rigorously derived within the canonical quantum formalism. The GUP and EUP proper then arise as limits of this more fundamental relation.

We stress that, in this model, $\Delta_\psi E_i \equiv \Delta_\psi X^i$ and $\Delta_\psi \mathbb{E}_j \equiv \Delta_\psi P_j$ represent the *physically observable* precisions, obtained from generalised position and momentum measurements with finite accuracies $\sigma_g > 0$ and $\tilde{\sigma}_g > 0$. By contrast, the canonical Hamiltonian is determined by the canonical (projective) position and momentum operators, $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$, *via* $\hat{H} = \hat{\mathbf{p}}^2/(2m) + V(\hat{\mathbf{x}})$, where the former obey the canonical Heisenberg algebra: $[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i{}_j\,\hat{\mathbb{I}}$, $[\hat{x}^i, \hat{x}^j] = 0$, $[\hat{p}_i, \hat{p}_j] = 0$. This leaves the canonical Heisenberg and Schrödinger equations unchanged and neatly evades the pathologies that afflict modified commutator models (Lake, 2020; Hossenfelder, 2013; Tawfik and Diab, 2014; Tawfik and Diab, 2015; Lake et al., 2023).

3 Discussion

We have shown that the three most common GURs studied in the quantum gravity literature, the GUP, EUP, and EGUP, can be derived from within the formalism of canonical quantum mechanics. A GUP-type uncertainty relation is obtained when the standard (projective) position operator is replaced by an appropriate POVM, representing finite-accuracy measurements with error bars of width $\sigma_g > 0$ in real space. In like manner, an EUP-type relation is obtained from finite-accuracy measurements with error bars of width $\tilde{\sigma}_g > 0$ in momentum space. These can be combined to give a relation that is formally analogous to the EGUP and the standard EGUP is recovered by setting $\sigma_g \simeq l_{\mathrm{Pl}}$, the Planck length, and $\tilde{\sigma}_g \simeq m_{\mathrm{dS}} c$, where $m_{\mathrm{dS}} = (\hbar/c)\sqrt{\Lambda/3}$ is the de Sitter mass.

This work suggests that GUP, EUP, and EGUP phenomenology can be understood in a physically intuitive way, as a simple and natural outcome of finite-accuracy measurements. Such measurements are capable of generating all three GURs and the same phenomenology is obtained, at the level of the uncertainty relations, regardless of whether the limits $(\Delta_\psi X^i)_{\mathrm{min}} = \sigma_g$ and $(\Delta_\psi P_j)_{\mathrm{min}} = \tilde{\sigma}_g$ are fundamental, or merely effective, as an outcome of an imperfect measurement scheme.

We propose that this should give pause for thought to the GUP community. If modified commutators are not *necessary* for GUP phenomenology, and, after nearly 30 years of research, we are no closer to resolving the pathologies that have afflicted these models since they were first proposed in the mid-1990s, then serious attempts should be made to find *alternative mathematical structures* that give rise to GURs. These should be capable of generating, *via* rigorous derivation, the uncertainty relations predicted by model-independent gedanken experiments, but without the problems associated with modified commutation relations.

In this paper, we have proposed one such model, within the context of canonical quantum theory. Another, more radical, alternative is to consider additional quantum mechanical degrees of freedom, not present in the canonical theory, which are capable of describing quantum fluctuations of the background geometry. Such a model was proposed in a recent series of works (Lake, 2019; Lake et al., 2019; Lake et al., 2020; Lake, 2021a; Lake, 2021b) and shares many features with the model described here, including the existence of a new action scale that relates the accuracies of generalised position and momentum measurements, $\beta := 2\sigma_g\tilde{\sigma}_g \simeq 10^{-61}\hbar$ (*). The fundamental difference between the two models is the existence of new degrees of freedom in the latter. From this, it follows that the new action scale β implies a modified de Broglie relation of the form $\mathbf{p}' = \hbar\mathbf{k} + \beta(\mathbf{k}' - \mathbf{k})$, where, here, $\mathbf{p}'$ denotes the *observable* momentum. Heuristically, the non-canonical term $\beta(\mathbf{k}' - \mathbf{k})$ can be interpreted as an additional momentum "kick," transferred to the canonical wave function by a quantum fluctuation of the background. The interested reader is referred to (Lake, 2020; Lake, 2019; Lake et al., 2019; Lake et al., 2020; Lake, 2021a; Lake, 2021b; Lake et al., 2023) for further details.

At first glance, this more radical alternative has nothing to do with the POVM approach described here. It requires extra degrees of freedom associated with the quantum state of the background geometry, contrary to the POVM formalism, which remains entirely within the context of canonical quantum theory. It follows from Stinespring's dilation theorem (Stinespring, 1955; Paulsen, 2003), however, that the two formalisms are equivalent if we assume the particular values, $\sigma_g \simeq l_{\rm Pl}$ and $\tilde{\sigma}_g \simeq m_{\rm dS}c$, and hence the relation (*) above. The POVM picture results from tracing out the $\mathbf{x}'$ ($\mathbf{p}'$) degrees of freedom associated with quantum fluctuations of the background and the $\mathbf{x}'$ ($\mathbf{p}'$) degrees of freedom appear as a consequence of dilating the POVM.

The POVM approach describes a quantum measurement of finite accuracy. The minimum resolution of the measurement may be due to technical limitations, or it can reflect the fact that the minimum length and momentum scales are fundamentally related. We postulate that in a universe with both fundamental and technological limitations to measurement accuracy, the complete description of a realistic quantum measurement should be a POVM extension of the model presented in (Lake, 2019; Lake et al., 2019). We expect that this would give rise to two additional contributions to the position and momentum variances, i.e., $\sigma_g^2 + \sigma_h^2$ and $\tilde{\sigma}_g^2 + \tilde{\sigma}_h^2$, respectively, where g is the fundamental smearing function that models the quantum indeterminacy of space-time, and h is the envelope function of a realistic detector. In the limit $\sigma_h \gg \sigma_g$, $\tilde{\sigma}_h \gg \tilde{\sigma}_g$, which corresponds to all present-day measurements, the latter are expected to dominate the former.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Problems with modified commutators

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The purpose of this paper is to challenge the existing paradigm on which contemporary models of generalised uncertainty relations (GURs) are based, that is, the assumption of modified commutation relations. We review an array of theoretical problems that arise in modified commutator models, including those that have been discussed in depth and others that have received comparatively little attention, or have not been considered at all in the existing literature, with the aim of stimulating discussion on these topics. We then show how an apparently simple assumption can solve, or, more precisely, evade these issues, by generating GURs without modifying the basic form of the canonical Heisenberg algebra. This simplicity is deceptive, however, as the necessary assumption is found to have huge implications for the quantisation of space-time and, therefore, gravity. These include the view that quantum space-time should be considered as a quantum reference frame and, crucially, that the action scale characterising the quantum effects of gravity, β, must be many orders of magnitude smaller than Planck's constant, $\beta \sim 10^{-61} \times \hbar$, in order to recover the present day dark energy density. We argue that these proposals should be taken seriously, as a potential solution to the pathologies that plague minimum length models based on modified commutators, and that their implications should be explored as thoroughly as those of the existing paradigm, which has dominated research in this area for almost three decades.

KEYWORDS

generalised uncertainty relations, generalised uncertainty principle, extended uncertainty principle, modified commutation relations, minimum length, minimum momentum, quantum geometry, quantum gravity

1 Introduction

Thought experiments in quantum gravity suggest the existence of generalised uncertainty relations (GURs) (Maggiore, 1993; Adler and Santiago, 1999; Scardigli, 1999; Bolen and Cavaglia, 2005; Bambi and Urban, 2008; Park, 2008) and two of the most widely studied GURs are known as the generalised uncertainty principle (GUP) and the extended uncertainty principle (EUP). These may be written as

$$\Delta x^i \gtrsim \frac{\hbar}{2\Delta p_j}\delta^i{}_j\left[1+\alpha_0\frac{2G}{\hbar c^3}(\Delta p_j)^2\right], \tag{1.1}$$

and

$$\Delta p_j \gtrsim \frac{\hbar}{2\Delta x^i}\delta^i{}_j\left[1+2\eta_0\Lambda(\Delta x^i)^2\right], \tag{1.2}$$

respectively, where α_0 and η_0 are numerical constants of order unity. The GUP implies the existence of a minimum length scale of the order of the Planck length (Maggiore, 1993;

Adler and Santiago, 1999; Scardigli, 1999) whereas the EUP implies a minimum momentum scale of the order of the de Sitter momentum (Bolen and Cavaglia, 2005; Bambi and Urban, 2008; Park, 2008) For later convenience, we define the Planck and de Sitter scales as

$$l_{\rm Pl} := \sqrt{\hbar G/c^3} \simeq 10^{-33}{\rm cm}, \quad m_{\rm Pl} := \sqrt{\hbar c/G} \simeq 10^{-5}{\rm g},$$
$$l_{\rm dS} := \sqrt{3/\Lambda} \simeq 10^{28}{\rm cm}, \quad m_{\rm dS} := (\hbar/c)\sqrt{\Lambda/3} \simeq 10^{-66}{\rm g}, \tag{1.3}$$

where $\Lambda \simeq 10^{-56}{\rm cm}^{-2}$ is the cosmological constant (Betoule et al., 2014; Aghanim et al., 2021). Assuming both minimum length and momentum scales suggests the extended generalised uncertainty principle (EGUP),

$$\Delta x^i \Delta p_j \gtrsim \frac{\hbar}{2}\delta^i{}_j\left[1 + \alpha_0\frac{2G}{\hbar c^3}(\Delta p_j)^2 + 2\eta_0\Lambda(\Delta x^i)^2\right], \tag{1.4}$$

but **Eqs. 1.1–1.4** are heuristic and it remains an open problem how to derive the GUP, EUP and EGUP rigorously, from a modified quantum formalism.

Until recently, the only method considered in the existing literature was to modifiy the canonical commutation relations such that (Tawfik and Diab, 2014; Tawfik and Diab, 2015)

$$[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i{}_j\,\hat{\mathbb{I}} \mapsto [\hat{X}^i, \hat{P}_j] = i\hbar\delta^i{}_j F(\hat{\mathbf{P}}, \hat{\mathbf{X}}), \tag{1.5}$$

which gives rise to GURs via the Schrödinger-Robertson relation (Robertson, 1929; Schrödinger, 1999) Throughout this work we use capital letters to denote modified operators, that give rise to modified commutators, and lower case letters to denote their canonical quantum counterparts. Unfortunately, this apparently reasonable assumption has been shown to give rise to a variety of pathologies (Hossenfelder, 2013; Hossenfelder, 2014; Tawfik and Diab, 2014). These strongly suggest that modified commutator models are not mathematically self-consistent (Lake, 2020; Lake et al., 2023).

In this paper, we review six fundamental problems encountered by GUR models based on modified commutation relations.

1. Violation of the equivalence principle,
2. Violation of Lorentz invariance in the relativistic limit,
3. The 'soccer ball' problem for multi-particle states,
4. The reference frame-dependence of the 'minimum' length,
5. The background geometry is *not* quantum,
6. The mathematical inconsistency of modified phase space volumes.

The first three of these have been discussed at length in the literature (see, for example (Hossenfelder, 2013; Hossenfelder, 2014; Tawfik and Diab, 2014) and references therein) and we review them only briefly. The fourth and fifth problems were discussed previously in (Lake, 2020) but, to the best of our knowledge, have not been discussed elsewhere. The sixth and final problem raised in this short review has, surprisingly, not been considered before. Nonetheless, we argue that it represents the most serious objection yet raised against the modified commutator paradigm.

We review each problem, sequentially, in **Sections 2.1–2.6**. In **Section 3**, we consider the relative importance of each, and ask whether or not such problems could instead be viewed as features, rather than bugs, of a viable extension of canonical quantum mechanics. An alternative model, that circumvents these issues without the use of modified commutation relations, is reviewed in **Section 4**. Our conclusions are summarised in **Section 5**.

2 Problems with modified commutators

2.1 Violation of the equivalence principle

In canonical quantum mechanics (QM), the Heisenberg equation for the time evolution of an arbitrary Hermitian operator $\hat{O}$ is

$$\frac{d}{dt}\hat{O}(t) = \frac{i}{\hbar}[\hat{H}, \hat{O}] + \left(\frac{\partial \hat{O}}{\partial t}\right)_H, \tag{2.1}$$

where $\hat{H} = \hat{p}^2/(2m) + V(\hat{\mathbf{x}})$. For the position operator $\hat{x}^i(t)$ this gives

$$\frac{d}{dt}\hat{x}^i(t) = \frac{\hat{p}^i}{m}, \tag{2.2}$$

where right-hand side follows from the form of the canonical position-momentum commutator, $[\hat{x}^i, \hat{p}_j] = i\hbar\delta^I{}_j\hat{\mathbb{I}}$. From **Eq. 2.2**, it follows that the acceleration of the position expectation value of a quantum particle is independent of its mass:

$$\hat{a}^i = \frac{1}{m}\frac{d\hat{p}^i}{dt} = \frac{d^2\hat{x}^i}{dt^2}. \tag{2.3}$$

For the generalised operators $\hat{X}^i$ and $\hat{P}^j$ satisfying the modified commutator

$$[\hat{X}^i, \hat{P}_j] = i\hbar\delta^i{}_j G(\hat{\mathbf{P}}), \tag{2.4}$$

the Heisenberg equation for $\hat{X}^i(t)$ is

$$\frac{d}{dt}\hat{X}^i(t) = \frac{\hat{P}^i}{m}G(\hat{\mathbf{P}}), \tag{2.5}$$

so that, for $G(\hat{\mathbf{P}}) \neq 1$, the particle experiences a mass-dependent acceleration:

$$\hat{A}^i = \frac{1}{m}\frac{d\hat{P}^i}{dt} = \frac{1}{G(\hat{\mathbf{P}})}\left[\frac{d^2\hat{X}^i}{dt^2} - \frac{\hat{P}^i}{m}\frac{dG(\hat{\mathbf{P}})}{dt}\right]. \tag{2.6}$$

Although there is no universally agreed upon formulation of the equivalence principle (EP) for quantum systems (Paunkovic and Vojinovic, 2022), it is clear that such acceleration violates any sensible definition of the EP in the quantum regime, and, crucially, no experimental evidence has yet been found to support its existence.

This argument assumes that the generalised Hamiltonian takes the form $\hat{H} = \hat{P}^2/(2m) + V(\hat{\mathbf{X}})$ and that a well defined Heisenberg picture exists in the generalised theory, but both of these are reasonable assumptions. Similar analyses demonstrate that the EP is also violated in models with modified commutators characterised by the functions $G(\hat{\mathbf{X}})$ and $G(\hat{\mathbf{X}}, \hat{\mathbf{P}})$. It is therefore impossible to obtain the GUP, EUP, or EGUP from modified commutator models without violating the founding principles of classical gravity and, most likely, any viable generalisation of the these principles that includes the quantum realm (Paunkovic and Vojinovic, 2022).

2.2 Violation of Lorentz invariance

We recall that the canonical Heisenberg algebra is simply an $\hbar$-scaled representation of the shift-isometry algebra of Euclidean space and of space-like slices of flat space-time in the relativistic limit, i.e., it is the translation subgroup of the full Poincaré group that characterises the symmetries of Minkowski space, including translations, Lorentz boosts and rotations. Therefore, any modification of the Heisenberg algebra implies the violation of translational symmetry *unless* we choose to interpret it as a manifestation of a modified de Broglie relation. In this case, the physical momentum **p** is a nonlinear function of the wavenumber **k**, but the latter may still be identified with the shift-isometry generator of the background space, $\hat{\mathbf{k}} \equiv \hat{\mathrm{d}}_{\mathbf{x}}$.

Unfortunately, both these scenarios lead to inconsistencies. In the first, in which we interpret the modified Heisenberg algebra as a manifestation of broken translation invariance, one faces a problem in defining the classical limit of the theory. Implementing a canonical quantisation scheme $\{O_1, O_2\}_{\rm PB} = \lim_{\hbar\to 0}\frac{1}{i\hbar}[\hat{O}_1, \hat{O}_2]$ and requiring the correspondence principle (Rae, 2002) to hold implies an equivalent modification of the canonical Poisson brackets. This violates Galilean invariance, even for classical macroscopic systems, and, hence, Poincaré invariance in the relativistic limit. To date, no evidence for the breaking of Poincaré invariance, including shift invariance, has been found, although bounds on the symmetry breaking parameters have been determined from a wide range of experiments (Gupta et al., 2022).

In the second scenario, one encounters problems related to the nonlinearity of $p(k)$, where $p = (p_0, -\mathbf{p})$ and $k = (k_0, -\mathbf{k})$ denote the relativistic 4-momentum and its corresponding wave number, respectively. (From here on, we neglect space-time indices for the sake of notational convenience.) When $p(k)$ is nonlinear it is unclear if we should require the physical momentum p or wave number k, also known as the pseudo-momentum, to transform under the Poincaré group. Choosing the wave number as the Lorentz invariant quantity, the Lorentz transformations become nonlinear functions of k and the transformation of the sum $k_1 + k_2$ is no longer equal to the sum of the transformations of k_1 and k_2, individually. Likewise, choosing p as the Lorentz invariant variable, which is physically more reasonable, a similar problem occurs and the transformation of $p_1 + p_2$ is no longer equal to the sum of the individual transformations of p_1 and p_2. Each case requires the definition of a new nonlinear addition law, either for the pseudo-momenta, or for the physical momenta, respectively (Hossenfelder, 2013).

In the latter case, the new sum rule for the physical momenta is independent of the chosen inertial frame, by construction, but a new problem is created. If the nonlinear composition function has a maximum at the Planck momentum, as implied by consistency with the GURs generated by the modified de Broglie relation, then the sum of momenta will never exceed this maximum value. The Planck momentum, $m_{\rm Pl}c \simeq 10^5$ g cm^{-1}, is large for fundamental particles with rest masses $m \ll m_{\rm Pl}$ but very small for macroscopic objects with rest masses $M \gg m_{\rm Pl}$, which may easily exceed it at ordinary non-relativistic velocities. The problem of reproducing a sensible multi-particle limit when choosing the physical momentum to transform under modified (nonlinear) Lorentz transformations is known as the 'soccer ball problem' (Hossenfelder, 2013; Hossenfelder, 2014). This will be considered in more detail in the following section, in which we outline a recently proposed solution (Amelino-Camelia, 2017), and its critique.

From the remarks above it is clear that the introduction of nonlinear de Broglie relations **p(k)** in non-relativistic QM requires that $p = (p_0, -\mathbf{p})$ must be a nonlinear function of $k = (k_0, -\mathbf{k})$ in the relativistic limit. This makes Lorentz violation unavoidable *unless* one introduces a new nonlinear composition law, either for the pseudo-momentum k, or the physical momentum $p(k)$. However, this leads to new problems, and it is not clear whether sensible multi-particle limits of such theories exist (Hossenfelder, 2013; Hossenfelder, 2014). In **Section 2.3** we argue that, despite valiant attempts (Amelino-Camelia, 2017), the soccer ball problem has not, in fact, been solved. This shows that a sensible relativistic limit of an arbitrary GUR model cannot be obtained by introducing a nonlinear composition law for Lorentz boosts, and one is left with Lorentz violation as the only possible outcome of such theories. Though not absolutely ruled out experimentally, the parameters characterising such violations are severely constrained by observations (Pérez de los Heros and Terzić, 2022).

Nonetheless, this does not necessarily mean that GURs imply Lorentz violation. The problem, here, is the derivation of GUP- and EUP-type relations from the assumption of a modified Heisenberg algebra. In **Section 3** we show how the GUP, EUP and EGUP can be derived from an alternative mathematical structure, which leaves the canonical Heisenberg algebra unchanged except for a simple rescaling of the form $\hbar \to \hbar + \beta$, and which, therefore, is compatible with Lorentz symmetry in the relativistic regime.

2.3 The soccer ball problem

A brief overview of the soccer ball problem was given in the previous section, in connection with the issue of Lorentz violation, and we will not repeat it here. Instead, we focus on the main proposal for a solution of the problem (Amelino-Camelia, 2017) and show that, unfortunately, this is not compatible with general GUR models.

