FRACTIONAL CALCULUS AND ITS APPLICATIONS IN PHYSICS

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FRACTIONAL CALCULUS AND ITS APPLICATIONS IN PHYSICS

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EDITORIAL
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Editorial: Fractional Calculus and Its Applications in Physics

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Editorial on the Research Topic

Fractional Calculus and Its Applications in Physics

Fractional calculus is deeply related to the dynamics of complicated real-world problems. Fractional operators are non-local and describe several natural phenomena in a better and systematic manner. Many mathematical models are accurately governed by fractional order differential equations. Since the classical mathematical models are special cases of the fractional order mathematical models, it implies that the results for the fractional mathematical model are more general and more accurate. The fractional derivatives and integrals are very helpful for engineers, mathematicians, scientists, and researchers working with the real-life phenomena. Thus, this Research Topic Ebook contains some recent investigations demonstrating the depth and breadth of ongoing studies in the area of fractional calculus and its applications in physics. It contains nine articles from 40 authors from all over the world.

Prakash and Verma presents a user-friendly technique using the theory of Adomian decomposition technique to obtain the analytical solutions of the Newell–Whitehead–Segel equations of fractional order. The fractional Newell–Whitehead–Segel equation finds its applications to interpret the formation of the stripe patterns in two-dimensional (2-D) systems. The authors show that the numerical results derived with the aid of the suggested scheme are very accurate.

Shat et al. present a fractional extension of the Laguerre differential equation. The authors used the conformable derivative of order $0 < \alpha < 1$. The authors used the Frobenius scheme together with the fractional power series expansion to derive two linearly independent solutions of the problem. The authors derive the fractional Laguerre functions in closed forms, and establish their orthogonality results.

Gill et al. show the computable solution of the advection-dispersion equation of the arbitrary order pertaining to Hilfer–Prabhakar fractional operator as well as the Laplace operator of fractional order. The technique for obtaining the solution is a mixed approach using the application of Sumudu and Fourier transforms. The authors derive the solution in compact and graceful forms in the form of the generalized Mittag–Leffler function, which is compatible for numerical evaluation of the results.

Hristov presents both the theory and formulation of linear viscoelastic response functions and their reasonable connection with the Caputo–Fabrizio (CF) fractional operator by using the Prony series decomposition (PSD). The author discusses the problem of interconversion with power and exponential laws and pays very special attention on the PSD approach, the connected interconversion problems, and the presentation of the viscoelastic constitutive equations in the form of CF operator of arbitrary order.

Cattani extends the sinc-fractional derivative to the Hilbert space established on Shannon wavelets. The author defines some novel operators of arbitrary order by using the concept of wavelets. The author's main work is to study the localization and compression nature of wavelets when working with operators of non-integer order.

Turalska and West present mathematical concept of the dynamical decision-making model and renewal events, and subordinate the nature of the individual to the mean field nature of the network. The authors proved that the dynamics of the individual is obtained by using the theory of subordination to be a tempered fractional differential equation. The authors reported the exact solution of fractional differential equation in the form of Mittag–Leffler function and computed the numerical results.

Chen et al. show that the Lévy flight is more effective than the Brownian motion if the targets are sparse. The authors consider that every flight of the forager is possibly interrupted by some unknown factors, such as hurdles on the direction of flight, natural opponents in the vision distance, and limitations in the energy storage for every flight, and suggested the tempered Lévy distribution $p(l) \sim e^{-\rho l} l^{-\mu}$. The authors validate both theoretical inspection and simulation outcomes and that a higher searching coherence can be achieved if the lower values of ρ or μ are selected. The authors demonstrate that by considering the flight time as the waiting time, the master equation (ME) of the random searching procedure can be determined. The authors construct two distinct kinds of MEs: one is the classical diffusion equation and second is the tempered diffusion equation of fractional order.

Agarwal et al. apply the fractional operators suggested by Marichev–Saigo–Maeda containing Appell function and establish many novel results of extended Lommel–Wright function. The authors also report that by using some integral transforms on the derived results, many more new and known results can be obtained.

Calcagni discusses the recent plans and agendas to be carried out in order to establish a workable definition of scale-dependent fractional operators and their applications to field theory and gravity. The author investigates distinct kinds of multifractional Laplacians and their properties.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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ORIGINAL RESEARCH
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Towards Multifractional Calculus

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After motivating the need of a multiscale version of fractional calculus in quantum gravity, we review current proposals and the program to be carried out in order to reach a viable definition of scale-dependent fractional operators. We present different types of multifractional Laplacians and comment on their known or expected properties.

Keywords: quantum gravity, fractional calculus, fractional derivatives, multiscale geometry, multifractional spacetimes

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1. INTRODUCTION

A branch of theoretical physics which has been attracting considerable attention in the last years is quantum gravity. Several independent theories, models and hypotheses are gathered under this broad name, from string theory to asymptotic safety, from non-local to loop quantum gravity, from causal dynamical triangulations to causal sets, and so on [1–4]. Most of these proposals aim to conciliate classical general relativity with the laws of quantum mechanics, in order to unify all forces of Nature under the same framework and to solve some problems left open in the traditional paradigms (for instance, the big-bang and cosmological constant problems [3]).

A surprising feature emerging from this variegated landscape is that the properties of spacetime geometry, such as the spectral or Hausdorff dimension and the way particles diffuse, change with the probed scale in all quantum gravities [5–7]. This so-called dimensional flow seems to be a manifestation of the impossibility to perform infinitely precise time and distance measurements in geometries with intrinsic uncertainties of quantum or stochastic origin [8, 9]. Some of these findings were made possible by assuming dimensional flow by default and treating spacetime geometry as fundamentally scale dependent. This general method can be embodied in a class of theories, called *multifractional*, where classical and quantum fields live on a spacetime characterized by a scale hierarchy, anomalous transport and correlation properties, and a multifractal geometry [10]. Surprisingly, all these features emerge automatically by assuming a slow dimensional flow at large scales (dimension in the infrared almost constant) [10, 11].

One can encode a multiscale geometry in the dynamics of particles and fields in several ways. The one followed by multifractional theories is a change in the integro-differential structure [12]. Integrals (such as dynamical actions) and derivatives (in kinetic terms) acquire a non-trivial scale dependence that can be illustrated in the prototype example of the scalar field theory

$$S = \int d\varrho(x) \left[\frac{1}{2}\phi\mathcal{K}\phi - V(\phi)\right], \tag{1}$$

where $\varrho(x)$ is the spacetime measure, $\mathcal{K}$ is a kinetic operator, and V is the scalar potential. In the standard case and in the absence of gravity (which will be ignored here), $\varrho(x) = d^D x$ is the usual Lebesgue measure in D topological dimensions and $\mathcal{K} = \Box = \partial_\mu\partial^\mu$ is the second-order Laplace–Beltrami operator. In the presence of dimensional flow, if the measure is factorizable in the coordinates (an assumption to make the problem tractable) then it takes the unique form [11]

$$\varrho(x) = \prod_\mu dq^\mu(x^\mu)\,, \qquad q^\mu(x^\mu) = x^\mu + \sum_{n=1}^{+\infty} \frac{\ell_n^\mu}{\alpha_{\mu,n}} \mathrm{sgn}(x^\mu) \left|\frac{x^\mu}{\ell_n^\mu}\right|^{\alpha_{\mu,n}} F_n(x^\mu)\,, \tag{2}$$

where

$$F_\omega(x^\mu) = 1 + A_{\mu,n}\cos\left(\omega_{\mu,n}\ln\left|\frac{x^\mu}{\ell^\mu_{\infty,n}}\right|\right) + B_{\mu,n}\sin\left(\omega_{\mu,n}\ln\left|\frac{x^\mu}{\ell^\mu_{\infty,n}}\right|\right), \tag{3}$$

all indices μ are inert (there is no Einstein summation convention), the first factor 1 in Equation (1) is optional [11] (it can be set to zero in the stochastic version of the theory [9]), ℓ_n^μ and $\ell^\mu_{\infty,n}$ are $2D$ length scales for each n, and $\alpha_{\mu,n}$, $A_{\mu,n}$, $B_{\mu,n}$, and $\omega_{\mu,n}$ are $4D$ real constants for each n.

Since the measure is factorized, in the following we can focus the discussion on the one-dimensional model

$$\begin{aligned} S &= \int dq(x)\left[\frac{1}{2}\phi\mathcal{K}\phi - V(\phi)\right],\\ q(x) &\simeq x + \frac{\ell_*}{\alpha}\mathrm{sgn}(x)\left|\frac{x}{\ell_*}\right|^\alpha F_\omega(x), \end{aligned} \tag{4}$$

where $\ell_* = \ell_1$, all μ indices are omitted and we can also ignore terms subleading both in the infrared and in the ultraviolet; this corresponds to consider only the $n = 1$ term in Equations (2) and (3).

In this paper, we will study the properties of three versions of the kinetic operator $\mathcal{K}$, expanding on the proposals sketched in Calcagni [10]. Since, in the context of quantum gravity, the integration measure is uniquely defined independently of the type of derivatives in the Lagrangian as in Equation (4) [11], here we are not interested in the formal properties of "multiscale integrals," the inverse of multiscale derivatives. Some of these operators are known, as is the case of Equations (5) and (35) below [10], while in the case of Equations (28) and (29) they are unknown and will require further work. On the other hand, there is no inverse operator for a linear combination of operators with different inverse, such as Equation (24). In all these cases, for our purposes it is sufficient to study the properties of multiscale derivatives with respect to the ordinary Lebesgue measure dx while, at the same time, taking into account the measure weight by inserting weight factors in the definitions of such derivatives to make them self-adjoint with respect to the measure.

2. *Q*-DERIVATIVES

While there is a unique parametric form of the measure $q(x)$, there is more freedom in the choice of kinetic operator $\mathcal{K}$. It turns out that there are three viable possibilities. One is a theory with so-called weighted derivatives, but this can be reduced to a system with ordinary derivatives and the spectral dimension of spacetime is constant in that case [10]. Another possibility is the second-order operator [13]

$$\mathcal{K} = \partial_q^2, \qquad \partial_q := \frac{\partial}{\partial q(x)} = \frac{1}{v(x)}\frac{\partial}{\partial x}, \tag{5}$$

where $v(x) = q'(x) := \partial_x q(x)$ and $q(x)$ is given by Equation (2). This "q-derivative" has a number of highly desirable properties:

1. It is multiscale, since the scale hierarchy is already encoded in the measure weight $v(x)$.
2. Its composition law is very simple:

$$\partial_q^2 := \partial_q\partial_q = (\partial_q)^2 - \frac{v'}{v^3}\partial_x. \tag{6}$$

3. It is linear. For any f and g in a suitably defined functional space,

$$\partial_q(f+g) = \partial_q f + \partial_q g \tag{7}$$

4. Its kernel is trivial and given by a constant:

$$\partial_q 1 = 0. \tag{8}$$

5. The Leibniz rule is extremely simple. For any f and g in a suitably defined functional space,

$$\begin{aligned}\partial_q(fg) &= \frac{1}{v}(f'g + fg')\\ &= (\partial_q f)\,g + f\,(\partial_q g). \end{aligned}\tag{9}$$

6. Integration by parts is straightforward. For any f and g in a suitably defined functional space,

$$\begin{aligned}\int dq\, f\,\partial_q g &\overset{(9)}{=} \int_{-\infty}^{+\infty} dx\, v\,\frac{1}{v}(fg)' - \int dq\,(\partial_q f)\,g\\ &= -\int dq\,(\partial_q f)\,g, \end{aligned}\tag{10}$$

where we threw away boundary terms. Consequently, $\mathcal{K}$ is self-adjoint:

$$\int dq\, f\,\partial_q^2 g = \int dq\,(\partial_q^2 f)\,g. \tag{11}$$

Notice that, in principle, these rules hold for an arbitrary $q(x)$, although in our case this profile is fixed as in Equation (4).

3. FRACTIONAL DERIVATIVES

The third extant multifractional theory is the least explored, but also the most interesting because it employs fractional calculus. This is by far the most obvious tool to implement an anomalous scaling in the geometry. The application of fractional derivatives to multiscale theories is not an easy task. Before seeing why, let us recall some basic aspects of fractional calculus.

There are different versions of fractional derivatives [14–16][1] and one must make a choice suitable for quantum gravity [12]. In particular, we believe that one cannot renounce to have a trivial kernel (Equation 8). Two fractional derivatives with this property are the Liouville derivative

$$\begin{aligned}{}_\infty\partial^\alpha f(x) &:= \frac{1}{\Gamma(m-\alpha)}\int_{-\infty}^{+\infty} dx'\,\frac{\theta(x-x')}{(x-x')^{\alpha+1-m}}\partial^m_{x'}f(x'),\\ &\quad m-1 \leqslant \alpha < m, \end{aligned}\tag{12}$$

[1] For bibliographic references, see [10, 12]. More recent applications and solving methods can be found in Baleanu et al. [17], Yang et al. [18], Sun et al. [19], Baleanu et al. [20, 21] and Inc et al. [22].

and the Weyl derivative

$$ {}_\infty\bar\partial^\alpha f(x) := \frac{1}{\Gamma(m-\alpha)}\int_{-\infty}^{+\infty} dx'\,\frac{\theta(x'-x)}{(x'-x)^{\alpha+1-m}}\partial_{x'}^m f(x')\,, \qquad m-1\leqslant\alpha<m\,, \tag{13}$$

where θ is Heaviside's step function. Obviously, these operators act linearly on f and ${}_\infty\partial_x^\alpha 1=0={}_\infty\bar\partial^\alpha 1$. One can also check that ${}_\infty\partial^\alpha{}_\infty\partial^\beta={}_\infty\partial^{\alpha+\beta}$ and ${}_\infty\bar\partial^\alpha{}_\infty\bar\partial^\beta={}_\infty\bar\partial^{\alpha+\beta}$ (these fractional derivatives commute) and that the Leibniz rule is

$$ {}_\infty\partial^\alpha(fg)=\sum_{j=0}^{+\infty}\binom{\alpha}{j}(\partial^j f)({}_\infty\partial^{\alpha-j}g)\,,\qquad \binom{\alpha}{j}=\frac{\Gamma(1+\alpha)}{\Gamma(\alpha-j+1)\Gamma(j+1)}\,, \tag{14}$$

and the same expression for the Weyl derivative, where $\partial^{\alpha-j}=I^{j-\alpha}$ are integrations for $j\geqslant 1$. Also, integration by parts with the Liouville derivative generates the Weyl derivative, and vice versa:

$$\int_{-\infty}^{+\infty} dx\, f\,{}_\infty\partial^\alpha g=\int_{-\infty}^{+\infty} dx\,({}_\infty\bar\partial^\alpha f)\,g\,. \tag{15}$$

3.1. Complicated Leibniz Rule

The importance to have the standard Leibniz rule (Equation 9) can be appreciated when trying to do physics with fractional calculus. In the theory with q-derivatives, integration by parts does not produce extra contributions and the kinetic terms $\int dq\,\phi\partial_q^2\phi$ or $-\int dq\,\partial_q\phi\partial_q\phi$ are completely equivalent. Therefore, the equation of motion $\partial_q^2\phi-V_{,\phi}=0$ can be determined easily by applying the variational principle on Equation (4). On the other hand, suppose we choose another type of multiscale derivative $\mathcal{D}$ such that $\mathcal{K}=\mathcal{D}^2$ and its Leibniz rule is more complicated:

$$\mathcal{D}(fg)=(\mathcal{D}f)\,g+f\,(\mathcal{D}g)+X\,, \tag{16}$$

where $X=X(f,g;x)$ is a function of f, g, their ordinary derivatives and the coordinate x. For consistency, if the kernel of $\mathcal{D}$ is trivial ($\mathcal{D}1=0$), then $X(f,1;x)=X(1,f;x)=0$ for any f. In particular, if $g=\mathcal{D}h$, then

$$\begin{aligned} f\mathcal{D}^2h &= \mathcal{D}(f\mathcal{D}h)-X(f,\mathcal{D}h;x)-(\mathcal{D}f)\,\mathcal{D}h\\ &=[\mathcal{D}(f\mathcal{D}h)-\mathcal{D}(h\mathcal{D}f)-X(f,\mathcal{D}h;x)+X(\mathcal{D}f,\mathcal{D}h;x)]\\ &\quad+(\mathcal{D}^2f)\,h=:Y(f,h;x)+(\mathcal{D}^2f)\,h\,. \end{aligned} \tag{17}$$

Therefore, when varying the action (Equation 4) with respect to ϕ one gets

$$\begin{aligned}\frac{\delta S}{\delta\phi} &= \int dq\left(\frac12\delta\phi\mathcal{D}^2\phi+\frac12\phi\mathcal{D}^2\delta\phi-\delta\phi V_{,\phi}\right)\\ &\overset{(17)}{=}\int dq\left[\delta\phi(\mathcal{D}^2\phi-V_{,\phi})+\frac12 Y(\phi,\delta\phi;x)\right]. \end{aligned}\tag{18}$$

Assuming that one could repeatedly integrate Y by parts to write it as $Y=2\delta\phi\,Z(\phi,x)$ up to some boundary term, we would end up with a dynamical equation

$$\mathcal{D}^2\phi-V_{,\phi}+Z(\phi,x)=0 \tag{19}$$

characterized by a term Z that can considerably hinder the study of solutions.

This is the main obstacle that prevented so far to consider multiscale theories with derivatives different from Equation (5) (barring the mathematically trivializable case of weighted derivatives). In fact, the only derivative with anomalous scaling such that $X=0$ in the Leibniz rule (Equation 16) is the q-derivative [23]. Genuine fractional derivatives always have $X\neq 0$.

3.2. Self-Adjoint Laplacian

Although $X\neq 0$, one could still obtain a clean integration by parts if, thanks to miraculous cancellations, Y were a total derivative or Z were zero on shell. This possibility is suggested by Equation (15), which implies that, for any combination

$$\tilde{\mathcal{D}}^\alpha:=c\,{}_\infty\partial^\alpha+\bar c\,{}_\infty\bar\partial^\alpha\,, \tag{20}$$

one has

$$\int dx\,f\,(c\,{}_\infty\partial^\alpha+\bar c\,{}_\infty\bar\partial^\alpha)g=\int dx\,g(\bar c\,{}_\infty\partial^\alpha+c\,{}_\infty\bar\partial^\alpha)f\,.$$

For instance, if $c=-\bar c=1/2$[2],

$$\int dx f\,\tilde{\mathcal{D}}^\alpha g=-\int dx\,(\tilde{\mathcal{D}}^\alpha f)\,g\,,\qquad \tilde{\mathcal{D}}^\alpha=\frac12({}_\infty\partial^\alpha-{}_\infty\bar\partial^\alpha)\,. \tag{21}$$

In the limit $\alpha\to 1$, ${}_\infty\partial^1=\partial$ and ${}_\infty\bar\partial^1=-\partial$, so that $\lim_{\alpha\to1}\tilde{\mathcal{D}}^\alpha=\partial$. Therefore, we can define an operator self-adjoint with respect to any measure weight $v(x)$:

$$\mathcal{K}_\alpha=\mathcal{D}^\alpha\mathcal{D}^\alpha\,,\qquad \mathcal{D}^\alpha:=\frac{1}{\sqrt v}\tilde{\mathcal{D}}^\alpha\left(\sqrt v\,\cdot\right)\,, \tag{22}$$

so that

$$\begin{aligned}\int dx\,v\,f\mathcal{K}_\alpha g &= \int dx\,(\sqrt v f)\tilde{\mathcal{D}}^\alpha\tilde{\mathcal{D}}^\alpha(\sqrt v g)\\ &=-\int dx\,[\tilde{\mathcal{D}}^\alpha(\sqrt v f)]\tilde{\mathcal{D}}^\alpha(\sqrt v g)\\ &=\int dx\,[\tilde{\mathcal{D}}^\alpha\tilde{\mathcal{D}}^\alpha(\sqrt v f)](\sqrt v g)\\ &=\int dx\,v\,(\mathcal{K}_\alpha f)\,g\,. \end{aligned}\tag{23}$$

Note that other, complex-valued choices of $c=(\bar c)^*$ may be more convenient when studying the spectrum of eigenvalues of these operators [24].

4. MULTIFRACTIONAL DERIVATIVES: THREE PROPOSALS

At this point, we can try to extend fractional calculus to a multiscale setting. We have found three ways to do that.

[2]A sign error in a similar expression in Calcagni [10] is corrected here.

4.1. Explicit Multiscaling

The most direct mean to induce a hierarchy of scales and a variable anomalous scaling is to consider a superposition of fractional derivatives of different order α [24]. In the mathematical literature, several authors [25–32] did propose a continuous superposition, the distributed-order fractional derivatives $\mathrm{D} := \int_0^1 d\alpha\, m(\alpha)\, \partial^\alpha$, where $m(\alpha)$ is a distribution on the interval $[0, 1]$. However, from previous experience in quantum gravity it may be more convenient, or just sufficient, to take a sum instead of an integral:

$$\mathcal{K} = \mathcal{D}^2\,, \qquad \mathcal{D} := \sum_n g_n \mathcal{D}^{\alpha_n}\,, \tag{24}$$

where $g_n = g_n(\ell_n)$ are some constant coefficients and $\mathcal{D}^{\alpha_n}$ is defined in Equations (22) and (21). A non-trivial dimensional flow is generated by just one scale, i.e., a sum of two terms: $\mathcal{D} = \partial + g_* \mathcal{D}^\alpha$. The equation of motion from the action (Equation 4) with kinetic operator (Equation 24) is

$$\mathcal{D}^2\phi - V_{,\phi} = 0\,. \tag{25}$$

This formulation of a multiscale theory with fractional derivatives is not exempt from problems. The operator $\mathcal{D}^2$ consists of many terms, even in the simplest case of only one scale where $\mathcal{D}^2$ is made of seven pieces (ignoring weight factors), $\partial^2 + 2g_*\tilde{\mathcal{D}}^{\alpha+1} + g_*^2\tilde{\mathcal{D}}^\alpha\tilde{\mathcal{D}}^\alpha = \partial^2 + g_*({}_\infty\partial^{\alpha+1} - {}_\infty\bar{\partial}^\alpha) + (g_*^2/4)({}_\infty\partial^{2\alpha} - {}_\infty\partial^\alpha{}_\infty\bar{\partial}^\alpha - {}_\infty\bar{\partial}^\alpha{}_\infty\partial^\alpha + {}_\infty\bar{\partial}^{2\alpha})$. Therefore, the dynamics (Equation 25) is deceptively clean and hides a rather messy multiorder fractional differential structure which may be very difficult to solve analytically. This eminently practical issue could be very important, or even fatal, at the time of studying the dynamics. To bypass it, one could consider another version of the kinetic operator [24]:

$$\mathcal{K} = \sum g_n \bar{\mathcal{D}}^{2\alpha_n}\,, \qquad \bar{\mathcal{D}}^{2\alpha_n} := \frac{1}{2}\frac{1}{\sqrt{v}}({}_\infty\partial^{2\alpha_n} + {}_\infty\bar{\partial}^{2\alpha_n})\left(\sqrt{v}\,\cdot\right)\,, \tag{26}$$

where $m = 2$ in Equations (12) and (13):

$$\begin{aligned}({}_\infty\partial^{2\alpha} + {}_\infty\bar{\partial}^{2\alpha})f(x) &= \\ &\frac{1}{\Gamma(2-2\alpha)}\int_{-\infty}^{+\infty} dx'\left[\frac{\theta(x-x')}{(x-x')^{2\alpha-1}} + \frac{\theta(x'-x)}{(x'-x)^{2\alpha-1}}\right]\partial_{x'}^2 f(x') \\ &= \frac{1}{\Gamma(2-2\alpha)}\int_{-\infty}^{+\infty}\frac{dx'}{|x-x'|^{2\alpha-1}}\,\partial_{x'}^2 f(x')\,.\end{aligned} \tag{27}$$

At the classical level, the great advantage of Equation (26) is that, in the single-scale case, it consists of just three terms $\partial^2 + (g_*/2)({}_\infty\partial^{2\alpha} + {}_\infty\bar{\partial}^{2\alpha})$ (again, weight factors are ignored) instead of seven. However, this $\mathcal{K}$ is not quadratic, since $\bar{\mathcal{D}}^{2\alpha} \neq \bar{\mathcal{D}}^\alpha\bar{\mathcal{D}}^\alpha$, which can lead to problems when quantizing the theory in Hamiltonian formalism: the kinetic term is not the square of a momentum operator.

At present, it is not clear which definition between Equations (24) and (26) will be more viable in the long run. They differ only in transient terms that can be dropped both at large and small scales, so that classically they give rise to the same physics. However, both have the added inconvenience of leading to a virtually symmetryless dynamics [10], a further point of concern if we want to do field theory and gravity with this formalism.

4.2. Implicit Multiscaling

The multiscaling characterizing Equation (24) is of a twofold nature, an explicit one in the sum over α_n and an implicit one in the measure weight $v(x)$. These two structures have been combined independently and we imposed that the sum over α_n in the combination of fractional derivatives is the same sum over α_n inside $v(x)$. There is nothing wrong with this construction, but there may be a more elegant formulation where the scale hierarchy is all included within the measure $q(x)$ [10]. Noting that the denominator $(x - x')^\alpha$ in Equations (12) and (13) for $m = 1$ $(0 < \alpha < 1)$ is the ultraviolet part of the profile in Equation (4), we can generalize those definitions as a left and right multifractional q-derivative:

$${}_q\mathcal{D} := \int_{-\infty}^{+\infty} dx'\,\frac{\theta(x-x')}{q(x-x')}\frac{\partial}{\partial x'}\,, \tag{28}$$

$${}_q\bar{\mathcal{D}} := \int_{-\infty}^{+\infty} dx'\,\frac{\theta(x'-x)}{q(x-x')}\frac{\partial}{\partial x'}\,, \tag{29}$$

where, again, the profile $q(x)$ is uniquely given by Equation (2). These expressions are similar to the so-called variable-order fractional derivatives proposed by Lorenzo and Hartley [30], although in our case $q(x - x')$ is completely fixed.

The kinetic operator in Equation (4) is then

$$\mathcal{K} = \frac{1}{\sqrt{v}}\left[\frac{1}{2}({}_q\mathcal{D} - {}_q\bar{\mathcal{D}})\right]^2\left(\sqrt{v}\,\cdot\right)\,. \tag{30}$$

To understand the dynamics, we first need to spell out the properties of these derivatives. At short scales, $q(x-x') \sim |x-x'|^\alpha$ and Equations (28) and (29) reduce to the Liouville and Weyl derivatives, respectively:

$$\text{small scales } (\ell \ll \ell_*): \qquad {}_q\mathcal{D} \sim {}_\infty\partial^\alpha\,, \qquad {}_q\bar{\mathcal{D}} \sim {}_\infty\bar{\partial}^\alpha\,, \tag{31}$$

while at large scales $q(x - x') \sim x - x'$ and Equations (28) and (29) give

$$\text{large scales } (\ell \gg \ell_*): \qquad {}_q\mathcal{D} \simeq \partial\,, \qquad {}_q\bar{\mathcal{D}} \simeq -\partial\,. \tag{32}$$

We have not made a formal proof of these statements, but it should not be difficult. The Leibniz and integration-by-parts rules are also unknown but they should coincide with those of Weyl and Liouville fractional derivatives in the limit of small scales or in any plateau region of dimensional flow $(q \sim x^{\alpha_n})$.

Therefore, we expect a complicated Leibniz and integration-by-parts rules everywhere at all scales of dimensional flow, except in plateau regions where a clean integration by parts of the type (Equation 15) emerges. For this reason, a variational principle valid at all scales may be ill defined in this case and an exact form of the equations of motion may be out of reach, although their asymptotic form at plateaux is obviously given by the limit of Equation (25) for one exponent α.

These considerations could eventually select the multifractional derivatives with explicit multiscaling as a simpler tool in quantum gravity, since they yield exact equations of motion.

4.3. Multiscale Differentials

A third alternative is to introduce a multiscale differential based on the geometric coordinate (Equation 2) or its simplified version in Equation (4) [10]:

$$\mathbb{d}q(x) = q(dx)\,, \tag{33}$$

which is a linear combination of the usual and fractional [12] differentials, $\mathbb{d}q \sim \mathbb{d}x + \mathbb{d}|x|^\alpha + \ldots = dx + |dx|^\alpha + \ldots$. [In D dimensions, this differential generates the line element $\mathbb{d}q(s) = \sqrt{g_{\mu\nu}\mathbb{d}q^\mu(x^\mu)\otimes\mathbb{d}q^\nu(x^\nu)} = q(ds) = \sqrt{g_{\mu\nu}q^\mu(dx^\mu)\otimes q^\nu(dx^\nu)}$, where $g_{\mu\nu}$ is the metric.] The operator $\mathbb{D}$ is a superposition of ordinary and fractional derivatives of the form (to be taken as indicative; coefficients are ignored)

$$\mathbb{d} = \mathbb{d}q\mathbb{D} \sim dx\,\partial + |dx|^\alpha\partial^\alpha + \ldots\,. \tag{34}$$

The following multiscale derivative and Laplacian are then defined implicitly:

$$\mathbb{D} := \frac{\mathbb{d}}{\mathbb{d}q}\,, \qquad \mathcal{K} = \mathbb{D}^2\,. \tag{35}$$

These operators are invariant under translations, since $\mathbb{D}_{x-\bar{x}} = \mathbb{D}_x$, while $\mathcal{K}$ is invariant also by "q-boosts" [10]. Therefore, this theory has more symmetries than the theories with multifractional derivatives with explicit or implicit multiscaling.

In any plateau of dimensional flow, $\mathbb{d}q \simeq (dx)^{\alpha_n}$ and $\mathbb{D} \simeq \frac{\mathbb{d}}{(dx)^{\alpha_n}} = \tilde{\mathcal{D}}^{\alpha_n}$. Notice that $\mathbb{D} \simeq \partial_q$ in the near-infrared limit $\mathbb{d} \to d$ where the non-linear part of q is subdominant. Therefore, the theory with q-derivatives can be regarded as an approximation of the theory with multiscale derivatives (and, presumably, also of the other two theories with fractional derivatives) when the anomalous scaling effects are weak. The experimental constraints on the scales and parameters of the theory with q-derivatives might thus miss some important effects present in the fractional versions of the multiscale paradigm.

To determine the Leibniz and integration-by-parts rules, one should first define the operator $\mathbb{D}$ appearing in the differential $\mathbb{d} = \mathbb{d}q\mathbb{D}$. Again, we expect these rules to reduce to the usual ones in the infrared and to those of fractional derivatives in the ultraviolet. Since the operator (Equation 24) with explicit multiscaling is already a well-defined linear combination of fractional derivatives, we reach the same conclusion of the previous section, namely, that the operator (Equation 24) may be the best candidate for the concrete realization of multifractional theories with fractional derivatives. However, the main problem of the definitions (33) and (35) is that they are too abstract, which is the reason why we used qualitative expressions marked by "$\sim$." Understanding their actual properties will require more work.

5. CONCLUSIONS

In this paper, we have further analyzed the proposals of Calcagni [10] for a multifractional calculus with viable applications to field theory and gravity. Without the pretense of being rigorous, we have considered some properties of scale-dependent derivative operators which, in physical applications to quantum gravity, are interpreted to encode the multiscaling of the underlying anomalous geometry. The conclusion is that the most promising multifractional theory possibly is the one with explicit multiscaling in fractional derivatives. However, only a full systematic study of the properties of all these operators will be able to confirm which is the most viable Laplacian from a theoretical and practical point of view. The value of the complex coefficients in Equation (20) will be especially important to determine a well-defined calculus and spectral theory [24]. We will analyze the associated dynamics in detail in a future publication.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and approved it for publication.

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A Remark on the Fractional Integral Operators and the Image Formulas of Generalized Lommel–Wright Function

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In this paper, the operators of fractional integration introduced by Marichev-Saigo-Maeda involving Appell's function $F_3(\cdot)$ are applied, and several new image formulas of generalized Lommel–Wright function are established. Also, by implementing some integral transforms on the resulting formulas, few more image formulas have been presented. We can conclude that all derived results in our work generalize numerous well-known results and are capable of yielding a number of applications in the theory of special functions.

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1. INTRODUCTION AND PRELIMINARIES

Fractional calculus (FC) represents a complex physical phenomenon in a more accurate and efficient way than classical calculus. In recent years, many researchers [1–7] have used fractional order integral models in real-world problems in various fields of science and technology. There exists several definitions of fractional order integrals in the literature that can be used to solve the fractional integral equations involving special functions. For an exhaustive literature review, one may refer to the paper by Srivastava and Saxena [8].

The generalized functions such as Bessel, Lommel, Struve, and Lommel–Wright functions have originated from concrete problems in applied fields of sciences viz mechanics, physics, engineering, astronomy, etc.

The *generalized Lommel–Wright function* $J^{\varphi,m}_{\omega,\vartheta}(z)$ is defined by de'Oteiza et al. [9] and is represented in the following manner:

$$\begin{aligned} J^{\varphi,m}_{\omega,\vartheta}(z) &= (z/2)^{\omega+2\vartheta} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(\Gamma(\vartheta+k+1))^m \Gamma(\omega+k\varphi+\vartheta+1)} \\ &= (z/2)^{\omega+2\vartheta} {}_1\psi_{m+1}[(1,1);\ \underbrace{(\vartheta+1,1),}_{m-times}\ (\omega+\vartheta+1,\varphi);\ -z^2/4] \end{aligned} \tag{1.1}$$

$$z \in \mathbb{C}\backslash(-\infty,0],\quad \varphi>0,\quad m\in\mathbb{N},\quad \omega,\vartheta\in\mathbb{C},$$

where ${}_p\psi_q$ denotes the Fox–Wright generalized hypergeometric function which is defined as given in Srivastava and Karlsson [10, p. 21] and Kilbas et al. [11, P. 56]

$$ {}_p\psi_q\left(\begin{matrix}(a_1,A_1),...(a_p,A_p);\\(b_1,B_1),...,(b_q,B_q);\end{matrix}\ z\right)=\sum_{n=0}^{\infty}\frac{\Pi_{j=1}^{p}\Gamma(a_j+nA_j)}{\Pi_{j=1}^{q}\Gamma(b_j+nB_j)}\frac{z^n}{n!}, \tag{1.2}$$

where $a_i,b_j\in\mathbb{C}$ and $A_i,B_j\in\mathbb{R}=(-\infty,\infty)$; $A_i,B_j\neq 0$, $i=1,2,...,p,j=1,2,...,q,\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j>-1$.

A useful generalization of the Lommel–Wright function and its special cases, $J_{\omega}^{\varphi}(z)J_{\omega,\vartheta}^{\varphi}(z)$, depending on the arbitrary fractional parameter $\varphi>0$ presents a fractional order extension of the Bessel function $J_{\omega}(z)$.

Prieto et al. [12] studied some useful results in the theory of fractional calculus operators of generalized Lommel–Wright function. The convergence of series involving generalized Lommel–Wright function was studied by Konovska [13].

When $m=1$, the following generalization of the Bessel function, introduced by Pathak [14] is obtained as a special case of generalized Lommel–Wright function (1.1) (see e.g., [15, p. 353]):

$$\begin{aligned}J_{\omega,\vartheta}^{\varphi}(z)&=J_{\omega,\vartheta}^{\varphi,1}(z)\\&=\left(\frac{z}{2}\right)^{\omega+2\vartheta}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k}}{\Gamma(\vartheta+k+1)\Gamma(\omega+k\varphi+\vartheta+1)},\\&\quad z\in\mathbb{C}\backslash(-\infty,0],\quad\varphi>0,\quad\omega,\vartheta\in\mathbb{C}.\end{aligned}\tag{1.3}$$

On taking $m=1,\varphi=1$, and $\vartheta=\frac{1}{2}$ in (1.1), we obtain the *Struve function* $H_{\omega}(\cdot)$ (see e.g., [16, p. 28, Equation (1.170)])

$$\begin{aligned}H_{\omega}(z)&=J_{\omega,1/2}^{1,1}\\&=\left(\frac{z}{2}\right)^{\omega+1}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\omega+\frac{3}{2})}z,\omega\in\mathbb{C}.\end{aligned}\tag{1.4}$$

If we take $m=1,\varphi=1$, and $\vartheta=0$ in (1.1), it gives the relationship with the *Bessel function* as follows (see e.g., [16, p. 27, Equation (1.161)]):

$$J_{\omega}(z)=J_{\omega,0}^{1,1}(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\omega+2k}}{\Gamma(\omega+k+1)k!}\qquad z,\omega\in\mathbb{C},\quad z\neq0,\quad\Re(\omega)>-1.\tag{1.5}$$

A generalization of the hypergeometric fractional integrals, including the Saigo operators [17, 18] has been introduced by Marichev [19]. The details of these fractional operators have been found in Samko et al. [5, p. 194, Equation (10.47)] and later extended and studied by Saigo and Maeda [20, p. 393, Equation (4.12) and Equation (4.13)] in terms of complex order Appell function $F_3(\cdot)$ of two variables (see [10, p. 23]) in the kernel

$$\begin{aligned}&F_3(\zeta,\zeta',\varrho,\varrho';\eta;x;y)\\&=\sum_{m,n=0}^{\infty}\frac{(\zeta)_m(\zeta')_n(\varrho)_m(\varrho')_n}{(\eta)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\qquad(\max\{|x|,|y|\}<1).\end{aligned}\tag{1.6}$$

The Appell function F_3 reduces to the Gauss hypergeomatric function ${}_2F_1$ and satisfies the system of two linear partial differential equations of the second order as follows (see [10, p. 301, Equation 9.4]):

$$F_3(\zeta,\eta-\zeta,\varrho,\eta-\varrho;\eta;x;y)={}_2F_1(\zeta,\varrho;\eta;x+y-xy).\tag{1.7}$$

Further, it is easy to see that

$$F_3(\zeta,0,\varrho,\varrho',\eta;x,y)={}_2F_1(\zeta,\varrho;\eta;x)\tag{1.8}$$

and

$$F_3(0,\zeta',\varrho,\varrho',\eta;x,y)={}_2F_1(\zeta',\varrho';\eta;y).\tag{1.9}$$

In this paper, we develop and study the image formulas involving the generalized Lommel–Wright function using fractional calculus integral operators. We use the generalized Marichev-Saigo-Maeda fractional integral operators, involving the Appell function, defined as follows:

$$\begin{aligned}\left(I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}f\right)(x)&=\frac{x^{-\zeta}}{\Gamma\kappa}\int_0^x(x-t)^{\kappa-1}t^{-\zeta'}\\&\quad\times F_3\left(\zeta,\zeta',\varrho,\varrho';\kappa;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)dt,\\&\quad\Re(\kappa)>0,\zeta,\zeta',\varrho,\varrho',\kappa\in\mathbb{C},x>0\end{aligned}\tag{1.10}$$

and

$$\begin{aligned}&\left(I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa}f\right)(x)\\&=\frac{x^{-\zeta}}{\Gamma\kappa}\int_x^{\infty}(t-x)^{\kappa-1}t^{-\zeta}F_3\left(\zeta,\zeta',\varrho,\varrho';\kappa;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)dt,\\&\quad\Re(\kappa)>0,\zeta,\zeta',\varrho,\varrho',\kappa\in\mathbb{C},x>0.\end{aligned}\tag{1.11}$$

respectively.

The power functions of left-hand sided and right-hand sided Marichev-Saigo-Maeda fractional integral operators as given in the Equations (1.10) and (1.11) (see Saigo et al. [6, 20]) are given by

$$\begin{aligned}&\left(I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}t^{\chi-1}\right)(x)\\&=\frac{\Gamma(\chi)\Gamma(\chi+\kappa-\zeta-\zeta'-\varrho)\Gamma(\chi+\varrho'-\zeta')}{\Gamma(\chi+\varrho')\Gamma(\chi+\kappa-\zeta-\zeta')\Gamma(\chi+\kappa-\zeta'-\varrho)}x^{\chi+\kappa-\zeta-\zeta'-1},\end{aligned}\tag{1.12}$$

where $\zeta,\zeta',\varrho,\varrho',\kappa\in\mathbb{C}$, $x>0$ and if $\Re(\kappa)>0,\Re(\chi)>\max\{0,\Re(\zeta+\zeta'+\varrho-\kappa),\Re(\zeta'-\varrho')\}$.

$$\begin{aligned}&\left(I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa}t^{\chi-1}\right)(x)\\&=\frac{\Gamma(1-\chi-\kappa+\zeta+\zeta')\Gamma(1-\chi+\zeta+\varrho'-\kappa)\Gamma(1-\chi-\varrho)}{\Gamma(1-\chi)\Gamma(1-\chi+\zeta+\zeta'+\varrho+\varrho'-\kappa)\Gamma(1-\chi+\zeta-\varrho)}\\&\quad\times x^{\chi-\zeta-\zeta'+\kappa-1},\end{aligned}\tag{1.13}$$

where $\zeta,\zeta',\varrho,\varrho',\kappa\in\mathbb{C}$ are such that $\Re(\kappa)>0$ and $\Re(\chi)<1+\min\{\Re(-\varrho),\Re(\zeta+\zeta'-\kappa),\Re(\zeta+\varrho'-\kappa)\}$.

1.1. Relation Among the Operators

In this section, we recall some relationships between the fractional integral operators.

If we set $\zeta' = 0$ then in view of the formula (1.8), the relationship between Marichev-Saigo-Maeda and the Saigo fractional integral operators is found by Saxena and Saigo [6, p. 93, Equation (2.15)] as

$$\left(I_{0,x}^{\zeta,0,\varrho,\varrho',\eta}f\right)(x) = \left(I_{0,x}^{\eta,\zeta-\eta,-\varrho}f\right)(x), \quad (\Re(\eta) > 0) \tag{1.14}$$

and

$$\left(I_{x,\infty}^{\zeta,0,\varrho,\varrho',\eta}f\right)(x) = \left(I_{x,\infty}^{\eta,\zeta-\eta,-\varrho}f\right)(x), \quad (\Re(\eta) > 0), \tag{1.15}$$

where the general operators $I_{0,x}^{\zeta,0,\varrho,\varrho',\eta}$ and $I_{0,x}^{\zeta,0,\varrho,\varrho',\eta}$ reduce, respectively, to the Saigo operators $I_{0,x}^{\zeta,\varrho,\eta}$ and $I_{x,\infty}^{\zeta,\varrho,\eta}$ [17] defined as follows:

$$\begin{aligned}\left(I_{0,x}^{\zeta,\varrho,\eta}f\right)(x) &= \frac{x^{-\zeta-\varrho}}{\Gamma(\zeta)}\int_0^x (x-t)_2^{\zeta-1} \\ &\quad \times F_1\left(\zeta+\varrho,-\eta;\zeta;1-\frac{t}{x}\right)f(t)dt, \quad (\Re(\zeta) > 0)\end{aligned} \tag{1.16}$$

and

$$\begin{aligned}\left(I_{x,\infty}^{\zeta,\varrho,\eta}f\right)(x) &= \int_x^\infty (t-x)^{\zeta-1}t^{-\zeta-\varrho}{}_2 \\ &\quad \times F_1\left(\zeta+\varrho,-\eta;\zeta;1-\frac{x}{t}\right)f(t)dt, \quad (\Re(\zeta) > 0)\end{aligned} \tag{1.17}$$

where integrals in (1.16) and (1.17) exist.

Let $\zeta, \varrho, \eta, \chi \in \mathbb{C}$ with $\Re(\zeta) > 0$. Then the following power function formulas involving the Saigo operators hold true:

$$\left(I_{0,x}^{\zeta,\varrho,\eta}t^{\chi-1}\right)(x) = \frac{\Gamma(\chi)\Gamma(\chi+\eta-\varrho)}{\Gamma(\chi-\varrho)\Gamma(\chi+\eta+\zeta)}x^{\chi-\varrho-1}, \tag{1.18}$$

$$\Re(\chi) > \max\left\{0, \Re(\varrho-\eta)\right\}$$

and

$$\left(I_{x,\infty}^{\zeta,\varrho,\eta}t^{\chi-1}\right)(x) = \frac{\Gamma(1-\chi+\varrho)\Gamma(1-\chi+\eta)}{\Gamma(1-\chi)\Gamma(1-\chi+\zeta+\varrho+\eta)}x^{\chi-\varrho-1}, \tag{1.19}$$

$$(\Re(\chi) < 1 + \min\left\{\Re(\varrho), \Re(\eta)\right\}).$$

On replacing $\varrho = -\zeta$ in the operators $I_{0,x}^{\zeta,\varrho,\eta}(\cdot)$ and $I_{x,\infty}^{\zeta,\varrho,\eta}(\cdot)$, these reduce to the Riemann-Liouville and the Weyl fractional integral operators, respectively, by means of the following relationships (see Kilbas [11]):

$$\left(\mathcal{R}_{0,x}^{\zeta}f\right)(x) = \left(I_{0,x}^{\zeta,-\zeta,\eta}f\right)(x) \tag{1.20}$$

and

$$\left(\mathcal{W}_{x,\infty}^{\zeta}f\right)(x) = \left(I_{x,\infty}^{\zeta,-\zeta,\eta}f\right)(x). \tag{1.21}$$

The Riemann-Liouville fractional integral operator and the Weyl fractional integral operator are defined as follows (see e.g., [21]):

$$\left(\mathcal{R}_{0,x}^{\zeta}f\right)(x) = \frac{1}{\Gamma(\zeta)}\int_0^x (x-t)^{\zeta-1}f(t)dt, \quad (\Re(\zeta) > 0) \tag{1.22}$$

and

$$\left(\mathcal{W}_{x,\infty}^{\zeta}f\right)(x) = \frac{1}{\Gamma(\zeta)}\int_0^x (t-x)^{\zeta-1}f(t)dt, \quad (\Re(\zeta) > 0), \tag{1.23}$$

provided both the integrals converge.

The operators $I_{0,x}^{\zeta,\varrho,\eta}(\cdot)$ and $I_{x,\infty}^{\zeta,\varrho,\eta}(\cdot)$ reduce to Erdélyi–Kober fractional integral operators on setting $\varrho = 0$ as follows:

$$\left(\mathcal{E}_{0,x}^{\zeta,\eta}f\right)(x) = \left(I_{0,x}^{\zeta,0,\eta}f\right)(x), \tag{1.24}$$

and

$$\left(\mathcal{K}_{x,\infty}^{\zeta,\eta}f\right)(x) = \left(I_{0,x}^{\zeta,0,\eta}f\right)(x), \tag{1.25}$$

where the Erdélyi–Kober type fractional integral operators are defined as follows (see [22]):

$$\left(\mathcal{E}_{0,x}^{\zeta,\eta}f\right)(x) = \frac{x^{-\zeta-\eta}}{\Gamma(\zeta)}\int_0^x (x-t)^{\zeta-1}t^{\eta}f(t)dt, \quad (\Re(\zeta) > 0) \tag{1.26}$$

and

$$\left(\mathcal{K}_{x,\infty}^{\zeta,\eta}f\right)(x) = \frac{x^{\eta}}{\Gamma(\zeta)}\int_x^\infty (t-x)^{\zeta-1}t^{-\zeta-\eta}f(t)dt, \quad (\Re(\zeta) > 0), \tag{1.27}$$

The function $f(t)$ is constrained so that both the defining integrals (1.26) and (1.27) converge.

The *Beta transform* (see, e.g.[23]) of a complex valued function $f(t)$ of a real variable t is defined as follows:

$$B\{f(t):a,b\} = \int_0^1 t^{a-1}(1-t)^{b-1}f(t)\,dt, \tag{1.28}$$

$$\Re(t) > 0,\ \Re(a), \Re(b) > 0.$$

Beta transform of the power function $t^{\chi-1}$ is given by:

$$\begin{aligned}B\left\{t^{\chi-1};a,b\right\} &= \int_0^1 t^{a+\chi-2}(1-t)^{b-1}dt \\ &= \frac{\Gamma(a+\chi-1)\Gamma(b)}{\Gamma(a+\chi+b-1)},\ \Re(t) > 0,\ \Re(a), \Re(b) > 0.\end{aligned} \tag{1.29}$$

The P_δ - *transform* of a complex valued function $f(t)$ of a real variable t denoted by $P_\delta[f(t);s]$ is a function $F(s)$ of a complex variable s, valid under certain conditions on $f(t)$, (given below is defined as (see Kumar [24])

$$P_\delta[f(t);s]=F(s)=\int_0^\infty [1+(\delta-1)s]^{-\frac{t}{\delta-1}}f(t)dt, \quad \delta>1. \tag{1.30}$$

Here $f(t)$ as a function is integrable over any finite interval (a,b), $0<a<t<b$; there exists a real number c such that

(i) if $b>0$ is arbitrary, then $\int_b^\Upsilon e^{-ct}f(t)dz$ tends to a finite limit as $\Upsilon\to\infty$
(ii) for arbitrary $a>0$, $\int_\omega^a |f(t)dt|$ tends to a finite limit as $\omega\to 0+$, then the P_δ-transform $P_\delta[f(t);s]$ exists for $\Re\left(\frac{\ln[1+(\delta-1)s]}{\delta-1}\right)>c$ for $s\in\mathbb{C}$.

P_δ - transform of the power function $t^{\chi-1}$ is given by

$$P_\delta[z^{\chi-1};s]=\left\{\frac{\delta-1}{\ln[1+(\delta-1)s]}\right\}^\chi\Gamma(\chi), \qquad \chi\in\mathbb{C},\ \Re(\chi)>0,\ \delta>1. \tag{1.31}$$

P_δ-transform has found many applications. The pathway transforms are the paths going from the binomial form $\ln[1+(\delta-1)s]^{-\frac{t}{\delta-1}}$ to the exponential from e^{-st}. In P_δ-transform, the variable t is shifted from the binomial factor $\ln[1+(\delta-1)s]^{-\frac{t}{\delta-1}}$ to the exponent, Hence, this form is more suitable for obtaining translation, convolution, etc. Recently, Agarwal et al. [25] found the solution of non-homogeneous time fractional heat equation and fractional Volterra integral equation using integral transform of pathway type. Also, Srivastava et al. [26] and [27] found some results involving generalized hypergeometric function and generalized incomplete gamma function by using P_δ-transform.

If we take $\delta\to 1$ in Equation (1.30), the P_δ-transform reduces to *Laplace integral transform* (Sneddon [23]):

$$L\left[f(t);s\right]=\int_0^\infty e^{-ts}f(t)dz;, \qquad \Re(s)>0. \tag{1.32}$$

The following relationship between the P_δ-transform is defined by (1.30) and the classical Laplace transform is defined by (1.32)

$$P_\delta[f(t):s]=L\left[f(t):\frac{\ln[1+(\delta-1)s]}{\delta-1}\right], \qquad (\delta>1) \tag{1.33}$$

or, equivalently,

$$L[f(t):s]=P_\delta\left[f(t):\frac{e^{(\delta-1)s}-1}{\delta-1}\right], \qquad (\delta>1), \tag{1.34}$$

which can be applied to convert the table of Laplace transforms into the corresponding table of P_δ-transforms and *vice versa*.