In (Amelino-Camelia, 2017) an ingenious solution to the soccer ball problem was proposed by Amelino-Camelia, who argued that the common formulation of the problem was, in fact, "a case of mistaken identity". In his proposal, the generalised momentum operators of a given modified commutator model are considered as the generators of 'generalised' spatial translations, by definition. This requires the unitary operator $\hat{U}(\mathbf{X}) := \exp[(i/\hbar)\mathbf{X}.\hat{\mathbf{P}}]$ to leave the modified $[\hat{X}^i, \hat{P}_j]$, $[\hat{X}^i, \hat{X}^j]$ and $[\hat{P}_i, \hat{P}_j]$ commutators, as well as the multi-particle Hamiltonian of the model, $\hat{H} := \sum_{I=1}^{N}\hat{\mathbf{P}}_I^2/(2m_I) + V(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2, \ldots \hat{\mathbf{X}}_N)$, where the subscript I labels the particle number, invariant.

Amelino-Camelia's key observation was that, if these invariances hold in a given model, then the corresponding Noether charge for an N-particle state is represented by the operator $\hat{\mathbf{P}}_{\rm Total} := \sum_{I=1}^{N}\hat{\mathbf{P}}_I$, which automatically commutes with the Hamiltonian: $[\hat{\mathbf{P}}_{\rm Total}, \hat{H}] = 0$. The usual law of linear momentum addition then holds for multi-particle states but a different nonlinear addition law, derived from the notion of spatial locality, holds for transfers of momentum between individual particles, due to the interactions specified by $\hat{H}$. In general, this requires the interaction potential $V(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2, \ldots \hat{\mathbf{X}}_N)$ to be carefully chosen so that $\sum_{I=1}^{N}\partial V/\partial \mathbf{X}_I = 0$, but this was shown to be possible in a specific example model containing two particles with equal masses, $m_A = m_B = m$, interacting in a 2-dimensional plane (Amelino-Camelia, 2017).

Unfortunately for GUP models, in the example system considered in (Amelino-Camelia, 2017), the definition of generalised spatial translation required to maintain the linear addition law for multi-requires the relation $[\hat{X}^i, \hat{P}_i] = 0$ to hold, for some $i \in \{A, B\}$, for both particles. In this case, there is no Heisenberg uncertainty principle, in at least one of the spatial dimensions, let alone a GUP, even though a minimum length-scale l still appears in the model via the position-position commutator, $[\hat{X}^1_I, \hat{X}^2_I] = il\hat{X}^1_I$.

This illustrates a more general point: it is by no means certain that a particular modified momentum operator, corresponding to a particular modification of the canonical Heisenberg algebra, and, therefore, a particular form of GUR, is compatible with a linear addition law for multi-particle states, derived via the procedure outlined in (Amelino-Camelia, 2017). In our view, this is not a weakness of Amelino-Camelia's method, which successfully demonstrates that *certain classes* of modified commutator models do not, in fact, suffer from a soccer ball problem after all. Instead, it is an inherent weakness of models that seek to derive the GUP, EUP or EGUP from modified commutation relations. In this respect, the analysis given in (Amelino-Camelia, 2017) still represents a huge step forward in understanding this problem, and we may apply Amelino-Camelia's procedure to any prospective GUP model based on modified commutators, using it to rule out the ones that give rise to these kinds of inconsistencies in the multi-particle limit.

In summary, although consistency with Amelino-Camelia's procedure represents a useful criterion for identifying physically viable theories, it is clear that arbitrary deformations of the canonical Heisenberg algebra are not consistent with the existence of a linear momentum addition law for multi-particle states. Further work is therefore needed to determine which GUR models truly suffer from a soccer ball problem and which ones do not. Though some GUP-type models may be free from this pathology, it is likely that a great many are still afflicted by it. It is therefore clear that the soccer ball problem has not been resolved, in general, for arbitrary GUP, EUP or EGUP models based on modified commutation relations.

2.4 Reference frame-dependence of the 'minimum' length

In their pioneering and hugely influential work (Kempf et al., 1995), Kempf, Mangano and Mann (KMM) gave the first truly rigorous treatment of modified commutator models, showing how they can be derived from the Hilbert space structure of a modified quantum formalism. Their key observation was that modified commutators correspond to modified phase space volumes. For example, the commutator

$$[\hat{X}^i, \hat{P}_j] = i\hbar\delta^i{}_j\left(1+\alpha\hat{\mathbf{P}}^2\right)\hat{\mathbb{I}}, \tag{2.7}$$

where $\alpha = \alpha_0(m_{\rm Pl}c)^{-2}$ and α_0 is a dimensionless constant of order one, which leads to the GUP-type relation

$$\Delta_\psi X^i \Delta_\psi P_j \geq \frac{\hbar}{2}\delta^i{}_j\left(1+\alpha\left[\left(\Delta_\psi \mathbf{P}\right)^2 + \langle\hat{\mathbf{P}}\rangle^2_\psi\right]\right), \tag{2.8}$$

corresponding to a modified normalisation condition and a modified resolution of the identity of the form

$$\langle\mathbf{P}|\mathbf{P}'\rangle = \left(1+\alpha\mathbf{P}^2\right)\delta^3\left(\mathbf{P}-\mathbf{P}'\right), \quad \int|\mathbf{P}\rangle\langle\mathbf{P}|\frac{{\rm d}^3{\rm P}}{\left(1+\alpha\mathbf{P}^2\right)} = \hat{\mathbb{I}}. \tag{2.9}$$

Similarly, the commutator

$$[\hat{X}^i, \hat{P}_j] = i\hbar\delta^i{}_j\left(1+\eta\hat{\mathbf{X}}^2\right)\hat{\mathbb{I}}, \tag{2.10}$$

where $\eta = \eta_0 l_{\rm dS}^{-2}$ and η_0 is a dimensionless constant of order unity, which leads to the EUP-type relation

$$\Delta_\psi X^i \Delta_\psi P_j \geq \frac{\hbar}{2}\delta^i{}_j\left(1+\eta\left[\left(\Delta_\psi \mathbf{X}\right)^2 + \langle\hat{\mathbf{X}}\rangle^2_\psi\right]\right), \tag{2.11}$$

corresponding to the modified phase space structure

$$\langle\mathbf{X}|\mathbf{X}'\rangle = \left(1+\eta\mathbf{X}^2\right)\delta^3\left(\mathbf{X}-\mathbf{X}'\right), \quad \int|\mathbf{X}\rangle\langle\mathbf{X}|\frac{{\rm d}^3{\rm X}}{\left(1+\eta\mathbf{X}^2\right)} = \hat{\mathbb{I}}. \tag{2.12}$$

In general, introducing a modified momentum space volume $G(\mathbf{P})^{-1}{\rm d}^3{\rm P}$ $(G(\mathbf{P}) \neq 1)$ yields a GUP-type relation, though in this case the position space representation is not well defined, whereas introducing a modified position space volume $G(\mathbf{X})^{-1}{\rm d}^3{\rm X}$ $(G(\mathbf{X}) \neq 1)$ yields an EUP-type relation, although the momentum space representation is not well defined. For an EGUP-type relation, characterised by the function $G(\mathbf{X},\mathbf{P}) \neq 1$, neither the position nor momentum space representations are well defined and one must instead introduce a generalised Bargman-Fock representation (Kempf, 1997).

To illustrate the problems with these type of constructions, we will focus on the most famous example, proposed in the original KMM paper (Kempf et al., 1995), i.e., the GUP-type relation **Eq. 2.8**. It is straightforward to show that **Eq. 2.8** implies the existence of a 'minimum' position uncertainty, $(\Delta_\psi X^i)_{\rm min}$, and a corresponding critical value of the momentum uncertainty, $(\Delta_\psi P_j)_{\rm crit}$, of the form

$$\left(\Delta_\psi X^i\right)_{\rm min} = \hbar\sqrt{\alpha\left(1+\alpha\langle\hat{\mathbf{P}}\rangle^2_\psi\right)}, \quad \left(\Delta_\psi P_j\right)_{\rm crit} = 1/\sqrt{\alpha\left(1+\alpha\langle\hat{\mathbf{P}}\rangle^2_\psi\right)}. \tag{2.13}$$

The problem with these expressions is that, while the standard deviations on the left-hand sides should be manifestly frame-independent, the quantities on the right are not, since $\langle\hat{\mathbf{P}}\rangle^2_\psi$ is not invariant under Galilean velocity boosts.

To show this more concretely, let us consider the action of the unitary operator (Lake, 2020)

$$\tilde{\mathcal{U}}(\mathbf{P}')|\mathbf{P}\rangle = \sqrt{\frac{1+\alpha\mathbf{P}^2}{1+\alpha(\mathbf{P}-\mathbf{P}')^2}}|\mathbf{P}-\mathbf{P}'\rangle. \tag{2.14}$$

This generates Galilean velocity boosts, which remain consistent with the modified momentum space volume **Eq. 2.9**, and reduces to the canonical boost operator $\hat{U}(\mathbf{p}')|\mathbf{p}\rangle = |\mathbf{p}-\mathbf{p}'\rangle$ when $\alpha = 0$. Its action on the moments of the generalised momentum operator,

$$\hat{P}_j = \int P_j|\mathbf{P}\rangle\langle\mathbf{P}|\frac{{\rm d}^3{\rm P}}{\left(1+\alpha\mathbf{P}^2\right)}, \tag{2.15}$$

gives

$$\hat{\mathbf{P}}^n \mapsto \tilde{\mathcal{U}}(\mathbf{P}')\hat{\mathbf{P}}^n\tilde{\mathcal{U}}^\dagger(\mathbf{P}') = \left(\hat{\mathbf{P}}+\mathbf{P}'\right)^n, \tag{2.16}$$

for $n \in \mathbb{N}$, and it is straightforward to demonstrate that this leaves $\Delta_\psi P_j$ unchanged. The modified commutator **Eq. 2.8** then transforms as

$$[\hat{X}^i, \hat{P}_j] \mapsto \tilde{\mathcal{U}}(\mathbf{P}')[\hat{X}^i, \hat{P}_j]\tilde{\mathcal{U}}^\dagger(\mathbf{P}') = i\hbar\delta^i{}_j\left(1+\alpha\left(\hat{\mathbf{P}}+\mathbf{P}'\right)^2\right)\hat{\mathbb{I}}, \tag{2.17}$$

which leads to $\mathbf{P}'$-dependence of the corresponding uncertainty principle. Since $\Delta_\psi P_j$ is invariant, it is clear that this must be due to the $\mathbf{P}'$-dependence of $\Delta_\psi X^i$.

Next, let us denote the boosted position uncertainty as $\Delta_\psi X'^i(\mathbf{P}')$, so that $\Delta_\psi X'^i(0) \equiv \Delta_\psi X^i$, where $\Delta_\psi X^i$ is the position uncertainty given in Eq. 2.8. We then have

$$\Delta_\psi X'^i(\mathbf{P}') \geq \frac{\hbar}{2\Delta_\psi P_j}\delta^i{}_j\left(1+\alpha\left[\left(\Delta_\psi \mathbf{P}\right)^2 + \left(\langle\hat{\mathbf{P}}\rangle_\psi + \mathbf{P}'\right)^2\right]\right). \tag{2.18}$$

Hence, even if $\psi(\mathbf{P})$ is symmetric in the original frame of observation, such that $\langle \hat{\mathbf{P}} \rangle_\psi = 0$, the minimum position uncertainty seen by an observer moving with relative velocity $\mathbf{V}' = \mathbf{P}'/m$ is

$$\left(\Delta_\psi X'^i\right)_{\min}(\mathbf{P}') \simeq \hbar\sqrt{\alpha}\left(1 + \frac{\alpha \mathbf{P}'^2}{2}\right). \quad (2.19)$$

For $|\mathbf{P}'| \ll 1/\sqrt{\alpha} \simeq m_{\rm Pl}c$ the boost-dependent term is, of course, very small, but its presence clearly violates the Galilean boost invariance that emerges from the low velocity limit of Lorentz invariance. Its presence is therefore at odds with the founding principles of both special and general relativity, even for one-particle states. Analogous reasoning demonstrates the frame-dependence of the 'minimum' momentum implied by the EUP-type relation Eq. 2.11, which is no longer invariant under spatial translations.

Though it is possible that boost and/or translation invariance may be broken due to quantum effects on the geometry of spacetime, we note that there is, intrinsically, nothing quantum mechanical about the physical space background of the KMM model. The geometry remains classical but its symmetries are unknown, as is the exact form of the classical metric, $g_{ij}(X)$, to which they correspond. In Section 2.5 and Section 2.6, we consider the implications of both these points and argue that they lead to further inconsistencies in the modified commutator paradigm.

2.5 The background geometry is *not* quantum

In the existing literature, there are many references to the 'quantum' geometry obtained by introducing modified phase space volumes, as in the KMM model (Kempf et al., 1995). This motivates a class of so-called nonlocal gravity models that, it is claimed, follow from the 'quantum gravity' corrections implied by GURs. In this section, we examine the link between modified commutation relations and the proposed nonlocality of the background geometry, and find that this claim does not hold up to scrutiny. Instead, we find that the background geometry implied by modified phase space volumes is certainly 'classical', in the sense that it does not admit quantum superpositions of states, but that, unlike classical geometries proper, it is not well defined by an appropriate class of symmetries or a metric function, $g_{ij}(X)$. These latter considerations are dealt with in detail in Section 2.6.

Much of the literature on nonlocal gravity models is motivated by the observation that modified momentum space volumes, such as those leading to the GUP-type model proposed in (Kempf et al., 1995), can be obtained by acting with an appropriate nonlocal operator on the position space representations of the canonical QM eigenfunctions, $\langle \mathbf{x}|\mathbf{x}'\rangle = \delta^3(\mathbf{x}-\mathbf{x}')$ and $\langle \mathbf{x}|\mathbf{p}\rangle = (2\pi\hbar)^{-3/2}e^{i\mathbf{p}.\mathbf{x}/\hbar}$. A simple example is the operator $e^{l^2\Delta}$, where l is a fundamental length scale, usually identified with the Planck length, and Δ is the Laplacian. For convenience, we rewrite this in the spectral representation as

$$\hat{\zeta} = e^{-\hat{H}_0\Delta t/\hbar}, \quad (2.20)$$

where $\hat{H}_0 = \hat{p}^2/2m$ is the canonical free particle Hamiltonian and $\Delta t = 2m\,l^2/\hbar$ is the characteristic time scale associated with l and the particle mass m. The operator $\hat{\zeta}$ reduces to $e^{l^2\Delta}$ in the wave mechanics picture but we may use Eq. 2.20 to define its action directly on the canonical eigenstates, $|\mathbf{x}\rangle$ and $|\mathbf{p}\rangle$, instead of the eigenfunctions, $\langle \mathbf{x}|\mathbf{x}'\rangle$ and $\langle \mathbf{x}|\mathbf{p}\rangle$.

This may not seem, at first, like an important distinction, but it is crucial to recognise that $|\mathbf{x}\rangle$ and $\psi(\mathbf{x})$ have dimensions of $(\text{length})^{-3/2}$ whereas $\langle \mathbf{x}|\mathbf{x}'\rangle$ and $|\psi(\mathbf{x})|^2$ have dimensions of $(\text{length})^{-3}$. Therefore, the position eigenfunction has the dimensions of a *probability density*, whereas the position eigenvector has the dimensions of a *quantum probability amplitude*. This matters because probability densities, including the Dirac delta, are inherently classical in nature, even when they are derived from an underlying quantum mechanical amplitude. It is elementary to rewrite any classical probability distribution as the square of a complex distribution, $\rho(\mathbf{x}) \equiv \rho_\psi(\mathbf{x}) = |\psi(\mathbf{x})|^2$, but this does not imply that it is quantum mechanical in origin. Furthermore, even if it *is* quantum mechanical in origin, measurements that depend on $|\psi(\mathbf{x})|^2$ alone destroy all phase information, and are operationally indistinguishable from outcomes that depend only on classical probabilities (i.e., those based on incomplete information about the system) (Isham, 1995).

Because of this, delocalising the canonical eigen*functions*, rather than the eigen*states*, maps classical point charges to classical charge-densities of nonzero volume, but does not introduce a genuine quantum state, i.e., a vector in a complex Hilbert space, corresponding to the quantum state of the background geometry. Though it is not very flattering to state it in this way, applying the operator $e^{l^2\Delta}$ to $\delta^3(\mathbf{x}-\mathbf{x}')$ blows up classical point-masses to classical golf balls or grapefruits, but does not achieve much else. Furthermore, it is not at all clear whether a classical geometry, in which each zero-dimensional point has somehow been blown up to the size of a three-dimensional grapefruit (or Planck volume), is really a well defined object. In the final subsection, Section 2.6, we will argue that modified phase space volumes cannot be consistently defined in a *physical* geometry, but, before that, we consider the action of $\hat{\zeta}$ on $\langle \mathbf{x}|\mathbf{x}'\rangle$ and $\langle \mathbf{x}|\mathbf{p}\rangle$ in more detail and show, explicitly, that the nonlocal geometry generated by this action is *classical*.

In the usual approach to nonlocal geometry models, the braket $\langle \mathbf{x}|\hat{\zeta}|\mathbf{x}'\rangle$ is used to define a set of generalised basis vectors, $|\mathbf{X}\rangle$ and $|\mathbf{P}\rangle$, such that

$$\langle \mathbf{x}|\hat{\zeta}|\mathbf{x}'\rangle = e^{\sigma^2\Delta}\langle \mathbf{x}|\mathbf{x}'\rangle = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^3 e^{-(\mathbf{x}-\mathbf{x}')^2/2\sigma^2}$$
$$\equiv \langle \mathbf{X}|\mathbf{X}'\rangle = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^3 e^{-(\mathbf{X}-\mathbf{X}')^2/2\sigma^2}, \quad (2.21)$$

and

$$\langle \mathbf{x}|\hat{\zeta}|\mathbf{p}\rangle = e^{\sigma^2\Delta}\langle \mathbf{x}|\mathbf{p}\rangle = \left(\frac{1}{\sqrt{2\pi\hbar}}\right)^3 e^{-\mathbf{p}^2/2\tilde{\sigma}^2}e^{i\mathbf{p}.\mathbf{x}/\hbar}$$
$$\equiv \langle \mathbf{X}|\mathbf{P}\rangle = \left(\frac{1}{\sqrt{2\pi\hbar}}\right)^3 e^{-\mathbf{P}^2/2\tilde{\sigma}^2}e^{i\mathbf{P}.\mathbf{X}/\hbar}, \quad (2.22)$$

where we have rewritten $\sigma \equiv l$ and defined $\tilde{\sigma} \equiv \hbar/\sqrt{2}l$, for later convenience. In other words, the nonlocal operator $\hat{\zeta}$ maps the Dirac delta to a finite-volume Gaussian of width σ via

$$\hat{\zeta}: \langle \mathbf{x}|\mathbf{x}'\rangle = \delta^3(\mathbf{x}-\mathbf{x}') \mapsto \langle \mathbf{X}|\mathbf{X}'\rangle = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^3 e^{-(\mathbf{X}-\mathbf{X}')^2/2\sigma^2}, \quad (2.23)$$

and Eq. 2.22 shows its corresponding action on the plane wave $\langle \mathbf{x}|\mathbf{p}\rangle = (2\pi\hbar)^{-3/2}e^{i\mathbf{p}.\mathbf{x}/\hbar}$. Consistency then requires the $|\mathbf{P}\rangle$ eigenstates to satisfy the modified normalisation condition and modified resolution of the identity

$$\langle \mathbf{P}|\mathbf{P}'\rangle = e^{\mathbf{P}^2/2\tilde{\sigma}^2}\delta^3(\mathbf{P}-\mathbf{P}'), \quad \int |\mathbf{P}\rangle\langle\mathbf{P}|e^{-\mathbf{P}^2/2\tilde{\sigma}^2}\mathrm{d}^3\mathrm{P} = \hat{\mathbb{I}}. \tag{2.24}$$

Expanding $e^{\mathbf{P}^2/2\tilde{\sigma}^2}$ to first order generates the GUP-type relation of the KMM model (Kempf et al., 1995), but the full expressions differ nonperturbatively.

It is then claimed that the link between GURs and nonlocal gravity is provided by the semi-classical approach (Nicolini and Niedner, 2011; Lake, 2022a) in which the classical curvature of space-time is sourced by the expectation value of the energy-momentum operator for the quantum matter fields, $\langle \hat{T}_{\mu\nu}\rangle_{\psi}$. In the weak field limit, the semi-classical field equations reduce to Poisson's equation, with the square of the wave function as a source term, neglecting the subdominant dark energy contribution (Møller, 1962; Rosenfeld, 1963; Kelvin et al., 2020),

$$\nabla^2\Phi = 4\pi Gm|\psi|^2. \tag{2.25}$$

It is then noted that the zero-width limit of the wave function, $\Delta_{\psi}\mathbf{X}\to 0$, yields a delta function source,

$$\lim_{\Delta_{\psi}\mathbf{X}\to 0}|\psi|^2 = \delta^3(\mathbf{X}-\mathbf{X}'). \tag{2.26}$$

The standard procedure, therefore, is to substitute the limiting value **Eq. 2.26** into **Eq. 2.25** and, interpreting the former as the position eigenfunction of a quantum mechanical wave function, to act on it with the nonlocal operator $e^{\sigma^2\Delta}$. On this basis, it is often claimed that GURs imply nonlocal gravity and, furthermore, that the latter arises from 'quantum' corrections to the classical geometry. There are two main problems with this interpretation. First, that in the semi-classical approach on which it is based the geometry is still classical and, second, that $\delta^3(\mathbf{X}-\mathbf{X}')$ is a limiting value of $|\psi|^2$ (not ψ), which is operationally indistinguishable from a classical finite-density mass distribution.

We stress, again, that the action of the nonlocal operator on a Dirac delta source term blows up a classical point mass into a classical blob of finite density. If the blow up is sufficiently strong, in some sense, the classical matter fluid acquires an effective equation of state which makes it stiff enough to resist gravitational collapse, curing the 'singularity problem' (Nicolini, 2012), but this has nothing to do with any quantum properties, either of the matter, or of the background geometry. The claim that the GURs imply nonlocal gravity is therefore somewhat inaccurate. Instead, it is more accurate to claim that classical nonlocal gravity models, such as those defined by Eq. 2.21 and Eq. 2.22, or similar constructions, and modified commutator models, such as the GUP- and EUP-type relations defined by Eq. 2.9 and Eq. 2.12, respectively, stem from the same underlying assumptions, that is, assumed modifications of the classical phase space volumes, over which both classical densities and quantum mechanical amplitudes (wave functions) must be integrated.

In both cases, the number of degrees of freedom remains the same as in the corresponding local theory and no new quantum degrees of freedom, capable of corresponding to the quantum state of the geometry, are introduced. Thus, although the background geometry of modified commutator models is 'nonlocal', in some sense, it is *not* nonlocal due to quantum effects. Furthermore, it is by no means clear whether such 'classical nonlocality' is well defined. We address this point, in detail, in the following section.

2.6 Mathematical inconsistency of modified phase space volumes

In the standard prescription, GURs require modified commutators and modified commutators require modified phase space volumes, yielding a one-to-one correspondence between the two. For example, for the KMM GUP-type model, this relation is as follows (Kempf et al., 1995):

$$[\hat{X}^i,\hat{P}_j] = i\hbar\delta^i{}_j(1+\alpha\hat{\mathbf{P}}^2)\hat{\mathbb{I}} \Leftrightarrow \hat{P}_j = \int P_j|\mathbf{P}\rangle\langle\mathbf{P}|\frac{\mathrm{d}^3\mathrm{P}}{(1+\alpha\mathbf{P}^2)}. \tag{2.27}$$

In this section, we explore a number of subtle points that, although *implicit* in the construction above, and others like it, have not been explicitly considered in the existing literature.

To begin, we note that models of this form are based on a canonical quantisation procedure that maps classical Poison brackets to commutators, $\{x^i,p_j\}_{\mathrm{PB}} \mapsto \frac{1}{i\hbar}[\hat{X}^i,\hat{P}_j]$, and classical Hamiltonians to quantum Hamiltonians, $H = |\mathbf{p}|^2/(2m) + V(\mathbf{x}) \mapsto \hat{H} = |\hat{\mathbf{P}}|^2/(2m) + V(\hat{\mathbf{X}})$. Next, we recall that the phase space of classical mechanics, on which these maps are defined, is a symplectic manifold, and, furthermore, that symplectic geometry is a notoriously 'loose' form of geometry. Unlike the more familiar Riemannian geometry, which corresponds to our experience of everyday life, symplectic structures do not carry any notion of distance, but volumes can be defined through the introduction of an appropriate symplectic 2-form (Frankel, 1997; Nakahara, 2003).

Defining a new symplectic 2-form defines a new volume element, but this in no way disturbs the symplectic structure of classical Hamiltonian systems (Frankel, 1997; Nakahara, 2003). When the canonical quantisation procedure is applied, this symplectic structure is taken over, unchanged, by the corresponding quantum theory. Hence, if the volume element in the classical phase space is, say, $(1+\alpha\mathbf{P}^2)^{-1}\mathrm{d}^3\mathrm{x}\mathrm{d}^3\mathrm{P}$, the phase space volume in the corresponding quantum theory is, simply, $(1+\alpha\mathbf{P}^2)^{-1}\mathrm{d}^3\mathrm{P}$. It is then assumed that the quantum state vector can be expanded in the 'usual' form, except for the modification of the volume element, $|\psi\rangle = \int\psi(\mathbf{P})|\mathbf{P}\rangle(1+\alpha\mathbf{P}^2)^{-1}\mathrm{d}^3P$, without issue. But is this really the case?

To answer this question, we must consider the meanings of the symbols $\mathbf{P}$ and $\hat{\mathbf{P}}$, appearing in Eq. 2.27, carefully. This is more difficult than it seems, since it is seldom explicitly stated, in the existing GUP literature, what exactly these symbols represent. The most probable reason for this is that, of course, everyone already 'knows' what they mean: $\mathbf{P}$ is a momentum space displacement vector and $\hat{\mathbf{P}}$ is its vector operator counterpart. The former may be written as

$$\mathbf{P} = P_j\mathbf{e}^j(\mathrm{X}) = P_X\mathbf{e}^X(\mathrm{X}) + P_Y\mathbf{e}^Y(\mathrm{X}) + P_Z\mathbf{e}^Z(\mathrm{X}), \tag{2.28}$$

where (X,Y,Z) denote *global* Cartesian coordinates and $(\mathbf{e}^X(\mathrm{X}),\mathbf{e}^Y(\mathrm{X}),\mathbf{e}^Z(\mathrm{X}))$ denote the tangent vectors, to the lines $X=$ const, $Y=$ const, and $Z=$ const, at any point $\mathrm{X}\in\mathbb{R}^3$ in physical space. $\hat{\mathbf{P}}$ is then constructed in like manner, by replacing the components P_j with the operators $\hat{P}_j$, defined in Eq. 2.27.

The problems with this construction are as follows.

1. *Global* Cartesian coordinates only exist in Euclidean space (Frankel, 1997; Nakahara, 2003),
2. Euclidean space is a Riemannian geometry, not a symplectic geometry,

3. It therefore possesses a metric, $g_{ij}(\mathrm{X}) = \langle \mathbf{e}_i(\mathrm{X}), \mathbf{e}_j(\mathrm{X})\rangle$, which generates both a notion of distance, $\mathrm{d}L = \sqrt{g_{ij}(\mathrm{X})\mathrm{d}X^i\mathrm{d}X^j}$, and a notion of volume, $\mathrm{d}V = \sqrt{g}\mathrm{d}^3X$, where $g(\mathrm{X}) = \det g_{ij}(\mathrm{X})$ is the determinant of the metric,
4. In the global Cartesians the metric of Euclidean space is $\delta_{ij} = \langle \mathbf{e}_i(\mathrm{X}), \mathbf{e}_j(\mathrm{X})\rangle$, since $\mathbf{e}_i(\mathrm{X}) = \mathbf{e}_i\,(\mathrm{X}')$ for all $\mathrm{X}, \mathrm{X}' \in \mathbb{R}^3$ and $i \in \{X, Y, Z\}$, giving $\det \delta_{ij}(\mathrm{X}) = 1$,
5. This gives rise to the distance element $L = \sqrt{X^2 + Y^2 + Z^2}$ and the volume element $\mathrm{d}V = \mathrm{d}X\mathrm{d}Y\mathrm{d}Z$,
6. The corresponding magnitude of the momentum vector **Eq. 2.18** is $P = \sqrt{P_X^2 + P_Y^2 + P_Z^2}$ and the volume element in momentum space is $\mathrm{d}\tilde{V} = \mathrm{d}P_X\mathrm{d}P_Y\mathrm{d}P_Z$, since the tangent space is isomorphic to the physical Euclidean space (Frankel, 1997; Nakahara, 2003),
7. Neither of these expressions are flexible.

Unlike volume elements derived from symplectic 2-forms, the Euclidean volume element is fixed by the underlying geometry, as well as the chosen coordinate system. The tangent space geometry in which the momentum vector **P** is defined is also fixed, and, in global Cartesians, the lines of constant X, Y and Z lie parallel to the trajectories for which P_X, P_Y and P_Z are conserved. Thus, for the coordinates assumed in the relation $\Delta_\psi X^i \Delta_\psi P_j \geq \frac{\hbar}{2}\delta^i{}_j(1 + \alpha[(\Delta_\psi \mathbf{P})^2 + \langle \hat{\mathbf{P}}\rangle_\psi^2])$ **Eq. 2.8**, which is derived from the modified phase space structure **Eq. 2.27**, i.e., $i, j \in \{X, Y, Z\}$, where (X, Y, Z) denote global Cartesians, the volume element of physical space is fixed as $\mathrm{d}X\mathrm{d}Y\mathrm{d}Z$ and the corresponding momentum space volume is $\mathrm{d}P_X\mathrm{d}P_Y\mathrm{d}P_Z$.

Put simply, if (X, Y, Z) represent global Cartesian coordinates in physical space, (P_X, P_Y, P_Z) represent global Cartesians in the conjugate momentum space. This leads to a contradiction, indicating the inconsistency of modified phase space volumes like the one outlined above. If we assume that the subscripts i and j in $\Delta_\psi X^i$ and $\Delta_\psi P_j$ refer to global Cartesian coordinates, then the momentum space integration measure is simply d^3P. Conversely, if we abandon this assumption and *define* $\mathbf{P}^2$ such that $\mathbf{P}^2 \equiv P^2 = \sum_{j=1}^3 P_j^2$, without specifying the coordinates (X^1, X^2, X^3), we are faced with an even bigger problem: in this case, we cannot make any physical predictions at all!

As stated in the Introduction, we believe that this constitutes the most serious criticism of the modified commutator/modified phase space paradigm yet formulated in the literature. It has immediate real world implications since, to the best of our knowledge, existing experimental bounds on the GUP parameter α have all been obtained by adopting two contradictory assumptions: 1) that the GUP arises from a modified commutation relation, and 2) that the modified commutation relation holds for the uncertainties $\Delta_\psi X^i$ and $\Delta_\psi P_j$ where $i, j \in \{X, Y, Z\}$ refer to global Cartesian coordinates (Pikovski et al., 2012; Bosso et al., 2017; Kumar and Plenio, 2018; Girdhar and Doherty, 2020; Cui et al., 2021; Sen et al., 2022).

In **Section 4**, we show how to derive the GUP, EUP and EGUP for Cartesian uncertainties, without introducing modified commutation relations or phase space volumes. The new model avoids the inconsistencies inherent in these models, in an almost trivial way, but is found to have exceedingly nontrivial implications for the quantisation of the background geometry. Before that, in **Section 3**, we consider whether the problems outlined here might instead be considered as `features', rather than `bugs', of modified commutator models.

3 Bug or feature?

It could be argued that a number of the problems discussed above represent features, rather than bugs, of the modified commutator paradigm. For example, it is widely accepted that some kind of breakdown of the EP and/or Lorentz invariance must occur due to quantum gravitational effects. Therefore, it may be regarded as unsurprising that minimum length models based on modified commutators generate both as a matter of course. From this viewpoint, such features acquire the status of 'smoking guns', i.e., predictions of new physics in the low-energy regime that any self-consistent model of high-energy quantum gravity must successfully reproduce.

These are strong claims, and, here, we argue that such claims require strong motivations, which are currently lacking in existing models. Furthermore, it is necessary to prove beyond doubt that such modifications of the canonical formalism do not give rise to internal inconsistencies, even within their domain of applicability.

As an analogy, there are an infinite number of ways to break the Poincaré group symmetries of Minkowski space, but only one of these gives rise to a self-consistent limit in the non-relativistic regime, i.e., the Galilean group of Euclidean space, with time as a parameter. By contrast, arbitrary violations of Poincaré symmetry do not give rise to well-defined geometries of any kind. Similarly, the breaking of a given symmetry, or equivalence, in a physical theory, is not the same as introducing a well-motivated generalisation of the original model. Since there are an infinite number of ways to break anything, why should this or that violation be preferred? Is the resulting model self-consistent? Below, we briefly consider the 'bug or feature' argument for each of the problems raised in **Sections 2.1–2.6**.

1. Violation of the EP: That the EP must break down due to quantum gravity effects is certain, but this is so for a very simple reason - the standard EP, as formulated in general relativity, is an inherently *classical* concept. It concerns the equivalence between classical gravity and classical accelerated frames of reference, which, by definition, excludes the concept of quantum superposition. In the quantum gravitational regime, we expect classical geometries to be replaced by quantum superpositions of geometries, and classical accelerated frames of reference to be replaced by quantum superpositions thereof, but is this the same as simply *breaking* the equivalence between the existing classical concepts? While it is possible, and even likely, that an appropriate course-graining over such superpositions could give rise to discrepancies with the existing classical theory, it is by no means clear what form these ought to take. In the absence of any indication from a specific model in the UV sector, we argue that the prediction of mass-dependent accelerations, in a classical background `geometry', without a well-defined symmetry group, should be treated with extreme caution.[1] Nonetheless, it must be admitted that this remains a theoretical possibility, which cannot be excluded *a priori*, at the present time.
2. Violation of Lorentz invariance: Similar arguments can be made regarding the violation of Lorentz invariance in the relativistic limit. We expect the quantisation of the geometric background

1 We recall that modifications of the canonical Heisenberg algebra imply the violation of Galilean symmetries.

to introduce space-time superpositions, generalising the classical concept of a sharp space-time point and, with it, the concept of a sharply-defined classical frame of reference, whether accelerated or inertial, to include superpositions. But this is not the same as simply breaking Lorentz symmetry in a still-classical background, especially when it is not clear whether the broken symmetry group corresponds to a well-defined geometry or not. In general, deforming an algebra of group generators does not yield a well-defined group, and, hence, the resulting operators do not correspond to a well-defined geometry. It is not analogous to the shift from relativistic space-time to non-relativistic space-plus-time, but a far more ambiguous procedure. Nonetheless, it must again be admitted that an appropriate course-graining over superpositions in the low-energy limit of quantum gravity may, in fact, violate Lorentz invariance in a way predicted by some classes of modified commutator models.

3. The soccer ball problem: It must also be admitted that the partial resolution of the soccer ball problem, presented in (Amelino-Camelia, 2017), provides a potential way out of the problems above. In this approach, the issue of the background geometry is moot, since the generalised momentum operators of the modified commutation relation are required to generate modified (non-Galilean) symmetries of the Hamiltonian. They may then be regarded as the symmetry generators of a modified (non-Euclidean) geometry, by definition. However, this places severe constraints on the form these generators may take, and, as shown in **Section 2.3**, not all such modifications are compatible with the existence of the GURs suggested by quantum gravity thought experiments. Therefore, at present, it is not clear whether this problem should be regarded as a bug, or a feature, of modified commutator models.
4. The reference frame dependence of the minimum length: The situation is different in the case of the reference frame dependence of the minimum length (momentum) scale, predicted by the GUP (EUP). This clearly represents a mathematical inconsistency of modified commutator models, since a mere shift in the coordinate origin, or Galilean velocity boost $v \to v' \ll c$, radically alters the values of the position and/or momentum uncertainties of the quantum wave packet. The standard measure of the statistical variance of a random variable, $(\Delta O)^2 = \langle O^2\rangle - \langle O\rangle^2$, is constructed to be manifestly invariant under such transformations. If this invariance no longer holds in modified commutator models, it is unclear how to measure the spread of the wave function in a coordinate-independent way. This undoubtably counts as a 'bug'.
5. The background geometry is not quantum: We may also ask if the *classically* nonlocal background geometry of the KMM model (Kempf et al., 1995), and others like it, could emerge from a more fundamental underlying quantum theory? Theoretically, the answer is again 'yes', but the same could be said of any (totally arbitrary) modification of the canonical quantum formalism. The most important point here is that, at present, no concrete proposal for such a model has been presented in the GUP literature (Hossenfelder, 2013; Tawfik and Diab, 2014; Tawfik and Diab, 2015). It had been hoped that the nonlocal operator, Eq. **2.20**, provided such a link, between the coarse-grained classical structure and an underlying probabilistic (i.e., quantum) model, but this assumption was debunked in **Section 2.5**. The confusion in the literature stemmed from a confusion between classical probability densities and quantum probability *amplitudes*. The status of this problem, therefore, resembles that of problems 1–2. Theoretically, the classically nonlocal geometry produced by the action of Eq. **2.20** may emerge from an appropriate coarse-graining over quantum probability amplitudes, but no concrete mathematical structure, able to reproduce this, has been discovered.
6. The mathematical inconsistency of modified phase space volumes: In the Introduction, we claimed that this, deceptively simple argument, represents the most serious objection yet raised to the modified commutator paradigm. In short, it is certainly not a feature, but a bug, since it represents a serious mathematical inconsistency of such models. In light of this, the previous five points raised above may be considered moot, since only one inconsistency is required to render a physical theory untenable. It is therefore of the utmost importance to establish whether or not this objection can be circumvented in some way.

4 Deriving the GUP, EUP and EGUP without modified commutators

It is straightforward to show that the GUP and EUP can be obtained, at least approximately, from far more 'natural' looking expressions, in which the variances of independent random variables add linearly. For example, let us consider the simplest scenario in which the back reaction of the wave function on the geometry is considered negligible, so that the latter undergoes quantum fluctuations which are independent of $\psi(\mathbf{x})$.

In this case, $(\Delta x_{\psi})^2$ denotes the variance of the canonical probability density $|\psi(\mathbf{x})|^2$, where $\mathbf{x} \in \mathbb{R}^3$ are the possible measured values of the particle's position in classical three-dimensional space. These are the canonical quantum degrees of freedom. In order to describe quantum fluctuations of the background we must introduce new degrees of freedom which are capable of describing *superpositions of geometries*, as expected in a viable theory of quantum gravity (Marletto and Vedral, 2017; Lake et al, 2019).

The additional fluctuations in the measured position of the particle, due to quantum fluctuations of the geometry in which it 'lives', may be described by an additional variance, $(\Delta_g x')^2$. This denotes the variance of the non-canonical probability density $|g(\mathbf{x}' - \mathbf{x})|^2$, where $|\mathbf{x}' - \mathbf{x}|$ quantifies the size of the fluctuation, i.e., the degree to which the measured position of the particle is perturbed by the quantum nature of the background. (For simplicity, we may imagine $|g(\mathbf{x}' - \mathbf{x})|^2$ as a three-dimensional Planck-width Gaussian distribution.) This corresponds to a new composite wave function, $\Psi(\mathbf{x}, \mathbf{x}')$, which describes the propagation of a quantum particle in a quantum background, rather than a fixed classical geometry:

$$\Psi(\mathbf{x}, \mathbf{x}') = \psi(\mathbf{x}) g(\mathbf{x}' - \mathbf{x}). \quad (4.1)$$

The possible measured positions of the particle are then given by the values of $\mathbf{x}' \in \mathbb{R}^3$, rather than $\mathbf{x}$, and $g(\mathbf{x}' - \mathbf{x})$ may be interpreted as the quantum probability amplitude for the coherent transition $\mathbf{x} \mapsto \mathbf{x}'$ in a smeared superposition of geometries (Lake et al., 2019; Lake et al., 2020). From here on, we refer to g as the 'smearing function'. The total variance for a position measurement in the smeared space is then, simply

$$(\Delta_{\Psi} x'^i)^2 = (\Delta_{\psi} x'^i)^2 + (\Delta_g x'^i)^2. \quad (4.2)$$

Setting $\Delta x'_g \simeq l_{\rm Pl}$, taking the square root of Eq. 4.2 and Taylor expanding to first order then yields the GUP. However, in this case, the relevant uncertainties are well defined, as the standard deviations of probability densities, unlike the heuristic uncertainties given in Eq. 1.1.