The following integral formula involving the Whittaker function (see Mathai et al. [16, p. 56]) is used in finding the image formula:

$$\int_0^\infty t^{\tau-1}e^{-\frac{t}{2}}W_{\sigma,\eta}(t)\,dt=\frac{\Gamma(\tau+\eta+\frac12)\Gamma(\tau-\eta+\frac12)}{\Gamma(\tau-\sigma+\frac12)}, \tag{1.35}$$

$$(\sigma\in\mathbb{C},\ \Re(\tau\pm\eta)>-1/2).$$

The Whittaker function (see e.g., Mathai et al. [16, p. 22]) is defined by

$$\begin{aligned}W_{\sigma,\eta}(z)&=\frac{\Gamma(-2\eta)}{\Gamma(\frac12-\sigma-\eta)}M_{\sigma,\eta}(z)+\frac{\Gamma(2\eta)}{\Gamma(\frac12-\sigma+\eta)}M_{\sigma,-\eta}(z)\\&=W_{\sigma,-\eta}(z),\end{aligned} \tag{1.36}$$

$$\sigma\in\mathbb{C},\ \Re(1/2+\eta\pm\sigma)>0$$

where

$$M_{\sigma,\eta}(z)=z^{\eta+\frac12}e^{-\frac{z}{2}}{}_1F_1\left(\frac12-\sigma+\eta;2\eta+1;z\right), \qquad \Re(1/2+\eta\pm\sigma)>0,\quad |argz|<\pi. \tag{1.37}$$

2. IMAGE FORMULA ASSOCIATED WITH FRACTIONAL INTEGRAL OPERATORS

Here, we establish image formulas for the generalized Lommel–Wright function involving Saigo-Maeda fractional integral operators (1.10) and (1.11), in terms of the Fox–Wright function.

Theorem 2.1. *Let* $\zeta,\zeta',\varrho,\varrho',\kappa,\vartheta\in\mathbb{C}, m\in\mathbb{N},\ \varphi>0$ *and* $x>0$ *be such that*

$$\begin{aligned}&\Re(\kappa)>0,\quad \Re(\omega)>-1,\\&\Re(\chi+\omega)>\max\{0,\Re(\zeta+\zeta'+\varrho-\kappa),\Re(\zeta'-\varrho')\}\end{aligned} \tag{2.1}$$

then there holds the formula

$$\begin{aligned}&I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left[t^{\chi-1}J_{\omega,\vartheta}^{\varphi,m}(tz)\right](x)=x^{A-\zeta-\zeta'+\kappa-1}\left(\frac{z}{2}\right)^{\omega+2\vartheta}\\&{}_4\psi_{4+m}\left[\begin{matrix}(A,2),(A+\kappa-\zeta-\zeta'-\varrho,2),(A+\varrho'-\zeta',2),(1,1)\\(A+\varrho',2),(A+\kappa-\zeta-\zeta',2),(A+\kappa-\zeta'-\varrho,2),(\omega+\vartheta+1,\varphi),\underbrace{(\vartheta+1,1)}_{m-times}\end{matrix}\middle|-\frac{(zx)^2}{4}\right]\end{aligned} \tag{2.2}$$

where $A=\chi+\omega+2\vartheta$.

Proof: Under the conditions stated with the Theorem 2.1, by taking the fractional integral of (1.1) using the equation (1.10) therein and changing the order of integration and summation, we get

$$\begin{aligned}&I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left[t^{\chi-1}J_{\omega,\vartheta}^{\varphi,m}(tz)\right](x)\\&=\sum_{k=0}^\infty\frac{(-1)^k(\frac{z}{2})^{\omega+2\vartheta+2k}\Gamma(k+1)}{(\Gamma(\vartheta+k+1))^m\Gamma(\omega+k\varphi+\vartheta+1)k!}\\&\times I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left(t^{\omega+2\vartheta+2k+\chi-1}\right)(x)\end{aligned} \tag{2.3}$$

Further, applying the result (1.12) with χ replaced by $\chi+\omega+2\vartheta+2k$, we obtain

$$I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left[t^{\chi-1}J_{\omega,\vartheta}^{\varphi,m}(tz)\right](x)$$
$$=x^{A-\zeta-\zeta'+\kappa-1}\left(\frac{z}{2}\right)^{\omega+2\vartheta}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(A+2k)\Gamma(k+1)}{\Gamma(A+\varrho'+2k)(\Gamma(\vartheta+1+k))^m}$$
$$\times\frac{\Gamma(A+\kappa-\zeta-\zeta'-\varrho+2k)\Gamma(A+\varrho'-\zeta'+2k)}{\Gamma(A+\kappa-\zeta'-\varrho+2k)\Gamma(\omega+\vartheta+1+\varphi k)\Gamma(A+\kappa-\zeta-\zeta'+2k)}$$
$$\times\frac{(zx)^{2k}}{4^k k!} \tag{2.4}$$

Here $A=\chi+\omega+2\vartheta$.

Interpreting the right-hand side of the above equation, in view of the definition (1.2), we arrive at the result (2.2).

Theorem 2.2. *Let $\zeta,\zeta',\varrho,\varrho',\kappa,\vartheta\in\mathbb{C}, m\in\mathbb{N},\ \varphi>0$ and $x>0$ be such that*

$$\Re(\kappa)>0,\quad \Re(\omega)>-1,$$
$$\Re(\chi-\omega)>1+\min\{\Re(-\varrho),\Re(\zeta+\zeta'-\kappa),\Re(\zeta+\varrho'-\kappa)\} \tag{2.5}$$

then there holds the formula

$$I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left[t^{\chi-1}J_{\omega,\vartheta}^{\varphi,m}(z/t)\right](x)=x^{\kappa-\zeta-\zeta'-A}\left(\frac{z}{2}\right)^{\omega+2\vartheta}$$
$${}_4\psi_{4+m}\left[\begin{array}{c|c}\begin{array}{c}(A-\kappa+\zeta+\zeta',2),(A+\zeta+\varrho'-\kappa,2),\\(A-\varrho,2),(1,1)\\(A,2)(A+\zeta+\zeta'+\varrho'-\kappa,2),(A+\zeta-\varrho,2),\\(\omega+\vartheta+1,\varphi),\underbrace{(\vartheta+1,1)}_{m-times}\end{array} & -\dfrac{z^2}{4x^2}\end{array}\right] \tag{2.6}$$

where $A=1-\chi+\omega+2\vartheta$.

Proof: Under the conditions stated with the Theorem 2.2, on making use of the definitions (1.11) and (1.1) and changing the order of integration and summation, we have

$$I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left[t^{\chi-1}J_{\omega,\vartheta}^{\varphi,m}(z/t)\right](x)$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{z}{2})^{\omega+2\vartheta+2k}\Gamma(k+1)}{(\Gamma(\vartheta+k+1))^m\Gamma(\omega+k\varphi+\vartheta+1)k!}$$
$$\times I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left(t^{\chi-\omega-2\vartheta-2k-1}\right)(x) \tag{2.7}$$

Here, on applying the formula (1.13) with χ replaced by $\chi-\omega-2\vartheta-2k$, we obtain

$$I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left[t^{\chi-1}J_{\omega,\vartheta}^{\varphi,m}(z/t)\right](x)$$
$$=x^{\kappa-\zeta-\zeta'-A}\left(\frac{z}{2}\right)^{\omega+2\vartheta}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(A-\varrho+2k)}{\Gamma(A+2k)(\Gamma(\vartheta+k+1))^m}$$
$$\times\frac{\Gamma(k+1)\Gamma(A-\kappa+\zeta+\zeta'+2k)\Gamma(A+\zeta+\varrho'-\kappa+2k)}{\Gamma(A+\zeta-\varrho+2k)\Gamma(\omega+k\varphi+\vartheta+1)\Gamma(A+\zeta+\zeta'+\varrho'-\kappa+2k)}$$
$$\times\frac{(z)^{2k}}{(4x^2)^k k!} \tag{2.8}$$

where $A=1-\chi+\omega+2\vartheta$.

So in view of the definition of the generalized Lommel–Wright function given by (1.1), the Equation (2.8) leads to the result (2.6).

For $m=1$ and in the light of Equation (1.3), Theorem 2.1 leads to the following corollaries:

Corollary 2.1. *Under the conditions stated with the Equation (2.1), the following image formula*

$$I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left[t^{\chi-1}J_{\omega,\vartheta}^{\varphi,1}(zt)\right](x)=x^{A-\zeta-\zeta'+\kappa-1}\left(\frac{z}{2}\right)^{\omega+2\vartheta}$$
$$\times{}_4\psi_5\left[\begin{array}{c|c}\begin{array}{c}(A,2),(A+\kappa-\zeta-\zeta'-\varrho,2),\\(A+\varrho'-\zeta',2),(1,1)\\(A+\varrho',2),(A+\kappa-\zeta-\zeta',2),(A+\kappa-\zeta'-\varrho,2),\\(\omega+\vartheta+1,\varphi),(\vartheta+1,1)\end{array} & -\dfrac{(zx)^2}{4}\end{array}\right] \tag{2.9}$$

$A=\chi+\omega+2\vartheta$, for generalized Bessel function $J_{\omega,\vartheta}^{\varphi,1}(zt)$ holds true.

Corollary 2.2. *Under the conditions stated with the Equation (2.5), the image formula*

$$I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left[t^{\chi-1}J_{\omega,\vartheta}^{\varphi,1}(z/t)\right](x)$$
$$=x^{\kappa-\zeta-\zeta'-A}\left(\frac{z}{2}\right)^{\omega+2\vartheta}$$
$$\times{}_4\psi_5\left[\begin{array}{c|c}\begin{array}{c}(A-\kappa+\zeta+\zeta',2),(A+\zeta+\varrho'-\kappa,2),\\(A-\varrho,2),(1,1)\\(A,2)(A+\zeta+\zeta'+\varrho'-\kappa,2),(A+\zeta-\varrho,2),\\(\omega+\vartheta+1,\varphi),(\vartheta+1,1)\end{array} & -\dfrac{z^2}{4x^2}\end{array}\right] \tag{2.10}$$

$A=1-\chi+\omega+2\vartheta$, for generalized Bessel function $J_{\omega,\vartheta}^{\varphi,1}(z/t)$ holds true.

If we take $m=1,\ \varphi=1$, and $\vartheta=\frac{1}{2}$ in (2.2), then we obtain the corresponding results for the Struve function $H_\omega(\cdot)$ [16] as

Corollary 2.3. *Under the conditions stated with the Equation (2.1), the following image formula*

$$I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left[t^{\chi-1}H_\omega(zt)\right](x)$$
$$=x^{A-\zeta-\zeta'+\kappa-1}\left(\frac{z}{2}\right)^{\omega+1}$$
$$\times{}_4\psi_5\left[\begin{array}{c|c}\begin{array}{c}(A,2),(A+\kappa-\zeta-\zeta'-\varrho,2),\\(A+\varrho'-\zeta',2),(1,1)\\(A+\varrho',2),(A+\kappa-\zeta-\zeta',2),\\(A+\kappa-\zeta'-\varrho,2),(\omega+\frac{3}{2},1),(\frac{3}{2},1)\end{array} & -\dfrac{(zx)^2}{4}\end{array}\right] \tag{2.11}$$

$A=\chi+\omega+1$, for Struve function $H_\omega(zt)$ holds true.

Corollary 2.4. *Under the conditions stated with the Equation (2.5), the following image formula*

$$
\begin{aligned}
&I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left[t^{\chi-1}H_\omega(z/t)\right](x)\\
&= x^{\chi-\omega-\zeta-\zeta'+\kappa-2}\left(\frac{z}{2}\right)^{\omega+1}\\
&\times {}_4\psi_5\left[\begin{matrix}(A-\kappa+\zeta+\zeta',2),(A+\zeta+\varrho'-\kappa,2),\\(A-\varrho,2),(1,1)\\ \\(A,2)(A+\zeta+\zeta'+\varrho'-\kappa,2),(A+\zeta-\varrho,2),\\(\omega+\frac{3}{2},1),(\frac{3}{2},1)\end{matrix}\middle| -\frac{z^2}{4x^2}\right]
\end{aligned}
\tag{2.12}
$$

where $A = 2-\chi+\omega$, for Struve function $H_\omega(z/t)$ holds true.

2.1. Special Cases

(1) On taking $\varphi = 1$, $m = 1$, $\vartheta = 0$, and $z = 1$ in Theorem 2.1, we obtain the image formula for the Bessel function considered by Purohit et al. [28, Theorem 1].

Corollary 2.5. *Under the conditions stated with the Equation (2.1), the following image formula*

$$
\begin{aligned}
&I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left[t^{\chi-1}J_\omega(t)\right](x)\\
&=\frac{x^{\chi+\omega-\zeta-\zeta'+\kappa-1}}{2^\omega}\\
&\times {}_3\psi_4\left[\begin{matrix}(\chi+\omega,2),(\chi+\omega+\kappa-\zeta-\zeta'-\varrho,2),\\(\chi+\omega+\varrho'-\zeta',2)\\ \\(\chi+\omega+\varrho',2),(\chi+\omega+\kappa-\zeta-\zeta',2),\\(\chi+\omega+\kappa-\zeta'-\varrho,2),(\omega+1,1)\end{matrix}\middle| -\frac{x^2}{4}\right]
\end{aligned}
\tag{2.13}
$$

for Bessel function $J_\omega(t)$ holds true.

(2) Further, on taking $\varphi = 1$, $m = 1$, and $\vartheta = 0$ in Theorem 2.2, we arrive the right-sided image formula for the Bessel function considered by Purohit et al. [28, Theorem 2].

Corollary 2.6. *Under the conditions stated with the Equation (2.5), the image formula*

$$
\begin{aligned}
&I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left[t^{\chi-1}J_\omega(1/t)\right](x)\\
&=\frac{x^{\kappa-\zeta-\zeta'-1+\chi-\omega}}{2^\omega}\\
&\times {}_3\psi_4\left[\begin{matrix}(1-\chi+\omega-\kappa+\zeta+\zeta',2),(1-\chi+\omega+\zeta+\varrho'-\kappa,2),(1-\chi+\omega-\varrho,2)\\(1-\chi+\omega,2)(1-\chi+\omega+\zeta+\zeta'+\varrho'-\kappa,2),(1-\chi+\omega+\zeta-\varrho,2),(\omega+1,1)\end{matrix}\middle| -\frac{1}{4x^2}\right]
\end{aligned}
\tag{2.14}
$$

for Bessel function $J_\omega(1/t)$ holds true.

3. IMAGE FORMULAS ASSOCIATED WITH INTEGRAL TRANSFORMS

In this section, we obtain the theorem involving the results obtained in previous sections associated with the integral transforms such as Beta transform, pathway transform, Laplace transform, and Whittaker transform.

3.1. Image Formulas for Beta Transform

Theorem 3.1. *Let $\zeta,\zeta',\varrho,\varrho',\kappa,\vartheta\in\mathbb{C}$, $m\in\mathbb{N}$, $\varphi>0$, and $x>0$ be such that*

$$
\begin{aligned}
&\Re(l)>0,\quad \Re(n)>0\quad \Re(\kappa)>0,\quad \Re(\omega)>-1,\\
&\Re(\chi+\omega)>\max\{0,\Re(\zeta+\zeta'+\varrho-\kappa),\Re(\zeta'-\varrho')\}
\end{aligned}
\tag{3.1}
$$

then the following Beta transform formula holds:

$$
\begin{aligned}
&B\left[I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left(t^{\chi-1}J_{\omega,\vartheta}^{\varphi,m}(tz)\right)(x):l,n\right]\\
&=\frac{x^{A-\zeta-\zeta'+\kappa-1}\Gamma(n)}{2^{\omega+2\vartheta}}\\
&{}_5\psi_{5+m}\left[\begin{matrix}(A,2),(A+\kappa-\zeta-\zeta'-\varrho,2),\\(A+\varrho'-\zeta',2),(C-n,2)(1,1)\\ \\(A+\varrho',2),(A+\kappa-\zeta-\zeta',2),(A+\kappa-\zeta'-\varrho,2),\\(\omega+\vartheta+1,\varphi),(C,2),\underbrace{(\vartheta+1,1)}_{m-times}\end{matrix}\middle| -\frac{x^2}{4}\right]
\end{aligned}
\tag{3.2}
$$

Here $A = \chi+\omega+2\vartheta$ and $C = l+\omega+2\vartheta+n$.

Proof: For our convenience, let the left-hand side of the formula (3.2) be denoted by ς. Applying (1.28) to Equation (3.2), we get

$$\varsigma = \int_0^1 z^{l-1}(1-z)^{n-1}\left[I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left(t^{\chi-1}J_{\omega,\vartheta}^{\varphi,m}(tz)\right)(x)\right]dz.$$

Here, applying Equation (2.2) to the integral, we obtain the following expression

$$
\begin{aligned}
\varsigma &= \int_0^1 z^{l-1}(1-z)^{n-1}z^{\omega+2\vartheta}\frac{x^{A-\zeta-\zeta'+\kappa-1}}{2^{\omega+2\vartheta}}\\
&\times\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(A+2k)\Gamma(k+1)}{\Gamma(A+\varrho'+2k)\Gamma(A+\kappa-\zeta-\zeta'+2k)}\\
&\times\frac{\Gamma(A+\varrho'-\zeta'+2k)\Gamma(A+\kappa-\zeta-\zeta'-\varrho+2k)}{\Gamma(A+\kappa-\zeta'-\varrho+2k)\Gamma(\omega+\vartheta+1+\varphi k)(\Gamma(\vartheta+1+k))^m}\\
&\times\frac{(zx^2)^k}{4^k k!}dz
\end{aligned}
$$

Here $A = \chi+\omega+2\vartheta$.

Interchanging the order of integration and summation, we have

$$\begin{aligned}\varsigma &= \frac{x^{A-\zeta-\zeta'+\kappa-1}}{2^{\omega+2\vartheta}}\sum_{k=0}^{\infty}\frac{\Gamma(A+2k)\Gamma(A+\kappa-\zeta-\zeta'-\varrho+2k)}{\Gamma(A+\kappa-\zeta-\zeta'+2k)\Gamma(A+\kappa-\zeta'-\varrho+2k)}\\ &\times\frac{\Gamma(A+\varrho'-\zeta'+2k)\Gamma(k+1)(-1)^k}{\Gamma(A+\varrho'+2k)\Gamma(\omega+\vartheta+1+\varphi k)(\Gamma(\vartheta+1+k))^m}\frac{(x^2)^k}{4^k k!}\times\int_0^1 z^{l+\omega+2\vartheta+2k-1}(1-z)^{n-1}dz\\ &= \frac{x^{A-\zeta-\zeta'+\kappa-1}}{2^{\omega+2\vartheta}}\sum_{k=0}^{\infty}\frac{\Gamma(l+\omega+2\vartheta+2k)\Gamma(n)\Gamma(A+2k)\Gamma(A+\kappa-\zeta-\zeta'-\varrho+2k)}{\Gamma(l+\omega+2\vartheta+2k+n)\Gamma(A+\varrho'+2k)\Gamma(A+\kappa-\zeta-\zeta'+2k)}\\ &\times\frac{\Gamma(A+\varrho'-\zeta'+2k)\Gamma(k+1)}{\Gamma(A+\kappa-\zeta'-\varrho+2k)\Gamma(\omega+\vartheta+1+\varphi k)(\Gamma(\vartheta+1+k))^m}\times\frac{(-x^2)^k}{4^k k!}\end{aligned} \tag{3.3}$$

Interpreting the right-hand side of the above equation, in the view of the definition (1.2), we arrive at the required result (3.2).

Theorem 3.2. *Let $\zeta,\zeta',\varrho,\varrho',\kappa,\vartheta,\omega\in\mathbb{C}, m\in\mathbb{N}, \varphi>0$, and $x>0$ be such that*

$$\Re(\kappa)>0,\quad \Re(\omega)>-1,\quad \Re(l)>0,\ \ \Re(n)>0,$$
$$\Re(\chi-\omega)>1+\min\{\Re(-\varrho),\Re(\zeta+\zeta'-\kappa),\Re(\zeta+\varrho'-\kappa)\} \tag{3.4}$$

then the following Beta transform formula holds:

$$\begin{aligned}&B\left[I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left(t^{\chi-1}J_{\omega,\vartheta}^{\varphi,m}(z/t)\right)(x):l,n\right\}\\ &=\frac{x^{\kappa-\zeta-\zeta'-A}\Gamma(n)}{2^{\omega+2\vartheta}}\\ &\times{}_5\psi_{5+m}\left[\begin{matrix}(A-\kappa+\zeta+\zeta',2),(A+\zeta+\varrho'-\kappa,2),\\ (A-\varrho,2),(C-n,2),(1,1)\\ (A,2)(A+\zeta+\zeta'+\varrho'-\kappa,2),(A+\zeta-\varrho,2),\\ (\omega+\vartheta+1,\varphi),(C,2),\underbrace{(\vartheta+1,1)}_{m-times}\end{matrix}\middle|-\frac{1}{4x^2}\right]\end{aligned} \tag{3.5}$$

where $A=1-\chi+\omega+2\vartheta$ and $C=l+\omega+2\vartheta+n$.

Proof: The proof of the fractional integral formula (3.5) is similar to the proof of the formula (3.2) given in Theorem 3.1.

Remark 3.1.

(1) For $m=1$, Theorem 3.1 and Theorem 3.2 leads to the corresponding results for fractional integral of generalized Bessel function defined by (1.3).
(2) If we take $m=1$, $\varphi=1$, and $\vartheta=\frac{1}{2}$ in (3.2) and (3.5), we get the corresponding results for fractional integral of Struve function defined in (1.4).
(3) On taking $m=1$, $\varphi=1$, and $\vartheta=0$, in (3.2) and (3.5), we get the results for fractional integral of Bessel function defined in (1.5).

3.2. Image Formulas for P_δ-Transform

Theorem 3.3. *Let $\zeta,\zeta',\varrho,\varrho',\kappa,\chi,\vartheta\in\mathbb{C}, m\in\mathbb{N}, \varphi>0$, $\Re(\chi)>0$, $\Re(s)>0$, $\delta>1$, and $x>0$ be such that*

$$\Re(\kappa)>0,\quad \Re(\omega)>-1,\quad \Re(s)>0,$$
$$\Re(\chi+\omega)>\max\{0,\Re(\zeta+\zeta'+\varrho-\kappa),\Re(\zeta'-\varrho')\} \tag{3.6}$$

then the following P_δ-transform formula holds:

$$\begin{aligned}&P_\delta\left[z^{l-1}\left(I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}t^{\chi-1}J_{\omega,\vartheta}^{\varphi,m}(tz)\right)(x):s\right]\\ &=\left(\Lambda(\delta;s)\right)^{l+\omega+2\vartheta}\ \frac{x^{A-\zeta-\zeta'+\kappa-1}}{2^{\omega+2\vartheta}}\\ &\times{}_5\psi_{4+m}\left[\begin{matrix}(A,2),(A+\kappa-\zeta-\zeta'-\varrho,2),\\ (A+\varrho'-\zeta',2),(l+\omega+2\vartheta,2),(1,1)\\ (A+\varrho',2),(A+\kappa-\zeta-\zeta',2),\\ (A+\kappa-\zeta'-\varrho,2),\\ (\omega+\vartheta+1,\varphi),\underbrace{(\vartheta+1,1)}_{m-times}\end{matrix}\middle|-\frac{(\Lambda(\delta;s)x)^2}{4}\right]\end{aligned} \tag{3.7}$$

where $A=\chi+\omega+2\vartheta$ and $\Lambda(\delta;s)=\left(\frac{\delta-1}{\ln[1+(\delta-1)s]}\right)$.

Proof: For our convenience, we let the left-hand side of the formula (3.7) be denoted as Ξ. Applying (1.30) to Equation (3.2) we get,

$$\Xi=\int_0^\infty[1+(\delta-1)s]^{-\frac{z}{\delta-1}}z^{l-1}I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left(t^{\chi-1}J_{\omega,\vartheta}^{\varphi,m}(tz)\right)(x)dz$$

Here, applying Equation (2.4) to the integral, we obtain the following expression:

$$\begin{aligned}\Xi &= \frac{x^{A-\zeta-\zeta'+\kappa-1}}{2^{\omega+2\vartheta}}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(A+2k)\Gamma(A+\kappa-\zeta-\zeta'-\varrho+2k)}{\Gamma(A+\varrho'+2k)\Gamma(A+\kappa-\zeta-\zeta'+2k)\Gamma(A+\kappa-\zeta'-\varrho+2k)}\\ &\frac{\Gamma(A+\varrho'-\zeta'+2k)\Gamma(k+1)}{\Gamma(\omega+\vartheta+1+\varphi k)(\Gamma(\vartheta+1+k))^m}\frac{(x)^{2k}}{4^k k!}\times\int_0^\infty[1+(\delta-1)s]^{-\frac{z}{\delta-1}}z^{\omega+2\vartheta+2k+l-1}dz\end{aligned}$$

Here making use of the result (1.31) and interchanging the order of integration and summation, we obtain,

$$\Xi = \left(\Lambda(\delta;s)\right)^{l+\omega+2\vartheta} \frac{x^{A-\zeta-\zeta'+\kappa-1}}{2^{\omega+2\vartheta}} \sum_{k=0}^{\infty} \frac{\Gamma(A+2k)\Gamma(A+\kappa-\zeta-\zeta'-\varrho+2k)}{\Gamma(A+\varrho'+2k)\Gamma(A+\kappa-\zeta-\zeta'+2k)}$$
$$\times \frac{\Gamma(\omega+2\vartheta+2k+l)\Gamma(A+\varrho'-\zeta'+2k)\Gamma(k+1)(-1)^k}{\Gamma(A+\kappa-\zeta'-\varrho+2k)\Gamma(\omega+\vartheta+1+\varphi k)(\Gamma(\vartheta+1+k))^m} \frac{\{\Lambda(\delta;s)x\}^{2k}}{4^k k!} \tag{3.8}$$

where $A=\chi+\omega+2\vartheta$ and $\Lambda(\delta;s)=\left(\frac{\delta-1}{\ln[1+(\delta-1)s]}\right)$.
In view of the definition (1.2), we arrive at the required result (3.7).

Theorem 3.4. *Let $\zeta,\zeta',\varrho,\varrho',\kappa,\vartheta\in\mathbb{C}, m\in\mathbb{N}, \varphi>0$ $\Re(\chi)>0$, $\Re(s)>0$, $\delta>1$, and $x>0$ be such that*

$$\Re(\kappa)>0,\ \Re(\omega)>-1,\ \Re(s)>0,$$
$$\Re(\chi-\omega)>1+\min\{\Re(-\varrho),\Re(\zeta+\zeta'-\kappa),\Re(\zeta+\varrho'-\kappa)\} \tag{3.9}$$

then the following P_δ- transform formula holds:

$$P_\delta\left(z^{l-1}\left[I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa} t^{\chi-1} J_{\omega,\vartheta}^{\varphi,m}(z/t)\right](x):s\right)$$
$$=\left(\Lambda(\delta;s)\right)^{l+\omega+2\vartheta} \frac{x^{\chi-\omega-2\vartheta-\zeta-\zeta'+\kappa-1}}{2^{\omega+2\vartheta}}$$
$$\times {}_5\psi_{4+m}\left[\begin{matrix}(A-\kappa+\zeta+\zeta',2),(A+\zeta+\varrho'-\kappa,2),\\(A-\varrho,2),(l+\omega+2\vartheta,2),(1,1)\\(A,2)(A+\zeta+\zeta'+\varrho'-\kappa,2),(A+\zeta-\varrho,2),\\(\omega+\vartheta+1,\varphi),\underbrace{(\vartheta+1,1)}_{m-times}\end{matrix}\middle| -\frac{\{\Lambda(\delta;s)\}^2}{4x^2}\right] \tag{3.10}$$

where $A=1-\chi+\omega+2\vartheta$ and $\Lambda(\delta;s)=\left\{\frac{\delta-1}{\ln[1+(\delta-1)s]}\right\}$.

Proof: Our demonstration of the P_δ-transform of generalized Lommel–Wright function (3.10) is based upon the known result (2.6).

A limit case of the Theorems 3.3 and 3.4 when $\delta\to 1$ yields the following corollaries for the Laplace transform in view of the (1.32).

Corollary 3.1. *Under the conditions stated with the Equation (3.6), the following Laplace transform formula holds true:*

$$P_\delta\left(z^{l-1}\left(I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa} t^{\chi-1} J_{\omega,\vartheta}^{\varphi,m}(tz)\right)(x):s\right)=\frac{x^{A-\zeta-\zeta'+\kappa-1}}{s^l\,2^{\omega+2\vartheta}}$$
$$\times {}_5\psi_{4+m}\left[\begin{matrix}(A,2),(A+\kappa-\zeta-\zeta'-\varrho,2),(A+\varrho'-\zeta',2),\\(l+\omega+2\vartheta,2),(1,1)\\(A+\varrho',2),(A+\kappa-\zeta-\zeta',2),(A+\kappa-\zeta'-\varrho,2),\\(\omega+\vartheta+1,\varphi),\underbrace{(\vartheta+1,1)}_{m-times}\end{matrix}\middle| -\frac{x^2}{s^{2l}4}\right] \tag{3.11}$$

where $A=\chi+\omega+2\vartheta$.

Corollary 3.2. *Under the conditions stated with the Equation (3.9), the following Laplace transform formula holds true:*

$$P_\delta\left(z^{l-1}\left[I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa} t^{\chi-1} J_{\omega,\vartheta}^{\varphi,m}(z/t)\right](x):s\right)=\frac{x^{\chi-\omega-2\vartheta-\zeta-\zeta'+\kappa-1}}{s^l\,2^{\omega+2\vartheta}}$$
$$\times {}_5\psi_{4+m}\left[\begin{matrix}(A-\kappa+\zeta+\zeta',2),(A+\zeta+\varrho'-\kappa,2),\\(A-\varrho,2),(l+\omega+2\vartheta,2),(1,1)\\(A,2)(A+\zeta+\zeta'+\varrho'-\kappa,2),(A+\zeta-\varrho,2),\\(\omega+\vartheta+1,\varphi),\underbrace{(\vartheta+1,1)}_{m-times}\end{matrix}\middle| -\frac{1}{s^{2l}\,4x^2}\right] \tag{3.12}$$

where $A=1-\chi+\omega+2\vartheta$.

Remark 3.2.

(1) On taking $m=1$, Theorems 3.3 and 3.4 lead to the P_δ-transform formulas for fractional integrals of generalized Bessel function.
(2) A limit case of the Theorems 3.3 and 3.4, when $\delta\to 1$ and $m=1$, yields the Laplace transform formulas for fractional integrals of generalized Bessel function.
(3) On taking $m=1$, $\varphi=1$, and $\vartheta=\frac{1}{2}$, Theorems 3.3 and 3.4 yield the P_δ-transform formulas for fractional integrals of Struve function.
(4) A limit case of Theorem 3.3 and 3.4, when $\delta\to 1$ and $m=1$, $\varphi=1$, and $\vartheta=\frac{1}{2}$, yield the Laplace transform formulas for fractional integrals of Struve function.
(5) On taking $m=1$, $\varphi=1$, and $\vartheta=0$, Theorem 3.3 and 3.4 yield the corresponding results for fractional integrals of Bessel function.
(6) A limit case of Theorem 3.3 when $\delta\to 1$ and $m=1$, $\varphi=1$, and $\vartheta=0$ yield the corresponding Laplace transform formulas for fractional integrals of Bessel function.

3.3. Image Formulas for Whittaker Transform

Theorem 3.5. *Let $\zeta,\zeta',\varrho,\varrho',\kappa,\vartheta,\eta,\sigma\in\mathbb{C}$, $m\in\mathbb{N}$, $\varphi>0$, and $x>0$ be such that*

$$\Re(\kappa)>0,\Re(\omega)>-1,\ \Re(\tau\pm\eta)>-1/2,$$
$$\Re(\chi+\omega)>\max\{0,\Re(\zeta+\zeta'+\varrho-\kappa),\Re(\zeta'-\varrho')\} \tag{3.13}$$

then the following Whittaker transform formula holds:

$$\int_0^\infty z^{\sigma-1}e^{-z/2}\left[W_{\sigma,\eta} I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa}\left(t^{\chi-1}J_{\omega,\vartheta}^{\varphi,m}(zt)\right)(x)\right]dz$$
$$=\frac{x^{A-\zeta-\zeta'+\kappa-1}}{2^{\omega+2\vartheta}}$$
$${}_6\psi_{5+m}\left[\begin{matrix}(A,2),(A+\kappa-\zeta-\zeta'-\varrho,2),(A+\varrho'-\zeta',2),\\(E+\eta,2),(E-\eta,2),(1,1)\\(A+\varrho',2),(A+\kappa-\zeta-\zeta',2),(A+\kappa-\zeta'-\varrho,2),\\(\omega+\vartheta+1,\varphi),(E-\sigma,2),\underbrace{(\vartheta+1,1)}_{m-times}\end{matrix}\middle| -\frac{x^2}{4}\right] \tag{3.14}$$

where $A = \chi + \omega + 2\vartheta$ and $E = \tau + \omega + 2\vartheta + 1/2$.

Proof: For simplicity, let ϖ be the left-hand side of the formula (3.14). Applying (1.35) to Equation (3.14), we have

$$\varpi = \int_0^\infty z^{\sigma-1} e^{-z/2} W_{\sigma,\eta} \left[I_{0+}^{\zeta,\zeta',\varrho,\varrho',\kappa} \left(t^{\chi-1} J_{\omega,\vartheta}^{\varphi,m}(zt) \right)(x) \right] dz. \tag{3.15}$$

Here, applying Equation (2.2) to the integral, we obtain the following expression:

$$\begin{aligned}\varpi = &\int_0^\infty z^{\sigma+\omega+2\vartheta-1} e^{-z/2} W_{\sigma,\eta} \\ &\left[\frac{x^{A-\zeta-\zeta'+\kappa-1}}{2^{\omega+2\vartheta}} \sum_{k=0}^\infty \frac{\Gamma(A+2k)\Gamma(A+\kappa-\zeta-\zeta'-\varrho+2k)}{\Gamma(A+\varrho'+2k)\Gamma(A+\kappa-\zeta-\zeta'+2k)} \right.\\ &\times \frac{\Gamma(A+\varrho'-\zeta'+2k)\Gamma(k+1)(-1)^k}{\Gamma(A+\kappa-\zeta'-\varrho+2k)\Gamma(\omega+\vartheta+1+\varphi k)(\Gamma(\vartheta+1+k))^m} \\ &\left. \times \frac{(zx)^{2k}}{4^k k!} \right] dz\end{aligned}$$

where $A = \chi+\omega+2\vartheta$. Interchanging the order of integration and summation, we have

$$\begin{aligned}\varpi = &\frac{x^{A-\zeta-\zeta'+\kappa-1}}{2^{\omega+2\vartheta}} \sum_{k=0}^\infty \frac{\Gamma(E+\eta+2k)\Gamma(E-\eta+2k)\Gamma(A+\kappa-\zeta-\zeta'-\varrho+2k)}{\Gamma(E-\sigma+2k)\Gamma(A+\kappa-\zeta'-\varrho+2k)} \\ &\times \frac{(-1)^k\Gamma(A+2k)\Gamma(A+\varrho'-\zeta'+2k)\Gamma(k+1)}{\Gamma(A+\varrho'+2k)\Gamma(A+\kappa-\zeta-\zeta'+2k)\Gamma(\omega+\vartheta+1+\varphi k)(\Gamma(\vartheta+1+k))^m} \frac{x^{2k}}{4^k k!}\end{aligned} \tag{3.16}$$

where $A = \chi + \omega + 2\vartheta$ and $E = \tau + \omega + 2\vartheta + 1/2$.
Interpreting the right-hand side of the above equation, in view of the definition (1.2), we arrive at the required result (3.14).

Theorem 3.6. *Let $\zeta, \zeta', \varrho, \varrho', \kappa, \vartheta, \eta, \sigma \in \mathbb{C}, m \in \mathbb{N}$, $\varphi > 0$, and $x > 0$ be such that*

$$\begin{gathered}\Re(\kappa) > 0,\ \ \Re(\omega) > -1,\ \ \Re(\tau \pm n) > -1/2, \\ \Re(\chi-\omega) > 1 + \min\{\Re(-\varrho), \Re(\zeta+\zeta'-\kappa), \Re(\zeta+\varrho'-\kappa)\}\end{gathered} \tag{3.17}$$

then there holds the formula

$$\begin{aligned}&\int_0^\infty z^{\sigma-1} e^{-z/2} W_{\sigma,\eta} \left[\left(I_{0-}^{\zeta,\zeta',\varrho,\varrho',\kappa} t^{\chi-1} J_{\omega,\vartheta}^{\varphi,m}(zt) \right)(x) \right] dz \\ &= \frac{x^{\chi-\omega-2\vartheta-\zeta-\zeta'+\kappa-1}}{2^{\omega+2\vartheta}} \\ &{}_6\psi_{5+m} \left[\begin{array}{c|c} \begin{array}{c} (A-\kappa+\zeta+\zeta',2), (A+\zeta+\varrho'-\kappa,2), \\ (A-\varrho,2), (E+\eta,2), (E-\eta,2), (1,1) \\ \\ (A,2)(A+\zeta+\zeta'+\varrho'-\kappa,2), (A+\zeta-\varrho,2), \\ (\omega+\vartheta+1,\varphi), (E-\sigma,2), \underbrace{(\vartheta+1)}_{m-times} \end{array} & -\dfrac{1}{4x^2} \end{array} \right]\end{aligned} \tag{3.18}$$

where $A = 1 - \chi + \omega + 2\vartheta$ and $E = \tau + \omega + 2\vartheta + 1/2$.

Proof: We can establish the result given in Theorem 3.6 similar to the proof of Theorem 3.5.

Remark 3.3.

(1) For $m = 1$, Theorems 3.5 and 3.6 lead to the corresponding results for fractional integral of generalized Bessel function defined in (1.3).
(2) If we take $m = 1$, $\varphi = 1$, and $\vartheta = \frac{1}{2}$, Theorems 3.5 and 3.6 yield the corresponding results for fractional integral of Struve function defined in (1.4).
(3) On taking $m = 1$, $\varphi = 1$, and $\vartheta = 0$, Theorems 3.5 and 3.6 yield the corresponding results for fractional integral of Bessel function defined in (1.5).

4. SPECIAL CASES AND CONCLUDING REMARKS

In this section, we consider some special cases of our main results involved in Theorems 2.1–3.6 which can be obtained by setting $\zeta' = 0$. These interesting corollaries of our results involve the Saigo fractional integral operators $I_{0,x}^{\zeta,\varrho,\eta}$ and $I_{x,\infty}^{\zeta,\varrho,\eta}$ and can be deduced from the Theorems 2.1–3.6 by appropriately applying the relationships given in the definitions (1.16) and (1.17). If we set $\varrho = -\zeta$ in the Theorems 2.1–3.6, then from the relationships (1.20) and (1.21) we obtain the corresponding results for the Riemann–Liouville and the Weyl fractional integral operators, respectively. Again, if we put $\varrho = 0$ in the Theorems 2.1–3.6, then from the relationships (1.24) and (1.25) we obtain the analogous results for Erdélyi-Kober type fractional integral operators.

In our present investigation, we establish the relationship between well-known fractional integral operators with novel integral transforms. The results obtained here are useful in deriving at various image formulas. The results presented here are very generic and can be specialized to give further potentially interesting and useful formulas involving fractional integral operators.

AUTHOR CONTRIBUTIONS

RPA devised the problem and supervised the manuscript by adding various results to it. RA and SJ worked on the mathematics in the manuscript. DB provided guidance, checked all calculations, and suggested language modifications to the article paper.

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On Optimal Tempered Lévy Flight Foraging

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Optimal random foraging strategy has gained increasing concentrations. It is shown that Lévy flight is more efficient compared with the Brownian motion when the targets are sparse. However, standard Lévy flight generally cannot be followed in practice. In this paper, we assume that each flight of the forager is possibly interrupted by some uncertain factors, such as obstacles on the flight direction, natural enemies within the vision distance, and restrictions in the energy storage for each flight, and introduce the tempered Lévy distribution $p(l) \sim e^{-\rho l} l^{-\mu}$. It is validated by both theoretical analyses and simulation results that a higher searching efficiency can be derived when a smaller ρ or μ is chosen. Moreover, by taking the flight time as the waiting time, the master equation of the random searching procedure can be obtained. Interestingly, we build two different types of master equations: one is the standard diffusion equation and the other one is the tempered fractional diffusion equation.

Keywords: optimal random search, foraging, tempered Lévy distribution, master equation, tempered fractional derivative

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1. INTRODUCTION

One common approach to the animal movement patterns is to use the scheme of optimizing random search [1–3]. In a random search model, single or multiple individuals search a landscape to find targets whose locations are not known *a priori*, which is usually adopted to describe the scenario of animals foraging for food, prey or resources. The locomotion of the individual has a certain degree of freedom which is characterized by a specific search strategy such as a type of random walk and is also subject to other external or internal constraints, such as the environmental context of the landscape or the physical and psychological conditions of the individual. It is assumed that a strategy that optimizes the search efficiency can evolve in response to such constraints on a random search, and the movement is a consequence of the optimization on random search.

Many researchers have concentrated on the study of different animals' foraging movements. It is shown that when the environment contains a high density of food items, foragers tend to adopt Brownian walks, characterized by a great number of short step lengths in random directions that maintain foragers in a small portion of the available space [4, 5]. In contrast, when the density of food items is low, individuals tend to exhibit Lévy flights, where larger step lengths occasionally occur and relocate the foragers in the environment. Due to the fact that the density of food items is often low, many animals behave a Lévy flight when foraging and their movements have been found to fit closely to a Lévy distribution (power law distribution) with an exponent close to 2 [6, 7]. For instance, the foraging behavior of the wandering albatross on the ocean surface was found to obey a power law distribution [8]; the foraging patterns of a free-ranging spider monkey in the forests was also found to be a power law tailed distribution of steps consistent with Lévy walks [9, 10].

On this basis, researchers mainly consider two issues: one is to model the foraging behavior as a Lévy flight and the other one is to study the searching efficiency theoretically or experimentally. It is assumed that the forager takes a random walk according to the distribution $p(l) \sim l^{-\mu}$, $1 < \mu < 3$. Then it is proven that the highest searching efficiency can be obtained when μ is close to 2 for the non-destructive case (the same target site can be visited many times). While the searching efficiency is higher when μ tends to 1 for the destructive case (the target site found by the forager becomes undetectable in subsequent flights). Later, many more complex situations are considered. Due to the fact that foragers are always searching in a bounded area, Dybiec et al. [11] and Zhao et al. [12] studied the searching efficiency of Lévy flight in a bounded area. Kerster et al. [5] took the spatial memory of foragers into consideration and concluded that the spatial information influenced the foraging behavior significantly according to the experimental results. Interestingly, it was claimed that the Lévy flight foraging behavior can also be interpreted by a composite search model [13, 14]. The model consists of an intensive search phase, followed by an extensive phase, if no food is found in the intensive phase. Particularly, Zeng and Chen [15] considered the waiting time between two successive flights and formulated the master equation for such foraging behavior.

Though many studies have proven that it is usually more efficient to utilize Lévy flight foraging strategy, standard Lévy flight cannot be followed in practice because of many uncertain factors. For instance, the forager may encounter obstacles or natural enemies and extremely large flight distance cannot be reasonable due to the restriction of the forager's flight ability. In this paper, we take these conditions into consideration and temper the Lévy distribution with an exponential decaying function, which results in a tempered Lévy distribution $p(l) \sim e^{-\rho l} l^{-\mu}$. It is then shown that a higher searching efficiency will be derived when a smaller ρ or μ is chosen, both by simulation and theoretical analyses. Further, two different types of master equations are derived: one is the standard diffusion equation and the other one is the tempered fractional diffusion equation. Since the first and second order moments exist, the foraging movement will finally result in a Gaussian motion, which indicates that the tempered fractional diffusion equation is in fact another expression for the standard diffusion.

The remainder of the paper is organized as follows. Section 2 provides the basic foraging model and some basic results are also given. In section 3, we study the searching efficiency when a tempered Lévy distribution is considered. Two different types of master equations are derived in section 4 after treating the flight time as the waiting time. The paper is concluded in section 5.

2. BASIC DEFINITIONS AND MODEL DESCRIPTION

In this section, we mainly recall the original model and basic results of Lévy flight optimal random search. Assume that target sites are uniformly distributed and the forager behaves as follows

(1) If a target site lies within a "direct vision" distance r_v, then the forager moves on a straight line to the nearest site. A finite value of r_v, no matter how large, models the constraint that no forager can detect a target site located an arbitrarily large distance away.
(2) If there is no target site within a distance r_v, then the forager chooses a direction uniformly and a distance l_j from a probability distribution. It then incrementally moves to the new point, constantly looking for a target within a radius r_v along its way. If it does not detect a target, it stops after traversing the distance l_j and chooses a new direction and a new distance l_{j+1}; otherwise, it proceeds to the target as rule (1).

In the case of non-destructive foraging, the forager can visit the same target site many times. In the case of destructive foraging, the target site found by the forager becomes undetectable in subsequent flights. Let λ be the mean free path of the forager between two successive target sites [for two dimensions $\lambda = (2r_v\phi)^{-1}$ where ϕ is the target-site area density].

On the basis of above behaviors, assume that the flight distance is distributed as the Lévy distribution

$$p(l) \sim l^{-\mu},\ l \geq r_v,\ 1 < \mu < 3. \tag{1}$$

As shown in **Figure 1** where η is the searching efficiency defined as (7), researchers find that $\mu \approx 2$ and $\mu \to 1$ will result in an optimal searching efficiency for the non-destructive case and destructive case, respectively. For more details about the model and existing results, one may refer to the works of Viswanathan et al. [6, 7] and references therein.

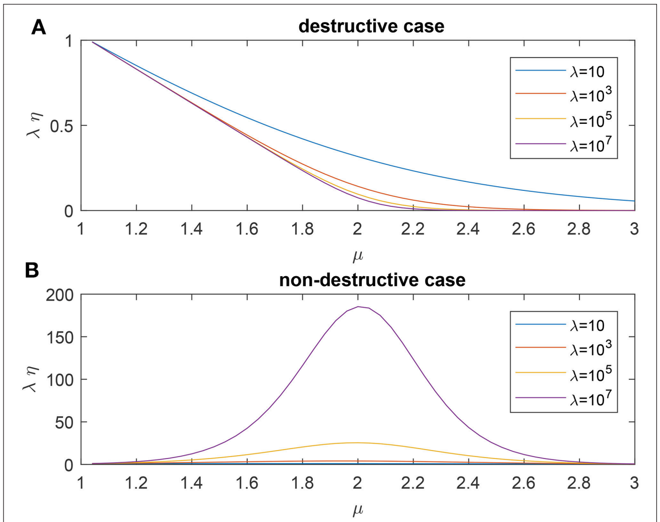

FIGURE 1 | Searching efficiency of standard Lévy flight for different mean free path λ: **(A)** the destructive case, **(B)** the nondestructive case.

3. SEARCHING EFFICIENCY WITH A TEMPERED LÉVY FLIGHT

In almost all the existing literatures about Lévy flight foraging, it is assumed that the flight distance at each step is independently distributed as (1). Distribution (1) is power-law decaying, which indicates that a large jump length will appear more frequently compared with the traditional Gaussian distribution. In practical foraging, after the forager determines the flight distance at some step, the flight will be interrupted by some unknown reasons, such as obstacles on the flight direction, natural enemies in the vision distance, and restrictions in the energy storage for each flight. Because of these reasons, we can assume that the flight distance is distributed as

$$p\left(l\right) \sim \mathrm{e}^{-\rho l} l^{-\mu},\ r_v \leq l,\ \rho > 0,\ \mu \geq 1, \tag{2}$$

which indicates that the forager can keep the flight direction with the probability of an exponential distribution. **Figures 2**, **3** show the probability density function (pdf) of a tempered Lévy distribution for different μ and ρ respectively. One can find that the density decreases slower with a smaller μ or a smaller ρ, which means that a larger jump length is more likely to happen. Particularly, the $\mu = 0$ case in **Figure 2** is included to show that the tempered Lévy distribution always has a shorter tail than the pure $\mu = 0$ Lévy distribution. The $\rho = 0$ case in **Figure 3** is the Lévy distribution which has a heavier tail compared with other cases.

Remark 1: The difference between (1) and (2) is that the power law distribution is tempered by an exponential decaying $\mathrm{e}^{-\rho l}$. The exponential part $\mathrm{e}^{-\rho l}$ can be viewed as the probability density that the forager can keep its flight direction before he completes one flight in the existence of some unknown factors and ρ is determined by the environment. Because Lévy distribution is now tempered by $\mathrm{e}^{-\rho l}$, the first and second order moments of distribution (2) exist for arbitrary $\mu \in \mathbb{R}$. In the paper, we will discuss the problem in a wider range $\mu \in [1, \infty)$ rather than $(1, 3)$ for the Lévy distribution.

3.1. The Non-destructive Case

In this part, we will borrow the idea from Viswanathan et al. [6] to optimize the searching efficiency. Given the pdf of the flight distance as (2), the mean flight distance can be calculated as

$$\begin{aligned}\langle l\rangle &= \frac{\int_{r_v}^{\lambda} e^{-\rho|x|}|x|^{-\mu+1}dx+\lambda\int_{\lambda}^{\infty} e^{-\rho|x|}|x|^{-\mu}dx}{\int_{r_v}^{\infty} e^{-\rho|x|}|x|^{-\mu}dx}\\ &= \frac{\Gamma_{up}(\rho r_v,2-\mu)-\Gamma_{up}(\rho\lambda,2-\mu)+\lambda\rho\Gamma_{up}(\rho\lambda,1-\mu)}{\rho\Gamma_{up}(\rho r_v,1-\mu)}\end{aligned} \tag{3}$$

where, the incomplete gamma function Γ_{up} is defined as

$$\Gamma_{up}\left(x, u\right) = \int_{x}^{\infty} t^{u-1}\mathrm{e}^{-t}\mathrm{d}t. \tag{4}$$

Remark 2: In Viswanathan et al. [6], the Lévy distribution is truncated by the mean free path λ because it is assumed that the forager must find a target after flight for distance λ. Different from the idea in Viswanathan et al. [6], we assume that the flight distance may be truncated according to an exponential distribution which is used to describe the probability of encountering some uncertain factors. In this paper, the truncation is also considered when calculating the mean flight distance since the jump length may be larger than the mean free path. We have also to declare that the mean free path is generally very large since the targets are sparse and the decaying speed of tempered Lévy distribution is much faster. Thus, the integral of $\lambda\int_{\lambda}^{\infty} e^{-\rho|x|}|x|^{-\mu}dx$ is very small such that it almost has no influence on the searching efficiency.