A similar construction in momentum space,

$$\tilde{\Psi}(\mathbf{p},\mathbf{p}') = \tilde{\psi}(\mathbf{p})\tilde{g}(\mathbf{p}'-\mathbf{p}), \tag{4.3}$$

yields

$$\left(\Delta_\Psi p'_j\right)^2 = \left(\Delta_\psi p'_j\right)^2 + \left(\Delta_g p'_j\right)^2. \tag{4.4}$$

Setting $\Delta p'_g \simeq m_{\rm dS}c$, taking the square root of Eq. 4.4 and Taylor expanding to first order then yields the EUP. Again, the relevant uncertainties are well defined, as the standard deviations of the probability densities $|\tilde{\psi}(\mathbf{p})|^2$ and $|\tilde{g}(\mathbf{p}'-\mathbf{p})|^2$, unlike the heuristic uncertainties in Eq. 1.2.

But what, exactly, are the functions $\tilde{\psi}(\mathbf{p})$ and $\tilde{g}(\mathbf{p}'-\mathbf{p})$? How are they related to $\psi(\mathbf{x})$ and $g(\mathbf{x}'-\mathbf{x})$? In canonical QM, $\tilde{\psi}(\mathbf{p})$ is the $\hbar$-scaled Fourier transform of $\psi(\mathbf{x})$ and, to emphasize this point, we rewrite it with an appropriate subscript as

$$\tilde{\psi}(\mathbf{p}) \equiv \tilde{\psi}_\hbar(\mathbf{p}) = \left(\frac{1}{\sqrt{2\pi\hbar}}\right)^3 \int \psi(\mathbf{x})\, e^{-\frac{i}{\hbar}\mathbf{p}.\mathbf{x}} {\rm d}^3{\rm x}. \tag{4.5}$$

The canonical de Broglie relation $\mathbf{p} = \hbar\mathbf{k}$ ensures that the exponent is independent of Planck's constant, but $\hbar$ necessarily appears in this expression through the normalisation constant $\sqrt{2\pi\hbar}^{-3}$. This is because $|\psi\rangle$ represents the state of a canonical quantum particle and $\hbar$ sets the (action) scale at which quantum effects become significant in canonical quantum matter (Rae, 2002).

However, were we to assume, likewise, that $\tilde{g}(\mathbf{p}'-\mathbf{p})$ is given by the canonical $\hbar$-scaled Fourier transform of $g(\mathbf{x}'-\mathbf{x})$, we would obtain the expression Eq. 4.4 with $\Delta_g p' \simeq m_{\rm Pl}c$ not $\Delta_g p' \simeq m_{\rm dS}c$! This leads to an EUP-type expression with a 'minimum' momentum equal to the Planck momentum and, hence, a minimum energy equal to the Planck energy, a minimum energy density equal to the Planck density, *etc.*, which is clearly at odds with empirical data. Therefore, we must instead assume a decomposition of the form

$$\tilde{g}(\mathbf{p}'-\mathbf{p}) \equiv \tilde{g}_\beta(\mathbf{p}'-\mathbf{p}) = \left(\frac{1}{\sqrt{2\pi\beta}}\right)^3 \int g(\mathbf{x}'-\mathbf{x})\, e^{-\frac{i}{\beta}(\mathbf{p}'-\mathbf{p}).(\mathbf{x}'-\mathbf{x})} {\rm d}^3{\rm x}', \tag{4.6}$$

with $\beta \neq \hbar$. It is straightforward to show that, in order to recover both the GUP Eq. 1.1 from Eq. 4.2 *and* the EUP Eq. 1.2 from Eq. 4.4, we must set

$$\Delta_g x'^i = \sigma_g := \sqrt{2\alpha_0}\, l_{\rm Pl}, \quad \Delta_g p'_j = \tilde{\sigma}_g := \sqrt{6\eta_0}\, m_{\rm dS}c, \quad \forall i,j, \tag{4.7}$$

together with

$$\beta := 2\sigma_g\tilde{\sigma}_g. \tag{4.8}$$

For α_0, $\eta_0 \sim \mathcal{O}(1)$, this gives

$$\beta = 2\sqrt{\frac{\rho_\Lambda}{\rho_{\rm Pl}}}\,\hbar \simeq 10^{-61} \times \hbar, \tag{4.9}$$

where $\rho_\Lambda = \Lambda c^2/(8\pi G) \simeq 10^{-30}$ g cm^{-3} is the dark energy density (Riess et al., 1998; Perlmutter et al., 1999) and $\rho_{\rm Pl} = c^5/(\hbar G^2) \simeq 10^{93}$ g cm^{-3} is the Planck density. Taken together, Eq. 4.5 and Eq. 4.6 are equivalent to imposing the modified de Broglie relation,

$$\mathbf{p} = \hbar\mathbf{k} + \beta(\mathbf{k}'-\mathbf{k}), \tag{4.10}$$

where the non-canonical term can be understood, heuristically, as the additional momentum 'kick' imparted to the canonical quantum state, by a fluctuation of the background (Lake et al., 2019; Lake et al., 2020).

Eq. 4.2 and Eq. 4.4 can also be recovered, with the appropriate minimum values Eq. 4.7, from the canonical-type braket constructions

$$(\Delta_\Psi X^i)^2 = \langle\Psi|(\hat{X}^i)^2|\Psi\rangle - \langle\Psi|\hat{X}^i|\Psi\rangle^2, \tag{4.11}$$

$$(\Delta_\Psi P_j)^2 = \langle\Psi|(\hat{P}_j)^2|\Psi\rangle - \langle\Psi|\hat{P}_j|\Psi\rangle^2, \tag{4.12}$$

where we have relabelled $\Delta_\Psi x'^i \equiv \Delta_\Psi X^i$ and $\Delta_\Psi p'_j \equiv \Delta_\Psi P_j$, for the sake of notational convenience. The appropriate generalised operators, $\hat{X}^i$ and $\hat{P}_j$, representing position and momentum measurements in the smeared superposition of geometries, are given by

$$\hat{X}^i := \int x'^i |\mathbf{x},\mathbf{x}'\rangle\langle\mathbf{x},\mathbf{x}'|{\rm d}^3{\rm x}{\rm d}^3{\rm x}', \tag{4.13}$$

$$\hat{P}_j := \int p'_j |\mathbf{p}\mathbf{p}'\rangle\langle\mathbf{p}\mathbf{p}'|{\rm d}^3{\rm p}{\rm d}^3{\rm p}', \tag{4.14}$$

where $|\mathbf{x},\mathbf{x}'\rangle := |\mathbf{x}\rangle \otimes |\mathbf{x}'\rangle$ and the basis $|\mathbf{p}\mathbf{p}'\rangle$ is entangled,

$$|\mathbf{p}\mathbf{p}'\rangle := \left(\frac{1}{2\pi\sqrt{\hbar\beta}}\right)^3 \int\int |\mathbf{x},\mathbf{x}'\rangle e^{-\frac{i}{\hbar}\mathbf{p}.\mathbf{x}} e^{-\frac{i}{\beta}(\mathbf{p}'-\mathbf{p}).(\mathbf{x}'-\mathbf{x})} {\rm d}^3{\rm x}{\rm d}^3{\rm x}'. \tag{4.15}$$

(We emphasize this by not writing a comma between $\mathbf{p}$ and $\mathbf{p}'$.)

However, the position and momentum space bases may be symmetrized, such that $|\mathbf{x},\mathbf{x}'\rangle \mapsto |\mathbf{x},\mathbf{x}'-\mathbf{x}\rangle$ and $|\mathbf{p}\mathbf{p}'\rangle \mapsto |\mathbf{p},\mathbf{p}'-\mathbf{p}\rangle$, by means of an appropriate unitary transformation (Lake et al., 2020). Formally, this is analogous to the unitary transformation defined in (Giacomini et al., 2019), which is intended to represent a switch between quantum reference frames (QRFs), but with the substitution $\beta \leftrightarrow \hbar$. This implies that, in the quantum mechanical 'smeared space' defined by our model, each 'smeared point' may be considered as a QRF, whose quantum uncertainties are controlled by the quantum of action for geometry, β, rather than that for canonical quantum matter, $\hbar$ (Lake, 2021a; Lake, 2021b; Lake, 2022b).

At first glance, the introduction of a second quantisation scale for geometry appears to contradict a rather large body of existing literature which claims that Planck's constant is unique. A closer look at this literature, however (for example, see (Sahoo, 2004; Deser, 2022) and references therein), shows that only quantisation schemes of the form $\mathbf{p} = \hbar\mathbf{k}$, $\mathbf{p}' = \hbar'\mathbf{k}'$, with $\hbar' \neq \hbar$, where $\mathbf{p}$ and $\mathbf{p}'$ refer to the momenta of different species of material particles, are ruled out by existing no-go theorems. The crucial mathematical difference between these models and the modified de Broglie relation Eq. 4.10 is the presence of relative variables, $\mathbf{k}' - \mathbf{k}$, in the latter. Physically, this is directly related to our treatment of the composite matter-geometry system as a QRF, in which the relative variable $\mathbf{x}' - \mathbf{x}$ describe quantum fluctuations of the smeared spatial points (Lake et al., 2019; Lake et al., 2020). This physical picture leads directly to a well-defined EGUP since Eqs. 4.2 and 4.4, 4.7 and 4.11, 4.12 can also be combined, directly, to give (Lake et al., 2020):

$$\Delta_\Psi X^i \Delta_\Psi P_j \gtrsim \frac{\hbar}{2}\delta^i{}_j\left[1 + \alpha_0 \frac{2G}{\hbar c^3}\left(\Delta_\Psi P_j\right)^2 + 2\eta_0 \Lambda\left(\Delta_\Psi X^i\right)^2\right]. \quad (4.16)$$

Crucially, it is straightforward to show that the smeared space position and momentum operators Eqs. 4.13-4.14 obey the following commutation relations:

$$\left[\hat{X}^i, \hat{P}_j\right] = i(\hbar + \beta)\,\delta^i{}_j \hat{\mathbb{I}}, \quad (4.17)$$

$$\left[\hat{X}^i, \hat{X}^j\right] = 0, \quad \left[\hat{P}_i, \hat{P}_j\right] = 0. \quad (4.18)$$

Since these are just a rescaled version of the canonical Heisenberg algebra, with $\hbar \mapsto \hbar + \beta$, we obtain the 'expected' quantum gravity phenomenology, i.e., the GUP, EUP and EGUP Eq. 4.16, without introducing any of the pathologies associated with standard modified commutator models (Lake, 2019; Lake, 2020). The physical interpretation of this rescaled algebra is subtle and the interested reader is referred to the more in-depth and complete works (Lake, 2020; Lake et al., 2020) for further details.

The price we have to pay for this neat solution is the introduction of a second quantisation constant for geometry, $\beta \ll \hbar$, which is directly related to the dark energy scale. With this in mind, we note that the product of the position and momentum uncertainties in smeared space, $\Delta_\Psi X^i \Delta_\Psi P_j$ in the EGUP Eq. 4.16, is minimized when (see (Lake et al., 2019) for details):

$$\Delta_\psi x'^i = \sqrt{\frac{\hbar}{2}\frac{\Delta_g x'^i}{\Delta_g p'_i}}, \quad \Delta_\psi p'_j = \sqrt{\frac{\hbar}{2}\frac{\Delta_g p'_j}{\Delta_g x'^j}}. \quad (4.19)$$

For the minimum values given by Eq. 4.7 this yields

$$\Delta_\Psi X \simeq l_\Lambda := \sqrt{l_{\rm Pl} l_{\rm dS}} \simeq 0.1\text{mm}, \quad c\Delta_\Psi P \simeq m_\Lambda c^2 := \sqrt{m_{\rm Pl} m_{\rm dS}}c^2 \simeq 10^{-3}\text{eV}, \quad (4.20)$$

where we have neglected to label dimensional indices. The corresponding energy density is

$$\mathcal{E}_\Psi \simeq \frac{c\Delta_\Psi P}{(\Delta_\Psi X)^3} \simeq \frac{m_\Lambda c}{l_\Lambda^3} \simeq \frac{\Lambda c^4}{G} \simeq \rho_\Lambda c^2, \quad (4.21)$$

so that any field which minimizes the smeared-space uncertainty relations must, necessarily, possess an energy density comparable to the present day dark energy density, $\rho_\Lambda \simeq 10^{-30}$ g cm^{-3}. In this scenario, the immense difference between the matter and geometry quantisation scales may be regarded as 'fundamental', while the immense difference between the Planck density and the observed vacuum density is an emergent phenomenon, stemming, ultimately, from the quantum properties of space-time (Lake et al., 2019; Lake et al., 2020; Lake, 2021a).

5 Discussion

In the first part of this paper we outlined six major pathologies that afflict models of generalised uncertainty relations (GURs) based on modified commutation relations. The first two of these, namely, violation of the equivalence principle and violation of Lorentz invariance in the relativistic limit, have been addressed at length in the existing literature (Hossenfelder, 2013; Hossenfelder, 2014; Tawfik and Diab, 2014) and we summarised them only briefly. The third, the so-called soccer ball problem for multi-particle states, has also been considered in detail and a would-be solution was proposed in (Amelino-Camelia, 2017). Though ingenious, and valid within its domain of applicability, we showed that the solution put forward in (Amelino-Camelia, 2017) does not apply, in general, to arbitrary GUR models based on modifications of the canonical Heisenberg algebra. The fourth and fifth problems, the reference frame-dependence of the would-be 'minimum' length and the inherently classical nature of the background geometry in modified commutator models, were considered previously in (Lake, 2020), but have not, to the best of our knowledge, been addressed elsewhere. Finally, we argued that there is, in fact, a sixth problem that appears in modified commutator models, which, remarkably, has not been considered at all the existing literature.

This problem is nothing less than the mathematical inconsistency of the modified phase space volumes from which modified commutators, and hence GURs, are usually derived. The essence of the problem is that the position-space coordinates X^i, corresponding to the quantum uncertainty ΔX^i, are assumed to represent global Cartesians, $X^i \in \{X, Y, Z\}$. The associated distance and volume measures in real space are then given by $L = \sqrt{X^2 + Y^2 + Z^2}$ and $\tilde{V} = \mathrm{d}P_X \mathrm{d}P_Y \mathrm{d}P_Z$, respectively. This immediately rules out the existence of modified X-space volumes and, hence, modified algebras leading to EUP-type uncertainty relations. Likewise, if $X^i \in \{X, Y, Z\}$ form a global Cartesian coordinate system in real space then the conjugate momenta $P_j \in \{P_X, P_Y, P_Z\}$, corresponding to the quantum uncertainty ΔP_j, must form a global Cartesian coordinate system in momentum space. The associated distance and volume measures are $P = \sqrt{P_X^2 + P_Y^2 + P_Z^2}$ and $\tilde{V} = \mathrm{d}P_X \mathrm{d}P_Y \mathrm{d}P_Z$. This immediately rules out the existence of modified P-space volumes and, hence, modified algebras leading to GUP-type uncertainty relations. To make matters worse, the usual approach in the literature is to assume the validity of the standard X- and P-space distance measures while simultaneously adopting modified volume forms. This procedure is mathematically inconsistent.

If, instead, we choose to abandon the assumption that X^i and P_j label global Cartesians in position and momentum space, respectively, we are faced with the following very difficult question: what, exactly, do they represent? This question is exceedingly difficult because, unless it can be answered concretely, abstract mathematical expressions of the form $\Delta X^i \Delta P_j \geq (\hbar/2)\delta^i{}_j G(\hat{\mathbf{X}}, \hat{\mathbf{P}})$ cannot be used to make *any* valid physical predictions. We believe that this deceptively simple observation represents the most serious objection to the modified commutator paradigm yet raised in the literature and that, taken together, the six pathologies described herein ought to signal the 'death knell' of modified commutator models. Though some have been discussed only recently, all six were inherent in such models from their conception nearly three decades ago, and appear no closer to being solved today than they were in the mid-1990s. Indeed, substantial evidence now suggests that at least some of these pathologies *cannot* be consistently resolved.

We believe that this, accumulated evidence, should strongly motivate the GUR research community to seek alternative mathematical structures which are capable of generating the same phenomenology, without the inconsistencies, ambiguities, and headaches associated with modified commutation relations. To this end we outlined one such formalsim, originally proposed in a series of works coauthored by one of us (Lake, 2019; Lake et al., 2019; Lake, 2020; Lake et al., 2020; Lake, 2021a; Lake, 2021b; Lake, 2022a; Lake, 2022b; Lake et al., 2023), in the second part of this paper. Whether, ultimately, this model has anything to do with physical reality, or not, is perhaps less important than what is demonstrates: that GUP, EUP and EGUP phenomenology can be obtained *without* assuming modified commutation relations of a non-Heisenberg type. This demonstrates, by means of an explicit example, the logical independence of GURs and modified algebras. The latter certainly do imply the former (though not, as we have seen, in a self-consistent formulation) whereas the former do not, in fact, *require* them. This is a common misconception in the existing literature, in which these two distinct mathematical structures are often conflated. There is no one-to-one correspondence between GURs and modified commutation relations and the two are *not* logically equivalent. This opens up an intriguing and exciting possibility, namely, that other mathematical structures, not yet discovered, are also capable of generating GURs without modified commutators. Potentially, these may tell us a great deal about the structure of low-energy quantum gravity and, hence, about the possible structure of a unified theory. We implore the phenomenological research community to search for them, earnestly, and to explore their implications as thoroughly as they have explored the implications of modified commutation relations, over the past quarter of a century.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding authors.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of interest

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Dimensionally-dependent uncertainty relations, or why we (probably) won't see micro-black holes at the LHC, even if large extra dimensions exist

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We present a simple gedanken experiment in which a compact object traverses a spacetime with three macroscopic spatial dimensions and n compact dimensions. The compactification radius is allowed to vary, as a function of the object's position in the four-dimensional space, and we show that the conservation of gravitational self-energy implies the dimensional dependence of the mass-radius relation. In spacetimes with extra dimensions that are compactified at the Planck scale, no deviation from the four-dimensional result is found, but, in spacetimes with extra dimensions that are much larger than the Planck length, energy conservation implies a deviation from the normal Compton wavelength formula. The new relation restores the symmetry between the Compton wavelength and Schwarzschild radius lines on the mass-radius diagram and precludes the formation of black holes at TeV scales, even if large extra dimensions exist. We show how this follows, intuitively, as a direct consequence of the increased gravitational field strength at distances below the compactification scale. Combining these results with the heuristic identification between the Compton wavelength and the minimum value of the position uncertainty, due to the Heisenberg uncertainty principle, suggests the existence of generalised, higher-dimensional uncertainty relations. These relations may be expected to hold for self-gravitating quantum wave packets, in higher-dimensional spacetimes, with interesting implications for particle physics and cosmology in extra-dimensional scenarios.

KEYWORDS
compactification, higher dimensions, compton wavelength, primordial black holes, generalised uncertainty relations contents, self-gravity

1 Introduction

For over 40 years, models with compact extra dimensions have attracted a great deal of attention in the theoretical physics literature. Much of this interest was motivated by superstring theory, which is only consistent in ten spacetime dimensions (Green et al., 1988a; Green et al., 1988b), requiring six space-like dimensions to be curled up on scales that make them inaccessible to current high-energy experiments. Theoretically, the compactification scale may be as low as the Planck length, placing it forever beyond the reach of terrestrial particle physics, but models with effective compactification scales as high a millimetre have also been proposed (Antoniadis et al., 1998; Arkani-Hamed et al., 1998). Prior to the start-up of the Large Hadron Collider (LHC), in 2010, interest in the phenomenology of higher dimensional models reached an all-time high. It peaked again following beam upgrades in 2015, but, since then, has been in decline.

In the heady days of the late nineteen-nineties and the first 2 decades of the 21th century, it was hoped, and, indeed, argued persuasively in the scientific literature, that the TeV scale experiments soon to be conducted at CERN would enable the direct detection of compact dimensions with length scales down to $\sim 10^{-19}$ m. It was claimed that these, so-called 'large' extra dimensions, could induce the formation of microscopic black holes (Arkani-Hamed et al., 1999; Bleicher et al., 2011; Khachatryan et al., 2011; Kiritsis and Taliotis, 2011; Bellagamba et al., 2012; Mureika et al., 2012; Park, 2012; Alberghi et al., 2013; Nicolini et al., 2013; Taliotis, 2013; Torres et al., 2013; Winstanley, 2013; Belyaev and Calmet, 2015; Hou et al., 2015; Sokolov and Pshirkov, 2017), also known as primordial black holes (PBH), in reference to their cosmic cousins (Carr, 2005; Carr and Kuhnel, 2020; Carr et al., 2021; Green and Kavanagh, 2021; Escrivà et al., 2022; Friedlander et al., 2022). These claims even attracted considerable attention in the popular press (American Physical Society, 2008; NASA, 2008; New York Times, 2008; BBC, 2013; Huffington Post, 2014; Forbes, 2016).

The argument behind this assertion was straightforward and reasonable. It is well known that the radius of an uncharged and non-spinning (Schwarzschild) black hole depends, not only on its mass, but also on the dimensionality it of the spacetime it inhabits. The higher-dimensional Schwarzschild radius varies as $\mathcal{R}_S \propto M^{\frac{1}{1+n}}$, where n is the number of space-like extra dimensions, over and above the three Hubble scale dimensions that make up the macroscopic Universe (Weinberg, 2008). Thus, assuming that the usual mass-dependence of the Compton wavelength, $R_C \propto M^{-1}$, remains unchanged in the presence of the compact space, the intersection between $\mathcal{R}_S$ and R_C occurs close to the critical values

$$R_{\rm crit} = \left(\frac{\hbar G_{4+n}}{c^3}\right)^{\frac{1}{2+n}}, \quad M_{\rm crit} = \left(\frac{\hbar^{1+n} c^{1-n}}{G_{4+n}}\right)^{\frac{1}{2+n}}. \tag{1}$$

For $n \geq 0$, these expressions serve as the definitions of the Planck length, and mass, respectively (Horowitz, 2012). Since, in spacetimes with n compact dimensions, the four-dimensional Newton's constant is related to its higher-dimensional counterpart, and to the compactification radius R_E, *via* (Maartens and Koyama, 2010)

$$G_{4+n} = G_4 R_E^n, \tag{2}$$

it follows that, for sufficiently large R_E, the mass-energy needed to create a black hole may be brought within the TeV range of the LHC.

More recently, new phenomenological models have been proposed, in which the possible dimensional dependence of the Compton wavelength has been explored (Lake and Carr, 2016; Carrr, 2018; Lake and Carr, 2018; Carr, 2022), *via* so-called black hole–uncertainty principle (BHUP) correspondence, which is also referred to as the Compton–Schwarzschild correspondence in the literature (Carr et al., 2011; Carr et al., 2015; Carr, 2016; Lake and Carr, 2015; Singh, 2017; Singh, 2018; da Silva and Silva, 2022). This modification alters the intersection with the Schwarzschild radius, and is capable of restoring complete symmetry to the (M, R) diagram, pushing the threshold for black hole formation back up to the four-dimensional Planck mass, $M_{\rm Pl} = \sqrt{\hbar c/G_4}$. However, despite the various arguments used to justify these models (Lake and Carr, 2016; Carrr, 2018; Lake and Carr, 2018; Carr, 2022), the proposed dimensional dependence lacks a clear physical motivation. In this work, we motivate them in a more direct way, by outlining a clear physical mechanism that is capable of altering the mass-radius relation of any compact object, including that of fundamental particles.

The structure of this paper is as follows. In the main body of the work, Sec. II, we present a simple gedanken experiment in a hypothetical Universe with three macroscopic spatial dimensions and n compact extra dimensions. The compactification radius is allowed to vary as a function of position in the four-dimensional subspace, which is divided into three regions. In the first region, the extra dimensions are compactified at the four-dimensional Planck-scale, $R_{\rm Pl} = \sqrt{\hbar G/c^3}$, while in the third they are compactified at a much larger radius, $R_E > R_{\rm Pl}$. The second region, in which the compactification scale grows monotonically, interpolates smoothly between the other two. We then consider a compact object, which passes from region 1 to region 3, and impose the conservation of gravitational self-energy. Roughly speaking, since gravity becomes stronger on scales $R_{\rm Pl} < R < R_E$, as we move through region 2, the radius of the object must increase, in order to keep its gravitational self-energy constant. Furthermore, since rest mass is conserved during this transition, it follows that the mass-radius relation must be modified.

In this study, we perform explicit calculations by assuming that the gravitational potential of the object can be approximated by the weak field (Newtonian) limit. However, despite this, our analysis correctly reproduces well-known results for strongly-gravitating objects, such as higher-dimensional black holes and neutron stars, up to numerical factors of order unity, which is consistent with the non-relativistic approximation. This gives us confidence in the method, which we then extend to the study of fundamental particles, for which the non-relativistic approximation is undoubtedly valid.

We verify that, beginning with an effectively four-dimensional black hole in region 1, we obtain the correct (order of magnitude) expression for the higher-dimensional Schwarzschild radius in region 3. This gives us confidence in our procedure, which we note is agnostic to the initial mass-radius relation of the object. We then consider a fundamental particle, by beginning instead with the standard formula for the Compton wavelength, and obtain an effective, higher-dimensional Compton radius, in the third region. Its implications for the (non-) formation of black holes at the LHC, as well as for the quantum mechanical uncertainty relations of self-gravitating wave packets in higher-dimensional spacetimes, are briefly discussed. We summarise our conclusions, and consider the prospects for future work on this model, in Sec. III.

2 The gedanken experiment

Let us assume, for simplicity, that the compact object we consider is spherically symmetric. In region 1, its internal energy is, therefore

$$E = Mc^2 - \alpha_4 \frac{G_4 M^2}{R}, \tag{3}$$

where α_4 is a numerical constant determined by the mass profile of the sphere, $M(r)$. For example, $\alpha_4 = 3/5$ for a sphere of uniform density and should be of order unity for all non-pathological profiles (Weisstein et al., 1973). Here, R denotes the effective macrosopic radius of the object and Eq. 3 holds for all $R \gtrsim R_E = R_{Pl}$. We note that distances below this scale cannot be probed directly, by either black holes or fundamental particles, due to the intersection of the Compton wavelength and Schwarzschild radius lines near the Planck point on the (M, R) diagram (Lake and Carr, 2015; Carr, 2016).

Requiring $E \le 0$, which implies a bound state, yields

$$R \le \alpha_4 \frac{G_4 M}{c^2}. \tag{4}$$

For $\alpha_4 = 2$, we then recover the condition

$$R \le R_S(M) = \frac{2G_4 M}{c^2}, \tag{5}$$

where R_S is the four-dimensional Schwarzschild radius. Thus, if Eq. 5 is satisfied, the object is a black hole in the first region. For $\alpha_4 = 9/4$, an analogous condition implies violation of the Buchdahl inequality (Buchdahl, 1959) and the sphere may be viewed as a compact star undergoing collapse. Conversely, for $E > 0$, $R > (9/4)G_4M/c^2$, the object is stable against its own self-gravity.

Setting $R = R_C$, where

$$R_C(M) = \frac{\hbar}{Mc} \tag{6}$$

is the standard Compton radius (Rae and Napolitano, 2015), Eq. 3 implies that a fundamental particle is stable against gravitational collapse ($E > 0$) when

$$M \lesssim M_{Pl}, \quad R_C(M) \gtrsim R_{Pl}, \tag{7}$$

where

$$R_{Pl} = \sqrt{\frac{\hbar G_4}{c^3}} \simeq 10^{-35}\,\text{m}, \quad M_{Pl} = \sqrt{\frac{\hbar c}{G_4}} \simeq 10^{-8}\,\text{kg}. \tag{8}$$

Equation 7 justifies our previous assertion that Eq. 3 holds, for $R \gtrsim R_{Pl}$, when the extra dimensions are compactified at the (four-dimensional) Planck scale. For fundamental particles, this corresponds to the region $M \lesssim M_{Pl}$, whereas, for black holes, it corresponds to

$$M \gtrsim M_{Pl}, \quad R_S(M) \gtrsim R_{Pl}. \tag{9}$$

The intersection of the standard Compton line and the four-dimensional Schwarzschild line near the Planck point then precludes the existence of *any* fundamental object with $R(M) \lesssim R_{Pl}$.

In the third region, the internal energy of the object is given by Eq. 3, for $R > R_E$, where $R_E > R_{Pl}$ is the compactification radius, but by

$$E = Mc^2 - \alpha_{4+n} \frac{G_{4+n} M^2}{\mathcal{R}^{1+n}} \tag{10}$$

for $R_{Pl} \le \mathcal{R} \le R_E$. Here, $\mathcal{R}$ denotes the $(4+n)$-dimensional radius in region 3 and α_{4+n} is a numerical constant determined by the mass profile of the object in the higher-dimensional space. For simplicity, we assume that all n extra dimensions are compactified on the same scale. The relation between G_4 and the higher-dimensional Newton's constant, G_{4+n}, is given by Eq. 2 (Maartens and Koyama, 2010).

By choosing appropriate values of α_{4+n}, we may recover the $(4+n)$-dimensional analogues of the Buchdahl bound (Burikham et al., 2015; Burikham et al., 2016) and the Schwarzschild radius (Horowitz, 2012), from the energy conditions $E < (>) 0$. In any number of dimensions, the Buchdahl radius is proportional to the Schwarzschild radius, and, neglecting numerical factors of order unity, the latter may be written as

$$\mathcal{R}_S(M) \simeq \left(\frac{G_{4+n} M}{c^2}\right)^{\frac{1}{1+n}} \simeq \left(R_S(M) R_E^n\right)^{\frac{1}{1+n}}. \tag{11}$$

where $R_S(M)$ again denotes the four-dimensional Schwarzschild radius, as in Eq. 5.

Let us now consider a non-relativistic, self-gravitating sphere, with arbitrary mass-radius relation, passing from region 1 to region 3. Furthermore, let us assume that, whatever its mass-radius relation in the four-dimensional space of the first region, the sphere remains small enough to be effectively $(4+n)$-dimensional in the third. Thus, in region 1, its radius in the three macroscopic spatial dimensions is $R(M) \gtrsim R_{Pl}$ and, in region 3, its higher-dimensional radius satisfies $R_{Pl} \lesssim \mathcal{R}(M) \lesssim R_E$. If its internal energy remains unchanged, energy conservation then implies

$$\mathcal{R}(M) \simeq \left(R(M) R_E^n\right)^{\frac{1}{1+n}}. \tag{12}$$

again ignoring numerical factors of order unity, which is consistent with the non-relativistic approximation. Note that we again use the calligraphic font, $\mathcal{R}$, to denote radii in $(4+n)$ dimensions, and the normal font R to denote four-dimensional radii.

Substituting $R(M) \simeq R_S(M)$ 5) into (12), we recover the correct expression for the higher-dimensional Schwarzschild radius, $\mathcal{R}_S(M)$ (11). Next, we note that, if $R_{Pl} < \mathcal{R}_S(M) < R_E$, then $R_{Pl} < R_S(M) < R_E$. It follows, immediately, that $\mathcal{R}_S(M) > R_S(M)$. This result can be understood intuitively as follows. Since, in the third region, the gravitational force is stronger than in the first on scales $\mathcal{R} < R_E$, the radius of the black hole can neither decrease, nor remain the constant, without increasing its internal energy. If this energy is conserved, the black hole must increase in size and the $(4+n)$-dimensional Schwarzschild radius, $\mathcal{R}_S(M)$, must be larger than the four-dimensional radius, $R_S(M)$. The relation between the two is fixed, by energy conservation, according to Eq. 11,

Clearly, we may repeat a similar argument for stable compact objects obeying the four-dimensional Buchdahl bound in region 1. The same compact spheres then obey the higher-dimensional Buchdahl bound in region 3. Hence, although the argument presented above is simple and heuristic, it allows us to recover the same relations (to within an order of magnitude) as those obtained by exactly solving the gravitational field equations in $(4+n)$-dimensional spacetime (Horowitz, 2012; Burikham et al., 2015; Burikham et al., 2016).

However, its greatest advantage is that is agnostic to the mass-radius relation of the compact object. We may therefore apply it to fundamental particles, as well as to black holes and conventional fluid spheres. Thus, substituting $R(M) = R_C(M) \propto M^{-1}$ 6) into Eq. 12, we obtain the higher-dimensional Compton wavelength,

$$\mathcal{R}_C(M) \simeq \left(R_C(M) R_E^n\right)^{\frac{1}{1+n}} \simeq R_* \left(\frac{M_{Pl}}{M}\right)^{\frac{1}{1+n}}. \tag{13}$$

where

$$R_* = \left(R_{Pl} R_E^n\right)^{\frac{1}{1+n}}, \tag{14}$$

so that $R_{Pl} < R_* < R_E$. It may be verified that the $(4+n)$-dimensional Compton and Schwarzschild lines intersect at the point $(M, R) \simeq (M_{Pl}, R_*)$, so that the production of PBHs still requires energies of the order of the Planck energy (Lake and Carr, 2016; Carrr, 2018; Lake and Carr, 2018; Carr, 2022).

This result also be understood, intuitively, in the same way as our heuristic derivation of the higher-dimensional Schwarzschild radius. Namely, if the rest mass of the particle remains constant as it traverses the path from region 1 to region 3, its radius cannot remain constant, or decrease, without increasing its gravitational binding energy. Therefore, if its total internal energy remains constant, its radius must expand as it enters the higher-dimensional region, in which gravity is stronger, on scales $\mathcal{R} < R_E$, than in four-dimensional space. Clearly, this relation must also hold for particles that were always confined to region 3.

To aid visualisation, a schematic representation of the gedanken experiment set up is given in **Figure 1**. In **Figure 2A**, the key length and mass scales of the standard scenario, corresponding to Eq. 1, are depicted on the (M, R) diagram, while the key scales for our scenario are depicted in **Figure 2B**. The important difference between the two scenarios is that the former does not account for the self-gravitational energy of the particle, whereas the latter does, to within the accuracy permitted by the non-relativistic, weak-field approximation, which we also apply to micro-black holes. Maintaining this approximation, we may apply the usual, heuristic identification between the Compton wavelength formula and the limiting values of the Heisenberg uncertainty principle (HUP),

$$(\Delta X)_{\min} \simeq \mathcal{R}_C(M), \quad (\Delta P)_{\max} \simeq Mc, \tag{15}$$

giving

$$\Delta X \gtrsim R_* \left(\frac{M_{Pl} c}{\Delta P}\right)^{\frac{1}{1+n}}. \tag{16}$$

We recall that, for $\Delta P \gtrsim Mc$, fundamental particles have sufficient energy to undergo pair-production, in interactions that conserve

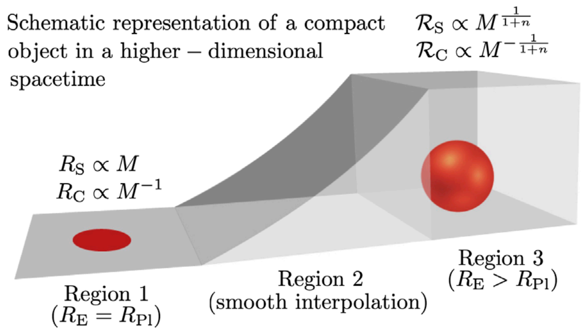

FIGURE 1
Schematic illustration of the three-part universe in our gedanken experiment. To enable the schematic representation of $(3+n)$-dimensional space, neglecting the time dimension of the $(4+n)$-dimensional spacetime, the three large dimensions are depicted as a two-dimensional plane and the n compact directions are depicted as a single extra dimension, extending into the z-direction of the diagram. Furthermore, since Planck-sized extra dimensions do not contribute correction terms, either to the higher-dimensional Schwarzschild radius, or to the Compton wavelength, we neglect them in this illustration. Hence, the region on the far left-hand side represents $(3+n)$-dimensional space, with n dimensions compactified at the Planck scale, while the region on the far right-hand side represents a space with three large dimensions and n extra dimensions, compactified on some scale $R_E > R_{Pl}$. The central region interpolates smoothly between the two, so that the gravitational radius of the compact body changes, according to the following scheme: In region 1 (left), the extra dimensions are compactified at the (four-dimensional) Planck scale and both black holes and fundamental particles are effectively four-dimensional, even in the presence of the higher-dimensional space. In region 3 (right), the compactification radius is much larger than the Planck length and all objects are effectively $(4+n)$-dimensional, on scales smaller than the compactification radius. Conservation of energy implies that, whatever the mass-radius relation of the object in the first region, $R(M)$, its radius in the third region, $\mathcal{R}(M)$, must be larger: $\mathcal{R}(M) > R(M)$. This is due to the increased strength of the gravitational field in higher dimensions. For black holes, $R_S \propto M$ in region 1 and $\mathcal{R}_S \propto M^{\frac{1}{1+n}}$ in region 3. Applying the same logic to the gravitational radius of fundamental particles, $R_C \propto M^{-1}$ in region 1, yielding $\mathcal{R}_C \propto M^{-\frac{1}{1+n}}$ in region 3, due to the conservation of gravitational self-energy.

the relevant quantum numbers (Peskin and Schroeder, 1995; Donoghue et al., 2014), yielding the limits in Eq. 15. These, in turn, correspond to the dimensionally-dependent uncertainty relation, Eq. 16.

Equation 16 may be expected to hold for self-gravitating wave packets, on scales $\mathcal{R} < R_E$, in spacetimes with compact extra dimensions. By contrast, on scales $\mathcal{R} > R_E$, or when $R_E \simeq R_{Pl}$, the standard HUP,

$$\Delta X \gtrsim \frac{R_{Pl} M_{Pl} c}{\Delta P}, \tag{17}$$

still holds, where we have rewritten $\hbar = R_{Pl} M_{Pl} c$.