Let N be the mean number of flights taken by a Lévy forager while traveling between two successive target sites. Since the first

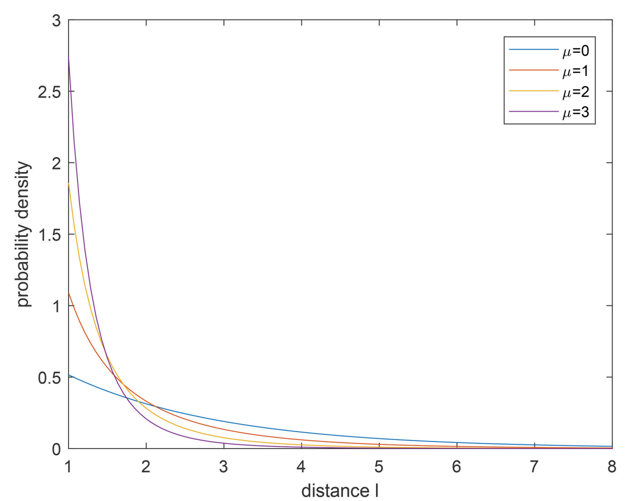

FIGURE 2 | Probability density for tempered Lévy distribution with $\rho = 0.5$ for different μ.

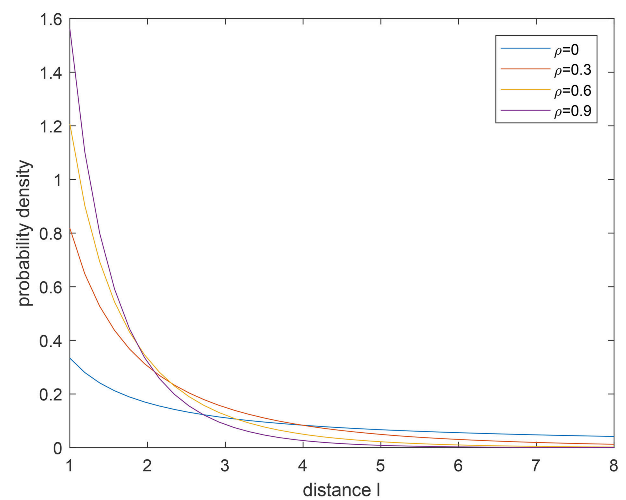

FIGURE 3 | Probability density for tempered Lévy distribution with $\mu = 1$ for different ρ.

and second order moments of tempered Lévy distribution exist, the trajectory of the forager will result in a Brownian motion. According to the existing results by Viswanathan et al. [6], for the non-destructive case, it follows that the mean flight number between two successive targets can be estimated as

$$N_n \approx \left(\frac{\lambda^2}{2D}\right)^{\frac{1}{2}} \tag{5}$$

where, D is the diffusion constant. According to the standard diffusion equation in section 4, it is found that the diffusion constant $D = \frac{a}{2b}$, where a is the second order moment of flight distance and b is the mean of the waiting time. Since we do not take the time into consideration, one can conclude that N_n is proportional to $\left(\frac{\lambda^2}{a}\right)^{\frac{1}{2}}$. Here, a can be calculated as

$$a = \int_{r_v}^{\infty} \mathrm{e}^{-\rho l} l^{2-\mu} \mathrm{d}l = \rho^{\mu-3}\Gamma_{up}\left(\rho r_v, 3-\mu\right). \tag{6}$$

Based on the above analyses, we can then calculate the searching efficiency which is defined as

$$\eta = \frac{1}{N\langle l\rangle}. \tag{7}$$

Take r_v as 1 when simulating and the results for different mean free path λ are shown in **Figure 4**. Following observations can be drawn

(1) For fixed mean free path λ and ρ, a smaller μ will result in a higher searching efficiency.
(2) For fixed mean free path λ and μ, a smaller ρ will result in a higher searching efficiency.
(3) The mean free path λ almost has no influence on the choice of μ and ρ to derive the highest searching efficiency.

As interpreted in the existing papers, the Lévy distribution can lead to a higher efficiency in a sparse area due to the higher probability of large jump lengths. For this issue, a smaller μ or ρ will both decrease the decaying speed of the probability density, which means that the large jump lengths are more likely to appear. Hence, observations (1) and (2) can be explained since frequently large jump lengths can help covering a wider range where it is more likely to find a target in a sparse area. Generally, the density of target site is sparse in practice which means that λ is usually large. Due to the exponential decaying of tempered Lévy distribution, the value of $\lambda \int_{\lambda}^{\infty} e^{-\rho|x|}|x|^{-\mu}dx$ is quite small and almost has no influence on the searching efficiency. It can then explain why the results of **Figure 4** with different λ are similar.

One can also interpret the observations from the practical perspective. As discussed before, the tempered item $\mathrm{e}^{-\rho l}$ can be viewed as the probability density that the forager can keep its flight direction before he completes one flight in the existence of some unknown factors. Thus, a smaller ρ means that the probability of a forager to encounter some uncertain factors is lower and the foraging efficiency should be higher.

3.2. The Destructive Case

For the destructive case, the mean number N can be expressed as

$$N_d \approx \frac{\lambda^2}{2D}. \tag{8}$$

Similar to the non-destructive case, one can then calculate the searching efficiency using (7). The results are shown in **Figure 5**, which is very similar to the non-destructive case. It is found that a smaller μ or ρ will both result in a higher search efficiency.

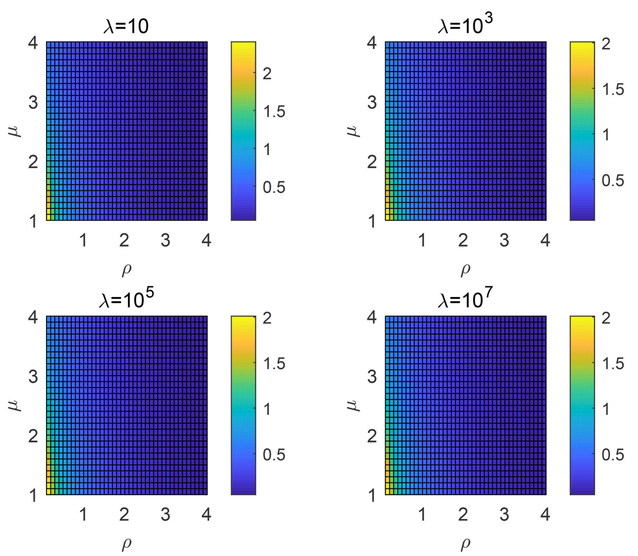

FIGURE 4 | Searching efficiency $\eta\lambda$ for different order μ and ρ: the non-destructive case with different λ.

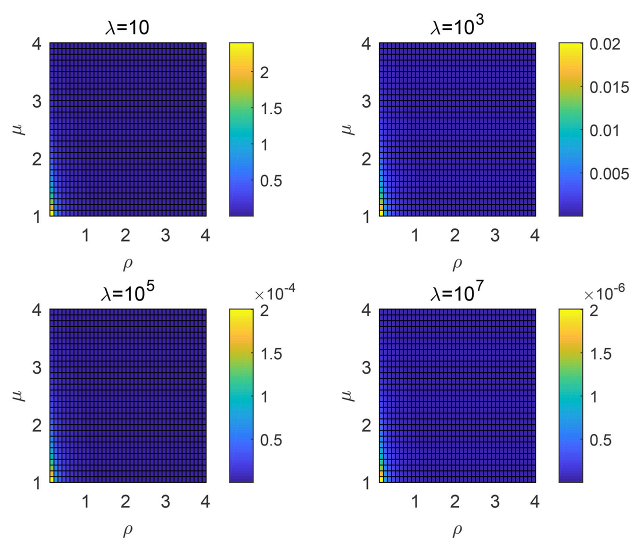

FIGURE 5 | Searching efficiency $\eta\lambda$ for different order μ and ρ: the destructive case with different λ.

The mean free path λ almost has no influence on the optimal choice of parameters μ and ρ. We have shown that for the Lévy distribution, $\mu \rightarrow 1$, where a large jump length appears more likely, will lead to a higher searching efficiency. Thus, a smaller ρ and μ will also result in a larger searching efficiency because large jump lengths are more likely to happen.

Remark 3: According to the results in **Figure 6** where we obtain the mean flight number after averaging 100 independent runs, one can find that a larger variance will lead to a smaller mean flight number. Moreover, it is shown that there is a linear property between them with a correlation coefficient -0.9781, which indicates that the estimation of mean flight number is fine.

3.3. Numerical Results

We also implement an experiment for validate the theoretical analyses. Consider a 200×200 area and 50 targets are uniformly distributed in this area. The vision distance is $r_v = 1$ and the total flight distance is no longer than 10,000 which can be viewed as the flight capability of the forager. The searching efficiency is estimated as $\frac{N_{num}}{L_{total}}$ where N_{num} is the number of found targets and L_{total} is the total flight distance. From **Figure 7** where the searching efficiency is derived by averaging 100 independent runs, one can find that a smaller ρ and μ will both lead to a higher searching efficiency, which is consistent with the theoretical analyses. Because a larger μ will make the density function decrease quickly, the range of jump lengths is then very tight. Thus, the searching efficiency is very close for a large μ where the jump lengths are all around the vision distance r_v. **Figures 8–10** give some typical foraging procedures for different parameters and one can find that all of them perform a Brownian motion which can be verified by the statistic results of averaging 100 independent generated jump lengths in **Figure 11**. Additionally, larger jump lengths frequently appear in **Figure 8** compared with the other two figures, for which the searching efficiency is the highest. It is also shown in **Figure 11**, where larger jump lengths are most likely to appear for the $\lambda = 0.5$ and $\mu = 1$ case.

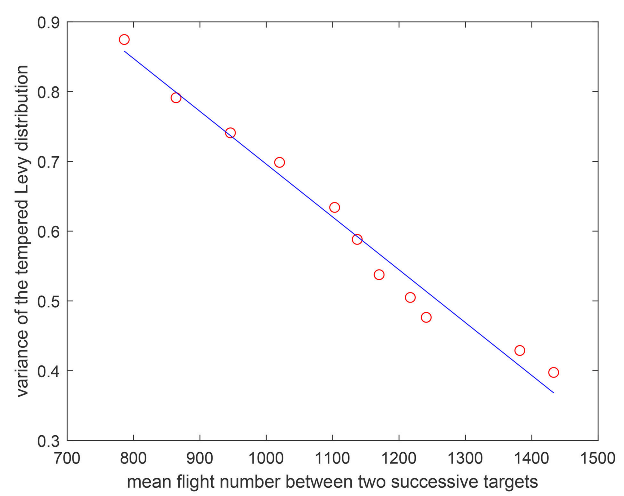

FIGURE 6 | The relation between the mean flight number and the variance of the tempered Lévy distribution.

Remark 4: In this paper, we numerically generate the jump lengths distributed as a tempered Lévy distribution and **Figure 12** shows the actual density function and the statistic result of generated jump lengths. It is found that the statistic result is very close to the actual density function. We have to declare here that all the foraging procedures in **Figures 8–10** will result in a Brownian motion because of the Central Limit Theorem.

Remark 5: Some conclusive remarks can be drawn as follows

1) Tempered Lévy model is to assume the uncertain factors during the foraging procedure may happen according to an exponential distribution. Whenever such uncertain factor happens, the forager has to stop its flight, which seems like that the flight distance is truncated.
2) If we take $\rho = 0$, the tempered Lévy fight will reduce to the standard Lévy flight.

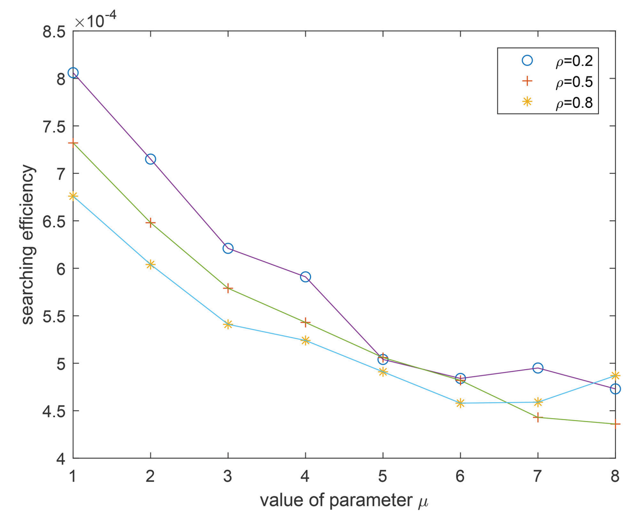

FIGURE 7 | Experimental results of searching efficiency for different λ and μ.

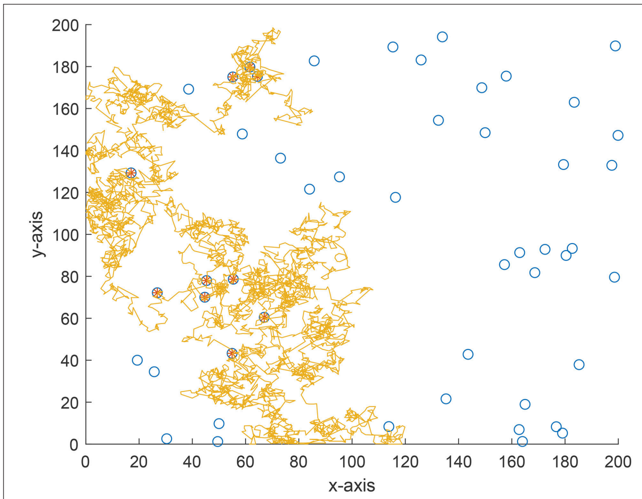

FIGURE 8 | A typical example of foraging procedure with $\lambda = 0.5$ and $\mu = 1$.

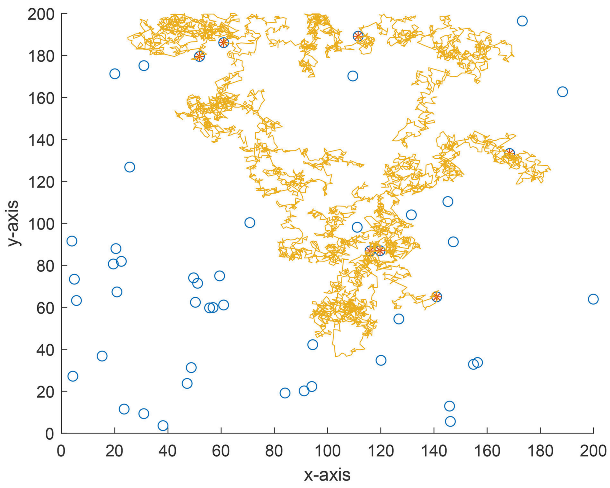

FIGURE 9 | A typical example of foraging procedure with $\lambda = 1$ and $\mu = 1$.

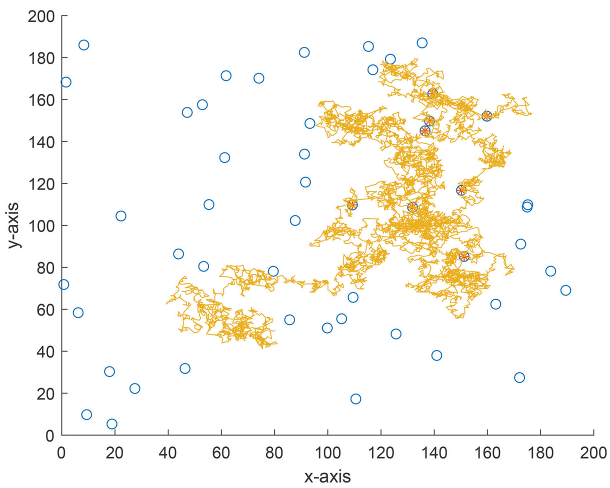

FIGURE 10 | A typical example of foraging procedure with $\lambda = 0.5$ and $\mu = 3$.

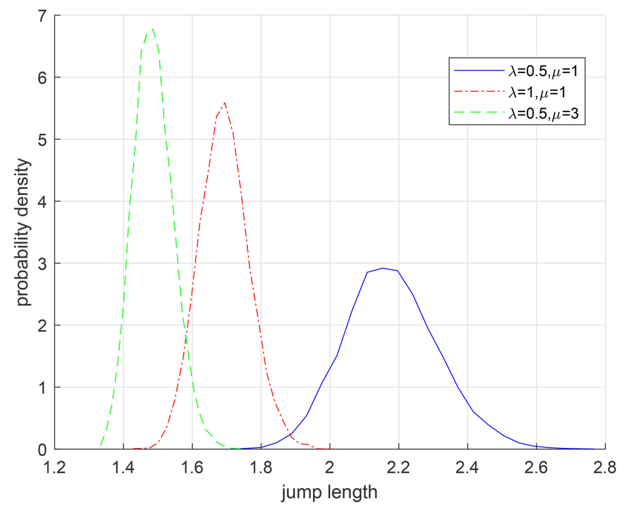

FIGURE 11 | The statistic property of jump length with different λ and μ.

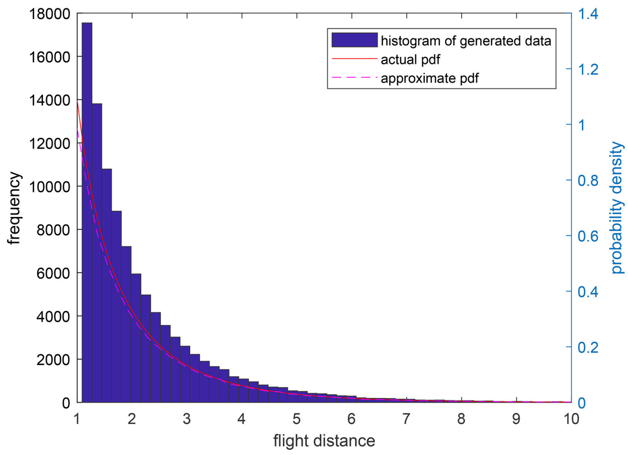

FIGURE 12 | The actual tempered Lévy density function with $\rho = 0.5$ and $\mu = 1$ and the statistic results of generated jump lengths.

3) According to the results in **Figure 5**, it is found that a smaller ρ leads to a higher searching efficiency. In fact, the searching efficiency with standard Lévy distribution is higher than the truncated Lévy distribution. It is easy to understand since the forager will have a higher searching efficiency if there is no interruption during the foraging procedure.

4. MASTER EQUATIONS

In the previous, we have not taken the flight time into consideration. Assume that the flight speed v is constant during the foraging process and treat the flight time between two flights as the waiting time. Then, the pdf of waiting time is the same as the flight distance with a scaling parameter v, which can be expressed as

$$p(t) \sim \mathrm{e}^{-\frac{\rho}{v}t} t^{-\mu}, \; t \ge \frac{r_v}{v}. \tag{9}$$

Let us introduce the Laplace transform for the waiting time as

$$\Psi(s) = \int_{r_v/v}^{\infty} \mathrm{e}^{-st} p(t)\, \mathrm{d}t. \tag{10}$$

The famous Montroll-Weiss Equation [16] in Fourier-Laplace space is in the following form

$$P(k, s) = \frac{1 - \Psi(s)}{s} \frac{1}{1 - W(k)\,\Psi(s)}, \tag{11}$$

where $W(k)$ is the Fourier transform for the flight distance in two dimensions which will be discussed later. Now consider the extreme distribution of $\Psi(s)$ with $s \to 0$. It is followed that

$$\begin{aligned} \Psi(s) &= \int_{r_v/v}^{\infty} \mathrm{e}^{-st} p(t)\,\mathrm{d}t \\ &= \int_{r_v/v}^{\infty} (1 - st + o(s))\, p(t)\,\mathrm{d}t \\ &= 1 - bs + o(s), \end{aligned} \tag{12}$$

where, b is the mean of flight time t and $o(\cdot)$ means the higher order infinitesimal. In the following, we will present two different types of master equations for this foraging procedure via different treatments to the extreme distribution of $W(k)$ with $k \to 0$.

4.1. The Standard Diffusion Equation Case

Assume that the searching direction θ is uniformly distributed in the interval $[0, 2\pi)$. If the waiting time and the flight distance are independent, then the flight distance of the forager in two dimensions can be formulated as $(x, y) = (l\cos\theta, l\sin\theta)$ and the following Fourier transform holds

$$\begin{aligned} W(k) &= \tfrac{1}{2\pi}\int_0^{2\pi}\int_{r_v}^{\infty} \mathrm{e}^{il(k_1\cos\theta + k_2\sin\theta)} p(l)\,\mathrm{d}l\mathrm{d}\theta \\ &= \tfrac{1}{2\pi}\int_0^{2\pi}\int_{r_v}^{\infty}\left(1 + il\Theta + (il\Theta)^2 + o(\Theta^2)\right) p(l)\,\mathrm{d}l\mathrm{d}\theta \\ &= 1 + \tfrac{1}{2\pi}\int_0^{2\pi}\int_{r_v}^{\infty}(il\Theta)^2 p(l)\,\mathrm{d}l\mathrm{d}\theta + o(\Theta^2) \\ &= 1 + \tfrac{a}{2}\left((ik_1)^2 + (ik_2)^2\right) + o(\Theta^2) \end{aligned} \tag{13}$$

where, k_1 and k_2 are the Fourier variables, $k = (k_1, k_2)$, $\Theta = k_1\cos\theta + k_2\sin\theta$ and a is the second order moment of the flight distance.

Substitute (12) and (13) into the Montroll-Weiss equation and ignore the higher order infinitesimal, yielding,

$$\begin{aligned} P(k,s) &= \frac{b}{bs - \frac{a}{2}(ik_1)^2 - \frac{a}{2}(ik_2)^2} \\ &= \frac{1}{s - \frac{a}{2b}\left((ik_1)^2 + (ik_2)^2\right)}. \end{aligned} \tag{14}$$

Perform inverse Fourier-Laplace transform and one can derive the master equation

$$\frac{\partial}{\partial t}p(x,y,t) = \frac{a}{2b}\frac{\partial^2}{\partial x^2}p(x,y,t) + \frac{a}{2b}\frac{\partial^2}{\partial y^2}p(x,y,t). \tag{15}$$

Remark 6: Unlike the master equation derived by Zeng and Chen [15], the master equation in this study is a normal diffusion equation since the first and second order moments exist. We have to mention that the master equation proposed by Zeng and Chen [15] should also be standard diffusion equation rather than fractional diffusion differential equation since the Lévy distribution is truncated by the mean free path λ. Moreover, the master equation should be two-dimensional rather than one-dimensional.

4.2. The Tempered Fractional Diffusion Equation Case

In this subsection, our purpose is to express the master equation as a tempered fractional diffusion equation and we have restrict μ varies from 1 to 2 to derive the tempered fractional derivative expression. The vector jump length can be described as $l\Theta$, where $\Theta = (\cos\theta, \sin\theta)$. From Equation (7.9) in the book of Meerschaert and Sikorskii [17], it shows that

$$\begin{aligned} W(k) &= \int_{\|\Theta\|=1}\int_{r_v}^{\infty} \mathrm{e}^{ik\cdot l\Theta} p(l)\,\mathrm{d}lM(\mathrm{d}\Theta) \\ &= 1 + \int_{\|\Theta\|=1}\int_{r_v}^{\infty}\left(\mathrm{e}^{ik\cdot l\Theta} - 1\right) p(l)\,\mathrm{d}lM(\mathrm{d}\Theta) \\ &= 1 + C\int_{\|\Theta\|=1}\left[(\lambda - ik\cdot\Theta)^{\mu-1} - \lambda^{\mu-1}\right] M(\mathrm{d}\Theta), \end{aligned} \tag{16}$$

where $k\cdot\Theta = k_1\cos\theta + k_2\sin\theta$, $M(\mathrm{d}\Theta)$ is a uniform distribution on a unit circle, and C is a constant relevant to coefficients ρ and μ.

Substitute (12) and (16) into the Montroll-Weiss Equation (11) and ignore the higher order infinitesimal, yielding,

$$P(k,s) = \frac{1}{s - \frac{C}{b}\int_{\|\Theta\|=1}(\lambda - ik\cdot\Theta)^{\mu-1} - \lambda^{\mu-1}M(\mathrm{d}\Theta)}. \tag{17}$$

Define

$$^{\lambda}\nabla_M^{\alpha} f(x) = \int_{\|\Theta\|=1} {}^{\lambda}D_M^{\alpha} f(x)\, M(\mathrm{d}\Theta) \tag{18}$$

where,

$$^{\lambda}D_{\Theta}^{\alpha} f(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty}\left[g(t) - g(t-r)\right] e^{-\lambda r} r^{-\alpha-1}\mathrm{d}r \tag{19}$$

with $g(t) = f(x + t\Theta)$ is the generator form for vector tempered fractional derivative.

Inverse (17) to derive the master equation

$$\frac{\partial p(L,t)}{\partial t} = \frac{C}{b}{}^{\lambda}\nabla_M^{\mu-1} p(L,t), \tag{20}$$

where, L is a vector (x, y).

Remark 7: Since the first order and second order moments of tempered Lévy distribution exist, the resulting standard diffusion equation (15) makes sense. Interestingly, we borrow the idea from Meerschaert and Sikorskii [17] and give another expression of the master equation, where vector tempered fractional derivative is used. In this paper, we do not give detailed proof for the derivation of vector tempered fractional derivative and one can refer to Chapter 6 and 7 in the book of Meerschaert and Sikorskii [17]. All these indicate that tempered fractional diffusion equation is in fact a different expression of the standard diffusion.

5. CONCLUSION

In this paper, we consider the optimal random foraging whose flight distance is distributed according to a tempered Lévy distribution $p(l) \sim \mathrm{e}^{-\rho l} l^{-\mu}$. It is found that a higher searching efficiency can be derived when we choose a smaller ρ or μ, which results in a slower decaying speed. Furthermore, we obtain the master equation of the random foraging. A standard diffusion equation is derived since the first and second order moments of the distribution for flight distance exist. Using the definition of tempered fractional derivative, a vector tempered fractional diffusion equation is then derived, which can be viewed as a special expression for the standard diffusion. A promising research topic can be directed to finding the optimal searching strategy for other types of flight distance distributions.

AUTHOR CONTRIBUTIONS

YC mainly contributed to the theoretical analysis and accomplishing the paper. DH mainly contributed to the numerical simulation. YW and YC contributed for providing the idea of using tempered Lévy distribution in foraging and helped revising the paper.

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Fractional Dynamics of Individuals in Complex Networks

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The relation between the behavior of a single element and the global dynamics of its host network is an open problem in the science of complex networks. We demonstrate that for a dynamic network that belongs to the Ising universality class, this problem can be approached analytically through a subordination procedure. The analysis leads to a linear fractional differential equation of motion for the average trajectory of the individual, whose analytic solution for the probability of changing states is a Mittag-Leffler function. Consequently, the analysis provides a linear description of the average dynamics of an individual, without linearization of the complex network dynamics.

Keywords: fractional calculus, subordination, inverse power law, complex networks, control

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1. INTRODUCTION

The last decade has witnessed the blossoming of two quite different strategies for the mathematical modeling of the complex systems, which are network science [1–3] and fractional calculus [4–6]. The widespread adoption of the network science perspective to study phenomena such as epidemic spreading of diseases [7], neuronal avalanches [8], or social dynamics [9] derives from the fact that these systems are composites of many simpler, interconnected, and dynamically interacting elements. Similarly, popularization of fractional calculus in research that concerns physical processes that are characterized by long-term memory and spatial heterogeneity [10, 11] stems from its particular mathematical formulation, based on a definition of the nonlocal differentiation and integration operators. Therefore, since memory effects and heterogeneity are frequently observed in biological, social, and man-made systems [12, 13], the application of fractional calculus in the domain of complex networks is a natural step toward providing novel analytical tools that are capable of addressing research questions arising in the field.

Despite the simplicity of their basic building blocks, complex systems, such as cooperative animal behavior [14], the flow of highway traffic [15], or the cascades of load shedding on power grids [16], are characterized by rich self-emergent behavior. However, since in most cases, solving a system of coupled nonlinear equations that trace the dynamics of a network composed of N units is not possible, the primary focus of investigations into complex networks has been on their global behavior [17]. This approach follows the path taken by classical statistical physics, with Boltzmann's realization that the description of the state of a gas or a solid state could be only achieved on the scale of the entire system [18]. Analogously, the ability to characterize the global behavior of a complex network comes at a price of not being able to quantify the dynamics of the components that give rise to it. Typically, one attempts to infer the global dynamics by averaging the behavior of single elements within the system, following a bottom-up approach of the mean field theory (see **Figure 1**).

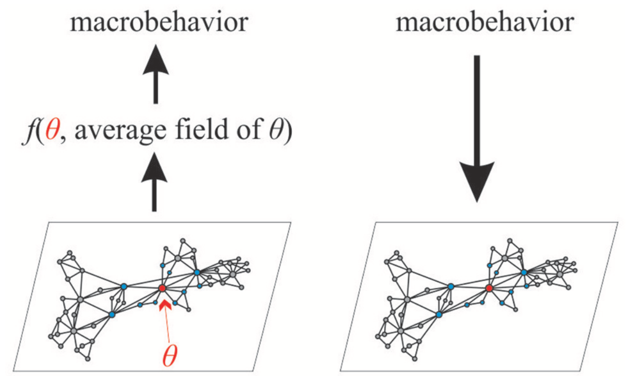

FIGURE 1 | Typical description of the dynamics arising from the interaction of numerous basic elements over a complex network that focuses on the global behavior of the system (*left*). Such an approach, however, comes with a price of not being able to quantify dynamics of individual elements within the system. In this paper, we address this problem by adopting statistical properties of the macroscopic dynamics in order to infer the behavior of individual units.

In this paper, we address this issue by posing the inverse question. Rather than inferring the global dynamics by combining the behavior of single elements within the dynamical system, we ask whether it is possible to construct a description of the dynamics of the individual elements, provided information about the network's global behavior. We approach the problem by considering statistical properties of the global variable.

Frequently, the macrovariables observed in complex networks display emergent properties of spatial and/or temporal scale-invariance. These are manifested by, for example, the inverse power scaling of waiting-time probability density functions (PDFs) between events, such as communication instances in human interactions or occurrence of earthquakes. At the same time, the inverse power laws (IPLs) that characterize the emergent macroscopic behavior are reminiscent of particle dynamics near a critical point, where a dynamic system undergoes a phase transition [19]. However, despite the advances made by the renormalization group approach and self-organized criticality theories that have shown how scale-free phenomena emerge at critical points, the issue of determining how the emergent properties influence the microdynamics of individual units of the system remains open.

Herein, we address the problem of quantifying the response of an individual unit to the dynamics of the collective. This is done by taking advantage of the fractional calculus apparatus, whose utility arises from its ability to seamlessly incorporate the IPL statistics into its dynamics. The phase transitions that characterize many complex systems suggest the wisdom of using a generic model from the Ising universality class to characterize system dynamics. It is then possible to demonstrate that the individual trajectory response to the collective dynamics of the system is described by a linear fractional differential equation. This is achieved through a subordination procedure without the necessity of linearizing the underlying dynamics. Following this procedure, it is shown that the analytic solution to the linear fractal differential equation retains the influence of the nonlinear network dynamics on the behavior of the individual. Moreover, the solution to the fractional equation of motion suggests a new direction for designing mechanisms to control the dynamics of complex networks.

In section 2, we sketch out the mathematics of the dynamical decision making model (DMM), introduce renewal events, and subordinate the behavior of the individual to the mean field behavior of the network. In section 2.2, the dynamics of the individual is determined from the subordination theory to be a tempered fractional differential equation. The exact solution to this equation is given by an attenuated Mittag-Leffler function, which is fitted to the numerical solution of the DMM equation. In section 4, we discuss some implications of the high quality convergence of the analytical and numerical results of this complex network.

2. COMPLEX NETWORK DYNAMICS

As demonstrated by Grinstein et al. [20], any discrete system, defined by means of local interactions, with symmetric transitions between states and randomness that originate from the presence of a thermal bath or internal causes belongs to the universality class of kinetic Ising models. One such system is the DMM [21–23] and is the one we implement herein. Each individual unit s_i of the model is a stochastic oscillator and can be found in either of the two states, $+1$ or -1. The dynamics are defined in terms of the probability of an individual to be in either state, and it is modeled by the coupled two-state master equation,

$$\frac{d\mathbf{p}(t)}{dt} = g_0\left[\mathbf{I} - 2\mathbf{1}\right]\mathbf{p}(t), \tag{1}$$

where $\mathbf{I}$ and $\mathbf{1}$ are the 2×2 identity and unit matrices, respectively. The probability of being in one of the two states $(+1, -1)$, $\mathbf{p}(t) = [p_1(t), p_2(t)]^{\intercal}$, defines a Markovian telegraph noise, with symmetric and constant rate of changing states $0 < g_0 < 1$.

Positioning N such individuals at the nodes of a complex network introduces coupling between them [21, 22], which, here, is limited to the nearest neighbor interactions. The influence that unit s_i experiences due to the presence of its neighbors is expressed by a modification of its transition rate

$$\begin{aligned} g_{12}^{(i)}(t) &= g_0 \exp\left[-\frac{K}{M^{(i)}}\left(M_1^{(i)}(t) - M_2^{(i)}(t)\right)\right], \\ g_{21}^{(i)}(t) &= g_0 \exp\left[\frac{K}{M^{(i)}}\left(M_1^{(i)}(t) - M_2^{(i)}(t)\right)\right], \end{aligned} \tag{2}$$

which becomes a time dependent variable. Here, K is the strength of the coupling between nodes, $0 < K < \infty$, constant for all nodes in the network. The variable $M^{(i)}$ denotes the degree of the node i, and $M_{1,2}^{(i)}(t)$ denotes, respectively, the count of the nearest neighbors in states $s_i(K, t) = 1$ and $s_i(K, t) = -1$ at time t. As single units s_i change their states, quantities $M_1^{(i)}(t)$ and $M_2^{(i)}(t)$ fluctuate in time, while their sum is always conserved $M_1^{(i)}(t) + M_2^{(i)}(t) = M^{(i)}$. In this paper, we consider the case of a regular two-dimensional lattice, where $M^{(i)} = 4$ and $0 \leq M_{1,2}^{(i)}(t)$ for all the nodes. The single unit in isolation corresponds to the case of $K = 0$. When the coupling constant $K > 0$, a unit in state $+1(-1)$ makes a transition to the state $-1(+1)$ faster or slower according to whether $M_2^{(i)}(t) > M_1^{(i)}(t)$ or $M_1^{(i)}(t) > M_2^{(i)}(t)$, respectively.

Time-dependent transition rates modify the two-state master (Equation 1) to take the form

$$\frac{d\mathbf{p}^{(i)}(t)}{dt} = \mathbf{G}_i(t)\mathbf{p}^{(i)}(t), \tag{3}$$

where the matrix of rates $\mathbf{G}_i(t)$ is defined as

$$\mathbf{G}_i(t) = \begin{pmatrix} -g_{12}^{(i)}(t) & g_{21}^{(i)}(t) \\ -g_{21}^{(i)}(t) & g_{12}^{(i)}(t) \end{pmatrix}, \tag{4}$$

and $\mathbf{p}^{(i)}(t)$ is the probability of the element $i = 1, 2, ..., N$ in the network at time t and is normalized such that $p_1^{(i)}(t) + p_2^{(i)}(t) = 1$ for every i.

Dynamics of an entire network is described by a system of N such coupled equations, resulting in a highly nonlinear system [23], containing $6N$ dynamic variables $\left(p_1^{(i)}(t), p_2^{(i)}(t), g_{12}^{(i)}(t), g_{21}^{(i)}(t), M_1^{(i)}(t), M_2^{(i)}(t)\right)$. This number of coupled variables prevents the successful application of analytic methods, as these are usually adopted to solve problems that involve only a few coupled time-dependent differential equations. Instead, extensive numerical calculations are supplemented by an analytic formulation of the evolution of a global variable.

As depicted in **Figure 2B**, the global behavior of the model, defined by the fluctuations of the mean field variable

$$\xi(K, t) = \frac{1}{N}\sum_{i=1}^{N} s_i(K, t), \tag{5}$$

shows a pronounced transition as a function of the control parameter K. While in **Figure 2A**, the single elements appear to be essentially unchanged by their interactions with the rest of the network, the global variable shifts from a configuration dominated by randomness to one in which strong interactions give rise to long-lasting majority states shown in **Figure 2B**. Note that the origin of the random fluctuation in the DMM is the finite size of the network, which has nothing to do with the thermal fluctuations in the Ising model of magnetization.

To characterize the changes in the temporal properties of the micro- and macro-variables, we evaluate the survival probability function, $\Psi(\tau)$, of time intervals τ between consecutive events defined as changes of the state or crossing of the zero-axis, for the single element or the global variable, respectively. These calculations unveil modest deviations of $\Psi(\tau)$ for a single individual from the exponential form, $\Psi(\tau) = exp(-g_0\tau)$, that characterizes single non-interacting elements, as shown in **Figure 2C**. Clearly, the influence of the network on the behavior of the individual does not appear to induce a significant change in the latter. Despite such a modest change in the behavior of the individual, the global variable manifests IPL statistics, as depicted in **Figure 2D**. Thus, the following question arises: To what extent are individual opinions within a complex network influenced by the network dynamics?

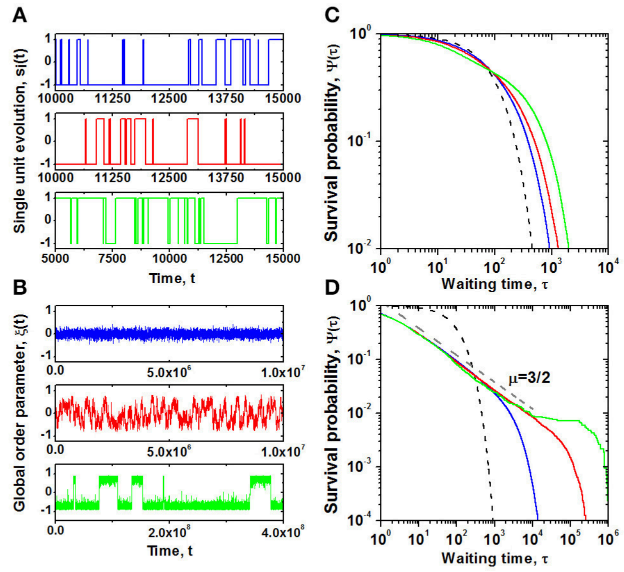

FIGURE 2 | Behavior of a discrete, two-state dynamic unit on a two-dimensional lattice. Temporal evolution and corresponding survival probability $\Psi(\tau)$ for the transitions between two states for the single unit $s_i(t)$ of the system, presented on panels **(A,C)**, respectively, are compared with the behavior and statistical properties of the global order parameter $\xi(t)$, showed on panels **(B,D)**. Simulations were performed on a lattice of size $N = 50 \times 50$ nodes, with periodic boundary conditions, for $g_0 = 0.01$ and increasing values of the control parameter K. Blue, red, and green lines correspond to $K = 1.50$, 1.70, and 1.90, respectively. The critical value of the control parameter is $K_C \approx 1.72$. Black dashed line on the plots of $\Psi(\tau)$ denotes an exponential distribution, with the decay rate g_0.

2.1. Renewal Events

Many physical processes, for example earthquakes, radioactive decay, and social processes, such as making a decision, can be viewed as particular events. A characteristic property of an event is that it's onset can be precisely localized in time, even if its occurrence has extended consequences in space. Thus, the dynamics of a process characterized by events is described in terms of the probability of an event occurring, rather than by a more traditional Hamiltonian approach.

The process of event occurrence is characterized by the waiting-time PDF $\psi(\tau)$, which specifies the distribution of times between consecutive events. The probability for an event to occur in the short time interval $[t, t + dt]$ is given by

$$\psi(t)\,dt = \Pr(t < \tau < t + dt), \tag{6}$$

where τ is measured from the occurrence of the previous event. Consequently, one can define the survival probability $\Psi(\tau)$ as the probability that no event occurs up to the time since the last event as

$$\Psi(\tau) \equiv \int_{\tau}^{\infty} \psi(t)\, dt. \quad (7)$$

As a consequence of this integral, the waiting-time PDF can be written as

$$\psi(t) = -\frac{d\Psi(t)}{dt}, \quad (8)$$

and the PDF $\psi(\tau)$ is a properly normalized function,

$$\int_{0}^{\infty} \psi(\tau)\, d\tau = 1,$$

since it is assumed that an event occurs somewhere within the time interval $(0, \infty)$. It is also true that no event occurs at time $t = 0$, which means that the survival probability $\Psi(0) = 1$.

A particular class of events can be defined, *renewal events*, that reset the clock of the system to an initial state instantaneously after their occurrence. After a renewal event takes place, the system evolves in time independently of whatever occurred earlier, having no memory of previous instances in which such an event occurred. Some examples of renewal events found in physics include anomalous diffusion of tagged particles inside living cells, blinking quantum dots, and defects arising in the weak turbulence regime of liquid crystals.

The renewal character of events is captured by the probability of n events occurring as follows. First, one assumes that an event occurs at time t = 0, thus, $\psi_0(t) = \delta(t)$. Next, the first event occurs at time $t > 0$, taking place with the probability $\psi_1(t) = \psi(t)$. Subsequently, the probability for event n in a sequence to occur at time t is expressed in terms of probabilities of earlier events by the correlation chain condition

$$\psi_n(t) = \int_{0}^{t} \psi_{n-1}(t')\, \psi_1(t - t')\, dt'. \quad (9)$$

Frequently, experimentally observed waiting-time PDFs are exponential, but quite often in complex networks they are IPLs. For the purpose of this paper, we define the waiting-time PDF in terms of the hyperbolic distribution

$$\psi(t) = \frac{(\mu - 1)\, T^{\mu-1}}{(T + t)^{\mu}}. \quad (10)$$

If the events are generated by an ergodic process, then $\mu > 2$, and the first moment of the hyperbolic PDF is

$$\langle t \rangle = \int_{0}^{\infty} t\psi(t)\, dt = \frac{T}{\mu - 2}. \quad (11)$$

In the framework of renewal theory, Equation (11) denotes the average time that one would have to wait between successive events. However, when $\mu < 2$, the process is non-ergodic, and the mean value of the distribution diverges. In the non-ergodic case, T becomes a characteristic time scale of the process.

2.2. Subordination of Time

The notion of different clocks associated with different physical systems arises naturally in physics; the linear Lorentz transformation in relativistic physics being probably the most familiar example. Thanks to the recent availability of time-resolved data, biological, and social sciences have also started adopting the notion of multiple clocks, distinguishing between cell-specific and organ-specific clocks in biology and person-specific and group-specific clocks in sociology. Of course, the notion of subjective and objective time dates back to the middle of the nineteenth century with the introduction of the empirical Weber-Fechner law [24].

However, the striking difference between the clocks of classical physics and natural sciences is that the relations between the latter clocks are nonlinear. While the global activity of an organ, such as the brain or the heart, might be characterized by quite regular, often periodic fluctuations, the activity of single neurons demonstrates burstiness and noisiness. Similarly, in a society, people operate according to their individual schedules, not always being able to perform particular actions in the same global time frame. Thus, owing to the stochastic behavior of one or both clocks, a probabilistic transformation between times is necessary. An example of such a transformation is the subordination procedure.

We begin by defining two clocks. The first clock records a discrete operational time n, which measures the time $T(n)$ of an individual. The second clock records the continuous chronological time t, which measures the time $T(t)$ that a system of individuals have agreed upon. If each advancement of the discrete clock n is thought of as an event, then the relation between the operational time and chronological time can be given by the waiting-time PDF of those events in chronological time $\psi(t)$. Assuming a renewal property for events, as given by chain condition from renewal theory (Equation 9), one can relate operational time to chronological time by

$$\langle T(t) \rangle = \sum_{n=1}^{\infty} \int_{0}^{t} \Psi(t - t')\, \psi_n(t')\, T(n)\, dt'. \quad (12)$$

Every advancement of the operational clock is an event, which in the chronological time occurs at time intervals drawn from the renewal waiting-time PDF. Because of this randomness, one needs to sum over all events, and the result is an average over many realizations of the transformation.

As an example, consider the behavior of a two-state operational clock, whose evolution is shown in **Figure 3**. In operational time, the clock switches back and forth between its two states at equal unit time intervals. In chronological time, however, this regular behavior is significantly distorted. In the figure, the time transformation was taken to be an IPL PDF of waiting times. Thus, a single time step in the operational time corresponds to a time interval being a random number drawn from $\psi(t)$ in chronological time. The long tail of the IPL PDF leads to especially strong distortions of the operational time trajectory, since there exist a non-zero probability of drawing very large time intervals between events.

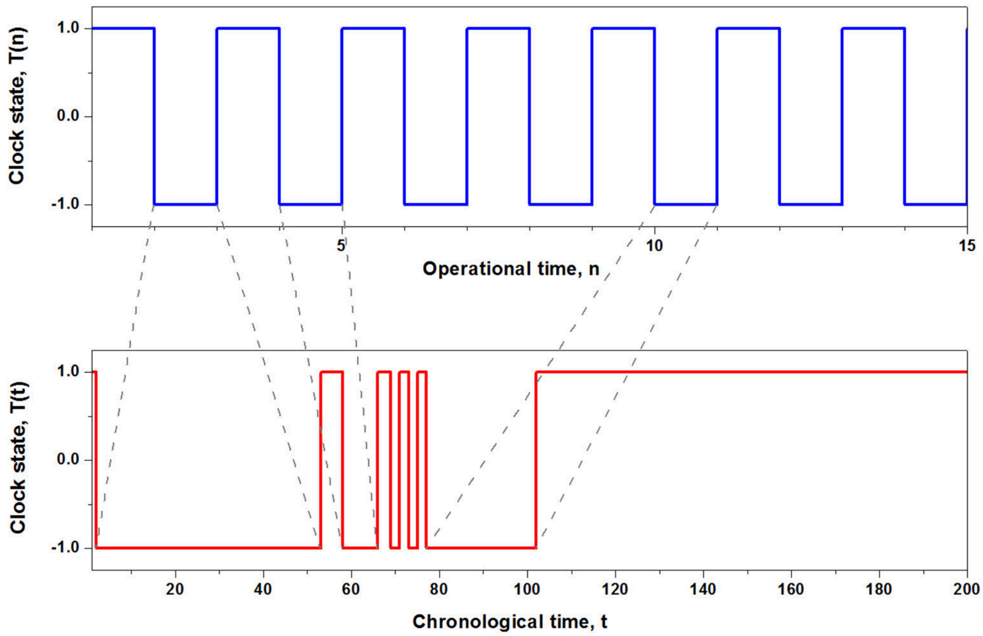

FIGURE 3 | The upper curve is the regular transition between the two states of the individual in operational time. The lower curve is the subordination of the transition times to an IPL PDF to obtain chronological time.

However, since the transformation between the operational and chronological time scales involves a random process, one needs to consider infinitely many trajectories in the chronological time, which leads to the average behavior of the clock in the chronological time denoted in Equation (12) by the bracket.

We note that the time subordination procedure can also be used to model communication delays in the system. However, contrary to frequently used approaches, where individual units of the system are subordinated to model the interaction delay, here, we adopt the statistics of the macroscopic variable to derive the behavior of the interacting individual units. The coupling between units causes them to deviate from the Poisson behavior of an individual non-interacting unit. However, as illustrated in **Figure 2**, the time scale of interacting units is orders of magnitude that are smaller than the time scale of the macroscopic variable. Thus, we use the statistical properties of the macroscopic variable to provide a first-order estimate of the single unit dynamics. As such, we adopt a top-down approach, which is different from the bottom-up approach adopted for the consideration of communication delays.

3. COMPLEX NETWORK SUBORDINATION

To determine the network's influence on the dynamics of the individual, we adapt the subordination argument of the preceding section and relate the time scale of the macro-variable $\xi(K,t)$ to the time scale of the micro-variable $s_i(K,t)$. The two-state master equation for a single isolated individual in discrete time n in steps of $\Delta\tau$ is

$$\varphi(n+1) - \varphi(n) = -g_0 \Delta\tau \varphi(n), \tag{13}$$

where the notations $\varphi(n) = \varphi(n\Delta\tau)$ and $\varphi = p_1 - p_2$ depict the difference in probabilities for the typical individual to assume one of the two states. The solution to this discrete equation is

$$\varphi(n) = (1 - g_0\Delta\tau)^n \varphi(0), \tag{14}$$

which, in the limit $g_0\Delta\tau << 1$, becomes an exponential. However, when the individual is a part of a network, the dynamics are not so simple.

Adopting the subordination interpretation, we define the discrete index n as an individual's operational time that is stochastically connected to the chronological time t, in which the global behavior is observed. We assume that the chronological time lies in the interval $(n-1)\Delta\tau \leq t \leq n\Delta\tau$ and, consequently, the equation for the average dynamics of the individual probability difference is given by [25]

$$\langle\varphi(t)\rangle = \sum_{n=1}^{\infty}\int_0^t \Psi\left(t-t'\right)\psi_n\left(t'\right)\varphi(n)\,dt'. \tag{15}$$

Here, the time t in the waiting-time PDF $\psi(t)$ is determined from the derivative of the survival probability. The empirically determined analytic expression for the survival probability is

$$\Psi(t) = \left(\frac{T}{T+t}\right)^{\mu-1} e^{-\epsilon t}. \tag{16}$$

The dominant behavior of the empirical survival probability is an IPL as indicated in **Figure 2D**. However, at early times, the probability of not making a transition approaches the constant value of one; at late times, the probability of not making a transition at a given time decays exponentially; it is in the middle range, where the probability is an IPL. The extent of the IPL range of the survival probability is determined by the empirical values of T, μ, and ϵ, and from **Figure 2D**, the value of ϵ is seen to become smaller as the control parameter K increases. The IPL functional form of the PDF results from the behavior of the survival probability $\Psi(\tau)$ of the global variable depicted in **Figure 2D**, with $\mu = 3/2$.