Finally, before concluding this section, we note that, although Eq. 16 represents a form of generalised uncertainty principle, which is valid for self-gravitating objects in higher-dimensional spacetimes, this is not the same as the 'generalised uncertainty principle' (GUP), commonly referred to in the quantum gravity literature (see, for example (Adler et al., 2001; Maziashvili, 2006; Xiang and Wen, 2009; Lake et al., 2019; Sakalli and Kanzi, 2022; Lake et al., 2023), and references therein). In fact, the derivation of

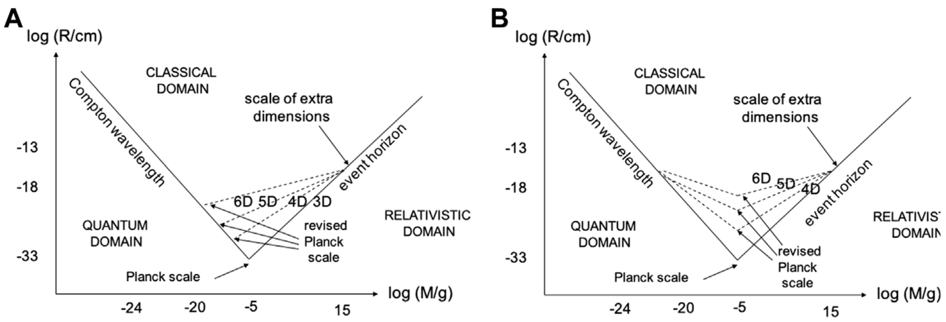

FIGURE 2
Figure 2A (left panel) shows the standard Compton line, $R_C \propto M^{-1}$, and the Schwarzschild radius lines for $n = 0$, $n = 1$, $n = 2$ and $n = 3$. These lines intersect near the higher-dimensional Planck point, $(M,R) = ((M_{Pl}^2 M_E^n)^{\frac{1}{2+n}}, (R_{Pl}^2 R_E^n)^{\frac{1}{2+n}})$, where R_{Pl} and M_{Pl} denote the four-dimensional Planck scales, $R_E > R_{Pl}$ is the compactification radius, and $M_E = \hbar/(R_E c) < M_{Pl}$ is the associated mass scale. The points of intersection are equivalent to the critical scales shown in Eq. **1**, due to Eq. **2**; **Figure 2B** (right panel) shows the modified scenario, in which we account for the increased self-gravity of the quantum particle in the presence of the extra dimensions, yielding $\mathcal{R}_C \propto M^{-\frac{1}{1+n}}$. The Compton and Scwarzschild lines now intersect at the point $(M,R) = (M_{Pl}, R_*)$, where R_* is defined in Eq. **14**. The restored symmetry of the mass-radius diagram precludes the formation of black holes at TeV scales, even if large extra dimensions exist. These figures are reproduced from (Lake and Carr, 2018), with permission.

Eq. **16** is based on two fundamental assumptions, namely, a) that the gravitational self-energy of the quantum wave packet is conserved in the presence of extra dimensions, and b) that the standard HUP holds in their absence.

By contrast, the usual GUP is derived, *via* a gedanken experiment in four-dimensional spacetime, by considering the gravitational interaction between a measured particle and a probing photon. This gives rise to a correction term, to the position uncertainty Δx, which is proportional to the effective four-dimensional Schwarzschild radius of the wave packet, $R_S \simeq G_4 \Delta p/c^3$, yielding

$$\Delta x \gtrsim \frac{\hbar}{2\Delta p} + \frac{2G_4}{c^3}\Delta p, \tag{18}$$

where α again denotes a numerical constant of order unity. Assuming, instead, that the GUP (18) holds in a four-dimensional Universe, in place of the HUP (17), we may expect a unification of the Compton and Schwarzschild lines, of the form

$$R_{C/S} \simeq \frac{\hbar}{2Mc} + \frac{2G_4}{c^2}Mc, \tag{19}$$

as predicted by the so-called BHUP correspondence, mentioned in the Introduction (Carrr, 2018; Lake and Carr, 2018; Carr, 2022; Lake and Carr, 2016; Carr et al., 2011; Carr et al., 2015; Carr, 2016; Lake and Carr, 2015; Singh, 2017; Singh, 2018; da Silva and Silva, 2022). Combing these expressions with the arguments presented above yields even richer phenomenology: rather than simply restoring symmetry to the (M,R) diagram higher dimensions, it may provide a way to unify the Compton and Schwarzschild lines, even in higher-dimensional spacetimes. Such an analysis lies outside the scope of the present, preliminary study, and is left to a future work.

3 Discussion

We have presented a simple gedanken experiment in a hypothetical spacetime with three macroscopic spatial dimensions and n compact extra dimensions. The compactification radius was allowed to vary as a function of spatial position, in the four-dimensional submanifold, which is divided into three regions. In the first region, the extra dimensions are Planck-scale, while in the third they are compactified at a much larger radius. The second region, in which the compactification scale grows monotonically, interpolates smoothly between the other two. We considered a spherical compact object that traverses a path from region 1 to region 3, and imposed the conservation of gravitational self-energy.

If the object is a black hole in the first region, with $R \propto M$, energy conservation alone yields the correct expression for the higher-dimensional Schwarzschild radius, $R \propto M^{\frac{1}{1+n}}$, in the third. However, this procedure is agnostic to the mass-radius relation of the object. Hence, considering a fundamental particle instead of a black hole, we instead imposed the standard formula for the Compton wavelength, $R \propto M^{-1}$, in the first region. Conservation of energy then implies the existence of a higher-dimensional Compton wavelength, $R \propto M^{-\frac{1}{1+n}}$, in the third region. Clearly, this relation must also hold for particles that have always been confined to region 3.

The new relation restores the symmetry between the Compton and Schwarzschild lines on the mass-radius diagram, in higher-dimensional spacetimes, and precludes the formation of black holes at TeV scales, even if large extra dimensions exist. We have shown how this follows, intuitively, as a direct consequence of the increased gravitational field strength at distances below the compactification scale. Combining these results with the usual, heuristic identification between the Compton wavelength and the minimum position uncertainty allowed by the Heisenberg uncertainty principle, $\Delta X \gtrsim R_C$ $(\Delta P \lesssim Mc)$, suggests the existence of generalised, higher-dimensional uncertainty relations.

Indeed, the possible dependence of the uncertainty relations on the dimensionality of the spacetime has already been explored in the literature, in the context of the so-called black hole-uncertainty principle (BHUP) correspondence (Lake and Carr, 2016; Carrr, 2018; Lake and Carr, 2018; Carr, 2022). If the usual uncertainty relation-Compton wavelength correspondence is still required to hold, in a higher-dimensional context, then the dimensional-dependence of the Compton wavelength is also (theoretically) necessary.

The difference between this and previous work is that, here, we present a clear physical argument for *why* this change should occur, and show, explicitly, that the effects of self-gravitation on quantum wave packets are precisely those required to maintain the, up to now conjectured, higher-dimensional BHUP correspondence. This is also known as the Compton-Schwarzschild correspondence, in some of the previous literature (Carr et al., 2011; Carr et al., 2015; Carr, 2016; Lake and Carr, 2015; Singh, 2017; Singh, 2018; da Silva and Silva, 2022).

In the present, preliminary analysis, we assumed throughout that the gravitational potential of the compact sphere can be well approximated by the Newtonian regime. Though this is undoubtedly a limitation of the current work, we were still able to recover, to within numerical factors of order unity, the well-known expressions for relativistic objects, such as higher-dimensional black holes and neutron stars (Burikham et al., 2015; Burikham et al., 2016). This strongly suggests that the dimensionally-dimensional uncertainty relations, which we derive for self-gravitating wave packets, are robust, since the weak field approximation is undoubtedly valid for fundamental particles.

As extensions of the current analysis, we should consider relativistic corrections, as well as the incorporation of modified uncertainty principles, obtained from the quantum gravity literature, such as the generalised uncertainty principle (GUP) (Adler et al., 2001; Maziashvili, 2006; Xiang and Wen, 2009; Lake et al., 2019; Sakalli and Kanzi, 2022; Lake et al., 2023), extended uncertainty principle (EUP), and extended generalised uncertainty principle (EGUP) (Bolen and Cavaglia, 2005; Bambi and Urban, 2008; Park, 2008). Furthermore, in order to consistently incorporate the latter, we must also consider the conditions for the formation of gravitational bound states, in higher dimensions, in the presence of a positive cosmological constant (Burikham et al., 2015; Burikham et al., 2016).

Previous studies suggest that these modifications may give rise to a unified description of the Compton and Schwarzschild radii, linking the properties of black holes and fundamental particles in higher-dimensional scenarios (Lake and Carr, 2016; Carrr, 2018; Lake and Carr, 2018; Carr, 2022). The present work represents a small, preliminary step towards understanding the physical mechanism behind this potentially important correspondence, which may have important phenomenological implications for black holes, cosmology, and high-energy particle physics, beyond the non-production of PBH at TeV scales.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding authors.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Corrigendum: Dimensionally-dependent uncertainty relations, or why we (probably) won't see micro-black holes at the LHC, even if large extra dimensions exist

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A Corrigendum on
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In the published article, there were three errors. **Equations 11–13** contained typos, which caused them to be mathematically incorrect.

Corrections have been made to **Section 2**: **Equation 11** previously stated

$$\mathcal{R}_S(M) \simeq \frac{G_{4+n}M}{c^2} \simeq \left(R_S(M)R_E\right)^{\frac{1}{1+n}}. \tag{1}$$

The corrected formula is given as follows:

$$\mathcal{R}_S(M) \simeq \left(\frac{G_{4+n}M}{c^2}\right)^{\frac{1}{1+n}} \simeq \left(R_S(M)R_E^n\right)^{\frac{1}{1+n}}. \tag{2}$$

Equation 12 previously stated

$$\mathcal{R}(M) \simeq \left(R(M)R_E\right)^{\frac{1}{1+n}}. \tag{3}$$

The corrected formula is given as follows:

$$\mathcal{R}(M) \simeq \left(R(M)\,R_{\mathrm{E}}^{n}\right)^{\frac{1}{1+n}}. \quad (4)$$

Equation 13 previously stated

$$\mathcal{R}_{\mathrm{C}}(M) \simeq \left(R_{\mathrm{C}}(M)\,R_{\mathrm{E}}\right)^{\frac{1}{1+n}} \simeq R_{*}\left(\frac{M_{\mathrm{Pl}}}{M}\right)^{\frac{1}{1+n}}. \quad (5)$$

The corrected formula is given as follows:

$$\mathcal{R}_{\mathrm{C}}(M) \simeq \left(R_{\mathrm{C}}(M)\,R_{\mathrm{E}}^{n}\right)^{\frac{1}{1+n}} \simeq R_{*}\left(\frac{M_{\mathrm{Pl}}}{M}\right)^{\frac{1}{1+n}}. \quad (6)$$

The authors apologize for these errors and state that they do not change the scientific conclusions of the article in any way. The original article has been updated.

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New deformed Heisenberg algebra from the μ-deformed model of dark matter

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Recently, the μ-deformation-based approach to modeling dark matter, which exploits μ-deformed thermodynamics, was extended to the study of galaxy halo density profile and of the rotation curves of a number of (dwarf or low brightness) galaxies. For that goal, μ-deformed analogs of the Lane–Emden equation (LEE) have been proposed, and their solutions describing density profiles obtained. There are two seemingly different versions of μ-deformed LEE which possess the same solution, and so we deal with their equivalence. From the latter property we derive new, rather unusual, μ-deformed Heisenberg algebra (HA) for the position and momentum operators, and present the μ-HA in few possible forms (each one at $\mu \to 0$ recovers usual HA). The generalized uncertainty relation linked with the new μ-HA is studied, along with its interesting implications including the appearance of the quadruple of both maximal and minimal lengths and momenta.

KEYWORDS

deformed BEC model of dark matter, deformed lane-emden equation, deformed heisenberg algebra, generalized uncertainty relation, maximal/minimal length uncertainty

1 Introduction

The suggestion of the existence of minimal nonzero (uncertainty of) length linked with generalized uncertainty principle (GUP) or relation (GUR) has been advanced in the context of string theory and quantum gravity (Gross and Mende, 1988; Amati et al., 1989; Adler and Santiago, 1999; Scardigli, 1999; Maziashvili, 2006), see also (Chang et al., 2011) and the reviews (Garay, 1995; Hossenfelder, 2013). It was shown to follow from modified or deformed

Extension (Kempf et al., 1995) of the Heisenberg algebra (HA). It is worth to mention that the concept of maximum observable momenta can play as well important role, see, e.g., (Ali et al., 2009). Such a quantity was predicted, in particular, within the doubly special relativity theory suggesting rather simple (with terms linear and quadratic in momentum) modification of the right hand side of commutators (Magueijo and Smolin, 2002; Magueijo and Smolin, 2005). Further it became clear that besides such a minimal extension of the original HA, a lot of generalizations are possible, suggesting diverse ways to generalize (or deform) the HA. As a tools to classify diverse forms of deformed HA, the concept of deformation function(s) is of importance, see, e.g., (Saavedra and Utreras, 1981; Jannussis, 1993; Gavrilik et al., 2010; Dorsch and Nogueira, 2012; Maslowski et al., 2012; Gavrilik and Kachurik, 2016a). Clearly, the choice of such function must determine the corresponding GUR. As usual, most of the authors deal with position-momentum commutation relations of deformed HA that involve particular function of X, p and deformation parameter(s) in its right hand side (Jannussis, 1993; Dorsch and Nogueira, 2012; Maslowski et al., 2012).

It is also possible that both the right and left hand sides of defining commutation relation are appropriately deformed (Gavrilik and Kachurik, 2012), although such approach may be overlapping with the case of standard commutator and the terms containing XP and PX in its right-hand side as it was considered in (Quesne and Tkachuk, 2007).

In the present paper, a special form of deformed HA will be derived in the context of the so-called μ-deformation based approach aimed to model (Gavrilik et al., 2018; Gavrilik et al., 2019) basic properties of dark matter that surrounds dwarf galaxies, and its consequences analyzed.

The case of GUR related with the minimal $(\Delta X)_{\min}$ is the best known and well studied one. In relation with this, due to the conjugated roles of position and momentum, the concept of $(\Delta P)_{\min}$ has appeared. As it was demonstrated in (Kempf, 1997), a single theory—single extended or generalized HA (GHA) and the corresponding GUR do exist which can jointly accommodate the both special quantities, $(\Delta X)_{\min}$ and $(\Delta P)_{\min}$.

Then, an interesting question arises whether it is possible that the opposite concept of *maximal* uncertainties for the momentum and/or the position does exist. Quite recently, it was shown in some papers that such a possibility indeed can be realized (Pedram, 2012; Perivolaropoulos, 2017; Bensalem and Bouaziz, 2019; Skara and Perivolaropoulos, 2019; Hamil and Lutfuoglu, 2021; Bensalem and Bouaziz, 2022; Pramanik, 2022). Moreover, as a generalization of the already mentioned unified treatment of A. Kempf, in the work (Perivolaropoulos, 2017) of L. Perivolaropoulos, it was explicitly shown that one can provide a theory (based on appropriate generalization of HA) which incorporates the whole quadruple of $(\Delta X)_{\min}$, $(\Delta P)_{\min}$, $(\Delta P)_{\max}$, and $(\Delta X)_{\max}$.

Usual treatments in the most of papers are in a sense model-independent, implying a kind of universality. That means, physical meaning of $(\Delta X)_{\min}$, $(\Delta P)_{\min}$, $(\Delta P)_{\max}$, and $(\Delta X)_{\max}$ is rather universal and depends on Planck length or its inverse, i.e., Planck energy scale (Planck mass).

On the contrary, our treatment is based on (related with) special deformed HA deduced in the framework of particular model of dark matter. It is remarkable that all the four quantities: $(\Delta X)_{\min}$, $(\Delta P)_{\min}$, $(\Delta P)_{\max}$, and $(\Delta X)_{\max}$ do appear. So it is clear and natural that the physical meaning of this quadruple is tightly linked with physics of the model, i.e., with properties of the halo of DM hosted by dwarf galaxies.

For our case (connection with DM) some motivation was due to the work (Perivolaropoulos, 2017), since therein the cosmology-related uncertainty relation was explored, along with clear meaning of maximal length: as suggested in (Perivolaropoulos, 2017), this quantity can be naturally interpreted as *cosmological horizon*.

The uncertainty relation in its initial form due to Heisenberg is linked with the standard commutation relation and *is shared* by different states. Unlike, for all the deformed versions of HA, explicit dependence of GUR on particular state does appear—for deformed oscillators this was noticed in the pioneer papers (Biedenharn, 1989; Macfarlane, 1989). In our present paper, just this fact/property is in the focus and exploited to full extent.

Unlike the approach perceived in (Harko, 2011) and some other papers also exploring galaxy rotation curves with the use of the well-known Lane–Emden equation (LEE), in our line of research we deal with the (μ-)Bose-condensate model of dark matter (Gavrilik et al., 2018), and with such tool as μ-deformed analogs (Gavrilik et al., 2019) of LEE. In general, as it is well-known, deformation of an object under study is not unique, and in (Gavrilik et al., 2019) we encountered two different possible forms of μ-deformed LEE, with the corresponding different sets of solutions, one of which being the deformed function $\sin_\mu(kr)/(kr)$. In the present work, the third form of LEE will be introduced that nevertheless possesses the indicated solution as well. Just from the requirement of equivalence of two seemingly different deformed versions of LEE, the new μ-deformed HA can be deduced and its basic properties and consequences explored.

The paper is structured as follows. In **Section 2**, some basics of μ-deformation and μ-deformed calculus are presented. In **Section 3.1** we describe relevant deformed analogs of LEE and, from the condition of their equivalence, obtain the μ-analog of HA which is the central object of this work. The corresponding GUR which involves the parameter μ is derived, and its main properties are explored in **Section 3.2**, including the appearance of minimal and maximal uncertainties of both position and momentum. **Section 3.3** is devoted to discussion of implications of these quantities for dark matter. The paper is ended with concluding remarks.

2 Deformed functions and calculus

2.1 Basis functions

The so-called μ-bracket of a number or operator X,

$$[X]_\mu \equiv \frac{X}{1+\mu X}; \qquad [X]_\mu \to X, \text{ if } \mu \to 0, \tag{1}$$

and the related μ-deformed oscillator have been introduced 3 decades ago in (Jannussis, 1993). More recently, there appeared some papers (Gavrilik et al., 2010; Gavrilik and Mishchenko, 2012; Gavrilik et al., 2013) in which the μ-deformation based approach was initiated and developed.

For our purposes we define the μ-deformed trigonometric function (see (Gavrilik et al., 2013; Gavrilik et al., 2019) and references therein) as

$$\sin_\mu x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{[2n+1]_\mu!}, \qquad \cos_\mu x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{[2n]_\mu!}, \tag{2}$$

where $[n]_\mu! = [1]_\mu\ [2]_\mu \ldots [n]_\mu$. Clearly, at $\mu \to 0$ one recovers customary sine and cosine.

For our purposes, we introduce the μ-deformed analogs of spherical Bessel functions, namely,

$$j_0^{(\mu)}(x) = \frac{\sin_\mu x}{x}, \qquad y_0^{(\mu)}(x) = \frac{\cos_\mu x}{x}. \tag{3}$$

At $\mu = 0$ these reduce to the familiar Bessel functions.

The physical motivation for introducing these functions is two-fold: the first one in Eq. (3) describes the density profile of the dark matter halo and also leads to the rotation curves within the μ-deformed extension (Gavrilik et al., 2019) of the Bose-condensate model, while both functions, taken jointly, are of importance for constructing the representation space of the position and momentum operators, see **Sections 3.1–3.2** below.

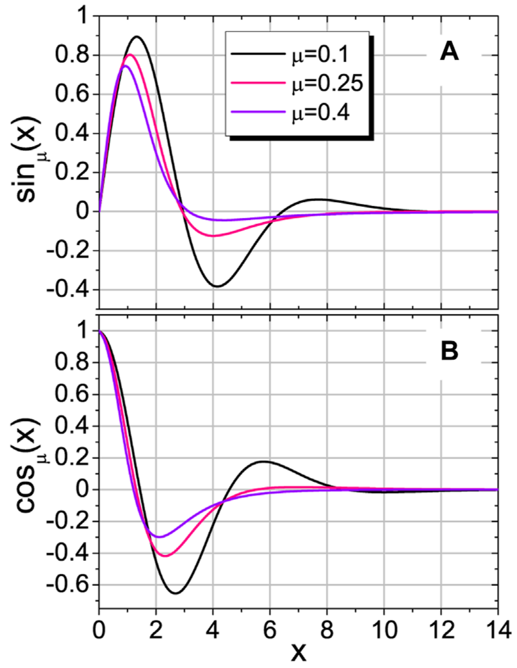

FIGURE 1
Basic μ-deformed trigonometric **(A, B)** and spherical **(C, D)** functions.

Since the applied deformation concerns mainly the basic trigonometric functions, let us study $\sin_\mu x$ and $\cos_\mu x$ in detail. Contracting the corresponding series to the Gaussian hypergeometric function, we can then represent them in the analytic form

$$\sin_\mu x = I(x,\mu)\sin\varphi(x), \qquad \cos_\mu x = I(x,\mu)\cos\varphi(x) \tag{4}$$

where

$$I(x,\mu) \equiv \left(1+\mu^2x^2\right)^{-\frac{1+\mu}{2\mu}}, \qquad \varphi(x) \equiv \frac{1+\mu}{\mu}\arctan(\mu x). \tag{5}$$

Therefore, in the case of μ-deformation, the main trigonometric identity is written as follows:

$$\sin^2_\mu x + \cos^2_\mu x = I^2(x,\mu), \qquad I(x,\mu) \le 1. \tag{6}$$

The behavior of the μ-deformed trigonometric and spherical functions is shown in Figure 1.

In principle, it is possible to express the deformed trigonometric functions in terms of the μ-deformed exponential function. The μ-analogs of exponential and logarithmic functions are

$$\mathrm{e}_\mu(x) = (1-\mu x)^{-\frac{1+\mu}{\mu}}, \qquad \ln_\mu(x) = \frac{1}{\mu}\left(1 - x^{-\frac{\mu}{1+\mu}}\right), \tag{7}$$

which give us the known functions at $\mu \to 0^-$ due to the asymptotic formulas:

$$\mathrm{e}(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n, \qquad \ln(x) = \lim_{n\to\infty} n\left(x^{1/n}-1\right). \tag{8}$$

Note that the μ-deformed functions exhibit a non-trivial property at $\mu > 0$:

$$\left(\mathrm{e}_\mu(x)\right)^n = \mathrm{e}_\mu\left(\frac{1-(1-\mu x)^n}{\mu}\right), \qquad \ln_\mu(x^n) = \frac{1-\left(1-\mu\ln_\mu(x)\right)^n}{\mu}. \tag{9}$$

Focusing on the problems with spherical symmetry, we need to define an inner product $\langle f|g\rangle$ in terms of which the real functions $u_1(x) = j_0^{(\mu)}(x)$ and $u_2(x) = y_0^{(\mu)}(x)$ become orthonormal on finite interval $x \in [0;\ R(\mu)]$:

$$\langle f|g\rangle = \int_0^{R(\mu)} f^*(x)\,g(x)\,w_\mu(x)\,\mathrm{d}x, \qquad \langle u_i|u_j\rangle = \delta_{i,j}, \tag{10}$$

where the asterisk means complex conjugation; the Latin indexes i, j run from 1 to 2.

For the orthogonality of $\sin\varphi$ and $\cos\varphi$ on the interval $\varphi \in [0;\ \pi]$, we constitute *ad hoc*

$$w_\mu(x)\,\mathrm{d}x = \frac{2}{\pi}x^2 I^{-2}(x,\mu)\,\mathrm{d}\varphi(x), \qquad \pi = \varphi(R(\mu)), \tag{11}$$

and obtain

$$w_\mu(x) = \frac{2(1+\mu)}{\pi}x^2\left(1+\mu^2x^2\right)^{\frac{1}{\mu}}, \qquad R(\mu) = \frac{1}{\mu}\tan\frac{\mu\pi}{1+\mu}, \tag{12}$$

where $R(\mu)$ coincides with the first zero of $\sin_\mu x$.

Expanding these as.

$$w_\mu(x) = \frac{2}{\pi}x^2\left[1+(1+x^2)\mu+\left(x^2+\frac{x^4}{2}\right)\mu^2+O(\mu^3)\right], \tag{13}$$

$$R(\mu) = \pi - \pi\mu + \left(\pi+\frac{\pi^3}{3}\right)\mu^2 + O(\mu^3), \tag{14}$$

We see that the known quantities are restored at $\mu = 0$.

TABLE 1 The μ-deformed derivatives.

	f(x)	$D_x^{(\mu)} f(x)$
1	x^n	$[n]_\mu x^{n-1}$
2	$e_\mu(px)$	$pe_\mu(px)$
3	$\ln_\mu(x)$	$(1+\mu-\mu^2)^{-1}x^{-2+\frac{1}{1+\mu}}$
4	$\sin_\mu(x)$	$\cos_\mu(x)$
5	$\cos_\mu(x)$	$-\sin_\mu(x)$
6	$j_0^{(\mu)}(x)$	$\frac{1+\mu}{1-\mu}y_0^{(\mu)}(x)-\frac{1-\mu^2x^2}{(1-\mu)x}j_0^{(\mu)}(x)$
7	$y_0^{(\mu)}(x)$	$-\frac{1+\mu}{1-\mu}j_0^{(\mu)}(x)-\frac{1-\mu^2x^2}{(1-\mu)x}y_0^{(\mu)}(x)$

2.2 Deformed differential calculus

We would like to define the μ-deformed derivative $D_x^{(\mu)}$ with respect to the positive variable x, and its inverse. Let the functions $f(x)$ and $\phi(x)$ admit expansion in the Taylor series and satisfy the relation

$$D_x^{(\mu)} f(x) = \phi(x). \tag{15}$$

The actions of $D_x^{(\mu)}$ and antiderivative $\left(D_x^{(\mu)}\right)^{-1}$ are respectively given as.

$$\phi(x) = \frac{d}{dx}\left[f(x) - x^{-\frac{1}{\mu}}\int_0^x f'(s)\, s^{\frac{1}{\mu}}\, ds\right], \tag{16}$$

$$f(x) = \mu x\,\phi(x) + \int_0^x \phi(s)\, ds + f(0), \tag{17}$$

Were the prime means ordinary differentiation.

We see that $\phi(x) = f'(x)$ at $\mu \to 0$ due to vanishing $(s/x)^{1/\mu}$ for $s < x$. By definition, the derivative $D_x^{(\mu)}$ lowers the exponent of the monomial x^n by one, namely, $D_x^{(\mu)} x^n = [n]_\mu x^{n-1}$. However, the operator $D_x^{(\mu)}$ violates the Leibniz rule: $D_x^{(\mu)}(f(x)g(x)) \neq g(x)D_x^{(\mu)}f(x) + f(x)D_x^{(\mu)}g(x)$.

The μ-deformed derivatives of some functions are collected in Table 1. To derive expressions 4–7, we have used the known auxiliary integrals:

$$\int \sin^{p-1}x \begin{Bmatrix}\sin((p+1)x)\\ \cos((p+1)x)\end{Bmatrix} dx = \frac{1}{p}\sin^p x \begin{Bmatrix}\sin(px)\\ \cos(px)\end{Bmatrix}. \tag{18}$$

On the base of relations 4–5 (not 6–7) in Table 1, we define the Hermitian momentum operator $\hat{P}$ as

$$\hat{P} = -\frac{i}{x}D_x^{(\mu)}x, \tag{19}$$

so that $\langle u_i|\hat{P}|u_i\rangle = 0$, and $\langle u_1|\hat{P}|u_2\rangle = \langle u_2|\hat{P}^*|u_1\rangle = i$ (see Eq. 10), using the imaginary unit i. This operator will play an important role in the study of the deformed Heisenberg algebra further on.

To demonstrate the action of $\hat{P}$ on some functions, note that $\hat{P}x^n = -i[n+1]_\mu x^{n-1}$ for $n \geq 0$, and then

$$\hat{P}\psi_p(x) = p\psi_p(x), \qquad \psi_p(x) = \frac{e_\mu(ipx)}{x}. \tag{20}$$

In addition, we consider the radial part $\Delta_r^{(\mu)}$ of μ-deformed Beltrami–Laplace operator and its inverse (up to the additive constant $C \sim f(0)$).

$$\Delta_r^{(\mu)} f(r) \equiv \frac{1}{r^2}D_r^{(\mu)}\left(r^2 D_r^{(\mu)} f(r)\right), \tag{21}$$

$$\left(\Delta_r^{(\mu)}\right)^{-1} f(r) = \mu^2 r^2 f(r) + (1+\mu)\int_0^r f(s)\, s\, ds - \frac{1-\mu}{r}\int_0^r f(s)\, s^2\, ds + C. \tag{22}$$

It is easy to verify for positive n that

$$\Delta_r^{(\mu)} r^n = [n]_\mu \cdot [n+1]_\mu\, r^{n-2}; \quad \left(\Delta_r^{(\mu)}\right)^{-1} r^n = \frac{r^{n+2}}{[n+2]_\mu \cdot [n+3]_\mu}, \quad C = 0. \tag{23}$$

We also verify that

$$\left(\Delta_r^{(\mu)}\right)^{-1} j_0^{(\mu)}(r) + j_0^{(\mu)}(r) = 0, \qquad C = -j_0^{(\mu)}(0), \tag{24}$$

by the use of the integrals

$$\int \cos^{p-1}x \begin{Bmatrix}\sin((p+1)x)\\ \cos((p+1)x)\end{Bmatrix} dx = \frac{1}{p}\cos^p x \begin{Bmatrix}-\cos(px)\\ \sin(px)\end{Bmatrix}. \tag{25}$$

3 Deformed Heisenberg algebra and uncertainty principle

3.1 Deformed equations and Heisenberg algebra

Here we are going to present the equations of some models using deformed differential calculus. The main model for us, from which the deformed Heisenberg algebra will follow, is described by the deformation of the Lane–Emden equation (LEE) for finite density function $\rho(r)$ in the two possible formulations.

$$\left(\Delta_r^{(\mu)}\rho(r) + k^2\right)\rho(r) = 0, \tag{26}$$

$$\left(D_r^{(\mu)}D_r^{(\mu)} + g_\mu(r)\frac{2}{r}D_r^{(\mu)} + h_\mu(r)k^2\right)\rho(r) = 0, \tag{27}$$

Where

$$g_\mu(r) = \frac{1}{1-2\mu}\left(1 - \frac{1-\mu}{1+\mu}\mu^2k^2r^2\right), \quad h_\mu(r) = \frac{1+2\mu}{1-2\mu} - 2\mu^2\frac{1-\mu^2k^2r^2}{(1+\mu)(1-2\mu)}. \tag{28}$$

Note that the version Eq. 26 of μ-deformed LEE was already dealt with earlier in (Gavrilik et al., 2019), whereas the version in Eq. 27 is completely new, unpublished one. As seen, $g_\mu(r) \to 1$ and $h_\mu(r) \to 1$ at $\mu \to 0$.

It is important that, due to the special form of $g_\mu(r)$ and $h_\mu(r)$, these two μ-deformed versions of LEE have the same physical solution $j_0^{(\mu)}(kr)$ (along with $y_0^{(\mu)}(kr)$) at $\mu < 0.5$, which means that the two versions are equivalent. To display this equivalence we have to explicitly transform Eq. 26 into Eq. 27. Setting $kr \equiv x$ for simplicity, we assume the permutation rule as $D_x^{(\mu)}x = \sigma(x)\,x\,D_x^{(\mu)} + \lambda(x)$, apply it twice to the operator $D_r^{(\mu)}\, r^2\, D_r^{(\mu)}$ of the μ-Laplace operator in Eq. 21, and find the functions $\sigma(x)$ and $\lambda(x)$. Then, the equivalence of Eq. 26 and Eq. 27 is seen, with the non-trivial commutation relation:

$$\sigma(x)\; x\, D_x^{(\mu)} - D_x^{(\mu)}\, x = -\lambda(x), \tag{29}$$

$$\begin{aligned} \sigma(x) &= \frac{1}{\sqrt{h_\mu}} = \left[\frac{(1-2\mu)(1+\mu)}{1+\mu(3+2\mu^3x^2)}\right]^{1/2}, \\ \lambda(x) &= \frac{2g_\mu}{(1+\sigma)\,h_\mu} = \frac{1+\mu-(1-\mu)\mu^2x^2}{\mu\,[2+\mu(1+\mu^2x^2)]}(1-\sigma). \end{aligned} \tag{30}$$

As result, we have come to *the new (μ-deformed) generalization of Heisenberg algebra*.

The functions $\lambda(x)$ and $\sigma(x)$ are real for $0<\mu<0.5$, tend to 1 at $\mu\to 0$, and are shown in Figure 2. We have $0\le\sigma(x)\le 1$, while the maximum of $\lambda(x)$ is determined by $\lambda(0)$ and is equal to

$$\lambda_{\max}(\mu) = \frac{1+\mu}{(2+\mu)\,\mu}\left(1-\sqrt{\frac{(1-2\mu)(1+\mu)}{1+3\mu}}\right). \tag{31}$$

Although the function $\lambda(x)$ has a tail in negative values for

$$x > x_{\max}(\mu) = \frac{1}{\mu}\sqrt{\frac{1+\mu}{1-\mu}}, \tag{32}$$

as shown in Figure 2, consideration of the problem over finite interval of $x\in[0;\,R(\mu)]$ with $R(\mu)\le x_{\max}(\mu)$ for $\mu\in(0;0.5]$ guarantees a positive value of $\lambda(x)$. Therefore, $R(\mu)$ varies between $R_{\min}\simeq 2.886$ and $R_{\max}=2\sqrt{3}\simeq 3.464$, and it is the minimum positive number that satisfies the condition $\sin_\mu R(\mu)=0$ (see Eq. 12).

It seems to be of interest to consider, elsewhere, the quantum-mechanical problem of the propagation of a particle, viewed as a spherical wave $\Psi(r)$ in a space curved due to μ-deformation. Without specifying the boundary condition, it can be formulated as follows:

$$\hat{P}^2\Psi(r) = k^2\Psi(r), \tag{33}$$

where the momentum operator Eq. 19 for $x=r$ is used. Let us remark again that the operator $D_r^{(\mu)}$ in $\hat{P}$ is a pseudohermitian one, see e.g., (Mostafazadeh, 2002; Bagchi and Fring, 2009; Gavrilik and Kachurik, 2016b; Gavrilik and Kachurik, 2019), but the "sandwiching" $\eta^{-1}D_r^{(\mu)}\eta$ with $\eta=r$ transforms it into Hermitian form as in Eq. 33.

In view of the definition Eq. 19 of the momentum operator, we formulate our μ-deformed Heisenberg algebra:

$$\sigma(x)\; x\,\hat{P} - \hat{P}\,x = \mathrm{i}\lambda(x). \tag{34}$$

In what follows, we will focus on the study of the uncertainty principle (relation) which follows from the algebra Eq. 34.

3.2 Generalized uncertainty principle

Denoting the standard deviations as

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}, \qquad \Delta P = \sqrt{\langle \hat{P}^2\rangle - \langle \hat{P}\rangle^2}, \tag{35}$$

we proceed to the analysis of the generalized uncertainty relation (GUP)

$$\Delta x\,\Delta P \ge \frac{1}{2}|\langle [x,\hat{P}]\rangle|, \tag{36}$$

where the commutator is taken from Eq. 34.

To gain insight into the general properties of Eq. 36 for the μ-deformed Heisenberg algebra Eq. 34, let us combine Eq. 34 with its Hermitian conjugate to obtain

$$[(1+\sigma(x))x,\hat{P}] = 2\mathrm{i}\lambda(x). \tag{37}$$

Applying Eq. 36 to this commutation relation, we get

$$\Delta\,[(1+\sigma)x]\;\Delta P \ge \langle\lambda(x)\rangle. \tag{38}$$

Taking into account that $1\ge\sigma(x)>0$ for $\mu<0.5$ in the left hand side, we come to the GUP

$$\Delta x\,\Delta P \ge \frac{1}{2}\langle\lambda(x)\rangle. \tag{39}$$

To evaluate the averages, we specify the states similarly to quantum ones. So, let us consider a normalized mixed state $|\xi\rangle$ for $\xi\in[0;2\pi)$ in a Hilbert space basis Eq. 3 endowed with the inner product from Eq. 10:

$$|\xi\rangle = \cos\xi|u_1\rangle + \sin\xi|u_2\rangle. \tag{40}$$

Here $u_1(x)=j_0^{(\mu)}(x)$ and $u_2(x)=y_0^{(\mu)}(x)$ as before.

In fact, the mixed state $|\xi\rangle$ represents a general solution to the μ-deformed LEE, given by Eq. 26 and Eq. 27. Since the μ-deformed LEE is formulated for the local density of matter and, therefore, basically differs from the complex-valued Schrödinger equation, it is natural to describe its solution from Eq. 40 in terms of real-valued functions. Although the state $|0\rangle$ for $\xi=0$, such that $\langle x|0\rangle=j_0^{(\mu)}(x)$, serves to describe the finite DM distribution in (Gavrilik et al., 2019), the case $\xi\ne 0$ admits the contribution of the cuspidal distribution $y_0^{(\mu)}(x)$ at $x\to 0$.

Thus, we define the mean:

$$\langle(\ldots)\rangle = \langle\xi|(\ldots)|\xi\rangle \tag{41}$$

for fixed $\xi\in[0;\,2\pi]$ and $0<\mu<0.5$.

In contrast to quantum mechanics, Eq. 41 suggests to evaluate a mean of some operator $(\ldots)$ in the basis generated by the μ-deformed LEE. There is no mathematical incorrectness in choosing basis functions coinciding with physical distributions. Only in turning to an interpretation, does one face the averaging (of powers) of the distribution function itself (this also happens in multifractal analysis).

The necessary matrix elements are given by.

$$\langle\xi|f(x)|\xi\rangle = A_f(\mu) - B_f(\mu)\cos(2\xi) + C_f(\mu)\sin(2\xi), \tag{42}$$

$$\langle\xi|f(x)|\xi+\pi/2\rangle = B_f(\mu)\sin(2\xi) + C_f(\mu)\cos(2\xi), \tag{43}$$

$$\begin{aligned} \begin{Bmatrix} A_f \\ B_f \\ C_f \end{Bmatrix} &= \frac{1}{\pi}\int_0^\pi \begin{Bmatrix} 1 \\ \cos(2\varphi) \\ \sin(2\varphi) \end{Bmatrix} f(X(\varphi))\,\mathrm{d}\varphi, \\ X(\varphi) &= \frac{1}{\mu}\tan\frac{\mu\varphi}{1+\mu}. \end{aligned} \tag{44}$$

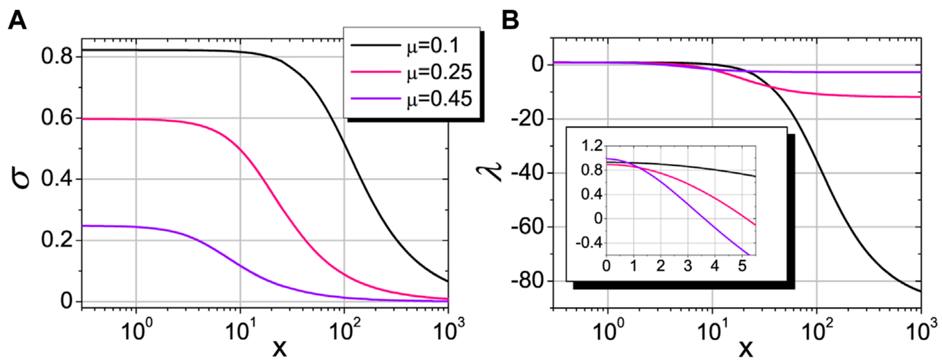

FIGURE 2
Functions σ(x) **(A)** and λ(x) **(B)** in a wide range of variable *x* and at fixed *μ*.

It is immediately seen that

$$\langle \hat{P} \rangle = \langle \xi | \hat{P} | \xi \rangle = -\mathrm{i} \langle \xi | \xi + \pi/2 \rangle = 0, \quad \langle \hat{P}^2 \rangle = \langle \xi | \hat{P}^2 | \xi \rangle = -\langle \xi | \xi + \pi \rangle = 1, \tag{45}$$

Therefore, the standard deviation $\Delta P = 1$ is fixed for the set of states $\{|\xi\rangle\}$.

On the other hand, let us introduce the functions

$$\Delta x(\xi,\mu) \equiv \sqrt{\langle \xi | x^2 | \xi \rangle - (\langle \xi | x | \xi \rangle)^2}, \quad \Lambda(\xi,\mu) \equiv \langle \xi | \lambda(x) | \xi \rangle, \tag{46}$$

which represent the averages Δx and $\langle \lambda(x) \rangle$, respectively.

Thus, in the basis of the μ-deformed spherical waves, one has $\Delta P = 1$, and it is required that

$$\Delta x(\xi,\mu) \geq \frac{1}{2}\Lambda(\xi,\mu). \tag{47}$$

This relation can be analyzed with the help of Figure 3A.