Using a renewal theory argument, Pramulkkul et al. [25] show that Equation (15) expressed in terms of Laplace transform variables indicated by $\widehat{f}(u)$ for the time-dependent function $f(t)$ has the form

$$\langle \widehat{\varphi}(s) \rangle = \frac{\varphi(0)}{u + \epsilon + \lambda_0 \widehat{\Phi}(u + \epsilon)} \tag{17}$$

where $\lambda_0 \equiv g_0 \Delta\tau$ and $\widehat{\Phi}(u+\epsilon)$ is the Laplace transform of the Montroll-Weiss memory kernel [25, 26],

$$\widehat{\Phi}(u+\epsilon) = \frac{(u+\epsilon)\,\widehat{\psi}(u+\epsilon)}{1 - \widehat{\psi}(u+\epsilon)}. \tag{18}$$

Note that u is replaced by $u + \epsilon$ in the Laplace transforms, because the exponential truncation of the empirical survival probability shifts the index on the Laplace transform operation. The asymptotic behavior of an individual in time is determined by considering the waiting-time PDF as $u \to 0$,

$$\widehat{\psi}(u+\epsilon) \approx 1 - \Gamma(1-\alpha)\, T^{\alpha} (u+\epsilon)^{\alpha} \quad ; \; 0 < \alpha = \mu - 1 < 1, \tag{19}$$

so that Equation (17) reduces to

$$\langle \widehat{\varphi}(u) \rangle = \frac{\varphi(0)}{u + \epsilon + \lambda^{\alpha}(u+\epsilon)^{1-\alpha}}. \tag{20}$$

The inverse Laplace transform of Equation (20) yields the tempered rate equation

$$(\partial_t + \epsilon)^{\alpha} \langle \varphi(t) \rangle = -\lambda^{\alpha} \langle \varphi(t) \rangle, \tag{21}$$

where the operator $\partial_t^{\mu-1}[\cdot]$ is the Caputo fractional derivative for $0 < \alpha = \mu - 1 < 1$ [11] and

$$\lambda T = \left[g_0 \Delta\tau / \Gamma(2-\mu) \right]^{\frac{1}{\mu-1}}. \tag{22}$$

Note that owing to the dichotomous nature of the states, $\langle \varphi(t) \rangle$ is the average opinion of the individual $s_i(K,t)$.

The solution of the asymptotic fractional master equation (Equation 21) for a randomly chosen unit within the network is given by an exponentially attenuated Mittag-Leffler function (MLF):

$$\langle \varphi(t)) \rangle = \varphi(0)) E_{\alpha}\left(-(\lambda t)^{\alpha}\right) \exp[-\epsilon t] \tag{23}$$

and the MLF is defined by the series

$$E_{\alpha}(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}. \tag{24}$$

The MLF is a stretched exponential at early times and an IPL at late times, with $\alpha = \mu - 1$ being the IPL index in both domains.

3.1. Comparisons With Numerics

We test the above analysis with numerical simulations of the dynamic network on a two-dimensional lattice with nearest-neighbor interactions in all three regions of DMM dynamics: subcritical, critical, and supercritical. The time-dependent average opinion of a randomly chosen individual is presented in **Figure 4**, where the average is taken over 10^4 independent realizations of the dynamics in the subcritical, critical, and supercritical regimes.

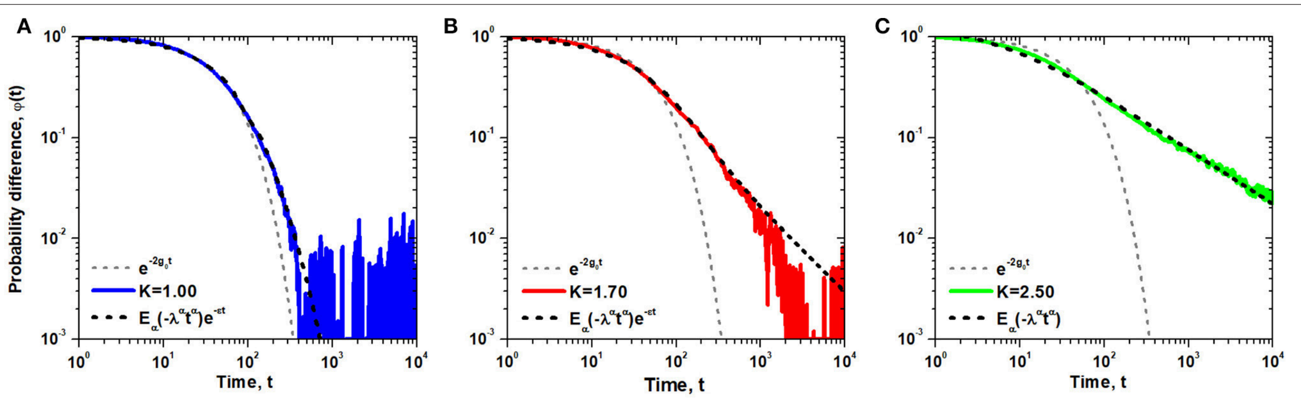

FIGURE 4 | The probability difference $\langle \varphi(t) \rangle$ estimated as an average over an ensemble of 10^4 independent realizations of single element trajectories. Each trajectory corresponds to evolution of a randomly selected node within a $N = 100 \times 100$ lattice network, with $g_0 = 0.01$ and the same initial condition $s_i(0) = 1$. The parameter values for the numerical data are given in **Figure 2** and from left to right $K = 1.0$ **(A)**, 1.7 **(B)**, 2.5 **(C)**, respectively. The fit of the exponentially truncated MLF to the numerical calculations is summarized in **Table 1**.

TABLE 1 | The probability difference $\langle\varphi(t)\rangle$ of **Figure 4** is fitted with the MLF using an algorithm developed by Podlubny [27].

	$K = 1.00$	$K = 1.70$	$K = 2.50$
μ	1.8920	1.8050	1.5580
λ	0.0147	0.0206	0.0293
ϵ	4.00×10^{-3}	1.40×10^{-11}	5.58×10^{-12}
R^2[1]	0.9910	0.9667	0.9725

Assuming $T = 0.10$, $\Delta\tau = 1$, and $g_0 = 0.01$, the parameters of an analytical solution are $\mu = 3/2$ and $\lambda = 0.0318$.

A comparison with the exponential form of $\langle\varphi(t)\rangle$ for an isolated individual indicates that the influence of the network on the individual's dynamics clearly persists for increasingly longer times with increasing values of the control parameter within the network. The parameters μ and λ of Equation (23) obtained through fitting numerical results of **Figure 4** with the MLF are summarized in **Table 1**. It is evident that the influence of the network dynamics on the individual is greatest at long times. The deviation of the analytic solution from the numerical calculation is evident for values of the control parameter at and below the critical value. The analytical prediction is least reliable at extremely long times in the subcritical domain. Consequently, the response of the individual to the group mimics the group's behavior most closely when the control parameter is equal to or greater than the critical value.

4. DISCUSSION

Herein, the subordination procedure provides an equivalent description of the average dynamics of a single individual within a complex network, in terms of a linear fractional differential equation. The fractional rate equation is solved exactly, determining the Poisson statistics of the isolated individual becomes attenuated Mittag-Leffler statistics, owing to the interaction of that individual with the other members of a complex dynamic network.

Consequently, an individual's simple random behavior, when isolated, is replaced with behavior that might serve a more adaptive role in social networks. We conjecture that the behavior of the individual is generic, given that the DMM network dynamics belong to the Ising universality class. Members of this universality class share the critical temporal behavior [28] that drives the subordination process. It is the renewal property of the event statistics, which, through the subordination process, gives rise to the linear fractional master equation for the typical individual's dynamics. The solution to the tempered fractional rate equation manifests the subsequent robust behavior of the individual; it remains to be determined just how robust the behavior of the individual is relative to control signals that might be driving the network.

As pointed out by Liu et al. [29], the ultimate understanding of complex networks is reflected in the ability to control them. Recent observations of the interconnectedness of infrastructure networks [30], facilitating the spread of failures [31] or the tight coupling between banking institutions, posing a danger to the stability of global financial system [32], demonstrate the importance of developing a systematic approach to influence and/or control the complex networks. The analysis presented here provides an alternative attempt to address this need directly. Subordination suggests a way to impose the conditions of traditional control theory [33] onto the complex network dynamics by, first, expressing the underlying nonlinear network dynamics in the form of a linear fractional equation of motion. This approach at addressing control will be pursued in a future publication.

AUTHOR CONTRIBUTIONS

BW developed the theoretical formalism and performed the analytic calculations. MT performed numerical simulations. Both authors discussed the results and contributed to the final manuscript.

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[1] Adjusted goodness of fit, $R^2 = 1 - \frac{SS_{reg}/(n-\kappa)}{SS_{tot}/(n-1)}$, is defined as the ratio of the sum of squared residues for the nonlinear fit with the MLF (SS_{reg}) and for the fit to the average value of data points (SS_{tot}), where n is the number of data points and κ is the number of free parameters being estimated.

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Sinc-Fractional Operator on Shannon Wavelet Space

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In this paper the sinc-fractional derivative is extended to the Hilbert space based on Shannon wavelets. Some new fractional operators based on wavelets are defined. One of the main task is to investigate the localization and compression properties of wavelets when dealing with the non-integer order of a differential operator.

Keywords: fractional calculus, shannon wavelet, sinc-function, operational matrix, connection coefficients

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1. INTRODUCTION

In recent years, fractional calculus has been growing fast both in theory and applications to many different fields. Several classical and fundamental problems have been revised by using fractional methods, thus showing unexpected new results [1–3], while more and more new problems were shaped to fit the theoretical models of fractional calculus [4–7].

In fractional calculus is based on two universally accepted principles: the first one is that the definition of fractional derivative is not unique, thus giving raise to a neverending controversial debate on the best fractional operator. The second principle is that, although the missing uniqueness of the fractional operator, fractional calculus is an essential tool for a deeper and more comprehensive investigation of complex , non-linear , local, or non-local problems.

Therefore according to the suitable choice of the fractional differential operator, there follows a corresponding model of analysis so that the physical model and the corresponding physical interpretation of the results it strongly depends on the chosen fractional operator.

In some recent papers [8–15] the classical Lie symmetry analysis has been combined with the Riemman-Liouville fractional derivative to solve time fractional partial differential equations. In these papers, Lie point symmetries have been used to convert a fractional partial differential equation into a non-linear ordinary differential equation, that can be solved by suitable methods. Some fractional operators have been used also to study non-differentiable functions [see e.g., [16] some of them are more suitable for the analysis of non-differentiable sets, or fractal sets like the Cantor fractal set [4–7] Some fractional operators have been specially defined to analyze complex functions [17–19]. For instance the chaotic decay to zero of the complex ζ-Riemann function was easily shown by using a suitable fractional derivative [19].

Among the many interesting definitions of fractional operators, some Authors have recenlty proposed a fractional differential operator based on the sinc-function [20]. This function is very popular in the signal analysis, also because it is a localized function with slow decay. Moreover, it is the fundamental basic function for the definition of the so-called Shannon wavelet theory, i.e., the multiscale analysis on Shannon wavelets [21–26].

This paper will focus on the definition of a fractional derivative by the Shannon wavelets. These functions belong to a special family of wavelets which have a sharp compact support in the frequency space, so that their Fourier transform are box-functions in frequencies. This is a great advantage because, the frequency domain of a signal can be easily decomposed in terms of scaled box-functions.

Wavelet theory has been growing very fast so that there has been also a wide spreading of wavelets for the solution of theoretical and applied problems. However, alike the various definition of fractional operators there exist also many different families of wavelets and this missing uniqueness it might be considered as a drawback because of the arbitrary choice. Nevertheless all families of wavelets enjoy two fundamental properties their localization in time (or frequency) and the multiscale decomposition. Due to their localization they can be used to detect, and single out, localized singularities and/or peaks, while the multiscale property enable to decompose the approximation space into separate scales [27]. Thanks to these properties wavelets have been used to solve non-linear problems and moreover they are the most suitable tool for the analysis of multiscale problems.

The sinc-fractional operator will be generalized in order to compute the fractional derivative of the $L_2(\mathbb{R})$-functions belonging to the Hilbert space defined by the Shannon wavelet. In doing so, we will be able to compute the fractional derivative of these functions by knowing only their wavelet coefficients. Moreover, with this approach we will be able to decompose the fractional derivative at different scales, thus showing the influence of a given scale in multiscale physical problems.

The organization of this paper is as follows: Preliminary remarks on fractional operators are given in section 2. In section 3 the sinc fractional derivative, as given by Yang et al. [20] is described. Section 4 gives the basic properties on the multiscale approximation defined on Shannon wavelet. The differential properties of the functions belonging to the Hilbert space based on Shannon wavelet are given in section 5, together with the explicit form of the integer order derivatives (see also [24, 26]). Section 6 deals with the sinc-fractional derivative on the Hilbert space based on Shannon wavelets , i.e., sinc-fractional derivative of functions which can be represented as Shannon wavelet series.

2. PRELIMINARY REMARKS

In this section some of the most popular definition of fractional derivatives [28–30] are given.

Let us start with the Riemann-Liouville derivative.

Definition 1. The Riemann-Liouville integral of fractional order $\nu \geq 0$ of a function $f(x)$, is defined as

$$\left(J^{\nu} f\right)(t) = \begin{cases} \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau, \nu > 0, \\ \\ f(t), \quad \nu = 0. \end{cases}$$

The Riemann-Liouville fractional operator J^{α} has the following properties:

$$\begin{aligned}
&(a)\ J^{\alpha}\left(J^{\beta} f(t)\right) && = J^{\beta}\left(J^{\alpha} f(t)\right), \\
&(b)\ J^{\alpha}\left(J^{\beta} f(t)\right) && = J^{\alpha+\beta} f(t), \\
&(c)\ J^{\alpha} t^{\nu} && = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)} t^{\nu+\alpha}, \qquad \alpha, \beta \geq 0, \nu > -1 \\
&(d)\ J^{\nu} e^{\lambda t} && = \frac{1}{\nu \Gamma(\nu)} e^{\lambda t} t^{\nu}, \ \ \nu > 0, \\
&(e)\ J^{\nu} c && = \frac{c}{\nu \Gamma(\nu)} t^{\nu}, \ \ \nu > 0.
\end{aligned}$$

From this definition there follows the corresponding derivative according to the following:

Definition 2. Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined as

$$D^{\alpha}_{RL} f(t) = \frac{d^n}{dt^n} J^{n-\alpha} f(t), \qquad n \in \mathbb{N}, \ \ n-1 < \alpha \leq n. \tag{2.1}$$

The main problem with this derivative is the unvanishing value for a constant function, therefore it was proposed by Caputo the following [28, 29].

Let $f(x) \in \mathcal{C}^n$ be a n-differentiable function, α a positive value, then

Definition 3. The α-order Caputo fractional derivative is defined as

$$D^{\alpha}_{C} f(x) = \begin{cases} \dfrac{d^n f(x)}{dx^n}, & 0 < \alpha \in \mathbb{N}, \\ \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \ t > 0, & 0 \leq n-1 \\ & < \alpha < n. \end{cases}$$

where n is an integer, $x > 0$, and $f \in \mathcal{C}^n$.

It can be easily shown that:

$$\begin{aligned}
&(a)\ J^{\alpha} D^{\alpha}_{C} f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad t > 0. \\
&(b)\ D^{\alpha}_{C} J^{\alpha} f(x) = f(x). \\
&(c)\ D^{\alpha}_{C} t^n = \begin{cases} 0, & \textit{for } n \in \mathbb{N}_0 \textit{ and } \alpha < n, \\ \dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha}, & \textit{otherwise}. \end{cases} \\
&(d)\ D^{\alpha}_{C} D^{\beta}_{C} f(x) = D^{\beta}_{C} D^{\alpha}_{C} f(x).
\end{aligned}$$

3. SINC-FRACTIONAL DERIVATIVE

Riemann-Liouville (RL) and Caputo (C) derivatives are the most popular derivatives and have been used in many applications (see e.g., [2, 3, 16, 18, 25, 26, 29, 31–41]), nevertheless they both suffer for some unavoidable drawbacks. In particular, the RL-derivative is unvanishing when $f(x) \neq constant$ while the C-derivative is defined on a singular kernel. Because of that, in recent years many efforts were devoted to find some more flexible non-singular derivatives. Moreover, due to the fact that the fractional derivative is not univocally defined, there have been proposed many alternative interesting new definitions.

Indeed the more general fractional derivative with a given kernel $K(x,\alpha)$, which generalizes the C-derivative is:

$$D^{\alpha}f(x) = \begin{cases} \dfrac{d^n f(x)}{dx^n}, & 0 < \alpha \in \mathbb{N}, \\ \displaystyle\int_0^x f^{(n)}(\tau)K(x-\tau,\alpha)d\tau, & x > 0, \quad 0 \leq n-1 < \alpha < n. \end{cases} \tag{3.1}$$

The kernel should be defined in a such a way that at least the two conditions

$$\lim_{\alpha\to 0} K(x-\tau,\alpha) = 1, \qquad \lim_{\alpha\to 1} K(x-\tau,\alpha) = \delta(x-\tau) \tag{3.2}$$

hold true, moreover, in order to be a non-singular kernel, it should be also

$$\lim_{x\to\tau} K(x-\tau,\alpha) \neq 0, \quad \forall\alpha\ . \tag{3.3}$$

Although there are several definitions of derivatives they all depend on a kernel. In particular, it can be easily seen that the C-derivative [42], the Caputo-Fabrizio (CF) derivative [34], and the Atangana-Baleanu (AB) derivative [43] are some special cases of (3.1) corresponding respectively to the kernels:

$$\begin{aligned} (C) \qquad K(x-\tau,\alpha) &= \frac{1}{\Gamma(n-\alpha)}(x-\tau)^{n-\alpha-1} \\ (CF) \qquad K(x-\tau,\alpha) &= \frac{M(\alpha)}{1-\alpha}e^{-\frac{\alpha}{1-\alpha}(x-\tau)} \\ (AB) \qquad K(x-\tau,\alpha) &= \frac{B(\alpha)}{1-\alpha}E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(x-\tau)\right), \end{aligned} \tag{3.4}$$

where the Mittag-Leffler function is taken as

$$E_{\alpha}(x) \overset{\text{def}}{=} \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(\alpha k+1)}.$$

It can be easily shown that all kernels (3.4) fulfill (3.2) while only (CF) and (AB) fulfill also the condition (3.3).

3.1. The Yang-Gao-Terneiro Machado-Baleanu Fractional Derivative [20]

With respect to the integration variable τ all kernels (3.4) have a decay to zero, in a such way that for a bounded $f^{(n)}(x)$ the integral $(3.1)_2$ converges.

Among the non-singular kernels with decay to zero a fractional derivative based on a sinc-function kernel was recently defined by Yang, Gao, Terneiro Machado, and Baleanu (YGTMB) [20].

The sinc-function, defined as Yang et al.[20]

$$\operatorname{sinc} x \overset{\text{def}}{=} \frac{\sin \pi x}{\pi x}, \tag{3.5}$$

owns a quite large amount of nice properties, so that it became a fundamental tools in applied science and signal analysis. In particular, it was shown (see e.g., [20]) that, for a given x

$$\lim_{\alpha\to 0} \frac{1}{\alpha}\operatorname{sinc}\left(\frac{x}{\alpha}\right) = \delta(x) \tag{3.6}$$

being $\delta(x)$ the Dirac-delta function

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}\ .$$

More in general from (3.6) it is

$$\lim_{\alpha\to 0} \frac{1}{\alpha}\operatorname{sinc}\left(\frac{x-\tau}{\alpha}\right) = \delta(x-\tau)\ . \tag{3.7}$$

By using the sinc-function, we have the following definition of the sinc fractional derivative [20].

Definition 4 (Yang-Gao-Tenreiro Machado-Baleanu). The YGTMB fractional derivative is defined as Yang et al. [20]

$$D^{\alpha}_{YGTMB}f(x) \overset{\text{def}}{=} \frac{\alpha P(\alpha)}{1-\alpha}\int_a^x \operatorname{sinc}\frac{\alpha(x-\tau)}{1-\alpha}f^{(n)}(x)d\tau, 0 \leq n-1 < \alpha < n\ . \tag{3.8}$$

We can see that also this kernel

$$(S) \qquad K(x-\tau,\alpha) = \frac{\alpha P(\alpha)}{1-\alpha}\operatorname{sinc}\frac{\alpha(x-\tau)}{1-\alpha} \tag{3.9}$$

belongs to the class of kernels (3.1). It can be also shown that this kernel fulfills the conditions (3.2),(3.3) (see [20]) being

$$\lim_{\alpha\to 0} \frac{\alpha P(\alpha)}{1-\alpha}\operatorname{sinc}\frac{\alpha(x-\tau)}{1-\alpha} = 1, \quad \lim_{\alpha\to 1} \frac{\alpha P(\alpha)}{1-\alpha}\operatorname{sinc}\frac{\alpha(x-\tau)}{1-\alpha} = \delta(x-\tau)\,,$$

and the normalization constant factor $P(\alpha)$ is such that

$$\lim_{\alpha\to 0}[K(x,\alpha)P(\alpha)] = \lim_{\alpha\to 1}[K(x,\alpha)P(\alpha)] = 1$$

In particular, there follows from (3.8)

$$D^{\alpha}_{YGTMB} f^{(n)}(x) = \begin{cases} f^{(n-1)}(x) - f^{(n-1)}(0), & \alpha = 0 \\ f^{(n)}(x) & , \quad \alpha = 1 \\ & (0 \le n-1 < \alpha < n). \end{cases}$$

3.1.1. Polynomial Approximation of the Kernel

The sinc kernel (3.9) can be written also as an infinite product as follows. Starting from the known product:

$$\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2 \pi^2}\right)$$

by taking into account (3.5) it is

$$\operatorname{sinc} x = \frac{\sin(\pi x)}{\pi x} = \frac{1}{\cancel{\pi x}} \cancel{(\pi x)} \prod_{k=1}^{\infty} \left(1 - \frac{(\pi x)^2}{k^2 \pi^2}\right)$$

so that

$$\operatorname{sinc} x = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right)$$

It should be noticed that in the interval $[-1, 1]$ the sinc-function can be approximated by

$$\operatorname{sinc} x \cong \prod_{k=1}^{n} \left(1 - \frac{x^2}{k^2}\right)$$

so that if we define as the error of approximation

$$\varepsilon(n) = \left| \operatorname{sinc} x - \prod_{k=1}^{n} \left(1 - \frac{x^2}{k^2}\right) \right|$$

we have

$\max \varepsilon(1) \le 0.14$, $\max \varepsilon(2) \le 0.08$, $\max \varepsilon(3) \le 0.055$, $\max \varepsilon(4) \le 0.04$,
so that already with $n = 1$:

$$\operatorname{sinc} x \cong \left(1 - x^2\right)$$

the error of approximation in $[-1, 1]$ is less that 15%.

It should be noticed that with this approximation the YGTMB-derivative (3.8) becomes

$$D^{\alpha}_{YGTMB*} f(x) = \frac{\alpha P(\alpha)}{1-\alpha} \int_a^x \left[1 - \frac{\alpha^2}{(1-\alpha)^2}(x-\tau)^2\right] f^{(n)}(x) d\tau, \\ \le n-1 < \alpha < n$$

that is

$$D^{\alpha}_{YGTMB*} f(x) = \frac{\alpha P(\alpha)}{1-\alpha} \left[f^{(n-1)}(x) - f^{(n-1)}(0)\right] - \frac{\alpha^3 P(\alpha)}{(1-\alpha)^3} \\ \int_a^x (x-\tau)^2 f^{(n)}(x) d\tau, \quad 0 \le n-1 < \alpha < n$$

By assuming as a normalization factor

$$P(\alpha) = \frac{(1-\alpha)^3}{\alpha}$$

we get

$$D^{\alpha}_{YGTMB*} f(x) = (1-\alpha)^2 \left[f^{(n-1)}(x) - f^{(n-1)}(0)\right] - \alpha^3 \\ \int_a^x (x-\tau)^2 f^{(n)}(x) d\tau, \quad 0 \le n-1 < \alpha < n$$

so that the fractional derivative can be seen as the interpolation between the function and its derivative (as shown e.g., in Cattani [25, 26]).

3.2. Sinc Fractional Derivative With Unbounded Domain

Let us consider the integral of sinc function over the unbounded domain $[-\infty, \infty]$. By a direct computation it can be shown that

$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} dx = 1, \quad \int_{-1}^{1} \frac{\sin \pi x}{\pi x} dx \cong 1.17$$

so that the sinc-function is a function mainly localized around the origin. In fact, the sinc function is known as a function with a decay to zero, therefore we can extend the definition (3.8) over the unbounded domain $\mathbb{R}$ so that we can define the sinc fractional derivative as the YGTMB fractional derivative on the unbounded domain $\mathbb{R}$, that is

Definition 5 (sinc fractional derivative). The sinc fractional derivative D^{α}_{S} of a function $f(x)$ is defined as

$$D^{\alpha}_{S} f(x) \overset{\text{def}}{=} \frac{\alpha P(\alpha)}{1-\alpha} \int_{-\infty}^{\infty} \operatorname{sinc} \frac{\alpha(x-\tau)}{1-\alpha} f^{(n)}(x) d\tau, \; 0 \le n-1 < \alpha < n \tag{3.10}$$

where the normalization factor $P(\alpha)$ is chosen to fulfill conditions (3.2), (3.3) and the kernel is

$$K(x-\tau, \alpha) = \frac{\alpha P(\alpha)}{1-\alpha} \operatorname{sinc} \frac{\alpha(x-\tau)}{1-\alpha} .$$

In particular, we can also assume

$$\frac{\alpha}{1-\alpha} = 2^{\beta}$$

so that

$$\alpha = \frac{2^{\beta}}{1-2^{\beta}}, \quad \beta = \log_2 \frac{\alpha}{1-\alpha}$$

and the derivative (3.10) can be written as

$$D^{\beta}_{S} f(x) \overset{\text{def}}{=} -2^{\beta} \int_{-\infty}^{\infty} \operatorname{sinc} (2^{\beta}\tau - 2^{\beta} x) f^{(n)}(\tau) d\tau, \quad 2^{\beta} \le \frac{n}{n+1} . \tag{3.11}$$

4. SHANNON WAVELETS

The sinc-function plays a fundamental role also in wavelet theory. In fact, the basic functions (scaling and wavelet) of the so-called Shannon wavelets (see e.g., [21–26]) can be defined by the sinc (3.5). In this section, some remarks on Shannon wavelets and connection coefficients are shortly summarized.

4.1. Preliminary Remarks

Shannon wavelet theory (see e.g., [21–24]) is based on the scaling function $\varphi(x)$, also known as sinc function, and the wavelet function $\psi(x)$ respectively defined as

$$\begin{cases} \varphi(x) &= \operatorname{sinc} x \stackrel{\text{def}}{=} \dfrac{\sin \pi x}{\pi x} = \dfrac{e^{\pi i x} - e^{-\pi i x}}{2\pi i x} \,. \\ \psi(x) &= \dfrac{\sin 2\pi(x-\frac{1}{2}) - \sin \pi(x-\frac{1}{2})}{\pi(x-\frac{1}{2})} \\ &= \dfrac{e^{-2 i \pi x}\left(-i + e^{i\pi x} + e^{3 i \pi x} + i\, e^{4 i \pi x}\right)}{2\pi\left(x-\frac{1}{2}\right)} \,. \end{cases} \tag{4.1}$$

The second function can be expressed in terms of the first, as

$$\psi(x) = 2\varphi(2x-1) - \varphi(x-\frac{1}{2}) \tag{4.2}$$

The families of translated and dilated Shannon scaling functions [21–24], are

$$\begin{aligned} \varphi^n_k(x) &= 2^{n/2}\varphi(2^n x - k) = 2^{n/2}\frac{\sin\pi\left(2^n x - k\right)}{\pi\left(2^n x - k\right)} \\ &= 2^{n/2}\frac{e^{\pi i(2^n x - k)} - e^{-\pi i(2^n x - k)}}{2\pi i\left(2^n x - k\right)} \,, \\ &= \frac{2^{n/2}}{2\pi i\left(2^n x - k\right)}\sum_{s=0}^{\infty}\frac{\pi^s i^s}{s!}\left[1-(-1)^s\right]\left(2^n x - k\right)^s \\ &= \frac{2^{n/2}}{2\pi i\left(2^n x - k\right)}\sum_{s=0}^{\infty}\frac{\pi^s i^s}{s!}\left(1-e^{\pi s}\right)\left(2^n x - k\right)^s \\ &= 2^{n/2-1}\sum_{s=1}^{\infty}\frac{\pi^{s-1} i^{s-1}}{s!}\left[1-(-1)^s\right]\left(2^n x - k\right)^{s-1} \,. \end{aligned} \tag{4.3}$$

By a direct computation it can be easily shown that this series can be also written as

$$\boxed{\varphi^n_k(x) = 2^{n/2}\sum_{s=0}^{\infty}(-1)^s\frac{\pi^{2s}}{(2s+1)!}\left(2^n x - k\right)^{2s}} \tag{4.4}$$

that is

$$\varphi^n_k(x) = 2^{n/2}\sum_{s=0}^{\infty}(-1)^s\frac{\pi^{2s}}{(2s+1)!}\sum_{j=0}^{2s}\binom{2s}{j}(2^n x)^j(-k)^{2s-j} \tag{4.5}$$

In the special case when $k=0$, from (4.4) we have

$$\varphi^n_0(x) = 2^{n/2}\sum_{s=0}^{\infty}(-1)^s\frac{\pi^{2s}}{(2s+1)!}2^{2ns}x^{2s} \tag{4.6}$$

while for the translated instances at the zero scale $n=0$ we obtain from (4.4)

$$\varphi_k(x) \stackrel{\text{def}}{=} \varphi(x-k) = \sum_{s=0}^{\infty}(-1)^s\frac{\pi^{2s}}{(2s+1)!}\left(x-k\right)^{2s} \tag{4.7}$$

Analogously, the translated and dilated instances of the Shannon wavelets are

$$\begin{aligned} \psi^n_k(x) &= 2^{n/2}\frac{\sin 2\pi(2^n x - k - \frac{1}{2}) - \sin\pi(2^n x - k - \frac{1}{2})}{\pi(2^n x - k - \frac{1}{2})} \,, \\ &= \frac{2^{n/2}}{2\pi(2^n x - k - \frac{1}{2})}\sum_{r=1}^{2} i^{1+r}e^{r\pi i(2^n x - k)} - i^{1-r}e^{-r\pi i(2^n x - k)} \,. \end{aligned} \tag{4.8}$$

or, by taking into account (4.2)

$$\psi^n_k(x) = 2\varphi^{n+1}_k(x) - \varphi^n_k(x-\frac{1}{2}) \tag{4.9}$$

and Equation (4.3), it is

$$\begin{aligned} \psi^n_k(x) &= 2^{n/2}\sum_{s=1}^{\infty}\frac{\pi^{s-1}i^{s-1}}{s!}\left[1-(-1)^s\right]\left(2^n x - k\right)^{s-1} \\ &- 2^{n/2-1}\sum_{s=1}^{\infty}\frac{\pi^{s-1}i^{s-1}}{s!}\left[1-(-1)^s\right]\left(2^n(x-\frac{1}{2}) - k\right)^{s-1} \,. \end{aligned}$$

From (4.9), by taking into account (4.4), it is

$$\begin{aligned} \psi^n_k(x) &= 2^{n/2}\sum_{s=0}^{\infty}(-1)^s\frac{\pi^{2s}}{(2s+1)!}\left\{2^{3/2}\left(2^{n+1}x - k\right)^{2s}\right. \\ &\left. -\left[\left(2^n x - k\right) - 2^{n-1}\right]^{2s}\right\} \end{aligned} \tag{4.10}$$

so that at the zero scale $n=0$ it is

$$\psi_k(x) \stackrel{\text{def}}{=} \psi^0_k(x) = \psi(x-k) = \sum_{s=0}^{\infty}(-1)^s\frac{\pi^{2s}}{(2s+1)!}\left\{2^{3/2}\left(2x-k\right)^{2s} - \left[\left(x-k\right)^{2s} - \frac{1}{2}\right]^{2s}\right\}$$

and, at the origin $k=0$

$$\psi^n(x) \stackrel{\text{def}}{=} \psi^n_0(x) = 2^{n/2}\sum_{s=0}^{\infty}(-1)^s\frac{\pi^{2s}}{(2s+1)!}\left\{2^{3/2}\left(2^{n+1}x\right)^{2s} - \left(2^n x - 2^{n-1}\right)^{2s}\right\}.$$

By assuming,

$$\varphi_0^0(x) = \varphi(x)\,,\ \psi_0^0(x) = \psi(x)\,,\ \varphi_k^0(x) = \varphi_k(x) = \varphi(x-k),$$
$$\psi_k^0(x) = \psi_k(x) = \psi(x-k),$$

and taking into account (4.4),(4.10) the fundamental functions $\varphi(x)$, $\psi(x)$, can be expressed as the power series

$$\begin{cases} \varphi(x) &= \displaystyle\sum_{s=0}^{\infty}(-1)^s \frac{\pi^{2s}}{(2s+1)!}x^{2s} \\ \psi(x) &= \displaystyle\sum_{s=0}^{\infty}(-1)^s \frac{\pi^{2s}}{(2s+1)!}\left[2^{2s+3/2}x^{2s} - \left(x-\frac{1}{2}\right)^{2s}\right]. \end{cases} \tag{4.11}$$

4.2. Properties of the Shannon Wavelet

Shannon wavelets enjoy some interesting properties. In particular, when they are evaluated at some special points they assume some very simple expressions. For instance, according to (4.3), it is

$$\varphi_k(h) = \varphi_h(k) = \varphi(h-k) = \varphi(k-h) = \delta_{kh}, \quad (h,\ k \in \mathbb{Z}), \tag{4.12}$$

so that

$$\varphi_k(h) = \delta_{kh} = \begin{cases} 0, & h \neq k, \quad (h,\ k \in \mathbb{Z}) \\ 1, & h = k, \quad (h,\ k \in \mathbb{Z}) \end{cases}$$

Analogously we have [24]

$$\begin{aligned} \psi_k^n(h) &= (-1)^{2^n h - k}\frac{2^{1+n/2}}{(2^{n+1}h-2k-1)\pi}, \ (2^{n+1}h - 2k - 1 \neq 0) \\ \psi_k^n(x) &= 0, \qquad x = 2^{-n}\left(k + \frac{1}{2} \pm \frac{1}{3}\right), \quad (n \in \mathbb{N},\ k \in \mathbb{Z}) \\ &\lim_{x\to 2^{-n}(h+\frac{1}{2})} \psi_k^n(x) = -2^{n/2}\delta_{hk}, \end{aligned} \tag{4.13}$$

being,

$$\psi_k^0(0) = (-1)^{k+1}\frac{2}{(2k+1)\pi}$$

and since $k \in \mathbb{Z}$, $2k+1 \neq 0$.

It can be shown (see e.g., [25]) that both scaling and wavelet functions are bounded, being:

$$\max[\varphi_k(x_M)] = 1, \quad x_M = k, \tag{4.14}$$

$$\max[\psi_k^n(x_M)] = 2^{n/2}\frac{3\sqrt{3}}{\pi}, \quad x_M = \begin{cases} -2^{-n}\left(k+\dfrac{1}{6}\right) \\ \dfrac{2^{-n-1}}{3}(18k+7), \end{cases} \tag{4.15}$$

and

$$\lim_{x\to\pm\infty}\varphi_k^n(x) = 0, \quad \lim_{x\to\pm\infty}\psi_k^n(x) = 0.$$

4.3. Shannon Wavelets in Fourier Domain

In order to define the multiscale analysis, based on Shannon wavelets, we need to define the Hilbert space of functions that can be reconstructed by them. The Shannon scaling function owns a very simple expression in the Fourier domain, therefore it would be easier to define the scalar product in Fourier domain. To this purpose we define the Fourier transform of the function $f(x) \in L_2(\mathbb{R})$, and its inverse transform as

$$\widehat{f}(\omega) = \widehat{f(x)} \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx, \quad f(x) = \int_{-\infty}^{\infty}\widehat{f}(\omega)e^{i\omega x}d\omega\,.$$

The Fourier transform of (4.1) give us [23]

$$\begin{cases} \widehat{\varphi}(\omega) &= \dfrac{1}{2\pi}\chi(\omega+3\pi) = \begin{cases} 1/(2\pi), & -\pi \le \omega < \pi \\ 0, & \text{elsewhere} \end{cases} \\ \widehat{\psi}(\omega) &= \dfrac{1}{2\pi}e^{i\omega/2}\left[\chi(2\omega) + \chi(-2\omega)\right] \end{cases} \tag{4.16}$$

with

$$\chi(\omega) = \begin{cases} 1, & 2\pi \le \omega < 4\pi \\ 0, & \text{elsewhere}\,. \end{cases}$$

The Fourier transform fulfills many interesting properties and among them the following:

$$\widehat{f(ax)} = \frac{1}{a}\widehat{f}(\frac{\omega}{a}), \quad \widehat{f(x-b)} = e^{-ib\omega}\widehat{f}(\omega), \quad \widehat{\frac{d^n}{dx^n}f(x)} = (i\omega)^n\widehat{f}(\omega). \tag{4.17}$$

So that for the dilated and translated instances of scaling/wavelet function, in the frequency domain, are

$$\begin{cases} \widehat{\varphi}_k^n(\omega) &= \dfrac{2^{-n/2}}{2\pi}e^{i\omega k/2^n}\chi(\omega/2^n + 3\pi) \\ \widehat{\psi}_k^n(\omega) &= \dfrac{2^{-n/2}}{2\pi}e^{i\omega(k+1/2)/2^n}\left[\chi(\omega/2^{n-1}) + \chi(-\omega/2^{n-1})\right]. \end{cases} \tag{4.18}$$

For the integer order derivatives of scaling and wavelet, according to (4.17), it is

$$\widehat{\frac{d^\ell}{dx^\ell}\varphi_k^n(x)} = (i\omega)^\ell\widehat{\varphi}_k^n(\omega), \qquad \widehat{\frac{d^\ell}{dx^\ell}\psi_k^n(x)} = (i\omega)^\ell\widehat{\psi}_k^n(\omega) \tag{4.19}$$

and, thanks to (4.18), we get

$$\begin{cases} \widehat{\dfrac{d^\ell}{dx^\ell}\varphi_k^n(x)} &= (i\omega)^\ell\dfrac{2^{-n/2}}{2\pi}e^{i\omega k/2^n}\chi(\omega/2^n+3\pi), \\ \widehat{\dfrac{d^\ell}{dx^\ell}\psi_k^n(x)} &= (i\omega)^\ell\dfrac{2^{-n/2}}{2\pi}e^{i\omega(k+1/2)/2^n}\left[\chi(\omega/2^{n-1})\right. \\ &\quad \left. + \chi(-\omega/2^{n-1})\right]. \end{cases} \tag{4.20}$$

The simple form of these derivative will help us to easily define also the fractional derivatives of these functions. Moreover, as we will see in the next section they form a basis for the $L_2(\mathbb{R})$-functions.

4.4. Wavelet Analysis and Synthesis

Both families of Shannon scaling and wavelet are $L_2(\mathbb{R})$-functions, therefore for each $f(x) \in L_2(\mathbb{R})$ and $g(x) \in L_2(\mathbb{R})$, the inner product is defined as

$$\left\langle f, g\right\rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f\,(x)\,\overline{g\,(x)}dx\,, \tag{4.21}$$

where the bar stands for the complex conjugate. By taking into account the Parseval theorem

$$\int_{-\infty}^{\infty} f\,(x)\,\overline{g\,(x)}dx = 2\pi \int_{-\infty}^{\infty} \widehat{f}\,(\omega)\,\overline{\widehat{g}\,(\omega)}d\omega,$$

it is

$$\left\langle f, g\right\rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f\,(x)\,\overline{g\,(x)}dx = 2\pi \int_{-\infty}^{\infty} \widehat{f}\,(\omega)\,\overline{\widehat{g}\,(\omega)}d\omega = 2\pi \left\langle \widehat{f}, \widehat{g}\right\rangle, \tag{4.22}$$

Shannon wavelets fulfill the following orthogonality properties (for the proof see e.g., [23, 24])

$$\begin{aligned} &\left\langle \psi_k^n\,(x)\,, \psi_h^m\,(x)\right\rangle = \delta^{nm}\delta_{hk}\,,\ \left\langle \varphi_k^0\,(x)\,, \varphi_h^0\,(x)\right\rangle = \delta_{kh}\,,\\ &\left\langle \varphi_k^0\,(x)\,, \psi_h^m\,(x)\right\rangle = 0, \qquad m \geq 0\,, \end{aligned} \tag{4.23}$$

δ^{nm}, δ_{hk} being the Kroenecker symbols.

Let $\mathcal{B} \subset L_2(\mathbb{R})$ the set of functions $f(x)$ in $L_2(\mathbb{R})$ such that the integrals

$$\begin{cases} \alpha_k \stackrel{\text{def}}{=} \langle f(x), \varphi_k(x)\rangle \stackrel{(4.22)}{=} \displaystyle\int_{-\infty}^{\infty} f(x)\overline{\varphi_k^0(x)}dx \\ \beta_k^n \stackrel{\text{def}}{=} \langle f(x), \psi_k^n(x)\rangle \stackrel{(4.22)}{=} \displaystyle\int_{-\infty}^{\infty} f(x)\overline{\psi_k^n(x)}dx\,, \end{cases} \tag{4.24}$$

exist with finite values, then it can be shown [23, 24, 27, 44], that the series

$$\boxed{f(x) = \sum_{h=-\infty}^{\infty} \alpha_h\,\varphi_h(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \beta_k^n \psi_k^n(x)\,,} \tag{4.25}$$

converges to $f(x)$. So that each function $f(x) \in \mathcal{B} \subset L_2(\mathbb{R})$ can be expressed as the wavelet series (4.25), and it is fully characterized by the wavelet coefficient α_h , β_k^n.

According to (4.22) the coefficients can be also computed in the Fourier domain [24] so that, together with (4.24) we can alternatively use the integrals

$$\begin{cases} \alpha_k &= \displaystyle\int_{-\pi}^{\pi} \widehat{f}(\omega)e^{i\omega k}d\omega\,, \\ \beta_k^n &= 2^{-n/2}\left[\displaystyle\int_{2^n\pi}^{2^{n+1}\pi} \widehat{f}(\omega)e^{i\omega(k+1/2)/2^n}d\omega \right. \\ &\left. \quad + \displaystyle\int_{-2^{n+1}\pi}^{-2^n\pi} \widehat{f}(\omega)e^{i\omega(k+1/2)/2^n}d\omega\right]. \end{cases} \tag{4.26}$$

In the frequency domain, Equation (4.25) gives [24]

$$\begin{aligned} \widehat{f}(\omega) =& \frac{1}{2\pi}\chi(\omega + 3\pi)\sum_{h=-\infty}^{\infty} \alpha_h e^{i\,\omega h} \\ &+ \frac{1}{2\pi}\chi(\omega/2^{n-1})\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} 2^{-n/2}\beta_k^n e^{i\,\omega(k+1/2)/2^n} \\ &+ \frac{1}{2\pi}\chi(-\omega/2^{n-1})\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} 2^{-n/2}\beta_k^n e^{i\,\omega(k+1/2)/2^n}\,. \end{aligned}$$

When the upper bound for the series of (4.25), is finite, then we have the approximation

$$f(x) \cong \sum_{h=-K}^{K} \alpha_h\,\varphi_h(x) + \sum_{n=0}^{N}\sum_{k=-S}^{S} \beta_k^n \psi_k^n(x)\,. \tag{4.27}$$

The error of the approximation has been estimated in Cattani [24, 26].

5. CONNECTION COEFFICIENTS AND DERIVATIVES

Let us assume that a function $f(x) \in \mathcal{B}$, so that $f(x)$ is a function belonging to the Hilbert space based on Shannon wavelets and thus being represented in the form of (4.25). In this section we will give the explicit form of the n-order integer derivative $f^{(n)}(x)$ and the sinc fractional order derivative $D_S^\alpha f(x)$. In order to get these derivatives we need to compute the ℓ-th integer order derivatives of the Shannon family (scaling and wavelet functions) $\varphi_h(x)$, $\psi_k^n(x)$ and the sinc-fractional derivative. The Equations (4.20) already give us the expression of the ℓ-order derivative in the Fourier domain. In the following sections we will give the explict form of these derivatives also in the space domain.

5.1. Integer Order Derivatives of the Shannon Wavelets

It can be shown that the integer order derivatives of the Shannon family can be expressed as orthogonal wavelet series [23, 24, 26] as follows:

Definition 6. The integer n-order derivative of the Shannon scaling and wavelet functions are

$$\left\{\begin{array}{rcl}\dfrac{d^\ell}{dx^\ell}\varphi_h(x) &=& \displaystyle\sum_{k=-\infty}^{\infty}\lambda^{(\ell)}_{hk}\,\varphi_k(x)\,,\\[2ex] \dfrac{d^\ell}{dx^\ell}\psi^m_h(x) &=& \displaystyle\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\gamma^{(\ell)mn}_{\ \ hk}\psi^n_k(x)\,,\end{array}\right. \tag{5.1}$$

being

$$\lambda^{(\ell)}_{kh}\equiv\left\langle\frac{d^\ell}{dx^\ell}\varphi^0_k(x)\,,\varphi^0_h(x)\right\rangle,\quad \gamma^{(\ell)nm}_{\ \ kh}\equiv\left\langle\frac{d^\ell}{dx^\ell}\psi^n_k(x)\,,\psi^m_h(x)\right\rangle, \tag{5.2}$$

the connection coefficients [21, 23, 45–50].

It should be noticed that the connection coefficients are not symmetric. In fact it is

$$\left\langle\frac{d^\ell}{dx^\ell}\varphi^0_k(x),\varphi^0_h(x)\right\rangle=\frac{d^\ell}{dx^\ell}\left\langle\varphi^0_k(x),\varphi^0_h(x)\right\rangle-\left\langle\varphi^0_k(x),\frac{d^\ell}{dx^\ell}\varphi^0_h(x)\right\rangle,$$

and by taking into account (4.23), there follows that

$$\lambda^{(\ell)}_{kh}=-\lambda^{(\ell)}_{hk}\qquad h\neq k$$

Analogously we have for the coefficients

$$\gamma^{(\ell)nm}_{\ \ kh}=-\gamma^{(\ell)nm}_{\ \ hk}$$

The connection coefficients can be easily computed so that it can be shown [21, 23, 24]

Theorem 1. *The connection coefficients* $(5.2)_1$ *of the Shannon scaling functions* $\varphi_k(x)$ *are*

$$\lambda^{(\ell)}_{kh}=\left\{\begin{array}{ll}(-1)^{k-h+\ell}\dfrac{i^\ell}{2\pi}\displaystyle\sum_{s=1}^{\ell}\frac{\ell!\pi^s}{s![i(k-h)]^{\ell-s+1}}\left[(-1)^s-1\right], & k\neq h\\[2ex] \dfrac{i^\ell\pi^{\ell+1}}{2\pi(\ell+1)}\left[1+(-1)^\ell\right], & k=h\,,\end{array}\right. \tag{5.3}$$

when $\ell\geq 1$. *When* $\ell=0$, *it is*

$$\lambda^{(0)}_{kh}=\delta_{kh}\,.$$

For the proof see e.g., [23].

Analogously, by defining the sign-function $\mu(x)=\mathrm{sign}(x)$, it can be shown that

Theorem 2. *The connection coefficients* $(5.2)_2$ *of the Shannon wavelets* $\psi^n_k(x)$ *are*

$$\left\{\begin{array}{rcl}\gamma^{(\ell)nm}_{\ \ kh} &=& \mu(h-k)\delta^{nm}\left\{\displaystyle\sum_{s=1}^{\ell+1}(-1)^{[1+\mu(h-k)](2\ell-s+1)/2}\frac{\ell!i^{\ell-s}\pi^{\ell-s}}{(\ell-s+1)!\,|h-k|^s}(-1)^{-s-2(h+k)}2^{n\ell-s-1}\right.\\[2ex] && \left.\times\left\{2^{\ell+1}\left[(-1)^{4h+s}+(-1)^{4k+\ell}\right]-2^s\left[(-1)^{3k+h+\ell}+(-1)^{3h+k+s}\right]\right\}\right\},k\neq h\\[2ex] \gamma^{(\ell)nm}_{\ \ kh} &=& \delta^{nm}\left[i^\ell\dfrac{\pi^\ell 2^{n\ell-1}}{\ell+1}(2^{\ell+1}-1)(1+(-1)^\ell)\right],k=h\end{array}\right. \tag{5.4}$$

for $\ell\geq 1$, *and*

$$\gamma^{(0)nm}_{\ \ kh}=\delta_{kh}\delta^{nm} \tag{5.5}$$

$\ell=0$ *respectively.*

For the proof see [23].

As a consequence of Equations (5.3),(5.8) the ℓ-order derivative of the basic functions (4.11) are

$$\left\{\begin{array}{rcl}\dfrac{d^\ell}{dx^\ell}\varphi(x) &=& \displaystyle\sum_{k=-\infty}^{\infty}\lambda^{(\ell)}_{0k}\,\varphi_k(x),\\[2ex] \dfrac{d^\ell}{dx^\ell}\psi(x) &=& \displaystyle\sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\gamma^{(\ell)0n}_{\ \ 0h}\psi^n_h(x)\\[2ex] &=& \displaystyle\sum_{h=-\infty}^{\infty}\gamma^{(\ell)00}_{\ \ 0h}\psi^0_h(x),\end{array}\right. \tag{5.6}$$

with

$$\lambda^{(\ell)}_{0k}=\left\{\begin{array}{ll}(-1)^{k+\ell}\dfrac{i^\ell}{2\pi}\displaystyle\sum_{s=1}^{\ell}\frac{\ell!\pi^s}{s!(ik)^{\ell-s+1}}\left[(-1)^s-1\right] & ,k\neq 0\\[2ex] \dfrac{i^\ell\pi^{\ell+1}}{2\pi(\ell+1)}\left[1+(-1)^\ell\right] & ,k=0\,,\end{array}\right. \tag{5.7}$$

and

$$\left\{\begin{array}{rcl}\gamma^{(\ell)00}_{\ \ 0h} &=& \mu(h)\left\{\displaystyle\sum_{s=1}^{\ell+1}(-1)^{[1+\mu(h)](2\ell-s+1)/2}\right.\\ && \times\frac{\ell!i^{\ell-s}\pi^{\ell-s}}{(\ell-s+1)!\,|h|^s}(-1)^{-s-2h}2^{-s-1}\times\left\{2^{\ell+1}\left[(-1)^{4h+s}\right.\right.\\ && \left.\left.\left.+(-1)^\ell\right]-2^s\left[(-1)^{h+\ell}+(-1)^{3h+s}\right]\right\}\right\},h\neq 0\\[2ex] \gamma^{(\ell)00}_{\ \ 0h} &=& \left[i^\ell\dfrac{\pi^\ell 2^{-1}}{\ell+1}(2^{\ell+1}-1)(1+(-1)^\ell)\right],h=0\end{array}\right. \tag{5.8}$$

In particular it is

$$\lambda^{(\ell)}\overset{\text{def}}{=}\lambda^{(\ell)}_{00}=\frac{i^\ell\pi^{\ell+1}}{2\pi(\ell+1)}\left[1+(-1)^\ell\right] \tag{5.9}$$

It can be easily shown that $\lambda^{(\ell)}=0$ for odd ℓ so that we have

$$\lambda^{(\ell)}=\left\{\begin{array}{ll}(-1)^s\dfrac{\pi^{2s}}{2s+1} & ,\ell=2s\\ 0 & ,\ell=2s+1\end{array}\right.\qquad(s=1,2,\ldots) \tag{5.10}$$

For instance according to (5.6), (5.7) a good approximation of the 2nd order derivative of $\varphi(x)$ is

$$\frac{d^2}{dx^2}\varphi(x) \cong \sum_{k=-2}^{2} \lambda_{0k}^{(\ell)}\, \varphi_k(x) = -\frac{1}{2}\varphi_{-2}(x) + 2\varphi_{-1}(x) - \frac{1}{3}\pi^2\varphi(x) + 2\varphi_1(x) - \frac{1}{2}\varphi_2(x)\,.$$

Also for higher derivatives with high amplitude, the approximation is quite good. For instance according to (5.6), (5.7) for the 7-th derivative of $\varphi(x)$ a quite good approximation is obtained with 15 terms

$$\frac{d^\ell}{dx^\ell}\varphi(x) \cong \sum_{k=-7}^{7} \lambda_{0k}^{(\ell)}\, \varphi_k(x)$$

5.2. Properties of Connection Coefficients

The connection coefficients own many interesting properties like e.g., the following for the scaling functions

Theorem 3. *The connection coefficients* (5.3) *are defined recursively by*

$$\lambda_{kh}^{(\ell+1)} = \begin{cases} \dfrac{\ell+1}{k-h}\lambda_{kh}^{(\ell)} - (-1)^{k-h}\dfrac{i^\ell \pi^{\ell+1}}{k-h}[(-1)^\ell + 1], & k \neq h \\[2ex] i\pi\dfrac{\ell+1}{\ell+2}\lambda_{kh}^{(\ell)} + \dfrac{(-i)^{\ell+1}\pi^{\ell+1}}{\ell+2}, & k = h\,, \end{cases} \tag{5.11}$$

Proof: see [26].