Let us introduce the auxiliary momentum variance, accounting for Eq. 47:

$$\delta P(\xi,\mu) = \frac{\Lambda(\xi,\mu)}{2\Delta x(\xi,\mu)} \leq 1. \tag{48}$$

We see that $\delta P(\xi,\mu) \leq \Delta P = 1$ and $\delta P(\xi,\mu)\, \Delta x(\xi,\mu) = \Lambda(\xi,\mu)/2$ by definition. The behavior of $\delta P(\xi,\mu)$ is shown in Figure 3B.

The admissible domain of variety of the running values of Δx and ΔP is shown in Figure 4. We see that the black and pink curves are in antiphase regime, as it should be. For comparison, the violet curve describes the change in the deviation Δx according to the hyperbolic law in accordance with the standard Heisenberg algebra.

3.2.1 Alternative approach

To confirm the validity of Eq. 39 for the algebra Eq. 34 non-linear in *x*, it is worth to develop an alternative calculation scheme applicable to various ways of writing the commutator for *x* and $\hat{P}$. For instance, there is a possibility to rewrite relation Eq. 34 in equivalent form as

$$[x,\hat{P}] = \mathrm{i}\frac{2\lambda(x)}{1+\sigma(x)} + \frac{1-\sigma(x)}{1+\sigma(x)}\{x,\hat{P}\}, \tag{49}$$

where $\{x,\hat{P}\} \equiv x\hat{P} + \hat{P}x$ is anticommutator.

To analyze the GUP given by Eq. 36 for this commutation relation, we assume that the brackets $\langle(\ldots)\rangle$ mean the quantum average over the state defined by the *real* wave function in the coordinate representation. Then, the action of the operator $\mathrm{i}\hat{P}$ on such a state results in a real-valued expression, what immediately yields

$$|\langle[x,\hat{P}]\rangle| = 2\left\langle\frac{\lambda(x)}{1+\sigma(x)}\right\rangle - \left\langle\frac{1-\sigma(x)}{1+\sigma(x)}\{x,\mathrm{i}\hat{P}\}\right\rangle, \tag{50}$$

when the positive first term on the right hand side dominates the second one.

In view of the inequality $|\langle\hat{A}\hat{B}\rangle| \leq |\langle\hat{A}\rangle|\,|\langle\hat{B}\rangle|$, we split the last term as

$$\left|\left\langle\frac{1-\sigma(x)}{1+\sigma(x)}\{x,\mathrm{i}\hat{P}\}\right\rangle\right| \leq \left\langle\frac{1-\sigma(x)}{1+\sigma(x)}\right\rangle|\langle\{x,\mathrm{i}\hat{P}\}\rangle|, \quad 0 \leq \sigma(x) \leq 1. \tag{51}$$

At this stage, we obtain

$$|\langle[x,\hat{P}]\rangle| \geq 2\left\langle\frac{\lambda(x)}{1+\sigma(x)}\right\rangle - \left\langle\frac{1-\sigma(x)}{1+\sigma(x)}\right\rangle|\langle\{x,\mathrm{i}\hat{P}\}\rangle|. \tag{52}$$

To evaluate $|\langle\{x,\mathrm{i}\hat{P}\}\rangle|$, we introduce the operators $\delta x = x - \langle x\rangle$ and $\delta\hat{P} = \hat{P} - \langle\hat{P}\rangle$, where the hat over $\hat{P}$ distinguishes the operator $\delta\hat{P}$ from the function δP in Eq. 48. Then one gets

$$\langle\{x,\mathrm{i}\hat{P}\}\rangle = 2\mathrm{i}\langle x\rangle\langle\hat{P}\rangle + \langle\{\delta x,\mathrm{i}\delta\hat{P}\}\rangle. \tag{53}$$

Due to the Cauchy–Schwartz inequality $|\langle\hat{A}\hat{B}\rangle|^2 \leq |\langle\hat{A}^2\rangle|\,|\langle\hat{B}^2\rangle|$, the following estimate holds:

$$|\langle\{\delta x,\mathrm{i}\delta\hat{P}\}\rangle| \leq 2\Delta x\,\Delta P. \tag{54}$$

Since the dark matter flux is assumed to be absent in the halo, one can put $\langle\hat{P}\rangle = 0$, which is confirmed by our direct calculations. Combining, we have the estimate

$$|\langle[x,\hat{P}]\rangle| \geq 2\left\langle\frac{\lambda(x)}{1+\sigma(x)}\right\rangle - 2\left\langle\frac{1-\sigma(x)}{1+\sigma(x)}\right\rangle\Delta x\,\Delta P. \tag{55}$$

Substituting it into Eq. 36 and accounting for $\langle 1\rangle = 1$, we arrive at.

$$\Delta x\,\Delta P \geq \frac{1}{2}\langle\lambda(x)\rangle_W, \tag{56}$$

$$\langle(\ldots)\rangle_W \equiv \frac{\langle W(x)(\ldots)\rangle}{\langle W(x)\rangle}, \quad W(x) = \frac{1}{1+\sigma(x)}, \tag{57}$$

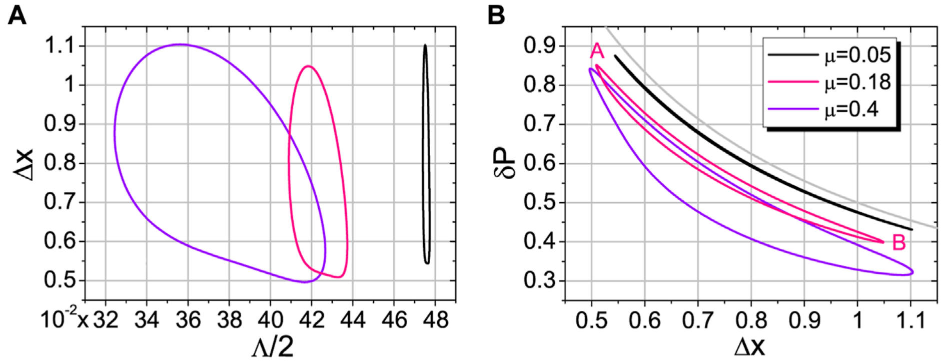

FIGURE 3
(A): Position deviation $\Delta x\ (\xi,\mu)$ *versus* mean $\langle\lambda(x)\rangle = \Lambda(\xi,\mu)$, which are computed in the state $|\xi\rangle$ at fixed μ **(B)**: Limiting momentum deviation $\delta P\ (\xi,\mu) = \Lambda/(2\Delta x)$ *versus* $\Delta x\ (\xi,\mu)$. Turning points of the pink banana-like curve are A (0.51;0.85), B (1.05;0.4). Grey line corresponds to $\delta P = 1/(2\Delta x)$.

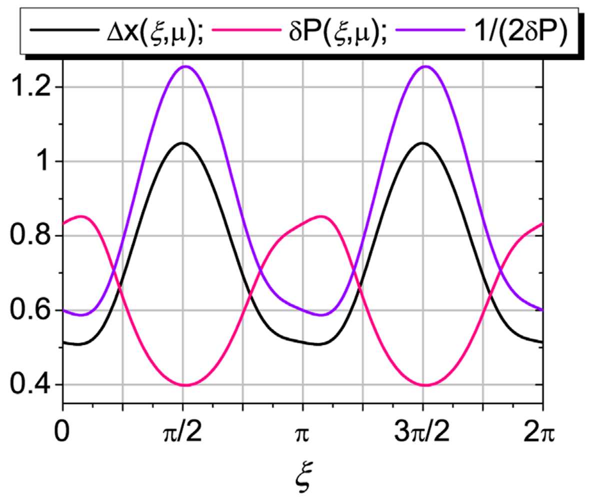

FIGURE 4
Dependence of the lower limit deviations $\Delta x\ (\xi,\mu)$ and $\delta P\ (\xi,\mu)$ on the state labeled by ξ at $\mu = 0.18$. The running deviations Δx and ΔP satisfy the GUP $\Delta x\Delta P \geq \Lambda/2$ and vary in the ranges $\Delta x\ (\xi,\mu) \leq \Delta x \leq R\ (\mu = 0.18) \simeq 2.89$ and $\delta P\ (\xi,\mu) \leq \Delta P \leq 1$.

Where the new mean $\langle(\ldots)\rangle_W$ with additional convex weighting function $W(x)$ arises.

For a given function $W(x)$ we get

$$\langle\lambda(x)\rangle_W = \langle\lambda(x)\rangle + \frac{\langle\delta W(x)\,\delta\lambda(x)\rangle}{\langle W(x)\rangle}, \tag{58}$$

where $\langle\delta W(x)\delta\lambda(x)\rangle$ is a covariance between the convex function $W(x)$ and concave $\lambda(x)$, and it determined by deviations like $\delta f(x) = f(x) - \langle f(x)\rangle$.

Since the function $\sigma(x)$ (and $W(x)$) changes only slightly over the interval $x \in [0; R(\mu)]$ in **Figure 2A**, it can be approximated by a constant close to $\sigma(0)$ when calculating integrals. This provides $\delta W(x) \to 0$ and numerically leads to expressions:

$$\left\langle\frac{\lambda(x)}{1+\sigma(x)}\right\rangle \simeq \frac{\langle\lambda(x)\rangle}{1+\langle\sigma(x)\rangle}, \qquad \left\langle\frac{1}{1+\sigma(x)}\right\rangle \simeq \frac{1}{1+\langle\sigma(x)\rangle}, \tag{59}$$

when we use $\langle(\ldots)\rangle = \langle\xi|(\ldots)|\xi\rangle$ in the range $0 < \mu < 0.5$.

This circumstance leads again to Eq. **39** for the states $|\xi\rangle$, that is just Eq. **47**.

Note that the appearance of the mean Eq. **57** is associated with the initial Eq. **49** for the commutation relation. In other cases, we may only encounter means of type Eq. **59**, where it would be justified to use the estimate $1 \geq \langle\sigma(x)\rangle$.

3.3 Application to dark matter

Let us remind the connection between the operators in terms of the dimensionless variable $x = kr$ and the operators of the physical radial coordinate r and the momentum $\hat{P}_r$:

$$r = \frac{x}{k}, \qquad \hat{P}_r = \hbar k\hat{P}, \tag{60}$$

where k is the parameter of Eqs **26** and **27** and has the dimension of inverse length; the operator $\hat{P}$ is given by Eq. **19**.

The most successful results of paper (Gavrilik et al., 2019) for describing the dark matter halo of dwarf galaxies based on the μ-deformed Lane–Emden equation were obtained in the following range of parameters:

$$\mu = 0.151\ldots0.18, \qquad k = 0.17\ldots2.64\ \mathrm{kpc}^{-1}. \tag{61}$$

Using the turning points $A\ ((\Delta x)_{\min};\ (\delta P)_{\max})$ and $B\ ((\Delta x)_{\max};\ (\delta P)_{\min})$ for fixed μ as in **Figure 3B**, we relate extreme physical values Δr and ΔP_r with dimensionless ones Δx and δP as.

$$\Delta r = \Delta x\left[\frac{k}{1\ \mathrm{kpc}^{-1}}\right]^{-1}\mathrm{kpc}, \tag{62}$$

$$\Delta P_r = \delta P\left[\frac{k}{1\ \mathrm{kpc}^{-1}}\right]\times 6.394\times10^{-27}\ \frac{\mathrm{eV}}{c}. \tag{63}$$

The calculation results are collected in **Table 2**. Therein, we present the obtained data for five dwarf galaxies (from the eight ones given in **Table 1** of (Gavrilik et al., 2019)), because just for these galaxies the μ-deformation based description of the rotation curves is most successful with respect to earlier approaches, as it provides the best agreement with observational data (certainly better then

TABLE 2 The parameters for the dark matter halos of dwarf galaxies.

Galaxy	μ	k, kpc^{-1}	$(\Delta r)_{max}$, kpc	$(\Delta r)_{min}$, kpc	$(\Delta P_r)_{max}$, $10^{-27} eV/c$	$(\Delta P_r)_{min}$, $10^{-27} eV/c$
M81dwB	0.18	2.64	0.398	0.193	14.38	6.75
DDO 53	0.18	0.97	1.082	0.526	5.28	2.48
IC 2574	0.179	0.17	6.18	3.0	0.926	0.435
NGC 2366	0.178	0.37	2.84	1.38	2.02	0.946
HO I	0.151	1.27	0.830	0.402	6.98	3.33

if one uses the profile from the usual Bose-condensate model of DM being the solution of non-deformed Lane–Emden equation as in (Harko, 2011), or uses the famous Navarro–Frenk–White profile (Navarro et al., 1997)).

Of course, the remaining three galaxies can also be considered, but the choice of five ones is both sufficient, trustful, and best suited for our treatment and conclusions.

Note that both the Lane–Emden equation and its μ-deformed extensions determine the distribution function $\rho(r)$ of the matter, not the wave function of single particle. Therefore, the mean Eq. 41 is a quadratic form in the distribution, related here with $|0\rangle$ which differs by a multiplicative constant defining $\rho(0)$ (Gavrilik et al., 2019). Generally speaking, the state $|0\rangle$ may not determine the turning points of a banana-like curve in the $(\Delta x, \delta P)$ plane in **Figure 3B**, along with the extreme values of the deviations Δr and ΔP_r. Nevertheless, the mathematically correct mean Eq. 41 can be used to obtain new additional information about dark matter, even by means of considering the moments $\langle \rho^n \rangle$ of the distribution similarly to multifractal analysis. Besides, the extreme deviations set the limits for the fluctuations of physical quantities at $\xi = 0$.

Without a deep study of the structure of averages here, let us analyze the physical consequences of the data in **Table 2**. We see that in the non-relativistic theory the momentum deviation $\Delta P_r = m\Delta v_r$, where m is the particle mass, Δv_r is the deviation of particle radial velocity v_r. Since the original work (Gavrilik et al., 2019) was using bosons with $m \sim 10^{-22}$ eV$/c^2$, we obtain from **Table 2** that $\Delta v_r \sim 10^{-5} c$ in units of the speed of light c. Moreover, deviation of the kinetic energy $\Delta E_K = m(\Delta v_r)^2/2$ can be used to determine the effective temperature of dark matter, namely, $T_{\text{eff}} = (\Delta P_r)^2/(2m) \sim 10^{-32}$ eV. This value is much smaller than the critical temperature of the Bose–Einstein condensation, as it should be in such a paradigm.

Due to the GUP given by Eq. 39, we relate the temperature T_{eff} of the spherical layer in the vicinity of $\langle r \rangle$ to its width $2\Delta r$:

$$(\Delta r)^2\, T_{\text{eff}} \geq \frac{\hbar^2}{8m} \langle \lambda \rangle^2. \quad (64)$$

This formula holds for a macroscopic system of finite volume when Δr does not exceed the radius of the system, and it shows that a smaller domain may have a higher temperature, and vice verse.

It is worth to note that the mean $\langle \lambda \rangle$ in Eq. 64 takes values in the limited interval $\langle \lambda \rangle = \Lambda(\xi, \mu) \in [\Lambda_{\text{min}}, \Lambda_{\text{max}}]$, where the positive Λ_{min} and Λ_{max} depend on μ (see Eq. 46; **Figure 3A**). For $\mu = 0.18$, we have $\Lambda_{\text{min}} \simeq 0.818$ and $\Lambda_{\text{max}} \simeq 0.875$.

4 Concluding remarks

In this work we have studied unusual consequences of the new (μ-deformed) generalization of the Heisenberg algebra Eqs 29 and 34 which is special as it was derived within the extension of Bose-condensate dark matter model based on μ-deformation. From the generalized algebra we obtained non-trivial GUR that generates minimal and maximal uncertainties of both positions (minimal/maximal lengths) and momenta. The obtained GUR is strictly dependent on the states (labeled by ξ) of the system, and such dependence was exploited to full extent.

In **Table 2**, the galaxies M81dwB and IC 2574 look as the two "extreme" cases. Namely, for the latter we have the largest $(\Delta r)_{\text{max}}$ and $(\Delta r)_{\text{min}}$, while for the former these quantities show smallest values. Clearly, the situation concerning $(\Delta P_r)_{\text{max}}$ and $(\Delta P_r)_{\text{min}}$ is quite opposite. Noteworthy, the value of μ (strength of deformation) for M81dwB and IC 2574 is almost the same. The relations Eq. 62, 63 show the defining role of the quantity k which involves scattering length a and particle mass m as $k \propto m^{3/2} a^{-1/2}$ (Harko, 2011).

For the considered galaxies (each labeled by its specific value of μ) we conclude: since the particle mass is same (namely, 10^{-22} eV$/c^2$), we have differing scattering lengths in halos of different galaxies (*vice versa*, would we assume same scattering length for all the five galaxies we would have somewhat differing masses of DM particle in different galaxies, though this second option seems to be less realistic). As already shown in (Harko, 2011), where the BEC DM model is also based on the LEE, there is no universality of model parameters when describing all admissible objects. In fact, this issue remains in our model, which improves the previously fitted rotation curves by including an additional parameter μ. Physically, we can only control the applicability conditions of our model: consider DM-dominated dwarf galaxies leaving aside their rigid rotation, which contributes to the distribution function (Zhang et al., 2018; Nazarenko, 2020). Therefore, giving clear physical meaning to differing scattering lengths in halos of different galaxies remains an interesting task for future study.

Note that the parameter k is related to the observed radius r_{gal} of the galactic halo by $kr_{\text{gal}} = R(\mu)$, where the right-hand side is determined by the parameter $R(\mu)$ from Eq. 12, replacing $\pi = R(0)$ in the non-deformed case. We can easily find a small difference (of several percent) between the values of r_{gal} in the deformed and non-deformed cases, by comparing these with the galaxy radii from (Harko, 2011). However, the simulation of rotation curves is more successful in the deformed case, as shown in (Gavrilik et al., 2019).

It is worth to remark that the values of $(\Delta r)_{\text{max}}$ and $(\Delta r)_{\text{min}}$ for the two galaxies M81dwB and IC 2574, and the others in **Table 2**, reside well within the observed sizes of DM halos as it should. Accordingly, the values of $(\Delta P_r)_{\max}$ and $(\Delta P_r)_{\min}$ for these same galaxies lie in the ranges completely consistent with DM being in the (μ-Bose) condensate state. Clearly this is in agreement with the above reasonings concerning the effective temperature.

It is of interest to analyze possible special meaning of our results on the existence of finite $(\Delta r)_{\max}$ and $(\Delta r)_{\min}$ in the context of treatment in (Lee and Lim, 2010; Lee, 2016) of minimum length scale of galaxies (note that for the candidate length scales one can take into consideration such concepts as coherence length, Compton wavelength, quantum Jeans length scale, gravitational Bohr radius, and de Broglie wavelength, see (Lee and Lim, 2010) and references therein). Time dependence of some of these quantities, e.g., characteristic length scale $\tilde{\xi}$ (minimum size of DM dominated galaxies) is studied in (Lee, 2016). Let us quote one of the interesting predictions of this work: with the mass of DM particles chosen as $m = 5 \times 10^{-22}$ eV$/c^2$, it follows that $\tilde{\xi}(z = 0) = 311.5$ pc while $\tilde{\xi}(z = 5) = 81.2$ pc, i.e., early dwarf galaxies were significantly more compact. In view of the extremely tiny mass of the particle from dark sector, a question may arise of possible (inter)relation of this entity with the cosmic microwave background (CMB). The very first answer which comes to one's mind could be that no relation is possible, because of the absence of interaction between visible and dark sectors. However, when considered in the framework of doubly special relativity, the properties of the photon gas at these special conditions can appear, see (Chung et al., 2019), much more interesting and non-trivial. Noteworthy, the treatment in (Chung et al., 2019), on one hand, is potentially applicable for studying some unclear features of CMB, and, on the other hand, involves a kind of deformation which is very similar to the μ-deformation explored herein. We hope to address the details of all these intriguing issues elsewhere.

Data availability statement

The original contributions presented in the study are included in the article/**Supplementary Material**, further inquiries can be directed to the corresponding author.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Supplementary material

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fspas.2023.1133976/full#supplementary-material

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