Analogously for the coefficients γ.

Theorem 4. *The connection coefficients* (5.8) *are recursively given by the matrix at the lowest scale level:*

$$\gamma^{(\ell)}{}^{nn}_{kh} = 2^{\ell(n-1)}\gamma^{(\ell)}{}^{11}_{kh}\,. \tag{5.12}$$

Proof: see [26].

Moreover we can easily check that

$$\gamma^{(2\ell+1)}{}^{nn}_{kh} = -\gamma^{(2\ell+1)}{}^{nn}_{hk}, \quad \gamma^{(2\ell)}{}^{nn}_{kh} = \gamma^{(2\ell)}{}^{nn}_{hk}\,.$$

5.3. Taylor Series

By using the connection coefficients, and taking into account that the basic functions, according to (5.1), are $\mathcal{C}^\infty$-functions, it is easy to show the following theorem:

Theorem 5. *Let $f(x) \in \mathcal{B} \subset L_2(\mathbb{R})$ the $\ell \geq 1$ order derivative is given by*

$$\boxed{f^{(\ell)}(x) = \sum_{h,k=-\infty}^{\infty} \alpha_h\, \lambda_{hk}^{(\ell)}\varphi_k(x) + \sum_{n,m=0}^{\infty}\sum_{k,s=-\infty}^{\infty} \beta_k^n \gamma^{(\ell)}{}^{mn}_{sk}\psi_s^m(x)} \tag{5.13}$$

where the coefficients α_h, β_k^n are given by (4.24) *(or* (4.26)*) and the connection coefficients are given by* (5.3), (5.8).

Proof: The proof easily follows from Equations (4.25), (5.1). □

Theorem 6. *If $f(x) \in B_\psi \subset L_2(\mathbb{R})$ and $f(x) \in \mathcal{C}^S$ the Taylor series of $f(x)$ in x_0 is*

$$f(x) = f(x_0) + \sum_{r=1}^{\infty}\left[\sum_{h,k=-\infty}^{\infty} \alpha_h\, \lambda_{hk}^{(r)}\varphi_k(x_0) + \sum_{n=0}^{\infty}\sum_{k,s=-\infty}^{\infty} 2^{r(n-1)}\beta_k^n\gamma^{(r)}{}^{11}_{sk}\psi_s^n(x_0)\right]\frac{(x-x_0)^r}{r!} \tag{5.14}$$

being α_h and β_k^n given by (4.24), (4.26).

Proof: From (4.25), the ℓ-order derivative ($\ell \leq S$) is

$$\begin{aligned} f^{(\ell)}(x) &= \sum_{h=-\infty}^{\infty} \alpha_h \frac{d^\ell}{dx^\ell}\varphi_h(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \beta_k^n \frac{d^\ell}{dx^\ell}\psi_k^n(x)\,, \\ &\overset{(5.1)}{=} \sum_{h=-\infty}^{\infty}\alpha_h \sum_{k=-\infty}^{\infty}\lambda_{hk}^{(\ell)}\varphi_k(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty}\beta_k^n \sum_{m=-\infty}^{\infty} \sum_{s=-\infty}^{\infty}\gamma^{(\ell)}{}^{mn}_{sk}\psi_s^m(x)\,, \\ &= \sum_{h,k=-\infty}^{\infty}\alpha_h\,\lambda_{hk}^{(\ell)}\varphi_k(x) + \sum_{n,m=0}^{\infty}\sum_{k,s=-\infty}^{\infty}\beta_k^n\gamma^{(\ell)}{}^{mn}_{sk}\psi_s^m(x)\,, \end{aligned}$$

so that by taking into account (5.12) the proof follows. □

By a suitable choice of the initial point x_0 Equation (5.14) can be simplified. For instance, at the integers, $x_0 = j$, $(j \in \mathbb{Z})$, according to Equations (4.12), (5.12) it is

$$f(x) \cong f(j) + \sum_{r=1}^{S}\left[\sum_{h=-\infty}^{\infty}\alpha_h\,\lambda_{hj}^{(r)} + \sum_{n=0}^{\infty}\sum_{k,s=-\infty}^{\infty}\frac{2^{r(n-1)+1+n/2}}{(2^{n+1}h-2s-1)\pi}\beta_k^n\gamma^{(r)}{}^{11}_{sk}\psi_s^n(h)\right]\frac{(x-j)^r}{r!}$$

In particular, for $x_0 = j = 0$, Equation (5.14) gives

$$\begin{aligned} f(x) &= f(0) + \sum_{r=1}^{\infty}\left[\sum_{h,k=-\infty}^{\infty}\alpha_h\,\lambda_{hk}^{(r)}\varphi_k(0) + \sum_{n=0}^{\infty}\sum_{k,s=-\infty}^{\infty}2^{r(n-1)}\beta_k^n\gamma^{(r)}{}^{11}_{sk}\psi_s^n(0)\right]\frac{x^r}{r!} \\ &= f(0) + \sum_{r=1}^{\infty}\left[\sum_{h=-\infty}^{\infty}\alpha_h\,\lambda_{h0}^{(r)} + \sum_{n=0}^{\infty}\sum_{k,s=-\infty}^{\infty}2^{r(n-1)}\beta_k^n\gamma^{(r)}{}^{11}_{sk}\psi_s^n(0)\right]\frac{x^r}{r!} \end{aligned} \tag{5.15}$$

and since

$$\psi_s^n(0) = (-1)^s \frac{2^{1+n/2}}{(-2s-1)\pi}, \qquad (-2k-1 \neq 0)$$

we get

$$\boxed{\begin{aligned} f(x) &= f(0) + \sum_{r=1}^{\infty}\left[\sum_{h=-\infty}^{\infty} \alpha_h \lambda_{h0}^{(r)} + \sum_{n=0}^{\infty}\sum_{k,s=-\infty}^{\infty} \right. \\ &\left. \frac{(-1)^{s+1}2^{n(r+1/2)+1-r}}{(2s+1)\pi} \beta_k^n \gamma^{(r)11}_{\ \ sk}\right] \frac{x^r}{r!} \end{aligned}} \tag{5.16}$$

with $\lambda_{h0}^{(r)}$ given by (5.7) and $\gamma^{(r)11}_{\ \ sk}$ by (5.8) respectively. So that each function $f(x) \in \mathcal{B} \subset L_2(\mathbb{R})$, can be easily expressed as a power series, when the finite values of the wavelet coefficients α_h, β_k^n are given, according to (4.24),(4.26).

There follows, in particular, the Taylor power series for the basic functions $\varphi(x)$, $\psi(x)$:

$$\begin{cases} \varphi(x) \ = \ 1 + \displaystyle\sum_{r=1}^{\infty}\left(\sum_{h=-\infty}^{\infty} \lambda_{h0}^{(r)}\right)\frac{x^r}{r!} \\ \psi(x) \ = \ -\dfrac{2}{\pi} + \displaystyle\sum_{r=1}^{\infty}\left(\sum_{n=0}^{\infty}\sum_{k,s=-\infty}^{\infty} \frac{(-1)^{s+1}2^{n(r+1/2)+1-r}}{(2s+1)\pi}\delta_k^n \gamma^{(r)11}_{\ \ sk}\right)\frac{x^r}{r!} \end{cases} \tag{5.17}$$

being $\psi(0) = -\dfrac{2}{\pi}$, according to (4.1).

For a fixed r the series

$$\Lambda^r \stackrel{\text{def}}{=} \sum_{h=-\infty}^{\infty} \lambda_{h0}^{(r)}, \quad r \geq 1, h \neq= 0$$

is converging, as can be easily shown by using Equation (5.7). In particular it is

$$\lambda_{h0}^{(1)} = -\frac{(-1)^h}{h}, \quad \lambda_{h0}^{(2)} = -2\frac{(-1)^h}{h^2}, \quad \lambda_{h0}^{(3)} = (-1)^h\left(\frac{\pi^2}{h} - \frac{6}{h^3}\right),$$

$$\lambda_{h0}^{(4)} = (-1)^h\left(\frac{4\pi^2}{h^2} - \frac{24}{h^4}\right), \quad \lambda_{h0}^{(5)} = (-1)^h\left(\frac{\pi^4}{h} - \frac{20\pi^2}{h^3} + \frac{120}{h^5}\right)$$

$$\lambda_{h0}^{(6)} = -(-1)^h\left(\frac{6\pi^4}{h^2} - \frac{120\pi^2}{h^4} + \frac{720}{h^6}\right), \quad \ldots$$

Moreover, since for odd r it is $\Lambda^{(r)} = 0$ while for even r it is

$$\Lambda^{2r} = 2\sum_{h=0}^{\infty} \lambda_{h0}^{(r)}, \quad r \geq 1, h \neq= 0$$

so that $\varphi(x)$ can be written as the power series

$$\varphi(x) = \sum_{r=0}^{\infty} \frac{\Lambda^r}{r!} x^r, \qquad (\Lambda^0 \stackrel{\text{def}}{=} 1)$$

The first (approximated) values of the coefficients Λ are:

$$\Lambda^0 \ = \ 1, \ , \Lambda^1 = 0.69, \ , \Lambda^2 = 1.64, \ , \Lambda^3 = -1.43, \ ,$$

$$\Lambda^4 = -9.74, \ , \Lambda^5 = 6.19$$

In particular, the Taylor series for the wavelet function $\psi(x)$ can be also easily computed as follows:

$$\begin{aligned} \psi(x) \ &= \qquad \psi(0) + \sum_{\ell=1}^{\infty}\left(\frac{d^\ell \psi(x)}{dx^\ell}\right)_{x=0} \frac{x^\ell}{\ell!} \\ &\stackrel{(5.1),(5.6)}{=} \ -\frac{2}{\pi} + \sum_{\ell=1}^{\infty}\left(\sum_{k=-\infty}^{\infty} \gamma^{(\ell)00}_{\ \ 0k}\psi_k^0(0)\right)\frac{x^\ell}{\ell!} \\ &\stackrel{(4.13)}{=} \ -\frac{2}{\pi} + \sum_{\ell=1}^{\infty}\left(\sum_{k=-\infty}^{\infty} \gamma^{(\ell)00}_{\ \ 0k}(-1)^{k+1}\frac{2}{(2k+1)\pi}\right)\frac{x^\ell}{\ell!} \end{aligned}$$

that is

$$\boxed{\psi(x) = -\frac{2}{\pi} + \sum_{\ell=1}^{\infty}\left(\sum_{k=-\infty}^{\infty}(-1)^{k+1}\frac{2}{(2k+1)\pi}\gamma^{(\ell)00}_{\ \ 0k}\right)\frac{x^\ell}{\ell!}} \tag{5.18}$$

6. SINC-FRACTIONAL DERIVATIVES FOR THE FUNCTIONS *F*(*X*) ∈ $\mathcal{B} \subset L_2(\mathbb{R})$

The sinc fractional derivative (3.10) is defined by a sinc kernel over an infinite domain. Although the sinc-function is the basic function for Shannon wavelet, this kernel is not a Shannon scaling function for the reason that the sinc function depends on the fractional (non-integer) order of derivative. On the other hand as shown by the Equation (5.13) the n-integer order derivative can be written as a linear combination of $\varphi_k(x)$, $\psi_k^m(x)$. Therefore, in order to give an explicit form to (3.10) as a function of Shannon wavelet and connection coefficients, we need to compute the scalar products of Shannon scaling and wavelet with sinc-function.

6.1. Scalar Products of the Shannon Scaling and Wavelet Functions With Sinc-Function

In this section we consider the scalar product of the sinc function with the Shannon scaling and wavelet functions and corresponding derivatives. We need these products to compute the sinc fractional derivatives.

6.1.1. Scalar Product of the Shannon Scaling Function With Sinc-Function

Let us assume $a, b \in \mathbb{R}$ and show the following theorem:

Theorem 7. *The scalar product ot the scaling functions $\varphi_k(\tau)$ with the sinc-function is*

$$\langle \text{sinc}\,(a\tau - b), \varphi_k(\tau)\rangle = \begin{cases} \dfrac{2\pi}{a}\text{sinc}\,(b+k), \quad a \geq 1 \\ \dfrac{2\pi}{a^2}\text{sinc}\,\dfrac{(b+k)}{a}, \ a < 1\,. \end{cases} \tag{6.1}$$

Proof: It is by definition

$$\langle \mathrm{sinc}\,(a\tau - b), \varphi_k(\tau)\rangle = \int_{-\infty}^{\infty} \mathrm{sinc}\,(a\tau - b)\varphi_k(\tau)d\tau\ .$$

According to (4.22) this product can be easily done in the Fourier domain,

$$\langle \mathrm{sinc}\,(a\tau - b), \varphi_k(\tau)\rangle = 2\pi\,\langle \widehat{\mathrm{sinc}\,(a\tau - b)}, \widehat{\varphi_k(\tau)}\rangle$$

from where by using the properties (4.17) it is

$$\widehat{\mathrm{sinc}\,(a\tau - b)} = \frac{1}{a}\widehat{\mathrm{sinc}}\,(\frac{\omega}{a})e^{-ib\omega}, \quad \widehat{\mathrm{sinc}}\,(\frac{\omega}{a}) \overset{(4.16)}{=} \frac{1}{2\pi}\chi(\frac{\omega}{a} + 3\pi)$$

so that by taking into account $(4.18)_1$

$$\langle \mathrm{sinc}\,(a\tau - b), \varphi_k(\tau)\rangle = 2\pi\frac{1}{a}\frac{1}{2\pi}\langle e^{-ib\omega}\chi(\frac{\omega}{a} + 3\pi), e^{-ik\omega}\chi(\omega + 3\pi)\rangle\ .$$

The integral can be easily computed, being

$$\begin{aligned}&\langle e^{-ib\omega}\chi(\frac{\omega}{a} + 3\pi), e^{-ik\omega}\chi(\omega + 3\pi)\rangle \\ &= \int_{-\infty}^{\infty} e^{-i(b+k)\omega}\chi(\frac{\omega}{a} + 3\pi)\chi(\omega + 3\pi)d\omega\rangle\ .\end{aligned}$$

There follows that, if $\frac{1}{a} \leq 1$ it is

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-i(b+k)\omega}\chi(\frac{\omega}{a} + 3\pi)\chi(\omega + 3\pi)d\omega &= \int_{-\infty}^{\infty} e^{-i(b+k)\omega}\chi(\omega + 3\pi)d\omega \\ &= \int_{-\pi}^{\pi} e^{-i(b+k)\omega}d\omega = 2\pi\,\mathrm{sinc}\,(b+k)\ .\end{aligned}$$

While for $\frac{1}{a} > 1$ it is

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-i(b+k)\omega}\chi(\frac{\omega}{a} + 3\pi)\chi(\omega + 3\pi)d\omega &= \int_{-\infty}^{\infty} e^{-i(b+k)\omega}\chi(\frac{\omega}{a} + 3\pi)d\omega \\ &= \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} e^{-i(b+k)\omega}d\omega = \frac{2\pi}{a}\mathrm{sinc}\,\frac{(b+k)}{a}\ .\end{aligned}$$

From where there easily follows the result (6.1).

□

In particular, according to (4.17), it is

$$\mathrm{sinc}\,\widehat{\frac{\alpha(x-\tau)}{1-\alpha}} = \mathrm{sinc}\,\widehat{\frac{\alpha}{\alpha-1}(\tau - x)} \overset{(4.17)}{=} e^{-i\frac{\alpha}{\alpha-1}x\omega}\mathrm{sinc}\,\widehat{\left(\frac{\alpha}{\alpha-1}\tau\right)}$$

that is

$$\mathrm{sinc}\,\widehat{\frac{\alpha(x-\tau)}{1-\alpha}} = \frac{\alpha-1}{\alpha}e^{-i\frac{\alpha}{\alpha-1}x\omega}\widehat{\mathrm{sinc}}\,\left(\frac{\alpha-1}{\alpha}\omega\right)\ .$$

Since we have

$$\widehat{\mathrm{sinc}\,(\tau)} \overset{(4.16)}{=} \frac{1}{2\pi}\chi(\omega + 3\pi)$$

there follows

$$\mathrm{sinc}\,\widehat{\frac{\alpha(x-\tau)}{1-\alpha}} = \frac{1}{2\pi}\frac{\alpha-1}{\alpha}e^{-i\frac{\alpha}{\alpha-1}x\omega}\chi\left(\frac{\alpha-1}{\alpha}\omega + 3\pi\right)$$

so that, by taking

$$a = \frac{\alpha}{\alpha-1}, \quad b = \frac{\alpha}{\alpha-1}x$$

from (6.1) we get

$$\begin{aligned}&\left\langle \mathrm{sinc}\,\left(\frac{\alpha}{\alpha-1}\tau - \frac{\alpha}{\alpha-1}x\right), \varphi_k(\tau)\right\rangle \\ &= \begin{cases} \dfrac{2\pi(\alpha-1)}{\alpha}\mathrm{sinc}\,\left(\dfrac{\alpha}{\alpha-1}x + k\right), & \alpha \geq 1 \\ \dfrac{2\pi(\alpha-1)^2}{\alpha^2}\mathrm{sinc}\,\left(x + k\dfrac{\alpha-1}{\alpha}\right), & \alpha < 1\ . \end{cases}\end{aligned} \tag{6.2}$$

Analogously we can give an explicit form to the scalar product of the integer n-order derivative.

Theorem 8. *The scalar product of the n-th order derivative $\varphi_k^{(n)}(\tau)$ with the sinc-function is*

$$\langle \mathrm{sinc}\,(a\tau - b), \varphi_k^{(n)}(\tau)\rangle = \begin{cases} \displaystyle\int_{-\pi}^{\pi}(i\omega)^n e^{-i(b+k)\omega}d\omega, & a \geq 1 \\ \displaystyle\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}(i\omega)^n e^{-i(b+k)\omega}d\omega, & a < 1 \end{cases} \tag{6.3}$$

Proof: It is

$$\langle \mathrm{sinc}\,(a\tau - b), \varphi_k^{(n)}(\tau)\rangle = \int_{-\infty}^{\infty} \mathrm{sinc}\,(a\tau - b)\varphi_k^{(n)}(\tau)d\tau\ .$$

According to (4.22) this product can be easily done in the Fourier domain,

$$\langle \mathrm{sinc}\,(a\tau - b), \varphi_k^{(n)}(\tau)\rangle = 2\pi\,\langle \widehat{\mathrm{sinc}\,(a\tau - b)}, \widehat{\varphi_k^{(n)}(\tau)}\rangle$$

from where by using the properties (4.17) it is

$$\begin{aligned}\widehat{\mathrm{sinc}\,(a\tau - b)} &= \frac{1}{2\pi a}e^{-ib\omega}\chi(\frac{\omega}{a} + 3\pi)\quad , \\ \widehat{\varphi_k^{(n)}(\tau)} &= (i\omega)^n\widehat{\varphi_k(\tau)} = (i\omega)^n e^{-ik\omega}\chi(\omega + 3\pi)\end{aligned}$$

so that by taking into account $(4.18)_1$

$$\begin{aligned}&\langle \mathrm{sinc}\,(a\tau - b), \varphi_k^{(n)}(\tau)\rangle \\ &= 2\pi\frac{1}{a}\frac{1}{2\pi}\langle e^{-ib\omega}\chi(\frac{\omega}{a} + 3\pi), (i\omega)^n e^{-ik\omega}\chi(\omega + 3\pi)\rangle\ .\end{aligned}$$

The integral can be easily computed, being

$$\langle e^{-ib\omega}\chi(\frac{\omega}{a} + 3\pi), e^{-ik\omega}\chi(\omega + 3\pi)\rangle$$

$$= \int_{-\infty}^{\infty} (i\omega)^n e^{-i(b+k)\omega} \chi(\frac{\omega}{a} + 3\pi)\chi(\omega + 3\pi)d\omega\rangle \ .$$

There follows that, if $\frac{1}{a} \leq 1$ it is

$$\begin{aligned}\int_{-\infty}^{\infty} (i\omega)^n e^{-i(b+k)\omega} \chi(\frac{\omega}{a} + 3\pi)\chi(\omega + 3\pi)d\omega \\ = \int_{-\infty}^{\infty} (i\omega)^n e^{-i(b+k)\omega} \chi(\omega + 3\pi)d\omega \\ = \int_{-\pi}^{\pi} (i\omega)^n e^{-i(b+k)\omega} d\omega \ .\end{aligned}$$

While for $\frac{1}{a} > 1$ it is

$$\begin{aligned}\int_{-\infty}^{\infty} (i\omega)^n e^{-i(b+k)\omega} \chi(\frac{\omega}{a} + 3\pi)\chi(\omega + 3\pi)d\omega \\ = \int_{-\infty}^{\infty} (i\omega)^n e^{-i(b+k)\omega} \chi(\frac{\omega}{a} + 3\pi)d\omega \\ = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} (i\omega)^n e^{-i(b+k)\omega} d\omega \ .\end{aligned}$$

From where there easily follows the result (6.1).

□

In particular, for the first derivative it is

$$\int_{-\pi}^{\pi} (i\omega) e^{-i(b+k)\omega} d\omega = \frac{2\pi}{b+k}\left[\mathrm{sinc}\,(b+k) - \cos(b+k)\pi\right]$$

and

$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} (i\omega)^n e^{-i(b+k)\omega} d\omega = \frac{2\pi}{a(b+k)}\left[\mathrm{sinc}\,\frac{b+k}{a} - a\cos\frac{b+k}{a}\pi\right]$$

so that

$$\begin{aligned}&\langle \mathrm{sinc}\,(a\tau - b), \varphi'_k(\tau)\rangle \\ &= \begin{cases} \frac{2\pi}{b+k}\left[\mathrm{sinc}\,(b+k) - \cos(b+k)\pi\right], & a \geq 1 \\ \frac{2\pi}{a(b+k)}\left[\mathrm{sinc}\,\frac{b+k}{a} - a\cos\frac{b+k}{a}\pi\right], & a < 1 \ . \end{cases}\end{aligned} \tag{6.4}$$

In general the scalar product of the n-order derivative (with $n > 1$) is given by the lengthly computation of the integrals (6.3). In the next section we will see that this computation can be avoided by using the connection coefficients.

6.1.2. Scalar Product of the Shannon Wavelets With Sinc Function

Analogously, for the derivative of the wavelet function it can be easily shown that

Theorem 9. *Let $a,\ b\ \in\ \mathbb{R}$, the scalar product ot the wavelet functions $\psi^n_k(\tau)$ with the sinc-function is*

$$\langle \mathrm{sinc}\,(a\tau - b), \psi^n_k(\tau)\rangle = \Gamma^n_k(\tau, a, b) \stackrel{\text{def}}{=} \frac{2^{n/2+1}}{a\pi(2^{n+1}b - 2k - 1)} \times$$
$$\times \begin{cases} 0 & ,\ a < 1 \\ \sin\left(\frac{1}{2}a(2b - 2^{-n}(1+2k))\right)\pi + \cos(-2^n b + k)\pi & ,\ 2^n < a < 2^{n+1} \\ \sin\left(-2^{n+1}b + 2k\right)\pi + \cos(-2^n b + k)\pi & ,\ 2^{n+1} \geq a \ . \end{cases} \tag{6.5}$$

Proof: It is

$$\langle \mathrm{sinc}\,(a\tau - b), \psi^n_k(\tau)\rangle = \int_{-\infty}^{\infty} \mathrm{sinc}\,(a\tau - b)\psi^n_k(\tau)d\tau \ .$$

According to (4.22) this product can be easily done in the Fourier domain,

$$\langle \mathrm{sinc}\,(a\tau - b), \psi^n_k(\tau)\rangle = 2\pi\,\langle \widehat{\mathrm{sinc}\,(a\tau - b)}, \widehat{\psi^n_k}\rangle$$

from where by using the properties (4.17), it is

$$\widehat{\mathrm{sinc}\,(a\tau - b)} = \frac{1}{a}\frac{1}{2\pi}\chi(\frac{\omega}{a} + 3\pi)e^{-ib\omega}$$

so that by taking into account $(4.18)_2$

$$\begin{aligned}\langle \mathrm{sinc}\,(a\tau - b), \psi^n_k(\tau)\rangle = \frac{2^{-n/2}}{2a\pi}\Big\langle e^{-ib\omega}\chi(\frac{\omega}{a} + 3\pi), e^{i\omega(k+1/2)/2^n} \\ \left[\chi(\omega/2^{n-1}) + \chi(-\omega/2^{n-1})\right]\Big\rangle\end{aligned}$$

that is

$$\begin{aligned}\langle \mathrm{sinc}\,(a\tau - b), \psi^n_k(\tau)\rangle = \frac{2^{-n/2}}{2a\pi}\Big\langle e^{i\omega(k+1/2-2^n b)/2^n}\chi(\frac{\omega}{a} + 3\pi), \\ \left[\chi(\omega/2^{n-1}) + \chi(-\omega/2^{n-1})\right]\Big\rangle \ .\end{aligned}$$

Let us notice that the value of the scalar product (and then of the integral) depends on the non-vanishing values of the characteristic function. On the other hands the characteristic function χ depends on the values $a,\ n\ ,k$. In fact the non-vanishing values of the characteristic functions are

$$\chi(\frac{\omega}{a} + 3\pi) = 1, \quad \text{if } -a\pi < \omega < a\pi$$

$$\chi(\omega/2^{n-1}) = 1, \quad \text{if } 2^n\pi < \omega < 2^n(2\pi)$$

$$\chi(-\omega/2^{n-1}) = 1, \ \text{if } -2^n(2\pi) < \omega < -2^n\pi \ .$$

There follow three cases:

1. $a\pi < \pi$. In this case $a < 1$, the characteristic functions have some disjoint intervals and the scalar product vanishes

$$\Big\langle e^{i\omega(k+1/2-2^n b)/2^n}\chi(\frac{\omega}{a} + 3\pi), \left[\chi(\omega/2^{n-1}) + \chi(-\omega/2^{n-1})\right]\Big\rangle = 0 \ .$$

2. $2^n\pi < a\pi < 2^n(2\pi)$. Here we have $2^n < a < 2^{n+1}$ the integral becomes

$$\left\langle e^{i\omega(k+1/2-2^n b)/2^n}\chi(\frac{\omega}{a}+3\pi),\left[\chi(\omega/2^{n-1})+\chi(-\omega/2^{n-1})\right]\right\rangle =$$

$$=\int_{-a\pi}^{-2^n\pi} e^{i\omega(k+1/2-2^n b)/2^n}d\omega+\int_{2^n\pi}^{a\pi} e^{i\omega(k+1/2-2^n b)/2^n}d\omega$$

$$=\frac{2^{n+2}}{2^{n+1}b-2k-1}\left[\sin\left(\frac{1}{2}a(2b-2^{-n}(1+2k))\right)\pi+\cos(-2^n b+k)\pi\right].$$

3. $2^n(2\pi)\leq a\pi$. We have $2^{n+1}\leq a$ so that the integral is

$$\left\langle e^{i\omega(k+1/2-2^n b)/2^n}\chi(\frac{\omega}{a}+3\pi),\left[\chi(\omega/2^{n-1})+\chi(-\omega/2^{n-1})\right]\right\rangle =$$

$$=\int_{-2^{n+1}\pi}^{-2^n\pi} e^{i\omega(k+1/2-2^n b)/2^n}d\omega+\int_{2^n\pi}^{2^{n+1}\pi} e^{i\omega(k+1/2-2^n b)/2^n}d\omega$$

$$=\frac{2^{n+2}}{2^{n+1}b-2k-1}\left[\sin\left(-2^{n+1}b+2k\right)\pi+\cos(-2^n b+k)\pi\right].$$

From where we obtain (6.5). □

6.2. Sinc-Fractional Derivative of Functions $f(x)\in\mathcal{B}\subset L_2(\mathbb{R})$

In order to define the sinc-fractional derivative for the functions $f(x)\in\mathcal{B}$, according to the reconstruction formula (4.25) we need to compute the sinc-fractional derivative of the scaling and wavelet functions. These derivatives are given by the following theorems.

Theorem 10. *The sinc-fractional derivative* (3.10) *of the scaling function $\varphi_k(x)$ is*

$$D_S^\alpha\varphi_h(x) = -2\pi P(\alpha)\sum_{k=-\infty}^{\infty}\lambda_{hk}^{(n)}\times\begin{cases}\operatorname{sinc}\left(\frac{\alpha}{\alpha-1}x+k\right), & \alpha\geq 1\\ \frac{\alpha-1}{\alpha}\operatorname{sinc}\left(x+\frac{\alpha}{1-\alpha}k\right), & \alpha<1\end{cases}\tag{6.6}$$

Proof: Starting from the definition (3.10) of the sinc-derivative it is

$$D_S^\alpha\varphi_h(x)\overset{(3.10)}{=}\frac{\alpha P(\alpha)}{1-\alpha}\int_{-\infty}^{\infty}\operatorname{sinc}\frac{\alpha(x-\tau)}{1-\alpha}\frac{d^n}{d\tau^n}\varphi_h(\tau)d\tau,\tag{6.7}$$

According to (4.22), the derivatives (6.7), can be written also as scalar product,

$$D_S^\alpha\varphi_h(x)=\frac{\alpha P(\alpha)}{1-\alpha}\left\langle\operatorname{sinc}\frac{\alpha(x-\tau)}{1-\alpha},\ \frac{d^n}{d\tau^n}\varphi_h(\tau)\right\rangle,\tag{6.8}$$

From here by using the integer order derivatives (5.1) we

$$D_S^\alpha\varphi_h(x)=\frac{\alpha P(\alpha)}{1-\alpha}\sum_{k=-\infty}^{\infty}\lambda_{hk}^{(n)}\left\langle\operatorname{sinc}\frac{\alpha(x-\tau)}{1-\alpha},\ \varphi_k(\tau)\right\rangle,\tag{6.9}$$

So that the computation of the sinc-fractional derivative of a function, that can be expressed as wavelet series, is reduced to the computation of the scalar product:

$$\left\langle\operatorname{sinc}\frac{\alpha(x-\tau)}{1-\alpha},\ \varphi_k(\tau)\right\rangle,\quad 0\leq n-1<\alpha<n\tag{6.10}$$

which is given by (6.1) with

$$a=\frac{\alpha}{\alpha-1},\quad b=\frac{\alpha}{\alpha-1}x.$$

It can be easily seen that these inequalities imply

$$a\geq 1\Longrightarrow\alpha\geq 1,\quad a<1\Longrightarrow\alpha<1.$$

From these inequalities, by taking into account (6.1),(6.9), there easily follows (6.6).

□

Analogously, we have for the Shannon wavelet fractional derivatives the following

Theorem 11. *The sinc-fractional derivative* (3.10) *of the Shannon wavelets $\psi_h^m(x)$ is*

$$D_S^\alpha\psi_h^m(x)=\frac{\alpha P(\alpha)}{1-\alpha}\sum_{s=0}^{\infty}\sum_{k=-\infty}^{\infty}\gamma^{(n)ms}_{\ \ hk}\frac{2^{s/2+1}(\alpha-1)}{\alpha\pi(2^{s+1}\frac{\alpha}{\alpha-1}-2k-1)}\times$$

$$\times\begin{cases}0, & a<1\\ \sin\left(\frac{1}{2}\frac{\alpha}{\alpha-1}(2\frac{\alpha}{\alpha-1}x-2^{-s}(1+2k))\right)\pi\\ \quad+\cos(-2^s\frac{\alpha}{\alpha-1}x+k)\pi, & 2^s<a<2^{s+1}\\ \sin\left(-2^{s+1}b+2k\right)\pi+\cos(-2^s b+k)\pi, & 2^{s+1}\geq a\end{cases}\tag{6.11}$$

Proof: From the definition (3.10) it is

$$D_S^\alpha\psi_h^m(x)\overset{(3.10)}{=}\frac{\alpha P(\alpha)}{1-\alpha}\int_{-\infty}^{\infty}\operatorname{sinc}\frac{\alpha(x-\tau)}{1-\alpha}\frac{d^n}{d\tau^n}\psi_h^m(\tau)d\tau,\tag{6.12}$$

According to (4.22), this derivative can be written also as scalar product,

$$D_S^\alpha\psi_h^m(x)=\frac{\alpha P(\alpha)}{1-\alpha}\left\langle\operatorname{sinc}\frac{\alpha(x-\tau)}{1-\alpha},\ \frac{d^n}{d\tau^n}\psi_h^m(\tau)\right\rangle,\tag{6.13}$$

so that by taking into account (5.1) which gives the integer order derivatives it is

$$D_S^\alpha\psi_h^m(x)=\frac{\alpha P(\alpha)}{1-\alpha}\sum_{s=0}^{\infty}\sum_{k=-\infty}^{\infty}\gamma^{(n)ms}_{\ \ hk}\left\langle\operatorname{sinc}\frac{\alpha(x-\tau)}{1-\alpha},\ \psi_k^s(\tau)\right\rangle,\tag{6.14}$$

and using (6.5) we get (6.11).

□

Equations (6.6) and (6.11) enable us to compute explicitly the sinc-fractional derivative of any function belonging to the Hilbert space $\mathcal{B} \subset L_2(\mathbb{R})$. In fact, let $f(x) \in \mathcal{B}$ a function such that it can be represented as the wavelet series (4.25). Its sinc-fractional derivative can be computed according to

Theorem 12. *The sinc-fractional derivative of the wavelet representation* (4.25) *of function $f(x) \in \mathcal{B} \subset L_2(\mathbb{R})$, is given by*

$$
\begin{aligned}
D^{\nu}_{S} f(x) &= -2\pi P(\nu) \sum_{h=-\infty}^{\infty} \alpha_h \sum_{k=-\infty}^{\infty} \lambda^{(n)}_{hk} \\
&\times \begin{cases} \operatorname{sinc}\left(\dfrac{\nu}{\nu-1} x + k\right), & \nu \geq 1 \\ \dfrac{\nu-1}{\nu} \operatorname{sinc}\left(x + \dfrac{\nu}{1-\nu} k\right), & \nu < 1 \end{cases} \\
&+ \frac{\nu P(\nu)}{1-\nu} \sum_{m=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta^m_h \sum_{s=0}^{\infty} \\
&\sum_{k=-\infty}^{\infty} \gamma^{(n)ms}_{hk} \frac{2^{s/2+1}(\nu-1)}{\nu\pi(2^{s+1}\frac{\nu}{\nu-1} - 2k - 1)} \times \\
&\times \begin{cases} 0 & , \nu < 1 \\ \sin\left(\dfrac{1}{2}\dfrac{\nu}{\nu-1}(2\dfrac{\nu}{\nu-1} x - 2^{-s}(1+2k))\right)\pi \\ + \cos(-2^s \dfrac{\nu}{\nu-1} x + k)\pi, & 2^s < \nu < 2^{s+1} \\ \sin\left(-2^{s+1}\dfrac{\nu}{\nu-1} x \right. \\ \left. +2k\right)\pi + \cos(-2^s \frac{\nu}{\nu-1} x + k)\pi, & 2^{s+1} \geq \nu \end{cases}
\end{aligned}
\tag{6.15}
$$

with $0 \leq n-1 < \nu < n$.

Proof : Let us start from Equation (3.10), and the representation (4.25), because of the linearity of the operator we have

$$
D^{\nu}_{S} f(x) = \sum_{h=-\infty}^{\infty} \alpha_h D^{\nu}_{S} \varphi_h(x) + \sum_{m=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta^m_h D^{\nu}_{S} \psi^m_h(x)
$$

where the wavelet coefficients α_h, β^m_h are given by (4.24) [or (4.26)]. From here, by using (6.6) and (6.11), we get (6.15).

□

In particular, with $n = 1$ we have

Theorem 13. *The sinc-fractional derivative of the wavelet representation* (4.25) *of function $f(x) \in \mathcal{B} \subset L_2(\mathbb{R})$, with order $0 < \nu < 1$, is*

$$
D^{\nu}_{S} f(x) = 2\pi P(\nu) \frac{1-\nu}{\nu} \sum_{h=-\infty}^{\infty} \alpha_h \sum_{k=-\infty}^{\infty} \lambda^{(1)}_{hk} \operatorname{sinc}\left(x + \frac{\nu}{1-\nu} k\right), \quad 0 < \nu < 1
\tag{6.16}
$$

Proof : Follows directly from Equation (6.15).

6.3. Example: Fractional Derivative of the Gaussian Function

In order to show the efficiency of the proposed method for the computation of a fractional derivative, let us consider the function e^{-x^2}. A good approximation of this function, in terms of Shannon wavelet expansion (4.25), can be obtained as

$$
\begin{aligned}
e^{-x^2} &\cong \sum_{h=-1}^{1} \alpha_h \varphi_h(x) + \sum_{n=0}^{0} \sum_{h=-1}^{1} \beta^n_h \psi^n_h(x) \\
&\cong \alpha_{-1}\varphi_{-1}(x) + \alpha_0 \varphi(x) + \alpha_1 \varphi_1(x) + \\
&+ \beta^0_{-1}\psi^0_{-1}(x) + \beta^0_0 \psi^0_0(x) + \beta^0_1 \psi^0_1(x)
\end{aligned}
$$

where

$$
\alpha_{-1} = \alpha_1 = 0.123,\ \alpha_0 = 0.30,\ \psi^0_{-1} = \psi^0_1 = 0.004,\ \psi^0_0 = 0.001\,.
$$

If we neglect also the detailed coefficients β^n_k the approximate Shannon wavelet representation is

$$
e^{-x^2} \cong 0.123\,\varphi_{-1}(x) + 0.30\,\varphi(x) + 0.123\,\varphi_1(x)\,.
$$

From (6.16) we have

$$
D^{\nu}_{S} e^{-x^2} \cong 2\pi P(\nu) \frac{1-\nu}{\nu} \sum_{h=-1}^{1} \alpha_h \sum_{k=-1}^{1} \lambda^{(1)}_{hk} \operatorname{sinc}\left(x + \frac{\nu}{1-\nu} k\right), \quad 0 < \nu < 1\,.
$$

The matrix $\lambda^{(1)}_{hk}$, according to (5.3) is

$$
\lambda^{(1)}_{-1-1} = \lambda^{(1)}_{00} = \lambda^{(1)}_{11} = 0\,,\ \lambda^{(1)}_{0-1} = -\lambda^{(1)}_{-10} = \lambda^{(1)}_{10} = -\lambda^{(1)}_{01} = 1\,,\quad \lambda^{(1)}_{-11} = -\lambda^{(1)}_{1-1} = \frac{1}{2}
$$

so that by simplifying we get

$$
D^{\nu}_{S} e^{-x^2} \cong 2\pi P(\nu) \frac{1-\nu}{\nu} (\alpha_0 - \frac{1}{2}\alpha_1) \left[\operatorname{sinc}\left(x - \frac{\nu}{1-\nu}\right) - \operatorname{sinc}\left(x + \frac{\nu}{1-\nu}\right) \right], 0 < \nu < 1
$$

that is

$$
D^{\nu}_{S} e^{-x^2} \cong 0.47\pi P(\nu) \frac{1-\nu}{\nu} \left[\operatorname{sinc}\left(x - \frac{\nu}{1-\nu}\right) - \operatorname{sinc}\left(x + \frac{\nu}{1-\nu}\right) \right], 0 < \nu < 1\,.
$$

CONCLUSION

Sinc function is playing a fundamental role in mathematics and physics. Due to the many properties of this function it deserves a special role in applications. In recent years some Authors have proposed [20] a fractional derivative based on this function. Moreover a wavelet theory based on the sinc function has been settled thus extending the many features of the Sinc. In this paper the sinc-fractional derivative has been extended to the Shannon wavelet space, in order to give the explicit analytical form of the fractional derivatives of functions belonging to the wavelet space. It has been shown that the sinc-fractional derivative is the most natural and suitable choice of fractional operator when dealing with functions that can be represented as Shannon wavelet series.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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Linear Viscoelastic Responses: The Prony Decomposition Naturally Leads Into the Caputo-Fabrizio Fractional Operator

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The study addresses the physical background and modeling of linear viscoelastic response functions and their reasonable relationships to the Caputo-Fabrizio fractional operator via the Prony (Dirichlet series) series decomposition. The problem of interconversion with power-law and exponential (single and multi-term functions) has been discussed. Special attentions have been paid on the Prony series decomposition approach, the related interconversion problems and the expression of the viscoelastic constitutive equations in terms of Caputo-Fabrizio fractional operator.

Keywords: linear viscoelasticity, response functions, interconversion, power-law response, non power-law responses, prony series, caputo-fabrizio operator

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Nothing is done in a vacuum; we must all stand on our forefathers to better ourselves and the world around us.

(Sir Isaac Newton, in his letter to rival Robert Hooke in 1676)

Science is built up of facts, as a house is built of stones; but an accumulation of facts is no more a science than a heap of stones is a house.

(Henri Poincare)

1. INTRODUCTION

This article addresses the physical background of modeling of dissipative phenomena, precisely response functions such as stress-strain relationships in the framework of the linear viscoelasticity and their reasonable relationships to the Caputo-Fabrizio fractional operator [1] via the Prony (Dirichlet series) series decomposition [2].

The appearance of the new definition of fractional derivatives with non-singular kernels was provoked by needs to model dissipative transport processes in many new materials appearing in modern technologies [1]. Recently the main achievements, especially results related to diffusion problems, were analyzed Hristov in [3] and we will avoid the thorough browsing and comments of published results (see also the rich list of reference in Hristov [3]). In the context of diffusion problems it was demonstrated that the Caputo-Fabrizio operator appears naturally in diffusion models [3, 4] when the flux-gradient relationship is expressed by a Jeffrey relaxation kernel and the Maxwell-Cattaneo concept of flux.

At the same time the Caputo-Fabrizio fractional operator was criticized [5–7] with points of view based on the classical fractional calculus with singular power-law memory kernel and with examples from the signal processing [6] and to some extent touching formal rheological relationship [5]. Despite the mathematical exactness of the counterexamples they do not focus the attention on the physical basis leading to exponential memory kernels. As a matter of fact, we need a clear answer what really this new operator models and how it appears in the modeling of physical problems, despite the fact that some properties known from the classical fractional calculus, such as the index law [6] , are not satisfied.

1.1. Motivation of This Study

The analysis done here and the principle task are oriented to modeling of viscoelastic constitutive relationships in terms of Caputo-Fabrizio operator naturally appearing through Prony series decomposition [2] of stress relaxation functions and not obeying power-law behaviors. That is, we focus on viscoelastic materials with dynamics which cannot be modeled with the classical fractional derivatives of Riemann-Liouville and Caputo [8].

This introduction avoids huge citations of works on Caputo-Fabrizio operators since without understanding of the physical background and the logic of its appearance any analysis of models created by formalistic fractionalization (see the comments in Hristov et al. [3]) is unproductive. Moreover, to the author personal experience as editor many manuscripts devoted to applications of the Caputo-Fabrizio derivative in formalistically fractionalized existing models are directly rejected by the reviewers with motivations based of the opinions in the criticizing articles [5–7]. This situation resembles that in the Catch 22 movie without perspective for escape. The existing situation is a consequence of some main reasons: (1) The Caputo-Fabrizio operator does not hold some properties such as the semi-groups, which are existing with the classical fractional derivatives with singular kernels, the strange form of the associated fractional integral and these issues cast doubts when it is applied inadequately (in blind manner) to various functions in a manner known from the integer-order calculus. (2) The formalistic approach by simple replacements of integer or fractional order (with singular kernels) derivatives with the Caputo-Fabrizio operator in existing models without taking into account the physics behind. (3) Last but not least, the human factor of author's rivaling which is dividing the scientific society into competing groups rejecting the achievements of each other. All these elements of the current situation create a discouraging atmosphere and disbelieve that this new fractional operator really cannot model natural phenomena and actually stops the further research, generally among the young researchers highly aspiring publications of submitted manuscripts.

The deep physics behind the fractional operator with exponential kernel (and the author's long time experience in science) motivated this study and the efforts are oriented to show that the existing knowledge and models, as well as techniques of data treatment, in the framework of linear viscoelasticity, lead naturally to formulation of the Caputo-Fabrizio fractional operator. This is in the context of the Sir Isaac Newton quote at the beginning of the article: the steps ahead on the shoulder of existing facts and results on the road to creations of new information are natural ways and actually the exciting moments in the beautiful journey in the world of science.

1.2. Aim and Paper Organization

This article is organized as follows: section 3 presents briefly the main properties of the Caputo-Fabrizio operator that will be used further in the analysis of the viscoelastic problems. Section 3 addresses the constitutive equations of viscoelasticity based on the Boltzmann superposition principle [9] and fading memory approach incorporated the hereditary stress and strain integrals. The principle problems of the quality and choice of the response function are discussed and the main properties requited are outlined in section 3.3. The interconversion of relaxation and creep response functions is discussed in section 4: the linear and non-linear scale-invariant (power-law) (related to application of Riemann-Liouville and responses are analyzed. Section 5 focuses on decompositions of experimental response functions by Prony series leading to discrete spectra and the related interconversions by examples with single term, two-terms and multi-term responses of exponential type, as well as interconversion of relaxation and creep compliance expressed as Prony series. Section 6 demonstrates how the constitutive equations (based on the Maxwell model) can be expressed in terms of the Caputo-Fabrizio fractional operator. The subsection 6.3 demonstrates that approximation of the response function by Bessel functions (of Maxwell-like materials) of first kind and expressed as infinite Dirichlet series naturally leads to incorporation of the Caputo-Fabrizio operator in the constitutive equations. Section 7 demodulates briefly how constitutive relationship (following the idea of Bagley and Torvik) with two fractional operators of Caputo-Fabrizio (of different orders) can be formulated.

2. CAPUTO -FABRIZIO OPERATOR

2.1. Definition

The Caputo -Fabrizio operator is defined as [1]

$$ {}_{cf}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] \frac{df(s)}{dt} ds, \quad 0 < \alpha < 1 \tag{1} $$

In Equation (1) $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. This definition is of Caputo-type because there is a convolution of the derivative $df(t)/dt$. The explanations in [1] relate the development of Equation (1) to the classical Caputo derivative [8] by mechanistic replacements of the singular kernel (by a non-singular exponential kernel) and the normalization function (see the explanations in Hristov [3].

From Equation (1) it follows that if $f(t) = C = const.$, then ${}_{cf}D_t^{\alpha} C = 0$ as in the classical Caputo derivative [8]. Actually, Equation (1) is a convolution of $f(t)$ and the convolution operator K [1]

$$ K = \exp\left[-\frac{\alpha}{1-\alpha}(t-s)\right] \frac{d}{ds} \tag{2} $$

The integration by parts in Equation (1) results in an alternative form Caputo et al. [10]

$$ {}_{cf}^{c}D_{t}^{\alpha}=\frac{1}{1-\alpha}f(t)-\frac{\alpha}{1-\alpha}\int_{a}^{t}f(s)\exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]ds,t>a \tag{3}$$

Futher, the integer order differentiation of ${}_{cf}^{c}D_{t}^{\alpha}$ follows the rule [1]

$$ {}_{cf}^{c}D_{t}^{(\alpha+n)}f(t)={}_{cf}^{c}D_{t}^{\alpha}\left(D_{t}^{(n)}f(t)\right),\quad n>0,\quad \alpha\in[0,1] \tag{4}$$

The associated fractional integral is Hristov [3] and Losada and Nieto [11]

$$ {}_{cf}^{c}D_{t}^{(\alpha+n)}f(t)={}_{cf}^{c}D_{t}^{\alpha}\left(D_{t}^{(n)}f(t)\right),\quad n>0,\quad \alpha\in[0,1] \tag{5}$$

With the assumption that $M(\alpha)=1$ used by Caputo and Fabrizio [1, 10]. The second definition of Losada and Nieto [11] is

$$ {}_{cf}D_{t}^{\alpha}f(t)=\frac{1}{1-\alpha}\int_{0}^{t}\exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]\frac{df(s)}{dt}ds \tag{6}$$

Hereafter we will use the definition (Equation 6).

2.2. Laplace Transform

The Laplace transform of ${}_{CF}D_{t}^{\alpha}$ with $a=0$ has the following Laplace transform L_T given with p variable [1] taking into account the general rule of Laplace transform of a convolution, namely

$$L_T\left[{}_{cf}^{c}D_{t}^{\alpha}f(t)\right]=\frac{1}{1-\alpha}L_T\left[f(t)\right]L_T\left[\exp\left(-\frac{\alpha}{1-\alpha}t\right)\right] \tag{7}$$

Hence,

$$L_T\left[{}_{cf}^{c}D_{t}^{\alpha}f(t)\right]=\frac{pL_T\left[f(t)-f(0)\right]}{p+\alpha(1-p)} \tag{8}$$

2.3. Fractional Derivative of Elementary and Transcendental Functions

Linear function $f(t)=Ct$ [1]

$$ {}_{cf}^{c}D_{t}^{\alpha}[Ct]=\frac{1}{1-\alpha}\int_{0}^{t}C\exp\left[-\frac{\alpha}{1-\alpha}(t-s)\right]ds=\frac{C}{\alpha}\left[1-\exp\left(-\frac{\alpha}{1-\alpha}t\right)\right],\quad 0<\alpha\leq 1 \tag{9}$$

Power-Law function $f(t)=Ct^{\beta}$ [3, 12].

For $f(t)=Ct^{\beta}$ and $\beta>0$ the fractional derivative ${}_{cf}^{c}D_{t}^{\alpha}\left[Ct^{\beta}\right]$ is

$$ {}_{cf}^{c}D_{t}^{\alpha}t^{\beta}=\left(C\beta t^{\beta-1}\right)\frac{1}{\alpha}\left[1-\exp\left(-\frac{\alpha}{1-\alpha}\right)t\right] \tag{10}$$

For $\alpha=1$ the expression (Equation 10) reduces to the classical (integer-order) result $C\beta t^{\beta-1}$.

Exponential function $f(t)=\exp(\beta t)$ [3, 12]

$$ {}_{cf}^{c}D_{t}^{\alpha}\exp(\beta t)=\frac{\beta\exp(\beta t)-\exp(-At)}{\beta+\alpha(1-\beta)},\quad A=\frac{\alpha}{1-\alpha} \tag{11}$$

For $\alpha\rightarrow 1$ the second exponential term in the nominator of Equation (11) goes to zero and therefore $\underbrace{{}_{cf}^{c}D_{t}^{\alpha}\exp(\beta t)}_{\alpha\rightarrow 1}\rightarrow\beta\exp(\beta t)$.

For $\beta<0$ we have

$$ {}_{cf}^{c}D_{t}^{\alpha}\exp(-\beta t)=\frac{-\beta\exp(-\beta t)+\exp(-At)}{A-\beta} \tag{12}$$

For $\alpha\rightarrow 1$ the second exponential term in the nominator of Equation (12) goes to zero and therefore $\underbrace{{}_{cf}^{c}D_{t}^{\alpha}\exp(-\beta t)}_{\alpha\rightarrow 1}\rightarrow -\beta\exp(-\beta t)$.

2.4. Caputo-Fabrizio Fractional Operator: Determination of the Fractional Parameter

In the Caputo-Fabrizio operator there is formal ambiguity because the stretched time is multiplied by a dimensional factor $\alpha/(1-\alpha)$ which should have a dimension s^{-1}. This contradicts the definition of the fractional order (parameter) because physically α is dimensionless. The answer given in [3, 13] resolved the problem by nondimesionalization of the exponential function by help of characteristic time scale of the relaxation process t_0 (the maximum time of the experiment in the sense of the rheological tests discussed here) , namely

$$\exp\frac{(t-s)}{\tau}=\exp\frac{(t/t_0-s/t_0)}{\tau/t_0}=\exp\frac{(\bar{t}-\bar{s})}{\bar{\tau}} \tag{13}$$

The nondimensalization does not change the meaning of the exponential relaxation function but avoid any doubts about the definition of the fractional order α as [3, 13].

$$\frac{1-\alpha}{\alpha}=\frac{\tau}{t_0}\Rightarrow\alpha=\frac{1}{1+\tau/t_0} \tag{14}$$

and relates it to data that can be really recovered from experimental data, such the relaxation and retardation times (see the sequel)

The relationship (Equation 14) says that for $\tau/t_0=1$ we get $\alpha=1/2$. Further, depending on the ratio τ/t_0 we may have fractional orders roughly arranged in two groups [14]: a) when $0\leq\tau/t_0\leq 1$ we have fractional orders $\alpha\in[0.5,1.0]$ and the relaxation time τ is shorter than the macroscopic process observation time t_0, and b) $1 \leq \tau/t_0 < \infty$ the fractional orders are $\alpha \in (0, 0.5]$ since the relaxation time τ is larger than the macroscopic process observation time t_0. Qualitatively, for $\tau/t_0 < 1$ the relaxations could be considered as *fast* (rapid) *relaxations*, while $\tau/t_0 > 1$ is related to *slow relaxations* (see in Hristov [14] the comments of numerical values of Prony decomposition and relaxation times)

Last to this point, but not least, as it was commented in Hristov [14], the ratio τ/t_0 is not integer and expressing the memory kernel as $\exp\left[-\beta\left(\bar{t}-\bar{s}\right)\right]$ where $\beta = (\tau/t_0)^{-1} = \alpha/(1-\alpha)$ we get a fractional operator. *Therefore, the memory kernel of the Caputo-Fabrizio operator is controlled by a non-integer parameter in the context of what is needed to say that this operator is fractional, despite the fact that it does not repeat exactly the properties of the Classical Riemann-Liouville and Caputo derivatives.*

3. CONSTITUTIVE EQUATIONS OF VISCOELASTICITY: FADING MEMORY APPROACH

3.1. Boltzmann Superposition Principle and Fading Memory Concept

The fading memory concept relating the flux to its gradient, for simple materials [15–17], is expressed by the following relation relating the flux and the gradient , namely

$$j(x,t) = -D_0 \nabla C(x,t) - D' \int\limits_{-\infty}^{t} R(t-\tau)\nabla C(x,\tau)\, d\tau \tag{15}$$

This definition is actually the Boltzmann linear superposition functional (Equation 16))

$$\varphi(x,t) = m\left[v_x(x,t)\right] + \lambda \int\limits_{0}^{t} R(t,\tau)\, v_x(\tau)\, d\tau \tag{16}$$

relating the present state of the flux to its history [9, 16–18] through the influence function (memory kernel) $R(t,z)$ during the time interval defined by τ. In Equation (16) m and λ are transport coefficients (diffusivities) with real physical meanings as it will be demonstrated in the sequel.

The memory function could be *unbounded and scale invariant* such as $R(t,\tau) = t^{-\mu}$ with integration singularity at or *bounded and not scale invariant* such as $R(t,\tau) = e^{-t/\tau}$ (see the analysis in Hristov [14] and the comments further in this article).

3.2. Stress-Strain Viscoelasticity Response and Hereditary Integral Construction and Response Functions

The linear theory of elasticity [19] (chapter 1) considers the stress in a sheared solid body as a quantity proportional to *the shear*, while in the liquids the shearing stresses are proportional to the *rate of shear* . Most solid materials, for example polymers, compromising both effects are called *viscoelastic*. When a slab of solid material under a shearing motion caused by a step change in the stress load applied to it (Heaviside unit step function $H_0(t)$) exhibits a strain (in one dimension) [19] (chapter 1)

$$\varepsilon(t) = \varepsilon_0 H_0(t) \tag{17}$$

With perfect elastic behavior of the body [19] $\sigma(t) = \sigma_0 H_0(t)$ for $t > 0$. On the contrary, in an ideal viscous fluid the stress is infinite and for $t > 0$ and the strain is $\varepsilon(t) = (\sigma_0/\eta)\, t$, thus introducing *the coefficient of viscosity* η. Real materials *do not shear with infinite speeds* that is the reason of the concept of *a finite relaxation time* τ [19]. Precisely, in solids the stresses attain finite values for long times. In contrast, in viscous fluids the stresses approach zero.

The task of this study addresses *the functional representations of the viscoelastic material responses*, that is : 1) the *stress relaxation function* $R(k,t)$, that is the stress history due to a shear step of size ε, and 2) *the creep function* $C(t)$ (shear history) due to unit stress σ applied. In the linear viscoelastic theory [19] the responses can be approximated as $R(\varepsilon,t) = G(t)\varepsilon + O(\varepsilon^3)$ and $C(\sigma,t) = J(t)\sigma + O(\sigma^3)$, thus defining the *linear stress relaxation modulus* $G(t)$ and the *linear creep compliance* $J(t)$. The common mechanistic models explaining the behavior of viscoelastic materials utilize linear springs and dashpots coming from the Maxwell interpretation [20–22] and modeling the *pure elastic behavior* and the *pure viscous state*, correspondingly.

Superposition of single-step material responses results in functional relationships of stress and strain in the linear viscoelasticity concept [19] incorporating the a time lag in $G(t)$ and $J(t)$ through hereditary stress (Equation 18) and creep integrals (Equation 19), namely

$$\sigma(t) = \int\limits_{0}^{t} G(t-s)\, d\varepsilon(s) \tag{18}$$

$$\varepsilon(t) = \int\limits_{0}^{t} J(t-s)\, d\sigma(s) \tag{19}$$

Both $\sigma(t)$ a and $\varepsilon(t)$ are causal functions and therefore the lower terminals in Equations (18, 19) are set at $t = 0$.

Applying the fading memory concept [9, 20] it is possible to relate *the instantaneous responses* G_∞ and J_∞ corresponding to equilibrium states (for long times) when the effects of memory terms (convolution integrals) fade out, namely

$$\sigma(t) = G_\infty + \int\limits_{0}^{t} G(t-s)\, d\varepsilon(s) \tag{20}$$

$$\epsilon(t) = J_\infty - \int\limits_{0}^{t} J(t-s)\, d\sigma(s) \tag{21}$$

3.3. Response Function: General Properties and Requirements

For a *linear isotropic viscoelastic body* exerting uniaxial stress the Boltzmann superposition principle formulates the constitutive equation relating the strain responses by help of the following hereditary integral [23]

$$\sigma(t) = \int_0^t G(t-s)\frac{d\varepsilon(s)}{d\tau}ds \tag{22}$$

The relaxation function $G(t)$ should decrease monotonically and account for short and long-time strains, thus the condition (Equation 23) should be obeyed [24–26]

$$(-1)^n \frac{\partial^n}{\partial t^n} G(t) \geq 0, \quad n = 1, 2, ... \tag{23}$$

Coleman and Noll, in their article on foundation principles of linear viscoelasticity [27], point out that there is no universal approach in definition of unique relaxation function $R(s)$ [the fading memory $R(s)$ in Equation (15) its is equivalent to $G(s)$ in Equation (22) . Despite this, two principle features are required to be obeyed by $R(s)$:

1) $R(s)$ is defined for $0 \leq s < \infty$ and $R(s) > 0$, and
2) $R(s)$ decays monotonically to zero for large s , that is $\lim_{s\to\infty} s^r R(s) = 0$.

Boltzmann (1874) in [9] (see also [14, 27]) suggested two memory functions widely applied so far, namely

a) Generalized power-law memory function $R(s) = s^r(s+1)^{-\mu}$ of order r when $r < \mu$. For $r = 0$ the memory function $R(s)$ is *unbounded* (singular) at t_{0+} and this *scale invariant* kernel forms the constructions of the classical fractional integrals and derivatives [8].

b) *Ordinary memory kernel* , *bounded* at t_{0+} as exponential function $e^{-\beta s}$, which is *not invariant with respect to the time scale*. The definition of Caputo-Fabrizio fractional operator utilizes this memory kernel.

The *adequate mathematical structure* in the construction of the relaxation (memory) function is the main problem that should be resolved in the modeling of viscoelastic responses. Generally, the relaxation function should adequately describe the natural process and therefore its structure should be tested (defined)by a data conversion algorithm (by fitting experimental data). Despite this intuitive and to greater extent logical approach the response function should satisfy some requirement defined in Garbarski [28] and Winter [29], and summarized in **Table 1**; with some author's comments related to application of fractional operators in the viscoelastic constitutive equations. Further, in the response of viscoelastic material to strain excitation, the stress relaxation spectrum (relaxation modulus) provides complete information concerning the time-dependent part of the material response. In the opposite situation, with materials undergoing stress excitations the retardation spectra (the compliances) [30] provide the required information. Hence, if the spectra are known it is possible to calculate the response of any excitation [20].

Besides, the thermodynamic theory of linear viscoelastic materials was developed extensively in the sixties of the last century [31]. In this context, the second law of thermodynamics yields severe restrictions on the constitutive properties [32–34]. This problem is beyond the scope of the present analysis but we may say that when the constitutive equation (Equation 20) contains exponential kernel it is compatible to second law of thermodynamics [34].

The principle relaxation functions used and the applicability of the aforementioned requirements will be analyzed next.

4. INTERCONVERSION OF RELAXATION AND CREEP: PROBLEM AND POWER-LAW RESPONSES

4.1. Interconversion Problem

In the linear viscoelastic materials the relationships between the creep compliance $J(t)$ and the relaxation response $G(t)$ are expressed as convolution integrals in accordance with the Boltzmann superposition principle [9] (see Equations 20, 21)

With the Laplace transforms of Equations (20, 21) we get equivalent relationships in the p -space (p is the transform variable)

$$\sigma(p) = pG(p)\,\varepsilon(p), \quad \varepsilon(p) = pJ(p)\,\sigma(p) \tag{24}$$

Hence, we may recast Equation (24) as

$$\frac{\sigma(p)}{\varepsilon(p)} = pG(p), \quad \frac{\sigma(p)}{\varepsilon(p)} = \frac{1}{pJ(p)} \tag{25}$$

Equating the right-hand sides of Equation (25) we get an implicit fundamental relationship

$$G(p)\,J(p) = \frac{1}{p^2} \Rightarrow J(t-s)\,G(t) = \int_o^t G(t-s)\,J(s)\,ds = t \tag{26}$$

with the constraints that

$$G(t_{0+})\,J(t_{0+}) = G(t\to\infty)\,J(t\to\infty) \tag{27}$$

Explicit form of Equation (26) can be obtained if the analytical forms of either $G(t)$ or$J(t)$ is known. Examples of interconversion related to power-law and exponential memory kernels are discussed in the sequel.

4.2. Scale-Invariant (Power-Law) Response

4.2.1. Physical Preliminaries Related to the Power-Law Response

Relatively short-time relaxation history modeled by time-dependent power-law can be observed in the time evolution of $G(t)$ from zero to the end of the time of observation t_0 [19]: commonly at the beginning of the relaxation process (short

TABLE 1 | Response function desired properties.

	Properties of the Response function [28]	Properties of the data conversion algorithm [29]	Comments (present work)
1	Be as simple as possible due to consequent practical applications	Good fit of the experimental data	
2	Should be positive and monotonically decreasing since non-monotonically decreasing functions have no physical meanings.	Avoidance of overfitting, that is the algorithm should be able to find optimum amount of details (i.e., parameter)	
3	After integration the function should be convergent to a limited value at infinite time	The format of the relaxation function should not be predetermined. It has to be freely adjustable during the data fitting	This is a general comment but working with fractional derivatives we have a limitation of kernels (response functions) that can be use
4	To have a simple Laplace transform	The resulting material parameter should have physical meaning	
5	Flexible to be adjusted to experimental data taken from relaxation tests.	Minimization of truncation error	
6	To allow calculations of consequent parameters of the viscoelastic relaxations such as spectra, moduli, etc.	Checking the of experimental data quality	This important since the parameter determinations are ill-posed problems
7	Possibility to be tabulated if analytical representation is not possible	For practical use $R(s)$ should be expressed by a function or a sum which can be easy integrable in various viscoelastic models and calculations	
8	The integral of the spectrum should be convergent to a finite value	For practical reasons the number of parameters should small as much as possible	This point is very important in the construction of relaxation kernels of fractional operators

times) and along the long tail (long times). The power law is *scale-invariant*. Materials exhibiting such behavior are called *power-law viscoelastic materials* [35–42] and can be easily detected by the linear behaviors of $J(t) \sim t^p$ and $G(t) \sim t^{-p}$ in logarithmic scales [43]. Actually, we have to memorize that this type of relaxation was modeled in the article of Caputo (1967) where the construction of the Caputo derivative was conceived [44]. The power-law functional relationships of the relaxation and the compliance, actually are the reasons to recognize the integral and derivatives of the classical fractional calculus [8] as adequate modeling tools since they are based on the same type of kernels [39, 42, 45], known also as weak singular kernels. As commented by Tschoegl [20] the scaling relationship $G(t) \sim t^{-p}$ is a fractional version of the Trouton law $\sigma(t) = \eta\dot{\varepsilon}(t)$. Here, we have to mention that Metzler et al. [39] discussing relaxations of filled /polymers stated that non-exponential (non-Debay) relaxation implies memory (Sic!). We may consider this declarative opinion (the results of Glockle and Nonnemacher [46]) as adequate to the situation in the 90s of the last century, when fractional calculus was only related to power-law kernels [47].

Moreover, as a valuable analysis showing when the *power-law* is *the adequate response function* we refer to Chapter 5 of Findley et al. [48] where it is clearly demonstrated that for many materials *the power-law response function* t^{μ} with $\mu < 0.5$ [48] is *a good short-time approximation* for materials such as plastics, metals and concrete. For short-time loading the creep of many different rigid plastics with sufficient accuracy can be presented as linear power-law $\varepsilon = \varepsilon_0 + \varepsilon_+ t^{\mu}$, where μ is stress-independent and nearly temperature-independent, too. The same approach following from the linear approximation approach of Pipkin [19] is commented in 3.2 . In this case the corresponding retardation spectrum is Findley et al. [48]

$$\varphi(t) \equiv \frac{\mu}{\Gamma(1-\mu)} t^{\mu-1} \equiv t^{-\alpha}, \quad 0 < \alpha = 1 - p < 1 \qquad (28)$$

The function (Equation 28) may be considered as *continuum spectrum of material retardation times* proportional to $t^{-\alpha}$. From this position, the step from Equation (28) toward modeling of the relaxation processes by the classical Riemann-Liouville derivative is straightforward. In this context, Bagley and Torvik [49] using (Equation 28) demonstrated that following constitutive equation holds

$$\sigma(t) = G_1\, {}^{RL}D_t^{\alpha}\left[k(t)\right] = \frac{G_1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{k(t-s)}{\alpha^k} ds, \quad t > 0 \qquad (29)$$

After application of the Leibniz rule to Equation (29) the result is

$$\sigma(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{G_1}{t^{\alpha}} \frac{dk(t-s)}{dt} ds + G(t)\, k(0) \qquad (30)$$

For $\varepsilon(0) = 0$, since no strain at $t = 0$ exists we get

$$\sigma(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{G_1}{(t-s)^{\alpha}} \frac{dk(s)}{ds} ds = G_1\, {}^{RL}D_t^{\alpha} k(t), \quad t > 0 \qquad (31)$$

Then, for $G_1 = G_0 = const$. the stress relaxation modulus $G(t)$ and the creep compliance $J(t)$ are

$$G(t) = \frac{G_1}{\Gamma(1-\alpha)} t^{-\alpha}, \quad J(t) = \frac{1}{G_1} \frac{1}{\Gamma(1-\alpha)} t^{\alpha} \tag{32}$$

4.2.2. Example 1. Interconversion of Power-Law Relaxation and Creep: Linear Case

Here, we consider again the power-law response in the context of the response function interconversion and in the sense of the relationships (Equations 26, 27). With $G(t) = At^{-\mu}$, where A is a data fitting coefficient and $0 < \mu < 1$, the Laplace transform in Equation (26) provides a power-law creep (Equation 33) compliance [50]

$$G(t) = \frac{G_1}{\Gamma(1-\alpha)} t^{-\alpha}, \quad J(t) = \frac{1}{G_1} \frac{1}{\Gamma(1-\alpha)} t^{\alpha} \tag{33}$$

Following Hanyga, neither experimental nor theoretical reasons lead to the assumption that the creep rate function and the stress relaxation function are bounded and regular at $t = 0$ [26]. The same point of view is valid for the unbounded kernels singular at $t = 0$ [14]. In the seminal study of Boltzmann [9] the value of $\alpha = 1$ was suggested. The condition $0 < \alpha < 1$ comes from the fact that for $\alpha > 1$ we get an infinite propagation speed [26]. Actually, the idea to apply fractional calculus, as is demonstrated by the model developed in the foundation studies of Scott-Blair [51–53] and Bagley and Torvik [49] comes from the experimental (empirical) findings of Nutting [54, 55]: precisely, for some viscoelastic materials the power-law relationship $[\varepsilon(t)/Const\ ] \sim \sigma^n t^{\alpha}$ is satisfied. Consequently, we may obtain (Equation 32).

Since the power-law relaxation is not the main problem discussed in this work, at the end of this point, we refer to the study of Carillo and Giorgi [56] (section 4 in this chapter) where it is clearly demonstrated that *the response of the material* may be for *short-time* or as *long tail* (for long times) *modeled by time-dependent power-law function*. Here, we may repeat again (see the comments in Hristov [14], too) that, *if the power-law is not exhibited by the material response functions then it is inadequate to apply the power-law memory kernels.*

4.2.3. Example 2.Interconversion of Power-Law Relaxation and Creep: Non-linear Case

If the material exhibits a non-linear viscoelastic behavior, for instance, as commented by Lakes et al. [50], that is

$$\sigma(t) = \int_0^t G[(t-s), \varepsilon(s)] \frac{d\varepsilon}{ds} ds,$$
$$\varepsilon(t) = \int_0^t J[(t-s), \sigma(s)] \frac{d\sigma}{ds} ds \tag{34}$$

The relaxation function $G(t,\varepsilon)$ can be expressed as a sum of products, precisely sum of products of functions of time and functions of strain, that is Lakes and Vanderby [50] $G(t,\varepsilon) = G_t(t)\, g(\varepsilon)$. Assuming no interaction between the steps (immediate and delayed Heaviside steps) strains in the summation series and may write [50]

$$\sigma_c = \varepsilon(0)\, G(t,s) + \sum_{i=0}^{N} \Delta\varepsilon_i G(t - t_i, \varepsilon) \tag{35}$$

Consequently, the creep compliance is

$$1 = J(0)\, G(t,\varepsilon) + \int_0^t G[(t-s), \varepsilon(s)] \frac{dJ(s)}{ds} ds \tag{36}$$

and for the linear case we will obtain the relationship (Equation 26).

To have explicit form of these relationships, if we assume *a power-law approximation in time* [50] $J(\sigma,t) = A(\sigma)\, t^{\mu}$, that is $J(\sigma,t) = \left(g_1 + g_2\sigma + g_3\sigma^2 + ...\right) t^{\mu}$, then we get

$$1 = J(0)\, G(t,\varepsilon) + \int_0^t G[(t-s), \varepsilon(s)] \frac{dJ(s)}{ds} ds \tag{37}$$

$$G(t,\varepsilon) = \left[f_1 t^{-\mu} + f_2 \varepsilon(t)\, t^{-2\mu} + f_3 t^{-3\mu}\right] \tag{38}$$

The nonlinear material exhibits a *relaxation response* which contains *a sum of power-law terms*, as given in equation (Equation 38) (see more comments in Lakes and Vanderby [50]).

Hereafter, non-linear behavior generally related to temperature-dependent and aging viscoelastic materials will not be discussed since the principle task of the present analysis is to show the correct origin and the physical background leading to the formulation of the Caputo-Fabrizio fractional operator.

5. INTERCONVERSION OF NON POWER-LAW RELAXATION AND CREEP

5.1. Non Power-Law Response: Relaxation Curve Approximation

The selection of approximation function which would fit the experimental points depends on the type of materials and would be established by simple trial error process. In this context, two principle issues arise when experimental data for linear viscoelastic materials should be treated [57]: (1) Appropriate relaxation curve approximation and (2) interconversion problem. The experimental data are commonly taken in the time-domain (or frequency-domain) tests. Now, we stress the attention on discrete spectrum approximation by Prony series [2] where the condition for the monotonicity of the functions (see the sequel) is satisfied.

5.2. Discrete Relaxation Spectrum by Prony Series Decompositions and Relevant Interconversions

Unlike the linear elasticity, where the material functions are related algebraically, the relationships in the linear viscoelasticity are time-dependent, as already was demonstrated in the previous section of the article. One known method is to find the unknown functions $G(t)$ and $J(t)$ by fitting the data points by a known function. However, when the focus is on the adequate construction of a fractional operator with a memory kernel obeying all mandatory conditions imposed on it,then the choice of approximating function is highly restricted. If the power-law is not the adequate choice then a series approximation is the more suitable solution. Since we address the construction of the Caputo-Fabrizio operator,thus it is natural to tackle data fitting by series of exponential terms (matching the function of the desired memory kernel) known as Prony series (also as Dirichlet series [58]).

5.2.1. Prony Series Decompositions

The viscoelastic relaxation function can be expressed by a discrete relaxation spectrum through a decomposition as a Prony series $g_P(t)$with N^{ϕ} terms [25, 59–64] with rate constants β_i, namely

$$g_P(t) = g_\infty + \sum_{i=1}^{N^\phi} g_i e^{-\beta_i t} = g_\infty + \sum_{i=1}^{N^\phi} g_i e^{-\frac{t}{\tau_i}}, \quad \beta_i = \frac{1}{\tau_i} \geq 0 \tag{39}$$

or through normalized weights (amplitudes or normalized relaxation moduli) λ_i [21, 59] as

$$\frac{g_P(t)}{g_\infty} = 1 + \sum_{i=1}^{N^\phi} \lambda_{gi}\left(e^{-\beta_i t} - 1\right), \quad \lambda_{gi} = \frac{g_i}{g_\infty} \tag{40}$$

In Equations (39, 40) the parameters g_∞ and g_i are *equilibrium* (at large times) values and the *relaxation moduli* (stiffness), respectively, are constrained according to Brinson and Brinson [21]

$$g_\infty + \sum_{1}^{N^\phi} g_i = 1 \tag{41}$$

The Prony series components have spectral strength g_i and relaxation time τ_i. This is the so-called *discrete Prony series representation* known also as *discrete relaxation spectrum* [20, 65–69]. Besides, the series approximation (Equation 39) may be substituted in formulation of the materials law such as Maxwell, Kelvin Voigt, etc. The relaxation time τ_i associated with the i_{th} element is related to the characteristic time of the spring-damper (dashpot) element and can be defined as *the ratio of the viscosity over the elastic modulus*, that is $\tau_i = g_i/\lambda_i$ [68].

The first derivative of the Prony series (Equation 42) for $t = 0$ is finite, a fact irrespective of the number of terms used in the approximation.

$$\frac{d\phi}{dt} = \frac{d}{dt}\sum_{i=1}^{N^\phi} g_i e^{-\beta_i t} \tag{42}$$

As commented by Bradshaw and Brinson [57], an exact solution of interconversion problem is possible if the known function is approximated by Prony series where all coefficients are positive [using the forms (Equation 39, 40)] and the basic interconversion relationship (Equation 26) is satisfied. In this case, by applying Laplace transform solutions, the resulting Prony series are analytically exact [70–73].

The Prony series approximations lead to the generalized Maxwell viscoelastic body (known also as Maxwell-Wiechert model) with N^{ϕ} spring-dashpot elements in parallel. The Prony series decompositions are applicable to any viscoelastic models through their time-dependent shear and bulk moduli [74–76]. For a deep thermodynamic analysis of such type of materials termed *viscoelastic solids of exponential type* (VESET) we refer to works of Fabrizio et al. [32, 33].

A common step in approximation by the truncated exponential sums is the definition of preliminary stipulated decay rates β_i , in a logarithmic scale that is , for $N_\phi = 4$, for instance $\beta_{1...4} = 100,\ 10,\ 1,\ 0.1$ [77]. This approach, is useful because the corresponding fractional parameters in the α_i in this case can be easily calculated (see further Equation 65) as $\alpha_{1...4} = 0.009,\ 0.09,\ 0.5,\ 0.9$, respectively. Alternatively, fixing the points (in normal or logarithmic scale) t_i along the time axis we may define the corresponding $\beta_i = 1/\tau_i$.

The parameter estimation is important and the first step was done in the seminal work of Prony (1795) [2] and several algorithms have been developed among them: graphically by log − log plots [78, 79], least squares method [80–82], nonlinear optimization methods [83], genetic based algorithm [84], from the continuous relaxation time spectrum [68, 72], logarithm equidistant distribution of relaxation times (known as R-method) [68, 77, 85], quasi-linearization for multi-exponential decay curves fitting [86], etc.

The number of the exponential terms in Equation (39) depends on the accuracy of data fitting. Commonly fourth-order Prony series fit adequately the stress-relaxation data in cases of non-linear viscoelastic behaviors [62, 87–93], while long-term relaxation tests need 10-15 terms [76, 85, 94–100]. In general, the problem corresponds to identifications of the kernel of an integral Fredholm equation which, actually, is ill-posed problem [68, 101–103] requiring Tikhonov regularization [104]. Comments on the required terms in the Prony decomposition are available in [14].

5.3. Examples of Interconversions With Exponential Terms

5.3.1. Example 3. Interconversion of a Single Exponential Model [105]

For a single exponential model (the Maxwell model) the relaxation and creep functions are

$$G_1 = 1 + g_1 \exp\left(-\frac{t}{\tau_1}\right) \tag{43}$$

$$J_1 = 1 - j_1 \exp\left(-\frac{t}{\lambda_1}\right) \tag{44}$$

The condition (Equation 27) is automatically satisfied, while for $t = 0$ in (Equation 43) (and taking into account ;Equation 27) the results is

$$G_1(0) J_1(0) = \left(1 + g_1\right)\left(1 - j_1\right) = 1 \tag{45}$$

The Laplace transforms of (Equations 43, 44) lead to an equation that should be solved

$$\left[\left(1 + g_1\right)\left(1 - j_1\right) - 1\right] p^2 + \left[\frac{g_1}{\lambda_1} - \frac{j_1}{\tau_1}\right] p = 0 \tag{46}$$

Since (Equation 46) should be satisfied for all p it follows that $\tau_1 = j_1\lambda_1$ and this condition with (Equation 46) leads to Anderssen et al. [105]

5.3.1.1. Interconversion from relaxation to creep

$$j_1 = \frac{g_1}{1 + g_1}, \quad \lambda_1 = \tau_1\left(1 + g_1\right) \tag{47}$$

5.3.1.2. Interconversion from creep to relaxation

$$g_1 = \frac{j_1}{1 - j_1}, \quad \tau_1 = \lambda_1\left(1 - j_1\right) \tag{48}$$

More details related to the sensitivities and relative errors of these two interconversions are available in Anderssen et al. [105]

5.3.2. Example 4. Interconversion of a Double Exponential Model [105]

In this case the relaxation and creep functions are

$$G_2(t) = 1 + g_1 \exp\left(-\frac{t}{\tau_1}\right) + g_2 \exp\left(-\frac{t}{\tau_2}\right) \tag{49}$$

$$J_2(t) = 1 - j_1 \exp\left(-\frac{t}{\lambda_1}\right) + j_2 \exp\left(-\frac{t}{\lambda_2}\right) \tag{50}$$

The values of the relaxation times τ_i are zeros of

$$\frac{1}{\tau} + \frac{j_1}{(\tau - \lambda_1)} + \frac{j_2}{(\tau - \lambda_2)} = 0 \tag{51}$$

and consequently

$$\tau_1\tau_2 = K\lambda_1\lambda_2, \quad K = \frac{1}{1 + j_1 + j_2} < 1 \tag{52}$$

5.3.3. Example 5. Interconversion of Multi-Term Exponential Model [105]

By Prony decomposition the relaxation and creep functions can be generally presented through discrete spectra as Anderssen et al. [105]

$$G_N(t) = G_\infty + \sum_{i=1}^{N} g_i \exp\left(-\frac{t}{\tau_i}\right), \quad g_i \geq 0 \tag{53}$$

$$J_N(t) = J_\infty - \sum_{i=1}^{N} j_i \exp\left(-\frac{t}{\lambda_i}\right), \quad j_i \geq 0 \tag{54}$$

The inverse times, that is the relaxation times τ_i and the retardation times λ_i satisfy

$$0 < \tau_1 < \tau_2 < \ldots\tau_N, \quad 0 < \lambda_1 < \lambda_2 < \ldots\lambda_N \tag{55}$$

The advantage of this (Prony decomposition) approach is that the monotonicity is automatically satisfied [105]

The attempts to solve (Equation 26) address many approaches for either $J(t)$ or $G(t)$ depending on the experimental data available [30, 106–109]

Now we recall that the Caputo-Fabrizio fractional operator [1, 3, 4, 10, 110] uses an exponential kernel matching the basic element of the Prony series.

5.3.4. Example 6. Interconversion of Prony Series of Relaxation and Creep

With Prony decomposition of the relaxation spectrum, the next step is to determine the creep modulus (compliance) $J(t)$. The interconversion equation (Equation 56) simple gives [22, 70, 71, 105, 111, 112].

$$[G(t) * J(t)] = \int_0^t G(t - s) J(s)\, ds = [J(t) * G(t)] = t \tag{56}$$

As a rule in the rheological studies, the discrete exponential models for $G(t)$ and $J(t)$, corresponding to associated relaxation and creep spectra $H(\tau)$ and $L(\tau)$ [105] (as sums of delta functions) are used that assures the monotonic behavior automatically and we have

$$G(t) = G(\infty) + \int_0^\infty \exp\left(-\frac{t}{\tau}\right)\frac{H(\tau)}{\tau} d\tau, \quad H(\tau) \geq 0 \tag{57}$$

$$\begin{aligned} J(t) &= J(0) + \int_0^\infty \exp\left(-\frac{t}{\tau}\right)\frac{L(\tau)}{\tau} \\ d\tau &= J(\infty) - \int_0^\infty \exp\left(-\frac{t}{\tau}\right)\frac{L(\tau)}{\tau} d\tau \end{aligned} \tag{58}$$

The constraint imposed [70, 105] by (Equations 26, 27) (see also Equation 56) allows $H(\tau)$ to be defined as function of $L(\tau)$ and *vice versa*. We will skip this general problem solved in [105] and to some extent demonstrated by Example 3 and 4, and will focus the attention on the discrete approximation of the relaxation (and compliance) spectra by Prony series.

The Prony series approximation of the stress relaxation (in a normalized form) is

$$G_N(t) = g_0 + \sum_{i=1}^{N} g_i \exp\left(-\frac{t}{\tau_i}\right), \quad g_0 > 0, \quad g_i \geq 0, \quad \tau_i \geq 0 \tag{59}$$

The corresponding form of $J(t)$ is

$$J_N(t) = j_0 - \sum_{i=1}^{N} j_i \exp\left(-\frac{t}{\lambda_i}\right), \quad j_0 > 0, \quad j_i \geq 0, \quad \lambda_i \geq 0 \tag{60}$$

The inverse times, that is the relaxation times τ_i and the retardation times λ_i satisfy the inequalities (Equation 55). The attempts to solve (Equation 56) address many approaches for either $J(t)$ or $G(t)$ depending on the type of experimental data available [30, 106–109].

6. CAPUTO-FABRIZIO OPERATOR IN THE CONSTITUTIVE VISCOELASTIC EQUATIONS

6.1. Relaxation Function in Terms of Caputo-Fabrizio Operator

Thus, applying the Prony approximation of the relaxation curve and substituting (Equation 61) in the convolution integral (Equation 22) the following approximation is obtained [14, 113]

$$\sigma = \int_0^t E_i \exp\left(-\frac{t-s}{\tau_i}\right) \frac{d\varepsilon}{ds} ds \tag{61}$$

Since we operate with a Prony series, that is with *a finite sum of exponentials*, then the inversion of the summation and the integral yields [113]

$$\sigma(t) = \int_0^t \sum_{i=0}^{N} E_i e^{-\frac{(t-s)}{\tau_i}} \frac{d\varepsilon}{ds} ds = \sum_{i=0}^{N} E_i \left[\int_0^t e^{-\frac{(t-s)}{\tau_i}} \frac{d\varepsilon}{ds} ds\right] \tag{62}$$

Thus, the memory effect from the convolution integral can be easy incorporated in each term of the Prony series [99],namely

$$\sigma(t) = \sum_{i=0}^{N} E_i k_i(t), \quad k_i(t) = \int_0^t e^{-\frac{(t-s)}{\tau_i}} \frac{d\varepsilon}{ds} ds \tag{63}$$

The finite sum of $N+1$ terms $\sigma_i(t)$ leads directly to the generalized Maxwell model, that is at any time t we have

$$\sigma_i(t) = E_i k_i(t) \tag{64}$$

$\sigma_i(t)$ is a product of the spring modulus E_i and its current strain $k_i(t)$, which to some extent could be considered as a hidden material variable. The clear physical meaning of this result is that the strain $k_i(t)$ at a given time t is expressed as convolution integral with an exponential kernel [14]. Therefore, the model parameters that should be defined, following the approximation of the right-hand side of (Equation 62), are [14, 113] : the single separate spring stiffness E_0 and *the spring stiffness* E_i as well as *the relaxations time* τ_i of each i^{-th} Maxwell element.

6.2. Relationships of the Relaxation Time Spectrum and Fractional Order Spectrum

As it was mentioned at the beginning, the fractional parameter α is related to the dimensionless relation time as $\alpha = 1/(1-\tau/t_0)$. Since we have a spectrum of relaxation times τ_i, then the spectrum of the fractional orders (parameters) is

$$\alpha_i = \frac{1}{1-\tau_i/t_0} \tag{65}$$

Therefore, $k_i(t)$ can be expressed in a form related to the construction of the Caputo-Fabrizio operator, namely

$$\begin{aligned} k_i(t) &= (1-\alpha_i)\left[\frac{1}{1-\alpha_i}\int_0^t e^{-\frac{\alpha_1}{1-\alpha_i}(\bar{t}-\bar{s})} \frac{d\varepsilon}{d\bar{s}} d\bar{s}\right] \\ &= (1-\alpha_i) D_t^{\alpha_i} \varepsilon(t) \end{aligned} \tag{66}$$

6.2.1. Stress Relaxation in Terms of Caputo-Fabrizio Operator

Thus, the constitutive equation of the stress relaxation can be presented as Hristov [14]

$$\sigma(t) = \sum_{i=0}^{N} E_i (1-\alpha_i) D_t^{\alpha_i} \varepsilon(t) \tag{67}$$

Now, we turn on the determination of the spectrum of fractional orders $\alpha_i = \tau_i/t_0$. From experimental data fittings there are a limited number of numerical values of relaxation times τ_i. Now, the principle problem at issue is the determination of the characteristic time t_0. Since the experiments last limited times then we may assume that t_0 equal the elapsed time t_e of the experiments. The literature data concerning Prony decompositions reveal (see the analysis in [14]) that the relaxation times form two groups (see the comments at the begging when the fractional order determination of the Caputo-Fabrizio operators was commented):: (1) *relaxation times less then the elapsed time* $\tau_i < t_e \Rightarrow \tau_i/t_e \leq 1$ and (2) *relaxation time greater than the elapsed time* $\tau_i > t_e \Rightarrow \tau_i/t_e \geq 1$.

Consequently, addressing the fractional orders α_i we have: i) fast relaxations for $\alpha_i \in [0.5-1)$ corresponding to $\tau_i/t_e \ \leq 1$ and ii) slow relaxations for $\alpha_i \in (0-0.5]$ when $\tau_i/t_e \ \geq 1$. Numerical examples supporting this estimates are reported in Hristov [14]

6.2.2. Creep Compliance in Terms of Caputo-Fabrizio Operator

By analogy of the stress relaxation expression in terms of Caputo-Fabrizio derivative, we may transform the Prony series decomposed compliance (Equation 54) as

$$J_N(t-s) = \sum_{i=0}^{N} j_i \exp\left(-\frac{t-s}{\lambda_i}\right) = \sum_{i=0}^{N} j_i \left\{\exp\left[-\frac{\beta_i}{1-\beta_i}\left(\bar{t}-\bar{s}\right)\right]\right\}, \quad \bar{\lambda}_i = \frac{\lambda_i}{t_0} = \frac{1-\beta_i}{\beta_i} \tag{68}$$

where $\bar{\lambda}_i = \lambda_i/t_0$ are the scaled (dimensionless) retardation times.

Hence, with the construction of the convolution integral describing the strain history we have

$$\int_0^t J(t-s)\frac{d\sigma}{dt} \Rightarrow \int_0^t \sum_{i=0}^{N} j_i \left\{\exp\left[-\frac{\beta_i}{1-\beta_i}\left(\bar{t}-\bar{s}\right)\right]\right\}\frac{d\sigma\left(\bar{s}\right)}{d\bar{s}} d\bar{s} = \sum_{i=0}^{N} j_i c_i(t) \tag{69}$$

$$c_i(t) = \int_0^t \exp\left[-\frac{\beta_i}{1-\beta_i}\left(\bar{t}-\bar{s}\right)\right]\frac{d\sigma\left(\bar{s}\right)}{d\bar{s}} d\bar{s} \tag{70}$$

Thus, the strain history is

$$\varepsilon(t-s) = \sum_{i=1}^{N} j_i\left(1-\beta_i\right)\left[\frac{1}{1-\beta}c_i(t)\right] = \sum_{i=1}^{N} j_i\left(1-\beta_i\right) D_t^{\beta}\sigma(t) \tag{71}$$

Then, the strain can be expressed as

$$\varepsilon(t) = J_{\infty} - \sum_{i=1}^{N} j_i\left(1-\beta_i\right)c_i(t) = J_{\infty} - \sum_{i=1}^{N} j_i\left(1-\beta_i\right)D_t^{\beta_i}\sigma(s) \tag{72}$$

where

$$D_t^{\beta_i}\sigma(s) = \left\{\frac{1}{\left(1-\beta_i\right)}\int_0^t \exp\left[-\frac{\beta_i}{1-\beta_i}(t-s)\right]\right\}\frac{d\sigma(s)}{ds} \tag{73}$$

Therefore,(Equation 73) defines a Caputo-Fabrizio operators with respect to $\sigma(t)$ and the fractional order β_i is related to the scaled retardation time $\bar{\lambda}_i$ (from the spectrum) as

$$\beta_i = \frac{1}{1+\lambda_i/t_0} \tag{74}$$

6.3. Description of Maxwell-Type Viscoelastic Media With Material Responses Modeled by Bessel Functions in Terms of Caputo-Fabrizio Operators

In the last two years (2016-2017) an interesting approach was developed by the group around Professor Mainardi [114–119] which could be considered as attempts to generalize the relaxation functions in the linear viscoelastic models, an approach also investigated in Colombaro et al. [120] and Guisti and Colombaro [121]. The main idea comes from the possibility to represent the relaxation function in a viscoelastic Maxwell-type body by infinite discrete spectrum with times related to the zeros of Bessel functions of the first kind [118, 119]. Here we will present some key points of these studies,in the context of the ideas developed in this article, that is to show the incorporation of the Caputo-Fabrizio operators in the relaxation functions expressed as infinite Dirichlet series (or infinite Prony series as the authors of these studies defined them).

The analysis developed in Colombaro et al. [114], Colombaro et al. [115], Colombaro and Guisti [116], Colombaro et al. [117], Guisti and Mainardi [118], and Guisti and Mainardi [119] is based on the power series representation of the modified Bessel function of the fist kind as Abramowitz and Stegun [122]

$$I_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma\left(\nu+k+1\right)}\left(\frac{z}{2}\right)^{2k} \tag{75}$$

with an asymptotic representation

$$I_{\nu}(z) \sim \frac{e^z}{\sqrt{2\pi}}\frac{1}{\sqrt{z}}, \quad |z| \rightarrow \infty, \quad \left|\arg z\right| < \frac{\pi}{2} \tag{76}$$

Now, if the function $F_{\nu}(t)$ has a Laplace transform [118]

$$\bar{F}(s) = \frac{2(\nu+1)}{s\sqrt{s}}\frac{I_{\nu+1}\left(\sqrt{s}\right)}{I_{\nu}\left(\sqrt{s}\right)} \tag{77}$$

It can be presented in the time domain as Guisti and Mainardi [118]

$$F_{\nu}(t) = 1 - 4(\nu+1)\sum_{n=1}^{\infty}\frac{\exp\left(-j_{\nu,n}^2 t\right)}{j_{\nu,n}^2}, \quad t > 0 \tag{78}$$

which is an expression as Dirichlet series, locally integrable, positive and increasing function for $t > 0$ [118], that is

$$F_{\nu}(t) \sim 4(\nu+1)\frac{\sqrt{t}}{\sqrt{\pi}}, \quad t > 0^{+} \tag{79}$$

because for $s \to \infty$ from Equation (77) we have

$$\bar{F}_{\nu}(s) \sim 2(\nu+1)\, s^{-\frac{3}{2}}, \quad \operatorname{Re}\{s\} \to +\infty \tag{80}$$

If the $\bar{F}_{\nu}(s)$ is considered as a part of the Laplace transformation of the relaxation memory function $\Phi_{\nu}(t)$ (here we use the original notations as that used in Guisti and Mainardi [118]), that is $\bar{\Phi}_{\nu}(s) = s\bar{F}_{\nu}(s)$, we have

$$\bar{\Phi}_{\nu}(s) = s\bar{F}_{\nu}(s) = \frac{2(\nu+1)}{\sqrt{s}} \frac{I_{\nu+1}\left(\sqrt{s}\right)}{I_{\nu}\left(\sqrt{s}\right)}, \quad \nu > -1 \tag{81}$$

In the time domain the relaxation function $\Phi_{\nu}(t)$ is [recall equation (Equation 78)]

$$\Phi_{\nu}(t) = \frac{dF_{\nu}(t)}{dt} = 4(\nu+1)\sum_{n=1}^{\infty} \exp\left(-j_{\nu,\ \ n}^{2} t\right) \tag{82}$$

Now, let us turn on to the interconversion problem. From $\sigma(s) = s\bar{G}_{\nu}(s)\,\bar{\varepsilon}(s)$ as it is suggested [118] that

$$\sigma(s) = s\bar{G}_{\nu}(s)\,\bar{\varepsilon}(s) = 1 - \bar{\Phi}_{\nu}(s)\,\bar{\varepsilon}(s) \tag{83}$$

In the time domain the relaxation modulus $G(t)$ is

$$G(t) = 1 - \int_{0}^{t} \Phi_{\nu}(\tau)d\tau = 1 - 4(\nu+1)\sum_{n=1}^{\infty} \frac{\exp\left(-j_{\nu,\ \ n}^{2} t\right)}{j_{\nu,n}^{2}} \tag{84}$$

Skipping details in calculation, the creep memory function denoted as $\bar{\Psi}(s)$ in the Laplace domain is Colombaro et al. [114], Colombaro and Guisti [116], and Guisti and Mainardi [119]

$$\bar{\Psi}(s) = \frac{2(\nu+1)}{\sqrt{s}} \frac{I_{\nu+1}\left(\sqrt{s}\right)}{I_{\nu+2}\left(\sqrt{s}\right)}, \quad \nu > -1 \tag{85}$$

and

$$1 + \bar{\Psi}(s) = \frac{1}{1 - \bar{\Phi}(s)} \tag{86}$$

which follows from the interconversion relationships.

For further deep reading in this elegant mathematical studies we refer to Colombaro et al. [114], Colombaro and Guisti [116], Colombaro et al. [117], Guisti and Mainardi [118], and Guisti and Mainardi [119] where it is was demonstrated that the asymptotic behaviors of the viscoelastic responses in the so-called Bessel medium are (precisely in Colombaro et al. [114])

$$\Phi_{\nu}(t) \propto \frac{2(\nu+1)}{\sqrt{\pi}} t^{-1/2}, \quad t \to 0$$
$$\Phi_{\nu}(t) \propto 4(\nu+1) exp(0 - j_{\nu,1}^{2} t), \quad t \to \infty \tag{87}$$

$$\Psi_{\nu}(t) \propto \frac{2(\nu+1)}{\sqrt{\pi}} t^{-1/2}, \quad t \to 0$$
$$\Psi_{\nu} \propto 4(\nu+1)(\nu+2), \quad t \to \infty \tag{88}$$

which actually follows from Equation (79).

The main idea to use a relaxation function expressed by modified Bessel functions of first kind comes from the possibility to calculate the sum of the infinite series of reciprocal positive zeros of the Bessel functions J_{ν} [118], namely

$$S_{\nu} = \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^{2}} = \frac{1}{4(\nu+1)}, \quad \nu > -1 \tag{89}$$

With (Equation 89) the relaxation modulus becomes [118]

$$G(t) = 4(\nu+1)\sum_{n=1}^{\infty} \frac{\exp\left(-j_{\nu,n}^{2} t\right)}{j_{\nu,n}^{2}} \tag{90}$$

Hence, the relaxation modulus is represented as an infinite Dirichlet series which are absolutely converging, which actually represent a discrete spectrum where the stiffness of the n^{th} element (spring-dashpot) is

$$_{B}E_{n} = \left(1/j_{\nu,n}^{2}\ \right)$$

(the prefix B means Bessel) and the relaxation times are $\tau_n = 1/j_{\nu,n}^{2}$. That is, in the context of the analysis done in this study, we have

$$G(t) = 4(\nu+1)\sum_{n=1}^{\infty} {}_{B}E_{n} \exp\left(-\frac{t}{\tau_n}\right) \tag{91}$$

Therefore, we get an expression in terms of an infinite sum of exponential functions, as in the Prony approximations, but now the relaxation times are defined by the zeros of the Bessel function. If a time scale t_0 of modeled relaxation process exists, then n^{th} dimensionless relaxation time is $0 < \tau_n/t_0 \ \le 1$, as it was demonstrated several times with the Prony series. Now, let us consider that $\tau_n/t_0 \ = (1-\beta_n)/\beta_n$,where $0 < \beta < 1$ and then we may re-write (Equation 91) as

$$G(t) = 4(\nu+1)\sum_{n=1}^{\infty} {}_{B}E_{n} \exp\left(-\frac{\beta_n}{1-\beta_n}\bar{t}\right), \quad 0 < \bar{t} = t/t_0 < 1 \tag{92}$$

Then, the construction the convolution integral of the stress relaxation with $G(t)$ yields

$$\sigma(t) = \int_{0}^{t} G(t-s)\frac{d\varepsilon(s)}{ds} ds$$

$$= \int_0^t 4(\nu+1)\sum_{n=1}^{\infty} {}_BE_n \exp\left[-\frac{\beta_n}{1-\beta_n}\left(\bar{t}-\bar{s}\right)\right]\frac{d\varepsilon(s)}{ds}ds \tag{93}$$

Interchanging the orders of the summation and the integral in equation (Equation 93) , we get

$$\sigma(t) = 4(\nu+1)\sum_{n=1}^{\infty} {}_BE_n \int_0^t \exp\left[-\frac{\beta}{1-\beta}\left(\bar{t}-\bar{s}\right)\right]\frac{d\varepsilon(s)}{ds}ds \tag{94}$$

Further, the step toward incorporation of the Caputo-Fabrizio operator is straightforward, namely

$$\sigma(t) = 4(\nu+1)\sum_{n=1}^{\infty} {}_BE_n(1-\beta_n) \left\{\frac{1}{1-\beta_n}\int_0^t \exp\left[-\frac{\beta_n}{1-\beta_n}\left(\bar{t}-\bar{s}\right)\right]\frac{d\varepsilon(s)}{ds}ds\right\} \tag{95}$$

or in a compact form as

$$\sigma(t) = 4(\nu+1)\sum_{n=1}^{\infty} {}_BE_n(1-\beta_n)\, {}_BD_t^{\beta_n}[\varepsilon(t)],$$
$$\beta_n = \frac{1}{1-\tau_n/t_0}, \quad \tau_n = 1/j_{\nu,n}^2 \tag{96}$$

with the condition

$$\sigma(t) = 4(\nu+1)\sum_{n=1}^{\infty} {}_BE_n(1-\beta_n) = 1 \tag{97}$$

Here ${}_BD_t^{\beta_n}(\bullet)$ denotes a Caputo-Fabrizio fractional operator with a fractional order β_n based on the positive zeros of J_ν.

Now, the naturally question coming to mind is: How this result can be applied to real data related to the stress relaxation and creep compliance of real viscoelastic materials? Unfortunately, no answers to this question exist in all works dealing with so-called Bessel media [114–119]. The first problem immediately appearing is: how the relaxation times $\tau_n = 1/j_{\nu,n}^2$ can be related to the real data? As mentioned in several points of this article, the relaxation times corresponds to measurements taken at equidistantly distributed (in normal scale or logarithmic scale) points along the time axis. How, this real approach could be related to the zeros of the Bessel function is a question still remaining open. The second problem comes from the impossibility to work with infinite sum in (Equations 90 , 95 , and 97) . Actually, all computer simulations should use finite number of terms that immediately leads to truncated Dirichlet series obeying the condition (Equation 97). This immediately, transforms the approximation of the relaxation function to approximation through Prony series, but the first problem formulated above, still remains unanswered.

Actually, the approximation by Prony series or infinite Dirichlet series, generally speaking, is an approach considering approximations of Non-Debye responses (relaxations) by superpositions of sub-processes (as Debye relaxations) with different relaxation times (an approach widely applied in the relaxation processes in glass transitions [123], for instance).

7. FINAL COMMENTS

The ideas and results developed in this work focused on the new fractional operator conceived in [1, 10] (2015) naturally appearing when we stayed on the shoulders of two classical results: the Prony series (1795) [2] and the Boltzmann superposition principle (1864) [9].

It was demonstrated that in many cases there are viscoelastic materials which experimental behaviors exhibit strong departures from the power-law. In such cases it is natural to rise the questions about the adequate modeling of the dynamic processes in such media and to ask for new fractional operators. This is, actually, the same question raised by Bagley and Torvik [49, 124] who in case of power-law media suggested the constitutive relationship

$$\sigma(t) = E_0\varepsilon(t) + E_1 \cdot D_t^{\mu}[\varepsilon(t)], \quad 0 < \mu < 1 \tag{98}$$

Following Bagley and Torvik [49, 124] for a homogeneous viscoelastic materials the constitutive equation is

$$\sigma(t) + \sum_{m=1}^{N} b_m D^{\beta_m}\sigma = E_0\varepsilon(t) + \sum_{n=1}^{N} E_n D^{\alpha_m}\varepsilon(t) \tag{99}$$

In (Equation 99) the derivatives have power-law kernels (Riemann-Liouville derivatives), which for $N = 1$ (one-term series of fractional derivatives) results in the simple expression

$$\sigma(t) + bD^{\beta}\sigma(t) = E_0\varepsilon(t) + E_1 D^{\alpha}\varepsilon(t) \tag{100}$$

containing two fractional derivatives with different orders.

This article does not focus on development of different viscoelastic models based on the Caputo-Fabrizio fractional operator since this is out of its scope and draws new problems to be resolved. Despite this, if the kernels in the convolution integrals departure from the power-law we may consider a similar (formally) constitutive equation, namely

$$\sigma(t) + \sum_{i=1}^{N} b_i\left[{}^{CF}D^{\beta_i}\sigma\right] = E_0\varepsilon(t) + \sum_{i=1}^{N} E_i\left[{}^{CF}D^{\alpha_i}\varepsilon(t)\right] \tag{101}$$

where, following the main idea of the Prony decompositions of the relaxation and the compliance, the series of fractional order of both sides of the equation have equal numbers of terms. Following [20, 107] the retardation times λ_i are satisfying the conditions

$$\tau_1 < \lambda_1 < .... < \tau_i < \lambda_i < ...\tau_N < \lambda_N \tag{102}$$

Therefore, from $\alpha_i = 1/(1 + \tau_i/t_0)$ and $\beta_i = 1/(1 + \lambda_i/t_0)$ it follows from Equation (102) that the fractional orders should satisfy the inequalities

$$0 < \beta_1 < \alpha_1 < ... < \beta_i < \alpha_i < ... < \beta_N < \alpha_N < 1 \qquad (103)$$

The example of Renardy [125] (see also [26] and [14]), related to polymer rheology, reveals that a discrete relaxation spectrum with accumulation point at zero *behaves as a power-law for short times*, namely

$$\sum_{i=0}^{\infty} exp(-i^{\gamma}\xi) \to t^{-\frac{1}{\gamma}}, \quad t \to 0, \quad \gamma > 1 \qquad (104)$$

Thus, in the asymptotic case (for $t \to 0$) we may expect that the model (Equation 101) would converge to the model (Equation 100).

If we suggest only, for example, that $N = 1$, which actually is a departure of the main idea of Prony series, we have

$$\sigma(t) + b^{CF}D^{\beta}\sigma = E_0\varepsilon(t) + E_1{}^{CF}D^{\alpha}\varepsilon(t), \quad 1 > \beta > \alpha > 0 \qquad (105)$$

This equation contains two fractional derivatives of different orders as in Equation (100). For instance, fractional viscoelastic models containing fractional derivatives and operators of two different order are thoroughly analyzed in Rossikhin and Shitikova [126] and Rossikhin and Shitikova [127] in the light of power-law materials. All these formal similarities and dissimilarities provoke new ideas that should be resolved.

To recapitulate, actually, we demonstrated that the Caputo-Fabrizio fractional operator naturally appears in the constitutive viscoelastic equations based on hereditary integrals when the material responses do not match the power-law. Moreover, it plays the same role as the fractional operators (derivatives) based on power-law memory kernels when they are inapplicable. At this moment the task of this study is accomplished and a lot of future works would be developed starting from results obtained here.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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Analytical Solution of Generalized Space-Time Fractional Advection-Dispersion Equation via Coupling of Sumudu and Fourier Transforms

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The objective of this article is to present the computable solution of space-time advection-dispersion equation of fractional order associated with Hilfer-Prabhakar fractional derivative operator as well as fractional Laplace operator. The method followed in deriving the solution is that of joint Sumudu and Fourier transforms. The solution is derived in compact and graceful forms in terms of the generalized Mittag-Leffler function, which is suitable for numerical computation. Some illustration and special cases of main theorem are also discussed.

Keywords: space-time fractional advection-dispersion equation, Fourier transforms, Sumudu transforms, Hilfer-Prabhakar fractional derivative, fractional laplacian operator, Mittag-Leffler function

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INTRODUCTION

In the last decade, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics, biology, engineering, and other areas. Several numerical and analytical methods have been developed to study the solutions of nonlinear fractional partial differential equations, for details, refer to the work in [1–6]. Fractional equations have enabled the investigation of the nonlocal response of multiple phenomena such as diffusion processes, electrodynamics, fluid flow, elasticity, and many more. Nowadays, fractional derivatives have gained a significant development to model some real life phenomena in the form of partial differential equations or the ordinary equations. Several researchers have performed the numerical simulation for fractional problems and revealed their applications in different directions include [7–12] and references therein. The exchange of heat, mass and momentum are considered to be the fundamental transfer phenomena in the universe. The mathematical framework for heat and mass transfer are of same kind, basically encompass by advection-dispersion equation. In recent work many authors have demonstrated the depth of mathematics and related physical issues of advection-dispersion equations. Schumer et al. [13] gave physical interpretation of space-time fractional advection-dispersion equation. Space-time fractional advection-dispersion equations are generalizations of classical advection-dispersion equations. The use of Hilfer-Prabhakar fractional derivative operator is gaining importance in physics because of their specific properties. The objective of this paper is to derive the solution of Cauchy type generalized fractional advection dispersion equation (18), associated with the Hilfer-Prabhakar fractional derivative. This paper provides an elegant extension of results, given earlier by Haung and Liu [14], Haubold et al. [15], Saxena et al. [16], and Agarwal et al. [17].

RESULTS REQUIRED IN THE SEQUEL

In early 90s, Watugala [18] introduced Sumudu transform, which is defined as,

$$A = \left\{ f(t) / \exists M,\ \tau_i > 0,\ \ i = 1,\ 2\ \ \left|f(t)\right| \leq M e^{\frac{|t|}{\tau_j}}\ \ if\ \ t \in (-1)^j \times [0,\ \infty) \right\}. \tag{1}$$

for all real $t \geq 0$ the Sumudu transform of function $f(t) \in A$ is defined as,

$$S\left[f(t);u\right] = F(u) = \int_0^\infty \frac{1}{u} e^{-\frac{t}{u}} f(t)\, dt, \qquad u \in (-\tau_1, \tau_2) \tag{2}$$

inversion formula of (2), is given by

$$S^{-1}[F(u)] = f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\frac{t}{u}} F(u)\, du, \tag{3}$$

where γ being a fixed real number.

Among others, the Sumudu transform was shown to have units preserving properties, and hence may be used to solve problems without resorting to the frequency domain. Further details and properties about this transform can be found in Belgacem [19], Belgacem et al. [20], and Katatbeh and Belgacem [21].

For a function $u(x,\ t)$, the Fourier transform of with respect to x is defined by

$$F[u(x,\ t)] = u^*(\eta,\ t) = \int_{-\infty}^{\infty} e^{i\eta x} u(x,\ t)\, dx, \quad (-\infty\ < \eta < \infty) \tag{4}$$

and for the function $u^*(\eta,\ t)$, inverse Fourier transform with respect to η is given by the formula

$$F^{-1}\left[u^*(\eta,\ t)\right] = u(x,\ t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\eta x} u^*(\eta,\ t)\, d\eta. \tag{5}$$

For more details of Fourier transform, see [Debnath and Bhatta [22]].

Mittag-Leffler function of two parameters is studied by Wiman [23] as

$$E_{\alpha,\,\beta}(z) = \sum\nolimits_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}\quad,\ \alpha,\ \beta \in C,\ R(\alpha) > 0. \tag{6}$$

Mittag-Leffler function of three parameter introduced by Prabhakar [24] as

$$E_{\alpha,\,\beta}^{\gamma}(z) = \sum\nolimits_{n=0}^{\infty} \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)\Gamma(\alpha n + \beta)} \frac{z^n}{n!},\ \alpha,\ \beta,\ \gamma \in C, R(\alpha) > 0. \tag{7}$$

Riemann-Liouville fractional integral (right-sided) of order α is defined in [25]

$$I_a^\alpha(u(x,\ t)) = {}_a^{RL}D_t^{-\alpha}(u(x,\ t)) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} u(x,\ t)\, d\tau, \quad (t > a),\ R(\alpha) > 0. \tag{8}$$

The right sided Riemann-Liouville fractional derivative of order α defined as

$${}_a^{RL}D_t^\alpha(u(x,\ t)) = \left(\frac{d}{dt}\right)^n \left(I_a^{n-\alpha} u(x,\ t)\right) \qquad (R(\alpha) > 0,\ n = [R(\alpha)] + 1), \tag{9}$$

here $[x]$ is the integral part of x.

Caputo [26], introduced fractional derivative of order $R(\alpha) > 0$ as

$${}_0^C D_t^\alpha(u(x,\ t)) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{u^m(x,\ \tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \\ m-1 < \alpha \leq m,\ R(\alpha) > 0,\ m \in N, \\ \frac{\partial^m}{\partial t^m} u(x,\ t), \quad if\ \ \alpha = m, \end{cases} \tag{10}$$

The Sumudu transform of (10) is given in [27], as

$$S\left[{}_0D_t^\alpha u(x,\ t);s\right] = s^{-\alpha}\bar{u}(x,\ s) - \sum\nolimits_{k=0}^{m-1} \frac{u^{(k)} u(x,\ 0)}{u^{\alpha-k}}, \quad (m-1 < \alpha \leq m) \tag{11}$$

where $\bar{u}(x,\ s)$ is the Sumudu transform of $u(x,\ t)$.

Hilfer [28], gave a fractional derivative operator of two parameters μ and ν, which is generalization of (9) and (10), in the form

$${}_0D_{0+}^{\mu,\nu}(u(x,\ t)) = I_t^{\nu(1-\mu)} \frac{\partial}{\partial t}\left(I_{0+}^{(1-\nu)(1-\mu)} u(x,\ t)\right),\ \ 0 < \mu < 1 \text{ and } 0 \leq \nu \leq 1 \tag{12}$$

For $\nu = 0$, equation (12) reduces into (9) and for $\nu = 1$, equation (12) reduces into (10).

The Sumudu transform of (12) is given in [29], as

$$S\left[{}_0D_{0+}^{\mu,\nu}(u(x,\ t));s\right] = s^{-\alpha}\bar{u}(x,\ s) - \sum\nolimits_{k=0}^{m-1} s^{k-m+\nu(m-\mu)} \frac{\partial^k}{\partial x^k} \left(I_{0+}^{(1-\nu)(1-\mu)} u(x, 0+\ )\right), \quad (m-1 < \mu \leq m). \tag{13}$$

Where the initial value term $I_{0+}^{(1-\nu)(1-\mu)} u(x, 0+\ )$ involves the Riemann-Liouville fractional derivative operator of order $(1-\nu)(1-\mu)$ as $t \to 0+$.

A generalization of Hilfer derivate is given in [30], known as Hilfer-Prabhakar derivative, is defined as:

Let $\mu \in (0,\ 1)$, $\nu \in [0,\ 1]$, and let f. belongs to the set of locally integrable real valued functions i.e., $f \in L^1[o, b], 0 < t < b \leq \infty, f * e_{\rho,\,(1-\nu),\omega}^{-\gamma(1-\nu)}(.) \in AC^1[0, b]$. The Hilfer-Prabhakar derivative is defined by

$${}_0D_{\rho,\omega.0+}^{\gamma,\mu,\nu}(u(x,\ t)) = E_{\rho,\ \nu(1-\mu),\omega,0+}^{-\gamma\nu} \frac{\partial}{\partial t}\left(E_{\rho,\ (1-\nu)(1-\mu),\omega,0+}^{-\gamma(1-\nu)}\right.$$

$$u(x,\ 0+))\,, \tag{14}$$

where $\gamma,\ \omega \in R,\ \rho > 0,$ and where $E^{0}_{\rho,0,\omega,0+}f = f$. We observe that (14) reduces to the Hilfer derivative for $\gamma = 0$. The Sumudu transform of this derivative operator (14) is given in [31], in the form:

$$S\left[{}_{0}D^{\gamma,\mu,\nu}_{\rho,\omega,0+}\,(u(x,\ t))\,;s\right] = s^{-\mu}\left(1-\omega s^{\rho}\right)^{\gamma}\bar{u}(x,\ s) - s^{\nu(1-\mu)-1}\left(1-\omega s^{\rho}\right)^{\gamma\nu} \left[E^{-\gamma(1-\nu)}_{\rho,\ (1-\nu)(1-\mu),\omega,0+}\,(x,\ 0+)\right] \tag{15}$$

For details of this derivative, refer to the work in [30, 31].

Brockmann and Sokolov [32], defined a fractional Laplace operator as:

$$\Delta^{\frac{\lambda}{2}} = \frac{1}{2\cos\left(\frac{\pi\lambda}{2}\right)}\left\{{}_{-\infty}D^{\lambda}_{x} + {}_{x}D^{\lambda}_{\infty}\right\},\quad (0 < \lambda \leq 2)\,,$$

where the operators are defined by

$${}_{-\infty}D^{\lambda}_{x}\left(u(x)\right) = \frac{1}{k-\lambda}\int_{-\infty}^{x}\frac{u^{k}(u)}{(x-u)^{\lambda+1-k}}du\,,\qquad \left(k=[\lambda]+1\right),$$

and

$${}_{x}D^{\lambda}_{\infty}\left(u(x)\right) = \frac{1}{k-\lambda}\int_{x}^{\infty}\frac{u^{k}(u)}{(x-u)^{\lambda+1-k}}du\,,\qquad \left(k=[\lambda]+1\right).$$

The Fourier transform of $\Delta^{\frac{\lambda}{2}}$ is given in [32], as

$$F\left\{\Delta^{\frac{\lambda}{2}}\,(u(x,\ t))\,;k\right\} = -\left|k\right|^{\lambda}F\left\{u(x,\ t)\right\},\ (0 < \lambda \leq 2)\,. \tag{16}$$

Inverse Sumudu transform of the following function is directly applicable in this sequel:

In the complex plane C, for any $R(\alpha) > 0,\ R(\beta) > 0,$ and $\omega \in C$

$$S^{-1}\left[u^{\gamma-1}\left(1-\omega u^{\beta}\right)^{-\delta}\right] = t^{\gamma-1}E^{\delta}_{\beta,\ \gamma}\left(\omega t^{\beta}\right). \tag{17}$$

SPACE-TIME FRACTIONAL ADVECTION-DISPERSION EQUATION

Here we will find, the solution of the generalized space-time Advection-Dispersion equation (18) under the conditions given in (19) and (20). Our main findings in the form of the following Theorem 3.1 and Corollary 3.2.

Theorem 3.1. Consider the generalized fractional order space-time advection-dispersion equation of Cauchy type

$${}_{0}D^{\gamma,\mu,\nu}_{\rho,\omega,t}\,(u(x,\ t)) = -\eta D_{x}u(x,\ t) + \varsigma\Delta^{\frac{\lambda}{2}}\,(u(x,\ t))\,, \tag{18}$$

where $\lambda \in (0,\ 2]\ x \in R,\ t \in R^{+},\ \mu \in (0,\ 1)\,,\ \nu \in [0,\ 1]\,,$ with initial condition,

$$E^{-\gamma(1-\nu)}_{\rho,\ (1-\nu)(1-\mu),\omega,0+}u(x,\ 0+) = g(x)\,,\ \gamma,\ \omega, x \in R,\ \rho > 0, \tag{19}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,\ t) = 0,\ \ t > 0, \tag{20}$$

where $\Delta^{\frac{\lambda}{2}}$ is the Laplace operator of fractional order $\lambda,\ \lambda \in (0,\ 2]$. The positive constant η represent the average fluid velocity and ς (positive constant) represent the dispersion coefficient. Subject to the above constraints, solution of equation (18), is

$$u(x,\ t) = \sum_{n=0}^{\infty}\frac{t^{\nu(1-\mu)+n\mu-1}}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}g(k)\left(i\eta k - \varsigma\left|k\right|^{\lambda}\right)^{n} E^{\gamma(n-\nu)}_{\rho,\ \nu(1-\mu)+n\mu}\left(\omega t^{\rho}\right)dk. \tag{21}$$

Proof: First, take the Fourier transform of equation (18) with respect to the space variable x, then

$${}_{0}D^{\gamma,\mu,\nu}_{\rho,\omega,t}\left(u^{*}\left(k,\ t\right)\right) = \eta i k u^{*}\left(k,\ t\right) - \varsigma\left|k\right|^{\lambda}u^{*}\left(k,\ t\right), \tag{22}$$

$u^{*}\left(k,\ t\right)$ represent Fourier transform of $u(x,\ t)$. Again, apply Sumudu transform on (22) with respect to time variable t, we get

$$s^{-\mu}\left(1-\omega s^{\rho}\right)^{\gamma}\overline{u^{*}}\left(k,\ s\right) - s^{\nu(1-\mu)-1}\left(1-\omega s^{\rho}\right)^{\gamma\nu} \left[E^{-\gamma(1-\nu)}_{\rho,\ (1-\nu)(1-\mu),\omega,0+}u\left(k,\ 0+\right)\right] = i\eta k\overline{u^{*}}\left(k,\ s\right) - \varsigma\left|k\right|^{\lambda}\overline{u^{*}}\left(k,\ s\right), \tag{23}$$

where $S\left[u\left(k,\ t\right);s\right] = \bar{u}\left(k,\ s\right)$.

Solve equation (23), by using conditions (19)-(20), we get

$$\left\{s^{-\mu}\left(1-\omega s^{\rho}\right)^{\gamma} - i\eta k + \varsigma\left|k\right|^{\lambda}\right\}\overline{u^{*}}\left(k,\ s\right) = s^{\nu(1-\mu)-1}\left(1-\omega s^{\rho}\right)^{\gamma\nu}g(k),$$

$$\Rightarrow \overline{u^{*}}\left(k,\ s\right) = \frac{s^{\nu(1-\mu)-1}(1-\omega s^{\rho})^{\gamma\nu}}{\left\{s^{-\mu}(1-\omega s^{\rho})^{\gamma} - i\eta k + \varsigma\left|k\right|^{\lambda}\right\}}g(k). \tag{24}$$

On taking inverse Sumudu transform of equation (24), and after little simplification, apply result (17), it gives

$$u^{*}\left(k,\ t\right) = \sum_{n=0}^{\infty}\left(i\eta k - \varsigma\left|k\right|^{\lambda}\right)^{n}g\left(k\right)t^{\nu(1-\mu)+n\mu-1} E^{\gamma(n-\nu)}_{\rho,\ \nu(1-\mu)+n\mu}\left(\omega t^{\rho}\right). \tag{25}$$

Taking inverse Fourier transform of (25), get our required result (21).

This completes the proof of the theorem 3.1.

0n taking $\eta = 0,\ \varsigma = \frac{ih}{2m}$ in Theorem 3.1, we arrive at:

Corollary 3.2. Consider the following one dimensional space-time Schrödinger equation of fractional order, for a free nature particle of mass m is

$${}_{0}D^{\gamma,\mu,\nu}_{\rho,\omega,t}\,(u(x,\ t)) = \frac{ih}{2m}\Delta^{\frac{\lambda}{2}}\,(u(x,\ t))\,, \tag{26}$$

where $\lambda \in (0,\ 2]\ ,\ x \in R,\ t \in R^{+},\ \mu \in (0,\ 1)\,,\ \nu \in [0,\ 1]\,,$ with initial condition

$$E^{-\gamma(1-\nu)}_{\rho,\ (1-\nu)(1-\mu),\omega,0+}u(x,\ 0+) = g(x)\,,\ \ \gamma,\ \omega \in R,\ \rho > 0, \tag{27}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,\ t)=0,\ \ t>0, \tag{28}$$

where $\Delta^{\frac{\lambda}{2}}$ is same as we defined earlier and $h = 6.625 \times 10^{-27} erg\ s = 4.21 \times 10^{-21} Mev\ s$ is the Planck constant. Subject to the above constraints, solution of equation (26), is

$$u(x,\ t) = \sum_{n=0}^{\infty} \frac{t^{\nu(1-\mu)+n\mu-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} g(k) \left(-\frac{ih}{2m}|k|^{\lambda}\right)^{n} E^{\gamma(n-\nu)}_{\rho,\ \nu(1-\mu)+n\mu}\left(\omega t^{\rho}\right) dk. \tag{29}$$

Proof: For obtaining the solution of Corollary 3.2, we follow same procedure, as we used in the proof of Theorem 3.1, and after little simplification, finally we obtain the desired result (29).

ILLUSTRATION

Example 4.1. To describe solute transport in aquifers, consider the following generalized fractional advection dispersion equation

$${}_0D^{\gamma,\mu,\nu}_{\rho,\omega,t}\left(u(x,\ t)\right) = -D_x u(x,\ t) + u'\Delta^{\frac{\lambda}{2}}\left(u(x,\ t)\right), \tag{30}$$

with initial condition

$$E^{-\gamma(1-\nu)}_{\rho,\ (1-\nu)(1-\mu),\omega,0+} u(x,\ 0+) = e^{-x},\ \ 0<x<1,\ \ t>0, \tag{31}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,\ t)=0,\ \ t>0, \tag{32}$$

where $\mu' = \frac{d}{\nu' L}$ and we consider a dimensionless parameter, called Peclet number, $Pe = \frac{1}{\mu'}$ where L is the packing length. The Peclet number determines the nature of the problem, that is, the Peclet number is low for dispersion-dominated problems and is large for advective dominated problems, d is the dispersion coefficient $\left[L^2T^{-1}\right]$ and ν' is the Darcy velocity $\left[LT^{-1}\right]$.

Our interest is in the solution of (30), for this we follow same procedure, as we applied in the proof of Theorem 3.1, and after little simplification, finally we obtain

$$u(x,\ t) = \sum_{n=0}^{\infty} \frac{t^{\nu(1-\mu)+n\mu-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} g(k) \left(ik-\mu'|k|^{\lambda}\right)^{n} E^{\gamma(n-\nu)}_{\rho,\ \nu(1-\mu)+n\mu}\left(\omega t^{\rho}\right) dk \tag{33}$$

Here $u(x,t)$ represent the analytical expression of solute concentration and $g(k) = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-(1+ik)}-1}{1+ik}\right]$.

Example 4.2. Consider the generalized fractional order space-time advection-dispersion equation

$${}_0D^{\gamma,\mu,\nu}_{\rho,\omega,t}\left(u(x,\ t)\right) = -D_x u(x,\ t) + u'\Delta^{\frac{\lambda}{2}}\left(u(x,\ t)\right), \tag{34}$$

with the initial condition

$$E^{-\gamma(1-\nu)}_{\rho,\ (1-\nu)(1-\mu),\omega,0+} u(x,\ 0+) = \delta(x), \tag{35}$$

Here $\delta(x)$ is Dirac-delta function and boundary condition

$$\lim_{|x|\to\infty} u(x,\ t)=0,\ \ t>0, \tag{36}$$

The solution of (34) can be obtained by same technique as we applied in proof of Theorem 3.1

$$u(x,\ t) = \sum_{n=0}^{\infty} \frac{t^{\nu(1-\mu)+n\mu-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \left(ik-\mu'|k|^{\lambda}\right)^{n} E^{\gamma(n-\nu)}_{\rho,\ \nu(1-\mu)+n\mu}\left(\omega t^{\rho}\right) dk. \tag{37}$$

SPECIAL CASES

Some interesting special cases of Theorem 3.1 are enumerated below:

If we set $\gamma = 0$, in (14), then Hilfer-Prabhakar derivative reduces to Hilfer derivative (12), and the Theorem 3.1 reduces to:

(I). Consider the generalized fractional order space-time advection-dispersion equation of Cauchy type

$${}_0D^{\mu,\nu}_{t}\left(u(x,\ t)\right) = -\eta D_x u(x,\ t) + \varsigma\Delta^{\frac{\lambda}{2}}\left(u(x,\ t)\right), \tag{38}$$

where $(0<\lambda\le 2),\ x\in R,\ t\in R^{+},\ \mu\in(0,\ 1),\ \nu\in[0,\ 1],$ with initial condition

$$I^{(1-\nu)(1-\mu)}_{0+} u(x,\ 0+) = g(x),\ \ x\in R, \tag{39}$$

and boundary condition

$$\lim_{|x|\to\infty} u(x,\ t)=0,\ \ t>0. \tag{40}$$

For obtaining the solution of (38), follow same procedure as we used in the proof of theorem 3.1, and use (13), after little simplification, obtain the following

$$u(x,\ t) = \frac{t^{\nu(1-\mu)+\mu-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} g(k)\, E^{1}_{\mu,\ \nu(1-\mu)+\mu}\left(\left(i\eta k-\varsigma|k|^{\lambda}\right)t^{\mu}\right) dk\,. \tag{41}$$

Again, use convolution theorem of the Fourier transform to (41), then we get solution of (38), in term of Green's function as

$$u(x,\ t) = \int_{-\infty}^{\infty} G(x-k,\ t)\, g(k)\, dk.$$

Here Green's function is given as

$$G(x,\ t) = \frac{t^{\nu(1-\mu)+\mu-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} E^{1}_{\mu,\ \nu(1-\mu)+\mu}\left(\left(i\eta k-\varsigma|k|^{\lambda}\right)t^{\mu}\right) dk.$$

If we set $\nu = 1$ in (12), then Hilfer fractional derivative reduces to Caputo fractional derivative operator (10) and the equation (38), yields the following:

(II). Consider the generalized fractional order space-time advection-dispersion equation of Cauchy type

$$ {}_0D_t^{\mu}\left(u\left(x,\ t\right)\right) = -\eta D_x u\left(x,\ t\right) + \varsigma \Delta^{\frac{\lambda}{2}}\left(u\left(x,\ t\right)\right), \tag{42} $$

where $(0 < \lambda \leq 2)$, $x \in R$, $t \in R^{+}$, $\mu \in (0,\ 1)$,
with initial condition

$$ u\left(x,\ 0+\right) = g\left(x\right),\ \ x \in R, \tag{43} $$

and boundary condition

$$ \lim_{|x|\to\infty} u\left(x,\ t\right) = 0,\ \ t > 0. \tag{44} $$

For obtaining the solution of (42), follow same procedure as we used in the proof of theorem 3.1, and use (11), after little simplification, obtain the following

$$ u\left(x,\ t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} g\left(k\right) E_{\mu,\ 1}^{1}\left(\left(i\eta k - \varsigma\left|k\right|^{\lambda}\right) t^{\mu}\right) dk. \tag{45} $$

Again, use convolution theorem of the Fourier transform to (45) then we get solution of (42), in term of Green's function as

$$ u\left(x,\ t\right) = \int_{-\infty}^{\infty} G\left(x-k,\ t\right) g\left(k\right) dk. $$

Here Green's function is given as

$$ G\left(x,\ t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx} E_{\mu,\ 1}^{1}\left(\left(i\eta k - \varsigma\left|k\right|^{\lambda}\right) t^{\mu}\right) dk. $$

(III). On giving suitable value to the parameters involved in Theorem 3.1, we can obtained same results, earlier given by Haung and Liu [14], Haubold et al. [15], Saxena et al. [16], and Agarwal et al. [17].

CONCLUSION

In this paper, we have presented a solution of generalized space-time fractional advection-dispersion equation. The solution has been developed in terms of Mittag-Leffler function with the help of Sumudu transform and Fourier transform. We can develop the efficient numerical techniques to find solution of various fractional partial differential equations arising in various fields by considering these analytic solutions as base. For future research, the methodology presented in this paper can serve as a good working template to solve any fractional advection-dispersion equations in higher dimensions.

AUTHOR CONTRIBUTIONS

VG, JS, and YS designed the study, developed the methodology, collected the data, performed the analysis, and wrote the manuscript.

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The Fractional Laguerre Equation: Series Solutions and Fractional Laguerre Functions

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In this paper, we propose a fractional generalization of the well-known Laguerre differential equation. We replace the integer derivative by the conformable derivative of order $0 < \alpha < 1$. We then apply the Frobenius method with the fractional power series expansion to obtain two linearly independent solutions of the problem. For certain eigenvalues, the infinite series solution truncate to obtain the singular and non-singular fractional Laguerre functions. We obtain the fractional Laguerre functions in closed forms, and establish their orthogonality result. The applicability of the new fractional Laguerre functions is illustrated.

Keywords: fractional differential equations, Laguerre equation, conformable fractional derivative, series solution, Frobenius method

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1. INTRODUCTION

In recent years, there are interests in studying fractional Sturm-Liouville eigenvalue problems. For instance, the fractional Bessel equation with applications was investigated in Okrasinski and Plociniczak [1, 2], where the fractional derivative is of the Riemann-Liouville type. In Abu Hammad and Khalil [3] the authors solved the fractional Legendre equation with conformable derivative and established the orthogonality property of the fractional Legendre functions. The applications of the fractional Legendre functions in solving fractional differential equations, were illustrated in Kazema et al. [4] and Syam and Al-Refai [5]. In this project we propose the following fractional generalization of the well-known Laguerre differential equation

$$x^\alpha D_0^\alpha D_0^\alpha y + (1 - x^\alpha) D_0^\alpha y + \lambda y = 0, \quad \frac{1}{2} < \alpha < 1, \quad x > 0, \tag{1.1}$$

where D_0^α is the conformable derivative of order α. The conformable derivative was introduced recently in Khalil et al. [6], and below are the definition and main properties of the derivative.

Definition 1.1. For a function $f:(0,\infty) \to \mathbb{R}$, the conformable derivative of order $0 < \alpha \le 1$ of f at $x > 0$, is defined by

$$(D_0^\alpha f)(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon},$$

and the derivative at $x = 0$ is defined by $(D_0^\alpha f)(0) = \lim_{x \to 0^+} (D_0^\alpha f)(x)$.

The conformable derivative is a local derivative which has a physical and a geometrical interpretations and potential applications in physics and engineering [7, 8]. It satisfies the nice properties of the integer derivative such as, the product rule, the quotient rule, and the chain rule, and it holds that

1. $D_0^\alpha C = 0,\ \ C \in \mathbb{R}$,
2. $D_0^\alpha x^p = p\, x^{p-\alpha}$,
3. $D_0^\alpha \sin(\frac{1}{\alpha}x^\alpha) = \cos(\frac{1}{\alpha}x^\alpha)$,
4. $D_0^\alpha \cos(\frac{1}{\alpha}x^\alpha) = -\sin(\frac{1}{\alpha}x^\alpha)$,
5. $D_0^\alpha e^{\frac{1}{\alpha}x^\alpha} = e^{\frac{1}{\alpha}x^\alpha}$.
6. $\int_0^a f(x)d\alpha(x) = \int_0^a x^{\alpha-1}f(x)dx$.

For more details about the conformable derivative we refer the reader to Abdeljawad [9] and Khalil et al. [6]. We mention here that even though the conformable is a nonlocal derivative (see [10, 11]), the simplicity and applications of the derivative make it of interests. Also, the applications of the obtained Fractional Leguerre functions are indicated in this manuscript. The rest of the paper is organized as follows: In section 2, we apply the Frobenius method together with the fractional series solution to solve the above equation and to obtain the fractional Laguerre functions. In section 3, we establish the orthogonality result of the fractional Laguerre functions and present the fractional Laguerre functions for several eigenvalues. Finally, we close up with some concluding remarks in section 4.

2. THE SERIES SOLUTION

The series solution is commonly used to solve various types of fractional differential equations (see [12–16]). Since $x = 0$, is α-regular singular point of Equation (1.1), see [17], we apply the well-known Frobenius method to obtain a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{\alpha(n+r)},$$

where the values of r will be determined. We have

$$\begin{aligned}
D_0^\alpha y &= \sum_{n=0}^{\infty} \alpha(n+r)a_n x^{\alpha(n+r-1)}, \\
&= \alpha a_0 r x^{\alpha(r-1)} + \sum_{n=0}^{\infty} \alpha(n+r+1)a_{n+1}x^{\alpha(n+r)}, \\
x^\alpha D_0^\alpha y &= \sum_{n=0}^{\infty} \alpha(n+r)a_n x^{\alpha(n+r)}, \\
D_0^\alpha D_0^\alpha y &= \sum_{n=0}^{\infty} \alpha^2(n+r)(n+r-1)a_n x^{\alpha(n+r-2)}, \\
x^\alpha D_0^\alpha D_0^\alpha y &= \sum_{n=0}^{\infty} \alpha^2(n+r)(n+r-1)a_n x^{\alpha(n+r-1)}, \\
&= \alpha^2 r(r-1)a_0 x^{\alpha(r-1)} \\
&\quad + \sum_{n=0}^{\infty} \alpha^2(n+r+1)(n+r)a_{n+1}\, x^{\alpha(n+r)}.
\end{aligned}$$

By substituting the above results in Equation (1.1) we have

$$\begin{aligned}
0 &= \alpha^2 r(r-1)a_0 x^{\alpha(r-1)} + \sum_{n=0}^{\infty} \alpha^2(n+r+1)(n+r)a_{n+1}\, x^{\alpha(n+r)} \\
&+ \alpha a_0 r x^{\alpha(r-1)} + \sum_{n=0}^{\infty} \alpha(n+r+1)a_{n+1}x^{\alpha(n+r)} \\
&- \sum_{n=0}^{\infty} \alpha(n+r)a_n x^{\alpha(n+r)} + \lambda \sum_{n=0}^{\infty} a_n x^{\alpha(n+r)}.
\end{aligned}$$

The coefficients of $x^{\alpha(r-1)}$ will lead to

$$a_0 \alpha r\Big(\alpha(r-1)+1\Big) = 0. \tag{2.1}$$

Because $\alpha \neq 0$, and $a_0 = 0$, will lead to the zero solution, we have

$$r = 0,\ r = 1 - \frac{1}{\alpha}. \tag{2.2}$$

We start with $r = 0$, we have

$$\alpha^2 n(n+1)a_{n+1} + \alpha(n+1)a_{n+1} - \alpha n a_n + \lambda a_n = 0,$$

or

$$a_{n+1} = \frac{\alpha n - \lambda}{\alpha(n+1)(\alpha n+1)}\, a_n,\ \ n \geq 0. \tag{2.3}$$

Lemma 2.1. *The coefficients a_n in Equation (2.3) satisfy*

$$a_{n+1} = \frac{\prod_{j=0}^{n}(j\alpha - \lambda)}{\alpha^{n+1}(n+1)!\, \prod_{j=0}^{n}(j\alpha+1)}\, a_0,\ \ n \geq 0. \tag{2.4}$$

Proof: The proof can be easily obtained by iterating the recursion in (2.3) and applying induction arguments. □

Remark 2.1. *For $\alpha = 1$, the recursion relation in (2.4) will reduce to*

$$a_{n+1} = \frac{-\lambda(1-\lambda)(2-\lambda)\cdots(n-\lambda)}{[(n+1)!]^2}\, a_0, \tag{2.5}$$

which is exactly the recursion relation that has been obtained in solving the Laguerre equation with integer derivative.

For $r = -\frac{1}{\alpha} + 1$, we have for $n \geq 0$,

$$\begin{aligned}
a_{n+1} &= \frac{\alpha(n+r) - \lambda}{\alpha(n+r+1)(\alpha[n+r]+1)}\, a_n, \\
&= \frac{\alpha(n+1) - (\lambda+1)}{\alpha(n+1)(\alpha[n+2]-1)}\, a_n.
\end{aligned} \tag{2.6}$$

By iterating the recursion in (2.6) and applying induction arguments, we have

Lemma 2.2. *The coefficients a_n in Equation (2.6) satisfy*

$$a_{n+1} = \frac{\prod_{j=1}^{n+1}(j\alpha - [\lambda + 1])}{\alpha^{n+1}(n+1)! \prod_{j=2}^{n+2}(j\alpha - 1)} a_0, \quad n \geq 0. \tag{2.7}$$

Remark 2.2. *By applying the Frobenius method to the regular Laguerre equation with integer derivative $\alpha = 1$, we obtain only one value of $r = 0$, which produces only one solution. Here with the fractional case, we obtain two values of $r = 0, 1 - \frac{1}{\alpha}$, that will produce two linearly independent solutions of the problem as we will see later.*

Now, in Equation (2.4), if we choose $\alpha = \alpha_m$ and $\lambda = \lambda_m$ such that

$$m\alpha_m = \lambda_m,$$

for some integer m, then

$$a_{m+1} = a_{m+2} = \cdots = 0,$$

and the infinite series solution will truncate to obtain the finite sum

$$\begin{aligned} u(x) &= \sum_{n=0}^{m} a_n x^{n\alpha_m} = a_0 \left(1 + \sum_{n=1}^{m} \frac{\prod_{j=0}^{n-1}(j\alpha_m - \lambda_m)}{\alpha_m^n n! \prod_{j=0}^{n-1}(j\alpha_m + 1)} x^{n\alpha_m} \right) \\ &= a_0 L^0_{m,\alpha_m}(x), \end{aligned}$$

where $L^0_{m,\alpha_m}(x)$ is the non-singular fractional Laguerre function of order m. Since

$$\begin{aligned} \prod_{j=0}^{n-1}(j\alpha_m - \lambda_m) &= \prod_{j=0}^{n-1}(j\alpha_m - m\alpha_m) = \prod_{j=0}^{n-1}\alpha_m \prod_{j=0}^{n-1}(j - m) \\ &= \alpha_m^n \prod_{j=0}^{n-1}(j - m), \end{aligned}$$

then

$$L^0_{m,\alpha_m}(x) = 1 + \sum_{n=1}^{m} \frac{\prod_{j=0}^{n-1}(j - m)}{n! \prod_{j=0}^{n-1}(j\alpha_m + 1)} x^{n\alpha_m}. \tag{2.8}$$

Analogously, in Equation (2.7), if we choose $\alpha = \alpha_m$ and $\lambda = \lambda_m$ such that

$$m\alpha_m = \lambda_m + 1,$$

then

$$a_m = a_{m+1} = \cdots = 0,$$

and the infinite series solution will truncate to obtain the solution

$$\begin{aligned} u(x) &= \sum_{n=0}^{m-1} a_n x^{\alpha_m(n - \frac{1}{\alpha_m} + 1)} = x^{-1} \sum_{n=0}^{m-1} a_n x^{\alpha_m(n+1)} \\ &= a_0 L^1_{m-1,\alpha_m}(x), \end{aligned}$$

where

$$\begin{aligned} L^1_{m-1,\alpha_m}(x) &= x^{-1}\left(x_m^{\alpha} + \sum_{n=1}^{m-1} \frac{\prod_{j=1}^{n}(j\alpha_m - (\lambda_m + 1))}{\alpha_m^n n! \prod_{j=2}^{n+1}(j\alpha_m - 1)} x^{\alpha_m(n+1)} \right), \\ &= x^{\alpha_m - 1}\left(1 + \sum_{n=1}^{m-1} \frac{\prod_{j=1}^{n}(j\alpha_m - m\alpha_m)}{\alpha_m^n n! \prod_{j=2}^{n+1}(j\alpha_m - 1)} x^{\alpha_m n} \right), \\ &= x^{\alpha_m - 1}\left(1 + \sum_{n=1}^{m-1} \frac{\prod_{j=1}^{n}(j - m)}{n! \prod_{j=2}^{n+1}(j\alpha_m - 1)} x^{\alpha_m n} \right), \end{aligned} \tag{2.9}$$

is the fractional singular Laguerre function of order $m - 1$.

Remark 2.3. *If we substitute $\alpha_m = 1$, then*

$$L^0_{m,1}(x) = L^1_{m,1}(x) = 1 + \sum_{n=1}^{m} \frac{\prod_{j=0}^{n-1}(j - m)}{n! \prod_{j=0}^{n-1}(j + 1)}.$$

Since $\prod_{j=0}^{n-1}(j + 1) = n!$, and $\prod_{j=0}^{n-1}(j - m) = (-1)^n \frac{m!}{(m-n)!}$, we have

$$L^0_{m,1}(x) = L^1_{m,1}(x) = 1 + \sum_{n=1}^{m} \frac{(-1)^m m!}{(n!)^2(m - n)!},$$

which is the expansion of the Laguerre polynomial $L_m(x)$.

3. THE FRACTIONAL LAGUERRE FUNCTIONS

We start with the orthogonality property of the fractional Laguerre functions $\Big(L_{m,\alpha_m}(x)\Big), m = 0, 1, 2, \cdots$. Here by $L_{m,\alpha_m}(x)$ we mean the non-singular and singular Laguerre functions obtained in (2.8) and (2.9).

Theorem 3.1. *The fractional Laguerre functions $\left(L_{m,\alpha_m}(x)\right), m = 0, 1, 2, \cdots$ are orthogonal on $(0, \infty)$ with respect to the weight function $\mu(x) = e^{-\frac{x^\alpha}{\alpha}}$, i.e.,*

$$\int_0^\infty e^{-\frac{x^\alpha}{\alpha}} L_{m,\alpha_m}(x) L_{n,\alpha_n}(x) dx = 0, \ m \neq n.$$

Proof: One can easily prove that Equation (1.1) can be written as

$$D_0^\alpha\left(xe^{-\frac{x^\alpha}{\alpha}} D_0^\alpha y\right) = -\lambda x^{1-\alpha} e^{-\frac{x^\alpha}{\alpha}} y. \tag{3.1}$$

Thus, the equation is of a special type of the fractional Sturm-liouville eigenvalue problem

$$D_0^\alpha\left(p(x) D_0^\alpha y\right) + q(x)y = -\lambda w(x)y,$$

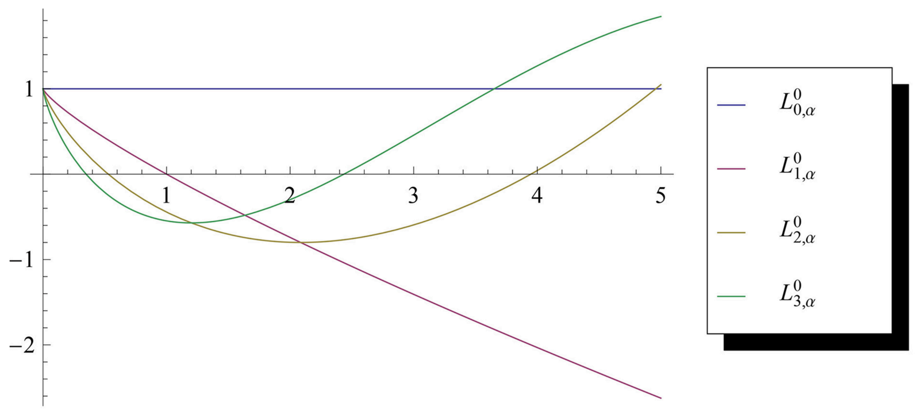

FIGURE 1 | A plot of $L^0_{0,\alpha}, L^0_{1,\alpha}, L^0_{2,\alpha}, L^0_{3,\alpha}$ for $\alpha = 0.8$.

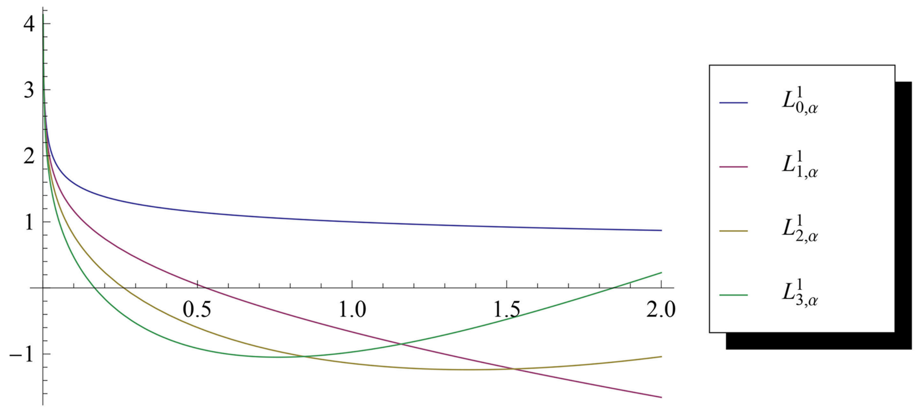

FIGURE 2 | A plot of $L^1_{0,\alpha}, L^1_{1,\alpha}, L^1_{2,\alpha}, L^1_{3,\alpha}$ for $\alpha = 0.8$.

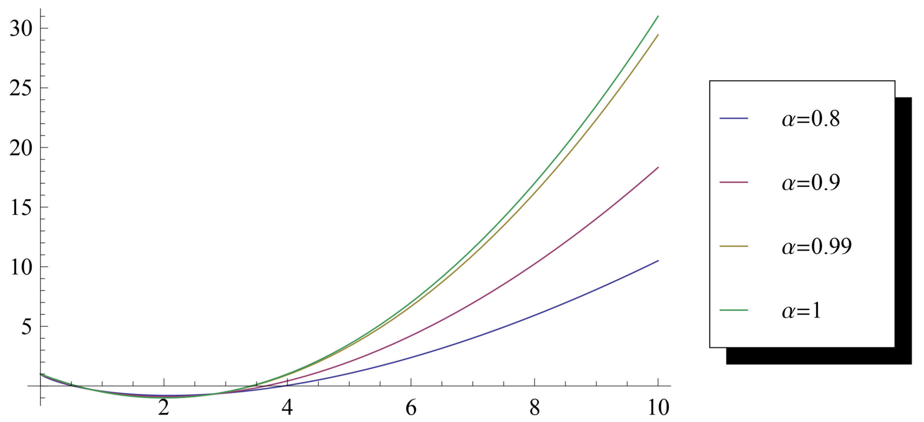

FIGURE 3 | A plot of $L^0_{2,\alpha}$, for $\alpha = 0.8, 0.9, 0.99, 1$.

where $p(x) = xe^{-\frac{x^\alpha}{\alpha}}$, $q(x) = 0$ and $w(x) = x^{1-\alpha}e^{-\frac{x^\alpha}{\alpha}}$. Using the fractional Lagrange Identity obtained in Al-Refai and Abdeljawad [18], we have

$$\begin{aligned} &-(\lambda_m - \lambda_n)\int_0^\infty w(x)L_{m,\alpha_m}L_{n,\alpha_n}d\alpha(x) \\ &= p(x)\left(L_{n,\alpha_n}D_0^\alpha L_{m,\alpha_m} - L_{m,\alpha_m}D_0^\alpha L_{n,\alpha_n}\right)\Big|_0^\infty. \end{aligned} \tag{3.2}$$

We have $p(0) = 0$, and

$$\lim_{x\to\infty} x^{1-\alpha}e^{-\frac{x^\alpha}{\alpha}}\left(L_{n,\alpha_n}D_0^\alpha L_{m,\alpha_m} - L_{m,\alpha_m}D_0^\alpha L_{n,\alpha_n}\right)(x) = 0.$$

Thus the right hand side of Equation (3.2) equals zero which together with $\lambda_m \neq \lambda_n$ will lead to

$$\begin{aligned} \int_0^\infty w(x)L_{m,\alpha_m}L_{n,\alpha_n}d\alpha(x) &= \int_0^\infty x^{1-\alpha}e^{-\frac{x^\alpha}{\alpha}}L_{m,\alpha_m}(x)L_{n,\alpha_n}(x)x^{\alpha-1}dx \\ &= \int_0^\infty e^{-\frac{x^\alpha}{\alpha}}L_{m,\alpha_m}(x)L_{n,\alpha_n}(x)dx = 0, \end{aligned} \tag{3.3}$$

and hence the result. □

Remark 3.1. *Since the fractional Laguerre functions are orthogonal, they can be used as a basis of the spectral method to study fractional differential equations analytically and numerically. They also can be used as a basis of the fractional Gauss-Laguerre quadrature for approximating the value of integrals of the form*

$$\int_0^\infty e^{-\frac{x^\alpha}{\alpha}}f(x)dx.$$

Remark 3.2. *New types of improper integrals are determined using the orthogonality property which are not known before, such as*

$$\int_0^\infty (1-x^\alpha)e^{-\frac{x^\alpha}{\alpha}}dx = 0,\ L^0_{0,\alpha_0}(x) = 1, L^0_{1,\alpha_1}(x) = 1 - x^{\alpha_1},$$
$$\int_0^\infty x^{2(\alpha-1)}(1 - \frac{1}{2\alpha-1}x^\alpha)e^{-\frac{x^\alpha}{\alpha}}dx = 0,$$
$$L^1_{0,\alpha_0}(x) = x^{\alpha_0-1}, L^1_{1,\alpha_1}(x) = x^{\alpha_1-1}(1 - \frac{1}{2\alpha_1-1}x^{\alpha_1}),$$

In the following we present the singular and non-singular fractional Laguerre functions of several orders.

$$\begin{aligned} L^0_{0,\alpha_0}(x) &= 1, \\ L^0_{1,\alpha_1}(x) &= 1 - x^{\alpha_1} \\ L^0_{2,\alpha_2}(x) &= 1 - 2x^{\alpha_2} + \frac{1}{\alpha_2+1}x^{2\alpha_2} \\ L^0_{3,\alpha_3}(x) &= 1 - 3x^{\alpha_3} + \frac{3}{\alpha_3+1}x^{2\alpha_3} - \frac{1}{(\alpha_3+1)(2\alpha_3+1)}x^{3\alpha_3}. \\ L^1_{0,\alpha_0}(x) &= x^{\alpha_0-1}, \\ L^1_{1,\alpha_1}(x) &= x^{\alpha_1-1}(1 - \frac{1}{2\alpha_1-1}x^{\alpha_1}), \\ L^1_{2,\alpha_2}(x) &= x^{\alpha_2-1}(1 - \frac{2}{2\alpha_2-1}x^{\alpha_2} + \frac{1}{(2\alpha_2-1)(3\alpha_2-1)}x^{2\alpha_2}), \\ L^1_{3,\alpha_3}(x) &= x^{\alpha_3-1}\left(1 - \frac{3}{2\alpha_3-1}x^{\alpha_3} + \frac{3}{(2\alpha_3-1)(3\alpha_3-1)}x^{2\alpha_3}\right. \\ &\quad \left. - \frac{1}{(2\alpha_3-1)(3\alpha_3-1)(4\alpha_3-1)}x^{3\alpha_3}\right). \end{aligned}$$

Figures 1, **2** depict the non-singular and singular fractional Laguerre functions of several orders for $\alpha = 0.8$. **Figure 3** depicts $L^0_{2,\alpha}$ for several values of α. One can see that, as α approaches 1, the non-singular fractional Laguerre functions approach the Laguerre polynomial of degree 2.

4. CONCLUSION

We have considered the fractional Laguerre equation with conformable derivative. We obtained two linearly independent solutions using the fractional series solution and Frobenius method. The first non-singular solution is analytic on $(0, \infty)$, and the second singular solution has a singularity at $x = 0$. For certain eigenvalues, these infinite solutions truncate to obtain the fractional Laguerre functions. Because of the orthogonality property of the fractional Laguerre functions, they can be used as a basis of the spectral method to study fractional differential equations, or as a basis of the Gauss-Laguerre quadrature for evaluating certain integrals. The obtained results coincide with the ones of the regular Laguerre polynomials as the derivative α approaches 1.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct and intellectual contribution to the work, and approved it for publication.

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Numerical Method for Fractional Model of Newell-Whitehead-Segel Equation

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The aim of the present work is to devote a friendly approach based on Adomian decomposition method (ADM) to find the numerical solution of the time-fractional Newell-Whitehead-Segel equation. Newell-Whitehead-Segel equation plays an efficient role in non-linear systems which describe the appearance of the stripe patterns in two dimensional systems. The numerical results obtained by proposed method are compared with exact solution for different values of fractional order α. Plotted graph illustrate the efficiency and accuracy of the proposed technique.

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Keywords: caputo fractional derivative, fractional newell-whitehead-segel equation, adomian decomposition method, fractional calculus, numerical method

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INTRODUCTION

Fractional calculus is a field of applied mathematics, three centuries old as the conventional calculus. Fractional calculus deals with derivatives and integrals of arbitrary orders. During the last decade, superb improvements have been visualized in the field of fractional calculus, very popular amongst science and engineering community. In recent year, differential equation containing fractional order derivatives has been contributed in various fields of science and engineering [1–4] such as diffusion equation, polarization, electro-magnetic waves, visco elasticity, electrode-electrolyte heat conduction, finance [5], control theory, biomedical engineering, biology [6] etc. In order to achieve the goal of highly accurate solution, many authors illustrate various techniques such as Adomian decomposition method [7], Finite difference method [8], Generalized differential transform method [9], Finite element method [10], Fractional differential transform method [11], Homotopy perturbation method [12, 13], Iterative methods [14], Variational iteration method [15], Homotopy analysis method [16], Differential quadrature method [17], Homotopy perturbation Sumudu transform method [18], Homotopy analysis transform method [19], Local fractional homotopy perturbation Sumudu transform method and Local fractional reduced differential transform method [20], Homotopy analysis Sumudu transform method [21] etc.

Recently various author used a new fractional derivative with Mittag-Leffler type kernel by different numerical method like Laplace decomposition method [22] and iterative method [23] etc.

The Newell-Whitehead-Segel equation model is the interaction of the effect of the diffusion term with the non-linear effect of the reaction term. Fractional Newell-Whitehead-Segel equation is written as

$$u_t^{\alpha} = ku_{xx} + au - bu^q, \quad t > 0, \; 0 < \alpha \le 1, \tag{1.1}$$

where $a,\ b$ and $k > 0$ are real numbers and q is a positive integers. First term on the left hand side in Equation (1.1) u_t^{α} represent the variation of $u(x,\ t)$ with time at a fixed location, first term on the right hand side u_{xx} represent the variation of $u(x,\ t)$ with spatial variable at a specific time and term $au - bu^q$ takes into account the effect of the source term. The function $u(x,\ t)$ may be non-linear distribution of temperature in an infinitely thin and long rod or fluid flow as a velocity in an infinitely long pipe with narrow diameter.

Mostly two types of patterns are observed. First is the roll pattern in which cylinders form by fluid stream lines. These cylinders may be bend and form spiral like patterns. Second pattern is the hexagonal in which liquid flow is divided into honey comb cells. The same patterns, stripes and hexagons appear in different physical system. For example, stripes patterns are notice in human fingerprints, on zebra skin and in a visual cortex. Hexagonal patterns are obtained from the propagation of laser beams through a non-linear medium and in systems with chemical reaction and diffusion species [24].

Recently Newell-Whitehead-Segel equations were solved by S. S. Nourazar, M. Soori, and A. Nazari-Golshan by homotopy perturbation method [25], A. Prakash and M. Kumar [26] by He's variational iteration method. Also fractional model of Newell-Whitehead-Segel were solved by Kumar et al. [27] and Prakash et al. [28] by homotopy analysis Sumudu transform method and fractional variational iteration method, respectively. But fractional model of Newell-Whitehead-Segel has not been solved by Adomian decomposition method. Adomian decomposition method is very powerful and efficient numerical method for handling non-linear fractional model. Adomian decomposition method (ADM) demonstrates fast convergence of the solution and therefore provides several significant advantages. This method attacks directly on non-linear term, in a straightforward fashion without using linearization, discretization, perturbation or any other restrictive assumption. Many studies have shown that few terms of decomposition series provide numerical result of high degree of accuracy which makes the method powerful when compared with other existing numerical techniques.

The outline of this paper is as follow. First section is introductory, in the Basic Definition of Fractional Calculus the basic definition of fractional calculus is discussed, in Proposed Adomian Decomposition Method solution process of non-linear Newell-Whitehead-Segel equation by Adomian decomposition method is discussed, in Error Analysis of The Proposed Method error analysis of proposed technique is discussed, in Application of ADM to Fractional Newell-Whitehead-Segel Equation five test examples of fractional Newell-Whitehead-Segel equation are given to elucidate the proposed method ADM and in last Conclusion of the work is drawn.

BASIC DEFINITION OF FRACTIONAL CALCULUS

In this section, we will introduce the basic definitions and properties of fractional calculus used to describe the proposed schemes.

Definition 2.1. A real function $f(t), t > 0$, is said to be in the space $C_\alpha, \epsilon\, \alpha R$, if there exists a real number p, $(p > \alpha)$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \epsilon C[0,\ \infty)$ and it is said to be in the space C_α^m $iff^{(m)} \epsilon C_\alpha,\ m \epsilon N \bigcup \{0\}$.

Definition 2.2. The Riemann-Liouville fractional integral of order $\alpha \geq 0$, of a function $f(t) \epsilon C_\beta,\ \beta \geq -1$ is defined as [29–31]:

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau = \frac{1}{\Gamma(\alpha+1)} \int_0^t f(\tau)(d\tau)^\alpha,$$
$$J^0 f(t) = f(t).$$

For the Riemann-Liouville fractional integral, we have

$$J^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\alpha+\beta},$$

where $\Gamma.$ is the well-known Gamma Function.

Definition 2.3. The Caputo fractional derivative of $f(t),\ f \in C_{-1}^m,\ m \in N,\ m > 0,$ is defined as [29–31]:

$$D^\alpha f(t) = I^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-x)^{m-\alpha-1} f^m(x)\, dx,$$

Where $m - 1 < \alpha \leq m$.

PROPOSED ADOMIAN DECOMPOSITION METHOD

In this section, we illustrate the basic idea of the Adomian Decomposition method (ADM) for the time-fractional Newell-Whitehead-Segel equation.

Consider time-fractional Newell-Whitehead-Segel equation as

$$u_t^\alpha = ku_{xx} + au - bu^q,\ \ t > 0,\ 0 < \alpha \leq 1, \tag{3.1}$$

where $a,\ b$ and $k > 0$ are real numbers and q is a positive integers with initial condition

$$u(x,\ 0) = f(x, t).$$

Applying the operator J_t^α on both sides of (3.1), we have

$$\begin{aligned} u(x,\ t) \ =\ & \sum_{k=0}^{m-1} \left(\frac{\partial^k u}{\partial t^k}\right)_{t=0} \frac{t^k}{\Gamma(k+1)} + J_t^\alpha f(x,t) \\ & - J_t^\alpha \left(ku_{xx} + au - bu^q\right). \end{aligned} \tag{3.2}$$

Next, we decompose the unknown function $u(x,\ t)$ into sum of an infinite number of components given by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,\ t), \tag{3.3}$$

and the non-linear term can be decomposed as

$$bu^q = \sum_{n=0}^{\infty} A_n, \tag{3.4}$$

where A_n are Adomian polynomial, given by

$$A_n = \frac{1}{\Gamma(n+1)}[\frac{d^n}{d\lambda^n}\{b\sum_{n=0}^{\infty}\lambda^i u_i(x,t)\}^q]_{\lambda=0}, \tag{3.5}$$

where $n = 0, 1, 2, 3, \ldots\ldots$.

Components $u_0, \ u_1, \ u_2, \ u_3, u_4, \ldots.$ are determined by substituting (3.3), (3.4), and (3.5) into (3.2) leading to

$$\sum_{n=0}^{\infty} u_n(x, t) = \sum_{k=0}^{m-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{\Gamma(k+1)} + J_t^{\alpha} f(x,t)$$
$$-J_t^{\alpha}\{k(\sum_{n=0}^{\infty} u_n(x, t))_{xx} + a(\sum_{n=0}^{\infty} u_n(x, t)) + \sum_{n=0}^{\infty} A_n\}. \tag{3.6}$$

This can be written as

$$u_0 + u_1 + u_2 + \ldots = \sum_{k=0}^{m-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{\Gamma(k+1)} + J_t^{\alpha} f(x,t)$$
$$-J_t^{\alpha}[k\left((u_0)_{xx} + (u_1)_{xx} + (u_2)_{xx} + \ldots\right) + a(u_0 + u_1 + u_2 + \ldots) + (A_0 + A_1 + A_2 + A_3 + \ldots)].$$

Adomian method uses the formal recursive relations as:

$$u_0 = \sum_{k=0}^{m-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{\Gamma(k+1)} + J_t^{\alpha} f(x,t),$$
$$u_{n+1} = -J_t^{\alpha}\{k(u_n)_{xx} + au_n + A_n\}, \qquad n \geq 0. \tag{3.7}$$

ERROR ANALYSIS OF THE PROPOSED METHOD

Theorem 4.1. If we can find a constant $0 < \varepsilon < 1$ such that $\left\|u_{m+1}(x,t)\right\| \leq \varepsilon \left\|u_m(x,t)\right\|$ for each value of m. Moreover, if the truncated series $\sum_{m=0}^{r} u_m(x,t)$ is employed as a numerical solution $u(x,t)$, then the maximum absolute truncated error is determined as

$$\left\|u(x,t) - \sum_{m=0}^{r} u_m(x,t)\right\| \leq \frac{\varepsilon^{r+1}}{(1-\varepsilon)}\left\|u_0(x,t)\right\|.$$

Proof. We have

$$\begin{aligned}
\left\|u(x,t) - \sum_{m=0}^{r} u_m(x,t)\right\| &= \left\|\sum_{m=r+1}^{\infty} u_m(x,t)\right\| \\
&\leq \sum_{m=r+1}^{\infty}\left\|u_m(x,t)\right\| \\
&\leq \sum_{m=r+1}^{\infty}\varepsilon^m\left\|u_0(x,t)\right\| \\
&\leq (\varepsilon)^{r+1}\left[1 + (\varepsilon)^1 + (\varepsilon)^2 + \ldots\right]\left\|u_0(x,t)\right\| \\
&\leq \frac{\varepsilon^{r+1}}{(1-\varepsilon)}\left\|u_0(x,t)\right\|.
\end{aligned}$$

Which proves the theorem.

APPLICATION OF ADM TO FRACTIONAL NEWELL-WHITEHEAD-SEGEL EQUATION

In this section, five test examples of fractional Newell-Whitehead-Segel equation demonstrate the efficiency of proposed ADM.

Ex. 5.1. We study the linear time-fractional Newell-Whitehead-Segel equation

$$u_t^{\alpha} = u_{xx} - 2u, \ t > 0, \ 0 < \alpha \leq 1, \tag{5.1}$$

with initial condition

$$u(x, 0) = e^x. \tag{5.2}$$

Applying the operator J_t^{α} on both side of above defined problem, we have

$$u(x, t) = \sum_{k=0}^{1-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{\Gamma(k+1)} + J_t^{\alpha}\{u_{xx} - 2u\}.$$

This gives the following recursive relation:

$$\begin{aligned}
u_0(x, t) &= \sum_{k=0}^{1-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{\Gamma(k+1)}, \\
u_{n+1}(x, t) &= J_t^{\alpha}\{(u_n)_{xx} - 2u_n\}, \qquad n \geq 0. \\
u_0 &= e^x, \\
u_1 &= -e^x\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\
u_2 &= e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\
u_3 &= -e^x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\
\sum_{n=0}^{\infty} u_n(x, t) &= e^x - e^x\frac{t^{\alpha}}{\Gamma(\alpha+1)} + e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
&\quad - e^x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots,
\end{aligned}$$

Now, for the standard case when $\alpha = 1$, we get $u(x, t) = e^{x-t}$, which is the exact solution of the classical Newell-Whitehead-Segel equation as obtained by HPM [25] and VIM [26]. Here the numerical results obtained by ADM upto eight terms of approximation and exact solution as shown in **Figures 1**, **2** are almost identical. It can be observed that as the value of t increases, u decreases, and as x increases, u also increases. Hence, the accuracy of ADM can be enhanced by increasing the number of iterations.

Ex. 5.2. We study the non-linear time-fractional Newell-Whitehead-Segel equation

$$u_t^{\alpha} = u_{xx} + 2u - 3u^2, \ t > 0, \ 0 < \alpha \leq 1, \tag{5.3}$$

with initial condition

$$u(x, 0) = \eta. \tag{5.4}$$

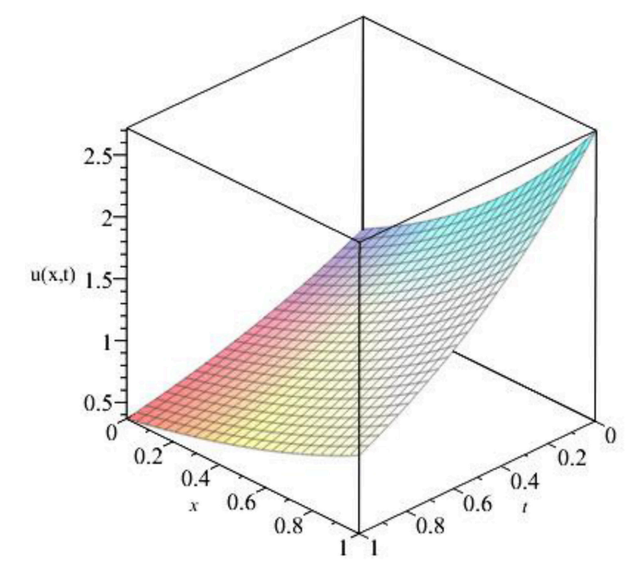

FIGURE 1 | Surface represents eight order approximate solution for $\alpha = 1$, for Ex. 5.1.

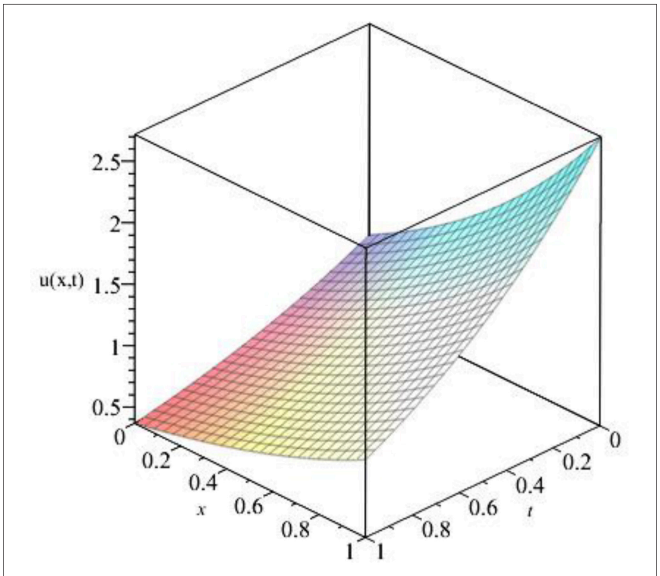

FIGURE 2 | Surface represents exact solution for $\alpha = 1$, for Ex. 5.1.

Applying the operator J_t^α on both side of above defined problem, we have

$$u\,(x,\ t) = \sum\nolimits_{k=0}^{1-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{\Gamma\left(k+1\right)} + J_t^\alpha\{u_{xx} + 2u + A_n\}.$$

This gives the following recursive relation:

$$u_0\,(x,\ t) = \sum\nolimits_{k=0}^{1-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{\Gamma\left(k+1\right)},$$

$$u_{n+1}(x,\ t) = J_t^\alpha\{(u_n)_{xx} + 2u_n + A_n\}, \qquad n \geq 0.$$

$$u_0 = \eta$$

$$u_1 = \eta(2-3\eta)\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2 = 2\eta\,(2-3\eta)\,(1-3\eta)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3 = 2\eta\,(2-3\eta)\,(18\eta^2-12\eta+2)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$-3\eta^2(2-3\eta)^2\frac{\Gamma\,(2\alpha+1)}{\Gamma\,(\alpha+1)^2}\frac{t^{3\alpha}}{\Gamma\,(3\alpha+1)},$$

$$u_4 = -12\eta^2\,(2-3\eta)\left(18\eta^2-12\eta+2\right)\frac{t^{4\alpha}}{\Gamma\,(4\alpha+1)}$$
$$+18\eta^3(2-3\eta)^2\frac{\Gamma\,(2\alpha+1)}{\Gamma\,(\alpha+1)^2}\frac{t^{4\alpha}}{\Gamma\,(4\alpha+1)}$$
$$-12\eta^2\,(2-3\eta)^2\,(1-3\eta)\,\frac{\Gamma\,(3\alpha+1)}{\Gamma\,(\alpha+1)\,\Gamma\,(2\alpha+1)}\frac{t^{4\alpha}}{\Gamma\,(4\alpha+1)}$$
$$+4\eta\,(2-3\eta)\,(18\eta^2-12\eta+2)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$
$$-6\eta^2(2-3\eta)^2\frac{\Gamma\,(2\alpha+1)}{\Gamma\,(\alpha+1)^2}\frac{t^{4\alpha}}{\Gamma\,(4\alpha+1)} + \ldots$$

$$\sum_{n=0}^{\infty} u_n\,(x,t) = \eta + \eta\,(2-3\eta)\,\frac{t^\alpha}{\Gamma\,(\alpha+1)}$$
$$+2\eta\,(2-3\eta)\,(1-3\eta)\,\frac{t^{2\alpha}}{\Gamma\,(2\alpha+1)}$$
$$+2\eta\,(2-3\eta)\left(18\eta^2-12\eta+2\right)\frac{t^{3\alpha}}{\Gamma\,(3\alpha+1)}$$
$$-3\eta^2(2-3\eta)^2\frac{\Gamma\,(2\alpha+1)}{\Gamma(\alpha+1)^2}\frac{t^{3\alpha}}{\Gamma\,(3\alpha+1)}$$
$$-12\eta^2\,(2-3\eta)\left(18\eta^2-12\eta+2\right)\frac{t^{4\alpha}}{\Gamma\,(4\alpha+1)}$$
$$+18\eta^3(2-3\eta)^2\frac{\Gamma\,(2\alpha+1)}{\Gamma(\alpha+1)^2}\frac{t^{4\alpha}}{\Gamma\,(4\alpha+1)}$$
$$-12\eta^2(2-3\eta)^2\,(1-3\eta)\,\frac{\Gamma\,(3\alpha+1)}{\Gamma\,(\alpha+1)\,\Gamma\,(2\alpha+1)}\frac{t^{4\alpha}}{\Gamma\,(4\alpha+1)}$$
$$+4\eta\,(2-3\eta)\,(18\eta^2-12\eta+2)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$$
$$-6\eta^2(2-3\eta)^2\frac{\Gamma\,(2\alpha+1)}{\Gamma(\alpha+1)^2}\frac{t^{4\alpha}}{\Gamma\,(4\alpha+1)} + \ldots$$

In particular when $\alpha = 1$, we get the solution in the form

$$u\,(x,\ t) = \eta + \eta\,(2-3\eta)\,t + 2\eta\,(2-3\eta)\,(1-3\eta)\,\frac{t^2}{\Gamma\,(3)}$$
$$+2\eta\,(2-3\eta)\left(27\eta^2-18\eta+2\right)\frac{t^3}{\Gamma\,(4)}$$
$$+12\eta\,(2-3\eta)\left(-54\eta^3+54\eta^2-14\eta+\frac{2}{3}\right)\frac{t^4}{\Gamma\,(5)}\ldots\ldots,$$

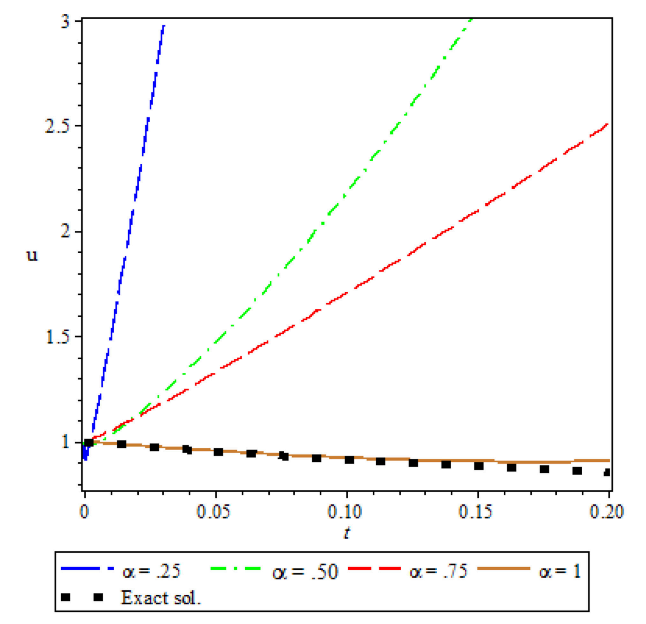

FIGURE 3 | Comparison of approx. sol. for different values of α and exact sol. at $\alpha = 1$, for Ex. 5.2.

Which converge to the exact solution of the classical Newell-Whitehead-Segel equation very fastly [25, 26].

$$u\,(x,\ t) = \frac{\frac{-2}{3}\eta e^{2t}}{-\frac{2}{3}+\eta-\eta e^{2t}}.$$

Figure 3 shows the comparison of approximate solution for different value of fractional order $\alpha = 0.25,\ 0.50,\ 0.75,\ 1$ and exact solution at $\alpha = 1$, when $\eta = 1$. It is observed from the **Figure 3** that there is a good agreement between exact solution and approximate solution at $\alpha = 1$. It is also noticed that solution depends on the time-fractional derivative. Accuracy and efficiency can be enhanced by increasing the number of iterations.

Ex. 5.3. We study the non-linear time-fractional Newell-Whitehead-Segel equation.

$$u_t^{\alpha} = u_{xx} + u - u^2 = 0,\ \ t>0,\ 0<\alpha\leq 1, \tag{5.5}$$

With initial condition,

$$u\,(x,\ 0) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^2}. \tag{5.6}$$

Applying the operator J_t^{α} on both side of above equation, we get

$$u\,(x,\ t) = \sum\nolimits_{k=0}^{1-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{\Gamma\,(k+1)} + J_t^{\alpha}\{u_{xx}+u+A_n\}.$$

This gives the following recursive relation:

$$u_0\,(x,\ t) = \sum\nolimits_{k=0}^{1-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{\Gamma\,(k+1)},$$

$$u_{n+1}(x,\ t) = J_t^{\alpha}\{(u_n)_{xx}+2u_n+A_n\}, \qquad n\geq 0.$$

$$u_0 = \frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^2},$$

$$u_1 = \frac{5}{3}\frac{e^{\frac{x}{\sqrt{6}}}}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3}\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2 = \frac{25}{18}\Bigg(\frac{e^{\frac{x}{\sqrt{6}}}\left(-1+2e^{\frac{x}{\sqrt{6}}}\right)}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^4}\Bigg)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$\begin{aligned}
u_3 = \Bigg\{&\frac{25}{18}\frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^5}\Big[\frac{8}{6}(e^{\frac{x}{\sqrt{6}}})^2 - 4(e^{\frac{x}{\sqrt{6}}})^3\\
&+\left(\frac{8}{6}(e^{\frac{x}{\sqrt{6}}})^2 - \frac{(e^{\frac{x}{\sqrt{6}}})}{6}\right)\left(1+e^{\frac{x}{\sqrt{6}}}\right)\\
&+\frac{4}{6}(e^{\frac{x}{\sqrt{6}}})^2 - \frac{16}{6}(e^{\frac{x}{\sqrt{6}}})^3 + \left(2(e^{\frac{x}{\sqrt{6}}})^2 - e^{\frac{x}{\sqrt{6}}}\right)\left(1+e^{\frac{x}{\sqrt{6}}}\right)\\
&+\frac{\frac{-20}{6}(e^{\frac{x}{\sqrt{6}}})^3 + \frac{40}{6}(e^{\frac{x}{\sqrt{6}}})^4}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)}\\
&-2\Bigg(\frac{(-e^{\frac{x}{\sqrt{6}}})^1 + 2(e^{\frac{x}{\sqrt{6}}})^2}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)}\Bigg)\Big]\Bigg\}\frac{t^{3\alpha}}{\Gamma\,(3\alpha+1)} - \frac{25}{9}\frac{(e^{\frac{x}{\sqrt{6}}})^2}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^6}\\
&\frac{\Gamma\,(2\alpha+1)\,t^{3\alpha}}{\Gamma\,(3\alpha+1)\,\Gamma\,(\alpha+1)^2}.
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{\infty}u_n\,(x,t) = &\frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^2} + \frac{5}{3}\frac{e^{\frac{x}{\sqrt{6}}}}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3}\frac{t^{\alpha}}{\Gamma\,(\alpha+1)}\\
&+\frac{25}{18}\left(\frac{e^{\frac{x}{\sqrt{6}}}\left(-1+2e^{\frac{x}{\sqrt{6}}}\right)}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^4}\right)\frac{t^{2\alpha}}{\Gamma\,(2\alpha+1)}\\
&+\Bigg\{\frac{25}{18}\frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^5}\Big[\frac{8}{6}(e^{\frac{x}{\sqrt{6}}})^2 - 4(e^{\frac{x}{\sqrt{6}}})^3\\
&+\left(\frac{8}{6}(e^{\frac{x}{\sqrt{6}}})^2 - \frac{(e^{\frac{x}{\sqrt{6}}})}{6}\right)\left(1+e^{\frac{x}{\sqrt{6}}}\right)\\
&+\frac{4}{6}(e^{\frac{x}{\sqrt{6}}})^2 - \frac{16}{6}(e^{\frac{x}{\sqrt{6}}})^3 + \left(2(e^{\frac{x}{\sqrt{6}}})^2 - e^{\frac{x}{\sqrt{6}}}\right)\left(1+e^{\frac{x}{\sqrt{6}}}\right)\\
&+\frac{\frac{-20}{6}(e^{\frac{x}{\sqrt{6}}})^3 + \frac{40}{6}(e^{\frac{x}{\sqrt{6}}})^4}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)}\\
&-2\Bigg(\frac{(-e^{\frac{x}{\sqrt{6}}})^1 + 2(e^{\frac{x}{\sqrt{6}}})^2}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)}\Bigg)\Big]\Bigg\}\frac{t^{3\alpha}}{\Gamma\,(3\alpha+1)} - \frac{25}{9}\frac{(e^{\frac{x}{\sqrt{6}}})^2}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^6}\\
&\frac{\Gamma\,(2\alpha+1)\,t^{3\alpha}}{\Gamma\,(3\alpha+1)\,\Gamma\,(\alpha+1)^2} + \ldots
\end{aligned}$$

In particular when $\alpha = 1$, we get the solution in the form

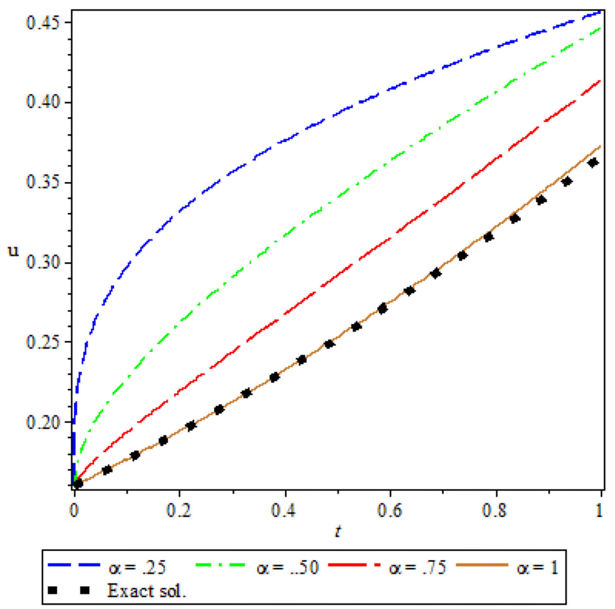

FIGURE 4 | Comparison of approx. sol. for different values of fractional order α and exact sol. at $\alpha = 1$, for Ex. 5.3.

$$u(x,t) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^2} + \frac{5}{3}\frac{e^{\frac{x}{\sqrt{6}}}}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^3}\frac{t}{1} + \frac{25}{18}\left(\frac{e^{\frac{x}{\sqrt{6}}}\left(-1+2e^{\frac{x}{\sqrt{6}}}\right)}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^4}\right)\frac{t^2}{2} + \left(\frac{125}{216}\frac{(e^{\frac{x}{\sqrt{6}}}(4\left(e^{\frac{x}{\sqrt{6}}}\right)^2-7e^{\frac{x}{\sqrt{6}}}+1)}{\left(1+e^{\frac{x}{\sqrt{6}}}\right)^5}\right)\frac{t^3}{3}+\dots.$$

Which converge to the exact solution of the classical Newell-Whitehead-Segel equation very fastly [25].

$$u(x,\ t) = \frac{1}{\left(1+e^{\frac{x}{\sqrt{6}}-\frac{5}{6}t}\right)^2}.$$

Figure 4 shows the comparison of third order approximate solution for different value of fractional order $\alpha = 0.25, 0.50, 0.75, 1$ and exact solution at $\alpha = 1$, for $x = 1$. It is observed from the **Figure 4** that there is a good agreement between exact solution and approximate solution at $\alpha = 1$. It is also noticed that solution depends on the time-fractional derivative. Accuracy and efficiency can be enhanced by increasing the number of iterations.

Ex. 5.4. We study the non-linear time-fractional Newell-Whitehead-Segel equation

$$u_t^\alpha = u_{xx} + u - u^4 = 0,\ \ t > 0,\ 0 < \alpha \le 1, \tag{5.7}$$

with initial condition

$$u(x,\ 0) = \left(\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}}\right)^{\frac{2}{3}}. \tag{5.8}$$

Applying the operator J_t^α on both side of above equation, we have

$$u(x,\ t) = \sum_{k=0}^{1-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{\Gamma(k+1)} + J_t^\alpha\{u_{xx}+u+A_n\}.$$

This gives the following recursive relation:

$$u_0(x,\ t) = \sum_{k=0}^{1-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{\Gamma(k+1)},$$

$$u_{n+1}(x,\ t) = J_t^\alpha\{(u_n)_{xx}+u_n+A_n\}, \qquad n \ge 0.$$

$$u_0 = \left(\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}}\right)^{\frac{2}{3}},$$

$$u_1 = \frac{7}{5}\left(\frac{e^{\frac{3x}{\sqrt{10}}}}{(1+e^{\frac{3x}{\sqrt{10}}})^{\frac{5}{3}}}\right)\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2 = \frac{49}{50}\left\{\frac{e^{\frac{3x}{\sqrt{10}}}\left(2e^{\frac{3x}{\sqrt{10}}}-3\right)}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)^{\frac{8}{3}}}\right\}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$\begin{aligned}u_3 = {}& \frac{1}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)^{\frac{11}{3}}}\Bigg\{\frac{3528}{500}\left(e^{\frac{3x}{\sqrt{10}}}\right)^2\left(1+e^{\frac{3x}{\sqrt{10}}}\right) - \frac{4704}{500}(e^{\frac{3x}{\sqrt{10}}})^3 \\ & -\frac{1323}{500}(e^{\frac{3x}{\sqrt{10}}})\left(1+e^{\frac{3x}{\sqrt{10}}}\right)+\frac{3528}{500}\left(e^{\frac{3x}{\sqrt{10}}}\right)^2-\frac{7056}{500}(e^{\frac{3x}{\sqrt{10}}})^3 \\ & +\frac{7056}{500}\left(e^{\frac{3x}{\sqrt{10}}}\right)^2+\frac{8624}{500}\frac{\left(e^{\frac{3x}{\sqrt{10}}}\right)^4}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)} \\ & -\frac{12936}{500}\frac{(e^{\frac{3x}{\sqrt{10}}})^3}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)} \\ & +\frac{49}{50}\left(2\left(e^{\frac{3x}{\sqrt{10}}}\right)^2-3e^{\frac{3x}{\sqrt{10}}}\right)\left(1+e^{\frac{3x}{\sqrt{10}}}\right) \\ & -\frac{\frac{196}{50}\left(2\left(e^{\frac{3x}{\sqrt{10}}}\right)^2-3e^{\frac{3x}{\sqrt{10}}}\right)}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)}\Bigg\}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ & -\frac{294}{25}\frac{(e^{\frac{3x}{\sqrt{10}}})^2}{(1+e^{\frac{3x}{\sqrt{10}}})^{\frac{14}{3}}}\frac{\Gamma(2\alpha+1)\,t^{3\alpha}}{\Gamma(3\alpha+1)\,\Gamma(\alpha+1)^2},\end{aligned}$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} u_n(x,t) = {} & \left(\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}}\right)^{\frac{2}{3}} + \frac{7}{5}\left(\frac{e^{\frac{3x}{\sqrt{10}}}}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)^{\frac{5}{3}}}\right) \\
& \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\
& + \frac{49}{50}\left(\frac{e^{\frac{3x}{\sqrt{10}}}\left(2e^{\frac{3x}{\sqrt{10}}}-3\right)}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)^{\frac{8}{3}}}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\
& + \frac{1}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)^{\frac{11}{3}}}\{\frac{3528}{500}\left(e^{\frac{3x}{\sqrt{10}}}\right)^{2}\left(1+e^{\frac{3x}{\sqrt{10}}}\right) \\
& - \frac{4704}{500}(e^{\frac{3x}{\sqrt{10}}})^{3} - \frac{1323}{500}(e^{\frac{3x}{\sqrt{10}}})\left(1+e^{\frac{3x}{\sqrt{10}}}\right) \\
& + \frac{3528}{500}\left(e^{\frac{3x}{\sqrt{10}}}\right)^{2} \\
& - \frac{7056}{500}(e^{\frac{3x}{\sqrt{10}}})^{3} + \frac{7056}{500}\left(e^{\frac{3x}{\sqrt{10}}}\right)^{2} + \frac{8624}{500}\frac{\left(e^{\frac{3x}{\sqrt{10}}}\right)^{4}}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)} \\
& - \frac{12936}{500}\frac{(e^{\frac{3x}{\sqrt{10}}})^{3}}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)} + \frac{49}{50}\left(2\left(e^{\frac{3x}{\sqrt{10}}}\right)^{2} - 3e^{\frac{3x}{\sqrt{10}}}\right)\left(1+e^{\frac{3x}{\sqrt{10}}}\right) \\
& - \frac{196}{50}\frac{\left(2\left(e^{\frac{3x}{\sqrt{10}}}\right)^{2} - 3e^{\frac{3x}{\sqrt{10}}}\right)}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\} \\
& - \frac{294}{25}\frac{\left(e^{\frac{3x}{\sqrt{10}}}\right)^{2}}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)^{\frac{14}{3}}}\frac{\Gamma(2\alpha+1)\,t^{3\alpha}}{\Gamma(3\alpha+1)\,\Gamma(\alpha+1)^{2}} \\
& + \ldots.
\end{aligned}
$$

Taking $\alpha = 1$, we get the solution in the form

$$
\begin{aligned}
u(x,t) = {} & \left(\frac{1}{1+e^{\frac{3x}{\sqrt{10}}}}\right)^{\frac{2}{3}} + \frac{7}{5}\left(\frac{e^{\frac{3x}{\sqrt{10}}}}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)^{\frac{5}{3}}}\right)\frac{t}{1} \\
& + \frac{49}{50}\left(\frac{e^{\frac{3x}{\sqrt{10}}}\left(2e^{\frac{3x}{\sqrt{10}}}-3\right)}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)^{\frac{8}{3}}}\right)\frac{t^{2}}{2}
\end{aligned}
$$

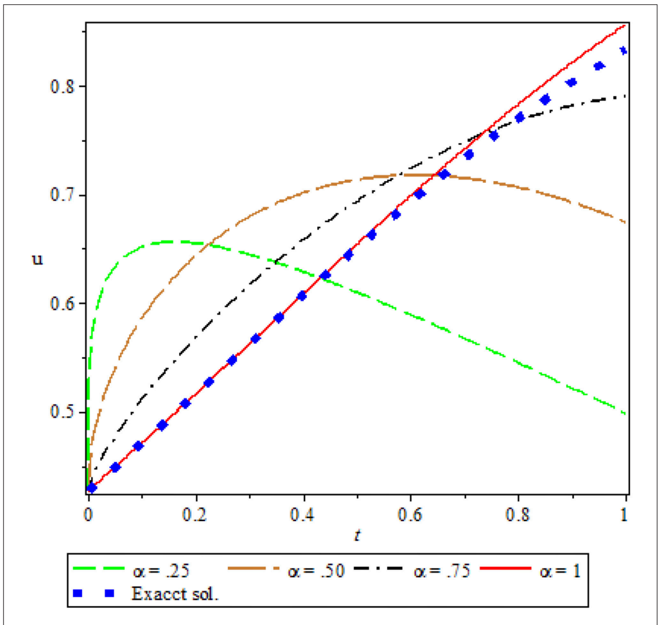

FIGURE 5 | Comparison of approx. sol. for different values of α and exact sol. at $\alpha = 1$, for Ex. 5.4.

$$
+ \frac{343}{1000}\left(\frac{\left(4\left(e^{\frac{3x}{\sqrt{10}}}\right)^{2} - 27e^{\frac{3x}{\sqrt{10}}} + 9\right)e^{\frac{3x}{\sqrt{10}}}}{\left(1+e^{\frac{3x}{\sqrt{10}}}\right)^{\frac{11}{3}}}\right)\frac{t^{3}}{3} + \ldots,
$$

Which converge to the exact solution of the classical Newell-Whitehead-Segel equation very fastly [25, 26].

$$
u(x,\ t) = [\frac{1}{2}\tanh\left(-\frac{3}{2\sqrt{10}}\left(x - \frac{7}{\sqrt{10}}t\right)\right) + \frac{1}{2}]^{\frac{2}{3}}.
$$

Figure 5 shows the comparison of third order approximate solution for different value of fractional order $\alpha = 0.25,\ 0.50,\ 0.75,\ 1$ and exact solution at $\alpha = 1$ for $x = 1$. It is observed from the **Figure 5** that there is a good agreement between exact solution and approximate solution at $\alpha = 1$. It is also noticed that solution depends on the time-fractional derivative. Accuracy and efficiency can be enhanced by increasing the number of iterations.

Ex. 5.5. We study the nonlinear time-fractional Newell-Whitehead-Segel equation of the form

$$
u_t^{\alpha} = u_{xx} + 3u - 4u^{4} = 0,\ \ t > 0,\ 0 < \alpha \le 1, \tag{5.9}
$$

with initial condition

$$
u(x,\ 0) = \sqrt{\frac{3}{4}\frac{e^{\sqrt{6}x}}{e^{\sqrt{6}x} + e^{\frac{\sqrt{6}}{2}x}}}. \tag{5.10}
$$

Applying the operator J_t^{α} on both side of above defined problem, we have

$$u\left(x,\ t\right)=\sum\nolimits_{k=0}^{1-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{\Gamma\left(k+1\right)}+J_t^{\alpha}\{u_{xx}+2u+A_n\}.$$

This gives the following recursive relation:

$$u_0\left(x,\ t\right)=\sum\nolimits_{k=0}^{1-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{\Gamma\left(k+1\right)},$$

$$u_{n+1}(x,\ t)=J_t^{\alpha}\{(u_n)_{xx}+3u_n+A_n\},\qquad n\geq 0.$$

$$u_0=\sqrt{\frac{3}{4}}\frac{e^{\sqrt{6}x}}{e^{\sqrt{6}x}+e^{\frac{\sqrt{6}}{2}x}},$$

$$u_1=\frac{9}{2}\sqrt{\frac{3}{4}}\frac{e^{\sqrt{6}x}e^{\frac{\sqrt{6}}{2}x}}{(e^{\sqrt{6}x}+e^{\frac{\sqrt{6}}{2}x})^2}\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2=\frac{81}{4}\sqrt{\frac{3}{4}}\Bigg(\frac{e^{\sqrt{6}x}e^{\frac{\sqrt{6}}{2}x}\left(-e^{\sqrt{6}x}+e^{\frac{\sqrt{6}}{2}x}\right)}{(e^{\sqrt{6}x}+e^{\frac{\sqrt{6}}{2}x})^3}\Bigg)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3=\frac{81}{4}\sqrt{\frac{3}{4}}\frac{1}{\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)^4}\{-\frac{3}{2}\left(e^{-\frac{\sqrt{6}}{2}x}\right)\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)$$
$$+\frac{9}{2}\left(e^{-\frac{\sqrt{6}}{2}x}\right)^2$$
$$+6\left(e^{-\frac{\sqrt{6}}{2}x}\right)^2\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)-9\left(e^{-\frac{\sqrt{6}}{2}x}\right)^3+9\left(e^{-\frac{\sqrt{6}}{2}x}\right)^2$$
$$-18\frac{\left(e^{-\frac{\sqrt{6}}{2}x}\right)^3}{\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)}$$
$$-\frac{27}{2}\left(e^{-\frac{\sqrt{6}}{2}x}\right)^3+18\frac{\left(e^{-\frac{\sqrt{6}}{2}x}\right)^4}{\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)}+3\Bigg(-e^{-\frac{\sqrt{6}}{2}x}$$
$$+\left(e^{-\frac{\sqrt{6}}{2}x}\right)^2\Bigg)\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)$$
$$-9\frac{e^{-\frac{\sqrt{6}}{2}x}(-1+e^{-\frac{\sqrt{6}}{2}x})}{(1+e^{-\frac{\sqrt{6}}{2}x})}\}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}-\frac{729}{4}$$
$$\sqrt{\frac{3}{4}}\frac{(-e^{-\frac{\sqrt{6}}{2}x})^2}{(1+e^{-\frac{\sqrt{6}}{2}x})^5}\frac{\Gamma\left(2\alpha+1\right)t^{3\alpha}}{\Gamma\left(3\alpha+1\right)\Gamma\left(\alpha+1\right)^2},$$

$$\sum_{n=0}^{\infty}u_n\left(x,t\right)=\sqrt{\frac{3}{4}}\frac{e^{\sqrt{6}x}}{e^{\sqrt{6}x}+e^{\frac{\sqrt{6}}{2}x}}$$
$$+\frac{9}{2}\sqrt{\frac{3}{4}}\frac{e^{\sqrt{6}x}e^{\frac{\sqrt{6}}{2}x}}{(e^{\sqrt{6}x}+e^{\frac{\sqrt{6}}{2}x})^2}\frac{t^{\alpha}}{\Gamma\left(\alpha+1\right)}$$
$$+\frac{81}{4}\sqrt{\frac{3}{4}}\frac{e^{\sqrt{6}x}e^{\frac{\sqrt{6}}{2}x}\left(-e^{\sqrt{6}x}+e^{\frac{\sqrt{6}}{2}x}\right)}{(e^{\sqrt{6}x}+e^{\frac{\sqrt{6}}{2}x})^3}\frac{t^{2\alpha}}{\Gamma\left(2\alpha+1\right)}$$
$$+\frac{81}{4}\sqrt{\frac{3}{4}}\frac{1}{\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)^4}\{-\frac{3}{2}\left(e^{-\frac{\sqrt{6}}{2}x}\right)$$
$$*\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)+\frac{9}{2}\left(e^{-\frac{\sqrt{6}}{2}x}\right)^2++6\left(e^{-\frac{\sqrt{6}}{2}x}\right)^2\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)$$
$$-9\left(e^{-\frac{\sqrt{6}}{2}x}\right)^3+9\left(e^{-\frac{\sqrt{6}}{2}x}\right)^2$$
$$-18\frac{\left(e^{-\frac{\sqrt{6}}{2}x}\right)^3}{\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)}-\frac{27}{2}\left(e^{-\frac{\sqrt{6}}{2}x}\right)^3+18\frac{\left(e^{-\frac{\sqrt{6}}{2}x}\right)^4}{\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)}$$
$$+3\left(-e^{-\frac{\sqrt{6}}{2}x}+\left(e^{-\frac{\sqrt{6}}{2}x}\right)^2\right)$$
$$*\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)-9\frac{e^{-\frac{\sqrt{6}}{2}x}\left(-1+e^{-\frac{\sqrt{6}}{2}x}\right)}{\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)}\}\frac{t^{3\alpha}}{\Gamma\left(3\alpha+1\right)}$$
$$-\frac{729}{4}\sqrt{\frac{3}{4}}\frac{\left(-e^{-\frac{\sqrt{6}}{2}x}\right)^2}{\left(1+e^{-\frac{\sqrt{6}}{2}x}\right)^5}\frac{\Gamma\left(2\alpha+1\right)t^{3\alpha}}{\Gamma\left(3\alpha+1\right)\Gamma\left(\alpha+1\right)^2}+\ldots$$

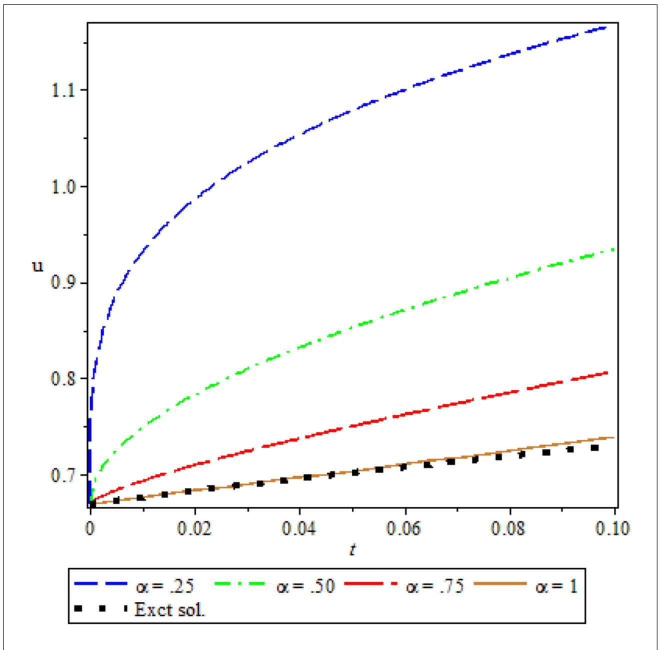

FIGURE 6 | Comparison of approx. sol. for different values of fractional order α and exact sol. at $\alpha = 1$, for Ex. 5.5.

Taking $\alpha = 1$, we get the solution in the form

$$u(x,t) = \sqrt{\frac{3}{4}}\frac{e^{\sqrt{6}x}}{e^{\sqrt{6}x}+e^{\frac{\sqrt{6}}{2}x}} + \frac{9}{2}\sqrt{\frac{3}{4}}\frac{e^{\sqrt{6}x}e^{\frac{\sqrt{6}}{2}x}}{\left(e^{\sqrt{6}x}+e^{\frac{\sqrt{6}}{2}x}\right)^2}\frac{t}{1}$$
$$+\frac{81}{4}\sqrt{\frac{3}{4}}\frac{e^{\sqrt{6}x}e^{\frac{\sqrt{6}}{2}x}\left(-e^{\sqrt{6}x}+e^{\frac{\sqrt{6}}{2}x}\right)}{\left(e^{\sqrt{6}x}+e^{\frac{\sqrt{6}}{2}x}\right)^3}\frac{t^2}{2}$$
$$+\frac{243}{16}\sqrt{\frac{3}{4}}\frac{e^{\sqrt{6}x}e^{\frac{\sqrt{6}}{2}x}\left(-4e^{\sqrt{6}x}e^{\frac{\sqrt{6}}{2}x}+\left(e^{\sqrt{6}x}\right)^2+\left(e^{\frac{\sqrt{6}}{2}x}\right)^2\right)}{\left(e^{\sqrt{6}x}+e^{\frac{\sqrt{6}}{2}x}\right)^4}\frac{t^3}{3}$$
$$+\ldots.$$

which converge to the exact solution of the classical Newell-Whitehead-Segel equation very fastly [25, 26].

$$u(x,\ t) = \sqrt{\frac{3}{4}}\frac{e^{\sqrt{6}x}}{e^{\sqrt{6}x}+e^{\left(\frac{\sqrt{6}}{2}x-\frac{9}{2}t\right)}}.$$

Figure 6 shows the comparison of third order approximate solution for different value of fractional order $\alpha = 0.25,\ 0.50,\ 0.75,\ 1$ and exact solution at $\alpha = 1$, for $x = 1$. It is observed from the **Figure 6** that there is a good agreement between exact solution and approximate solution at $\alpha = 1$. It is also noticed that solution depends on the time-fractional derivative. Accuracy and efficiency can be enhanced by increasing the number of iterations.

CONCLUSION

In this article, we have successfully applied the ADM to obtain the approximate analytic solutions of fractional model of Newell-Whitehead-Segel equation. The plotted graph and numerical result shows the accuracy of proposed method. We observed an excellent agreement between ADM and the exact solution. The results reveal that ADM is an efficient and computationally very attractive approach to investigate non-linear fractional model. Therefore, ADM can be further applied to solve various types of linear and non-linear fractional model arising in the field of science and engineering.

AUTHOR CONTRIBUTIONS

AP and VV designed the study, collected the data, performed the analysis, and wrote the manuscript.

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